Volume 19, Number 1 July 2015
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

An international publication of Eudoxus Press, LLC
(twelve times annually)

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A new consensus protocol of the multi-agent systems

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Abstract

In consensus protocols of multi-agent system, we can often find state differences between some agents and a reference agent. However, we find that taking the relative states value may not be best. Conversely, using the differences between some agents states and several times state of the reference agent can achieve quickly convergence by choosing a appropriate parameter r. Therefore, in this paper a new consensus protocol taking r times state of reference agent is proposed. Based on this idea, two convergence protocols of single-integrator kinematics are considered, and some sufficient conditions on consensus protocols of double-integrator dynamics are derived. Finally, numerical examples illustrate these theoretical results.

Keywords: Consensus protocol; Multi-agent system; Graph; Continuous-time system; Time-delay

1 Introduction

Recently, multi-agent systems have received significant attention due to their potential impacts in numerous civilian, homeland security and military applications, etc. Consensus plays an important role in achieving distributed coordination. The basic idea of consensus is that a team of vehicles reaches an agreement on a common

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value by negotiating with their neighbors. Consensus algorithms are studied for both single-integrator kinematics [1–5] and high-order-integrator dynamics [6–10].

Formal study of consensus problems in groups of experts originated in management science and statistics in 1960s. Distributed computation over networks has a tradition in systems and control theory starting with the pioneering work of Borkar et. al. Vicsek et. al [11] provided a formal analysis of emergence of alignment in the simplified model of flocking, which has an important influence on the developing of consensus theories of multi-agent systems. On the study of consensus of continnuous-time systems, classical model of consensus is provided by Olfati-Saber and Murray [12], which can be described by the following linear dynamic system:

$$\dot{x}_i(t) = u_i(t), \tag{1}$$

the consensus protocol is given as

$$u_i(t) = \sum_{j \in \mathcal{N}_i(t)} a_{ij}(t)(x_j(t) - x_i(t)), \tag{2}$$

and the model with time-delays is given as

$$u_i(t) = \sum_{j \in \mathcal{N}_i(t)} a_{ij}(t)(x_j(t-\tau_{ij}) - x_i(t-\tau_{ij})). \tag{3}$$

However, in the movement of agents for protocol (2) and (3), in order to achieve quick convergence, taking the relative state may not be the best, conversely, taking several times state values of reference agents may quickly achieve convergence (in Section 4, this phenomena can be verified by simulation experiments). In this paper, on basis of (2) and (3), we present the following protocol:

$$u_i(t) = \sum_{j \in \mathcal{N}_i(t)} a_{ij}(t)(x_j(t) - r x_i(t)),\ r > 1, \tag{4}$$

and the corresponding time-delayed protocol is given as

$$u_i(t) = \sum_{j \in \mathcal{N}_i(t)} a_{ij}(t)(x_j(t-\tau) - r x_i(t-\tau)),\ r > 1, \tag{5}$$

respectively, where $x_i(t) = x_i(0)$, for $t \in [-\tau, 0),\ i = 1, \cdots, n$. Next we consider double-integrator dynamics consensus protocols based on (4) and (5), respectively.

The paper is organized as follows. In Section 2, we introduce basic concepts and preliminary results. In Section 3, we firstly derive sufficient conditions on protocols for two single-integrator kinematics, then we obtain some sufficient conditions on consensus protocols for double-integrator dynamics. In Section 4, numerical examples are presented to illustrate our theoretical results. Conclusions are drawn in Section 5.

2 Preliminaries

Suppose that the multi-agent system under consideration consists of n agents. If there exists an available information channel from the agent j to the agent i, then agent j is a neighbor of agent i. The set of all neighbors of agent i is denoted by $\mathcal{N}_i(t)$. A directed graph (digraph) $G = (V, E)$ of order n consists of a set of nodes $V = \{v_1, \ldots, v_n\}$ and a set of edges $E = V \times V$. (i, j) is an edge of G if and only if $(i, j) \in E$. Each agent is regarded as a node in a directed graph G. Each available information channel from the agent i to the agent j corresponds to an edge $(i, j) \in E$. The weighted adjacency matrix $A \in \mathbb{R}^{n\times n}$ is defined as $a_{ii} = 0$ and $a_{ij} \geq 0$. $a_{ij} > 0$ if only and if $(j, i) \in E$. In this paper, we only consider that the weight $a_{ij}, i, j = 1, \cdots, n$, of digraph of multi-agent systems are unchanged. The Laplacian matrix $L \in \mathbb{R}^{n\times n}$ is defined as

$$l_{ii} = \sum_{j\neq i} a_{ij}, \quad l_{ij} = -a_{ij}, \text{ for } i \neq j.$$

A graph with the property that $(i, j) \in E$ implies $(j, i) \in E$ is said to be undirected. Moreover, matrix L is symmetric if an undirected graph has symmetric weights, i.e., $a_{ij} = a_{ji}$. A directed path is a sequence of edges in a directed graph with the form $(v_1, v_2), (v_2, v_3), \cdots$, where $v_i \in V$. A directed graph has a directed spanning tree if there exists at least one node that has a directed path to all other nodes.

Lemma 1 [12, 13].

(i) All the eigenvalues of Laplacian matrix L have nonnegative real parts;

(ii) Zero is an eigenvalue of L with 1_n (where 1_n is the $n \times 1$ column vector of all ones) as the corresponding right eigenvector. Furthermore, zero is a simple eigenvalue of L if and only if graph G has a directed spanning tree.

Lemma 2. *If $L \in \mathbb{R}^{n\times n}$ is a Laplacian matrix with the directed graph G that has a directed spanning tree, and $D \in \mathbb{R}^{n\times n}$ is a diagonal matrix with $d_i < 0$ $(i = 1, \cdots, n)$, then all the eigenvalues of $-L + D$ have negative real parts.*

Proof. Notice that the directed graph has a directed spanning tree, it follows from Lemma 1 that L has a simple zero eigenvalue and all other eigenvalues have positive real parts. With the definition of L, we know that $l_{ii} = \sum_{j\neq i} a_{ij}$. Let $\lambda_i, i = 1, \cdots, n$, be eigenvalues of $-L + D$. By Gerschgorin disk theorem,

$$|\lambda_i + l_{ii} - d_i| < l_{ii}, \quad i = 1, \cdots, n. \tag{6}$$

It follows from (6) that all the eigenvalues of $-L + D$ lie on a circle with center at $(d_i - l_{ii}, 0)$ and radius l_{ii}. Therefore, $\lambda_i, i = 1, \cdots, n$, have negative real parts. □

Lemma 3. *Let $A \in \mathbb{R}^{n\times n}$ with eigenvalue γ_i and associated right and left eigenvectors s_i and q_i, respectively. Let $B = \begin{pmatrix} 0_{n\times n} & I_n \\ A & A-\alpha I_n \end{pmatrix}$ $(\alpha > 0)$. Then the eigenvalues of B are given by*

$$\xi_{2i-1} = \frac{-\alpha - \gamma_i + \sqrt{(\gamma_i - \alpha)^2 + 4\gamma_i}}{2}$$

with associated right and left eigenvectors $(s_i^T, \xi_{2i-1}s_i^T)^T$ and $(\frac{\gamma_i}{\xi_{2i-1}}q_i^T, q_i^T)^T$, respectively, and

$$\xi_{2i} = \frac{-\alpha - \gamma_i - \sqrt{(\gamma_i - \alpha)^2 + 4\gamma_i}}{2}$$

with associated right and left eigenvectors $(s_i^T, \xi_{2i}s_i^T)^T$ and $(\frac{\gamma_i}{\xi_{2i}}q_i^T, q_i^T)^T$, respectively, $i = 1, \cdots, n$; $(\alpha q_i^T, q_i^T)^T$ is a left eigenvector of zero eigenvalue of B.

Proof. Suppose that ξ is an eigenvalue of B with an associated right eigenvector $(f^T, g^T)^T$, where $f, g \in \mathbb{C}^n$. It follows that

$$\begin{pmatrix} 0_{n\times n} & I_n \\ A & A-\alpha I_n \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix} = \xi\begin{pmatrix} f \\ g \end{pmatrix},$$

which implies

$$g = \xi f \text{ and } Af + Ag - \alpha g = \xi g.$$

Thus

$$(1+\xi)Af = (\xi^2 - \alpha\xi)f.$$

Let $f = s_i$. By $As_i = \gamma_i s_i$, we get $\gamma_i(1+\xi) = \xi^2 - \alpha\xi$. That is, each eigenvalue of A, γ_i, corresponds to two eigenvalues of B, denoted by

$$\xi_{2i-1,2i} = \frac{-\alpha - \gamma_i \pm \sqrt{(\gamma_i - \alpha)^2 + 4\gamma_i}}{2}.$$

Because $g = \xi f$, it follows that the right eigenvectors associated with ξ_{2i-1} and ξ_{2i} are, respectively, $(s_i^T, \xi_{2i-1}s_i^T)^T$ and $(s_i^T, \xi_{2i}s_i^T)^T$. A similar analysis can be used to find the left eigenvectors of B associated with ξ_{2i-1} and ξ_{2i}. $\square$

Lemma 4 [14]. *Let $\rho_\pm = \frac{-\alpha+\gamma u \pm\sqrt{(\gamma u-\alpha)^2+4u}}{2}$, where $\rho, u \in \mathbb{C}$. If $\alpha > 0, Re(u) < 0$ and $\gamma > \sqrt{\frac{2}{|u||\cos(\frac{\pi}{2} - \arctan\frac{-Re(u)}{Im(u)})|}}$, then $Re(\rho_\pm) < 0$, where $Re(\cdot)$ represents the real part of a number.*

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3 Consensus of continuous-time systems

In this section, we give our main results. In Subsection 3.1, two convergence protocols of multi-agent systems with single-integrator kinematics are considered. In Subsection 3.2, sufficient conditions on consensus protocols of multi-agent systems with double-integrator dynamics are derived.

3.1 Convergence protocols of multi-agent systems with single-integrator kinematics

Firstly, we consider the protocol (4). The result is given as follows:

Theorem 1. *Assume that there are n agents with $n \geq 2$. For the protocol (4), the state $x(t)$ of agents converge exponentially to zero if the directed graph G has a directed spanning tree.*

Proof. From (1) and (4), we can obtain

$$\dot{x}_i(t) = \sum_{j \in \mathcal{N}_i(t)} a_{ij}(t)(x_j(t) - r x_i(t)), \quad r > 1. \tag{7}$$

We take the vector form of the above equation:

$$\dot{x}(t) = -Lx(t) - (r-1)\mathrm{diag}(l_{11}, \cdots, l_{nn})x(t), \tag{8}$$

(8) can be rewritten as

$$\dot{x}(t) = (-L + D)x(t), \tag{9}$$

where $D = -(r-1)\mathrm{diag}(l_{11}, \cdots, l_{nn})$. By Lemma 2, all the eigenvalues of $-L + D$ have negative real parts. Therefore, the system is stable. Then we get

$$x(t) = e^{(-L+D)t}x(0).$$

So $x(t)$ converges exponentially to zero, and the convergence is asymptotically achieved. □

Below we consider the following equation with time-delay:

$$\dot{x}_i(t) = \sum_{j \in \mathcal{N}_i(t)} a_{ij}(t)(x_j(t-\tau) - r x_i(t-\tau)), \quad r > 1. \tag{10}$$

Furthermore, (10) can be rewritten as

$$\dot{x}(t) = (-L + D)x(t - \tau), \tag{11}$$

where $D = -(r-1)\mathrm{diag}(l_{11}, \cdots, l_{nn})$. We obtain the following result:

Theorem 2. *Assume that there are n agents with $n \geq 2$ and the directed graph G has a directed spanning tree. For the system (11), the state $x(t)$ of agents asymptotically converge to zero if*

$$\tau < \min_i \frac{\theta_i - \frac{\pi}{2}}{|\lambda_i|}, \quad i = 1, \cdots, n, \tag{12}$$

where λ_i is the eigenvalue of $-L + D$ and $\theta_i \in (\frac{\pi}{2}, \frac{3\pi}{2})$ denotes the phase of λ_i.

Proof. Let $\widetilde{L} = L - D$. By Lemma 2, the eigenvalues of $-\widetilde{L}$ have negative real parts, so we can assume that $\theta_i \in (\frac{\pi}{2}, \frac{3\pi}{2})$. Moreover, (11) can be rewritten as

$$\dot{x}(t) = -\widetilde{L}x(t-\tau). \tag{13}$$

Taking the Laplace transform of the system (13), and let $x(0) = 0$, we get

$$sX(s) = -e^{-s\tau}\widetilde{L}X(s), \tag{14}$$

where $X(s)$ is the Laplace transform of $x(t)$. Thus, we get the characteristic equation about $X(s)$ as follows:

$$\det(sI + e^{-s\tau}\widetilde{L}) = 0. \tag{15}$$

Moreover, (15) is equivalent to

$$\prod_{i=1}^{n}(s - e^{-s\tau}\lambda_i) = 0, \tag{16}$$

where λ_i is the eigenvalue of $-L + D, i = 1, \cdots, n$. Since all the eigenvalues of $-\widetilde{L}$ have negative real parts, obviously, $s = 0$ is not a root of the above equation. Thus, we consider the roots of the following equation

$$1 - \frac{e^{-s\tau}\lambda_i}{s} = 0. \tag{17}$$

Based on the Nyquist stability criterion, the roots of (17) lie on the open left half complex plane if and only if the Nyquist curve $-\frac{e^{-i\omega\tau}\lambda_i}{j\omega}$ does not enclose the point $(-1, j0)$ for $\omega \in \mathbb{R}$.

When $\omega \in (0, \infty)$, $|-\frac{e^{-i\omega\tau}\lambda_i}{j\omega}| = \frac{|\lambda_i|}{\omega}$ and $\arg(-\frac{e^{-i\omega\tau}\lambda_i}{j\omega}) = \theta_i - \omega\tau + \frac{\pi}{2}$ are all monotonously decreasing, where $\arg(\cdot)$ denotes the phase, and $\theta_i \in (\frac{\pi}{2}, \frac{3\pi}{2})$ denotes the phase of λ_i. Since $-\frac{e^{-i\omega\tau}\lambda_i}{j\omega}$ crosses the negative real axis for the first time at

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$\arg(-\frac{e^{-i\omega\tau}\lambda_i}{j\omega_0}) = \theta_i - \omega_0\tau + \frac{\pi}{2} = \pi$, the roots of (17) all lie on the open left half complex plane if and only if $\frac{|\lambda_i|}{\omega_0} < 1$. So we get

$$\tau < \frac{\theta_i - \frac{\pi}{2}}{|\lambda_i|}.$$

Similarly, when $\omega \in (-\infty, 0)$, we get

$$\tau < \frac{\frac{5\pi}{2} - \theta_i}{|\lambda_i|}.$$

By $\theta_i \in (\frac{\pi}{2}, \frac{3\pi}{2})$ and the above analysis, it is clear that

$$\tau < \min_i \frac{\theta_i - \frac{\pi}{2}}{|\lambda_i|}.$$

Therefore, the system (11) asymptotically converge to zero if (12) is satisfied. So the proof is completed. □

3.2 Consensus protocols of multi-agent systems with double-integrator dynamics

In this subsection, for the second-order multi-agent system, based on (4) and (5), we consider the following dynamics equations:

$$\dot{x}_i(t) = v_i(t), \tag{18}$$

$$\dot{v}_i(t) = u_i(t), \tag{19}$$

where $x_i(t) \in \mathbb{R}, v_i(t) \in \mathbb{R}$ and $u_i(t) \in \mathbb{R}$ are the position, velocity and acceleration. We consider the following dynamical consensus protocols:

$$u_i(t) = \sum_{j\in\mathcal{N}_i(t)} a_{ij}(t)(x_j(t) - x_i(t)) + \sum_{j\in\mathcal{N}_i(t)} a_{ij}(t)(v_j(t) - rv_i(t)), r > 1, \tag{20}$$

$$u_i(t) = \sum_{j\in\mathcal{N}_i(t)} a_{ij}(t)(x_j(t-\tau) - x_i(t-\tau)) + \sum_{j\in\mathcal{N}_i(t)} a_{ij}(t)(v_j(t-\tau) - rv_i(t-\tau)), r > 1, \tag{21}$$

where $x_i(t) = x_i(0), v_i(t) = 0, t \in [-\tau, 0),\ i = 1, \cdots, n$.

Furthermore, (18), (19) and (20) can be rewritten as

$$\dot{x}(t) = v(t), \tag{22}$$

$$\dot{v}(t) = -Lx(t) + (-L + D)v(t), \tag{23}$$

where $D = -(r-1)\mathrm{diag}(l_{11}, \cdots, l_{nn})$. For the consensus protocol (20), we obtain the following theorem:

Theorem 3. *Assume that there are n agents with $n \geq 2$ and the directed graph G has a directed spanning tree. For the consensus protocol (20), consensus is achieved if $l_{11} = l_{22} = \cdots = l_{nn}$ and*

$$\max_i \sqrt{\frac{2}{|\lambda_i|| \cos(\frac{\pi}{2} - \arctan \frac{-Re(\lambda_i)}{Im(\lambda_i)})|}} < 1 \text{ (when } \lambda_i \neq 0),\ i = 1, \cdots, n,$$

where l_{ii} is the diagonal element of Laplacian matrix L and λ_i is eigenvalue of $-L$. The state convergence values of agents are

$$x_i(t) = (r-1)l_{11}w_1^T x(0) + w_1^T v(0),\ v_i(t) = 0,\ t \to \infty,$$

where w_1 is a left eigenvector of zero eigenvalue of L, and $(r-1)l_{11}w_1^T 1_n = 1$.

Proof. Equations (22) and (23) can be rewritten in matrix form as

$$\begin{pmatrix} \dot{x(t)} \\ \dot{v(t)} \end{pmatrix} = \begin{pmatrix} 0_{n\times n} & I_n \\ -L & -L-(r-1)l_{11}I_n \end{pmatrix} \begin{pmatrix} x(t) \\ v(t) \end{pmatrix}. \tag{24}$$

Let $\Gamma = \begin{pmatrix} 0_{n\times n} & I_n \\ -L & -L-(r-1)l_{11}I_n \end{pmatrix}$, $\alpha = (r-1)l_{11}$ and λ_i be eigenvalue of $-L$ with associated right and left eigenvectors q_i and w_i, respectively, $i = 1, \cdots, n$. We can assume without loss of generality that $\lambda_1 = 0$ and $\lambda_i,\ i = 2, \cdots, n$, have negative real parts. Note from Lemma 3 that each eigenvalue λ_i of L corresponds to two eigenvalues of Γ given by $\xi_{2i-1,2i} = \frac{-\alpha-\lambda_i \pm \sqrt{(\lambda_i-\alpha)^2+4\lambda_i}}{2}$. Moreover, we get $\xi_1 = 0$ and $\xi_2 = (1-r)l_{11}$. By Lemma 4, if

$$\sqrt{\frac{2}{|\lambda_i|| \cos(\frac{\pi}{2} - \arctan \frac{-Re(\lambda_i)}{Im(\lambda_i)})|}} < 1,$$

then $\xi_{2i-1,2i}$ have negative real parts with associated right and left eigenvectors given by

$$(q_i^T, \xi_{2i-1}q_i^T)^T,\ (q_i^T, \xi_{2i}q_i^T)^T,\ (\frac{\lambda_i}{\xi_{2i-1}}w_i^T, w_i^T)^T \text{ and } (\frac{\lambda_i}{\xi_{2i}}w_i^T, w_i^T)^T,$$

respectively, $i = 2, \cdots, n$.

Moreover, Γ can be written in Jordan canonical form as SJS^{-1}, where the columns of S, denoted by $s_k,\ k = 1, \cdots, 2n$, can be chosen to be the right eigenvectors or generalized right eigenvectors of Γ associated with eigenvalue ξ_k, h_k^T, $k =$

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$1, \cdots, 2n$, which are the rows of S^{-1} can be chosen to be the left eigenvectors or generalized left eigenvectors of ξ_k satisfing $h_k^T s_k = 1,\ h_k^T s_l = 0\ (k \neq l)$, and J is the Jordan block diagonal matrix with ξ_k being the diagonal entries. Thus, we can choose $s_1 = (1_n^T, 0_n^T)^T$ and $h_1 = ((r-1)l_{11}\omega_1^T, \omega_1^T)^T$. Moreover, we can get

$$\begin{aligned}
\lim_{t\to\infty}\begin{pmatrix} x(t) \\ v(t) \end{pmatrix} &= e^{\Gamma t}\begin{pmatrix} x(0) \\ v(0) \end{pmatrix} \\
&= \begin{pmatrix} s_1 & \cdots & s_{2n} \end{pmatrix}\begin{pmatrix} 1 & 0_{1\times n} \\ 0_{n\times 1} & 0_{n\times n} \end{pmatrix}\begin{pmatrix} h_1^T \\ \vdots \\ h_{2n}^T \end{pmatrix}\begin{pmatrix} x(0) \\ v(0) \end{pmatrix} \\
&= \begin{pmatrix} 1_n \\ 0 \end{pmatrix}\begin{pmatrix} (r-1)l_{11}w_1^T & w_1^T \end{pmatrix}\begin{pmatrix} x(0) \\ v(0) \end{pmatrix},
\end{aligned}$$

which implies that $x_i(t) \to (r-1)l_{11}w_1^T x(0) + w_1^T v(0)$ and $v_i(t) \to 0$ as $t \to \infty$. So the theorem is proved. □

Remark 1. Our consensus protocol (20) has the following advantages compared with the classic one which is consensus protocol (20) at $r = 1$:

1) In (20) , if $r \geq 1$, then our consensus protocol can include the classic one as a special case;

2) Our consensus protocol has quicker convergence speed;

3) Our consensus protocol has wider convergence range.

The advantages of 2) and 3) can be seen in Example 1 and 3 of Section 4, respectively.

Recently many new protocols [4,5,7] are proposed on basis of the protocols in [12]. Though our protocol is got by modifying the protocols in [12], these new protocols can also be modified according to our method to improve convergence speed. This will be further considered in our future work.

Theorem 4. *Assume that there are n agents with $n \geq 2$, and the undirected graph G is connected and symmetric. For the protocol (21), consensus is achieved if $l_{11} = l_{22} = \cdots = l_{nn}$,*

$$\tau < \frac{\pi}{2(r-1)l_{11}} \quad \text{and} \quad \max_i \frac{\sqrt{\lambda_i^2 + \omega_0^2(\lambda_i + (r-1)l_{11})^2}}{\omega_0^2} < 1\ (\lambda_i \neq 0,\ i =, \cdots, n),$$

where $\omega_0 = \frac{\arctan(\frac{\omega_0(\lambda_i + (r-1)l_{11})}{\lambda_i})}{\tau}$, l_{ii} and λ_i are the diagonal element and eigenvalue of Laplacian matrix L, respectively. The state convergence values of agents are

$$x_i(t) = (r-1)l_{11}w_1^T x(0) + w_1^T v(0),\ v_i(t) = 0,\ t \to \infty,$$

where w_1 is a left eigenvector of zero eigenvalue of L, and $(r-1)l_{11}w_1^T 1_n = 1$.

Proof. (18), (19) and (21) can be rewritten as

$$\dot{x}(t) = v(t), \tag{25}$$

$$\dot{v}(t) = -Lx(t-\tau) + (-L + (1-r)l_{11}I_n)v(t-\tau), r > 1. \tag{26}$$

Taking the Laplace transform of the equation (25) and (26), we get

$$sX(s) = V(s), \tag{27}$$

$$sV(s) = -LX(s)e^{-s\tau} + (-L + (1-r)l_{11}I_n)V(s)e^{-s\tau}, \tag{28}$$

where $X(s)$ and $V(s)$ are the Laplace transforms of $x(t)$ and $v(t)$, respectively, and let $x(0) = v(0) = 0$. The above equation can be rewritten as

$$s^2X(s) = -LX(s)e^{-s\tau} + (-L + (1-r)l_{11}I_n)sX(s)e^{-s\tau}. \tag{29}$$

Thus, we get the characteristic equation about $X(s)$ as follows:

$$\det(s^2I_n + Le^{-s\tau} - (-L + (1-r)l_{11}I_n)se^{-s\tau}) = 0. \tag{30}$$

By the undirected graph G is connected and symmetric, we get L is positive semidefinite. Without loss of generality, we can assume that the eigenvalues of L, $\lambda_1 = 0$ and $\lambda_i > 0,\ i = 2, \cdots, n$. Thus, (30) equals

$$\prod_{i=1}^{n}(s^2 + e^{-s\tau}\lambda_i - (-\lambda_i + (1-r)l_{11})se^{-s\tau}) = 0. \tag{31}$$

Furthermore, we consider the roots of the following equation

$$s^2 + e^{-s\tau}\lambda_i - (-\lambda_i + (1-r)l_{11})se^{-s\tau} = 0,\ i = 1, \cdots, n. \tag{32}$$

When $\lambda_1 = 0$, (32) becomes

$$s^2 - (1-r)l_{11}se^{-s\tau} = 0. \tag{33}$$

Obviously, $s = 0$ is a root of the above equation. When $s \neq 0$, we consider the roots of the following equation

$$1 - \frac{(1-r)l_{11}e^{-s\tau}}{s} = 0. \tag{34}$$

By Nyquist stability criterion, the roots of (34) lie on the open left half complex plane if and only if the Nyquist curve $-\frac{(1-r)l_{11}e^{-j\omega\tau}}{j\omega}$ does not enclose the point $(-1, j0)$ for $\omega \in \mathbb{R}$. Similar to the proof of Theorem 2, we get

$$\tau < \frac{\pi}{2(r-1)l_{11}}.$$

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When $\lambda_i \neq 0$, we consider the roots of the following equation

$$1 + \frac{\lambda_i + s(\lambda_i + (r-1)l_{11})}{s^2} e^{-s\tau} = 0. \tag{35}$$

Similarly, the roots of (35) lie on the open left half complex plane if and only if the Nyquist curve $z = \frac{\lambda_i + j\omega(\lambda_i + (r-1)l_{11})}{-\omega^2} e^{-j\omega\tau}$ does not enclose the point $(-1, j0)$ for $\omega \in \mathbb{R}$. We can get

$$|z| = \frac{\sqrt{\lambda_i^2 + \omega^2(\lambda_i + (r-1)l_{11})^2}}{\omega^2},$$

$$\arg(z) = -\omega\tau + \pi + \arctan \frac{\omega(\lambda_i + (r-1)l_{11})}{\lambda_i}.$$

Obviously, $|z|$ is monotonously decreasing for $\omega \in (\omega_0, \infty)$. Since curve z crosses the real axis for the first time at

$$\omega_0 = \frac{\arctan(\frac{\omega_0(\lambda_i + (r-1)l_{11})}{\lambda_i})}{\tau}$$

and we can find $\arg(z)$ is monotonous function, the roots of (35) lie on the open left half complex plane if we let

$$\max_i \frac{\sqrt{\lambda_i^2 + \omega_0^2(\lambda_i + (r-1)l_{11})^2}}{\omega_0^2} < 1.$$

According to the above analysis, the roots of (30) all lie on the open left half complex plane except for a root at $s = 0$. Therefore, $x(t)$ and $v(t)$ of the system converge to a steady state, and consensus is achieved.

Equations (25) and (26) can be rewritten as

$$\begin{pmatrix} \dot{x}(t) \\ \dot{v}(t) \end{pmatrix} = \begin{pmatrix} 0_{n\times n} & I_n \\ -L & -L - (r-1)l_{11}I_n \end{pmatrix} \begin{pmatrix} x(t-\tau) \\ v(t-\tau) \end{pmatrix}. \tag{36}$$

Let

$$\Gamma = \begin{pmatrix} 0_{n\times n} & I_n \\ -L & -L - (r-1)l_{11}I_n \end{pmatrix}, z(t) = \begin{pmatrix} x(t) \\ v(t) \end{pmatrix}.$$

Taking the Laplace transform of the equation (36)

$$sZ(s) - z(0) = e^{-s\tau}\Gamma Z(s), \tag{37}$$

where $Z(s)$ is the Laplace transform of $z(t)$. Moreover, by the Proof of Theorem 3, Γ can be written in Jordan canonical form as $SJS^{-1} = S \begin{pmatrix} 0 & 0_{1\times n} \\ 0_{n\times 1} & \Gamma_0 \end{pmatrix} S^{-1}$,

where S is the same as one in the Proof of Theorem 3, and the eigenvalues of Γ_0 only have negative real parts. Thus

$$\begin{aligned}
\lim_{t\longrightarrow\infty} z(t) &= \lim_{s\longrightarrow 0_+} sZ(s) = \lim_{s\longrightarrow 0_+} s(sI_n - e^{-s\tau}\Gamma)^{-1}z(0) \\
&= \lim_{s\longrightarrow 0_+} S(I_n - \begin{pmatrix} 0 & 0_{1\times n} \\ 0_{n\times 1} & \frac{e^{-s\tau}}{s}\Gamma_0 \end{pmatrix})^{-1}S^{-1}z(0) \\
&= \lim_{s\longrightarrow 0_+} S\begin{pmatrix} 1 & 0_{1\times n} \\ 0_{n\times 1} & (I_{n-1} - \frac{e^{-s\tau}}{s}\Gamma_0)^{-1} \end{pmatrix} S^{-1}z(0) \\
&= \begin{pmatrix} s_1 & \cdots & s_{2n} \end{pmatrix}\begin{pmatrix} 1 & 0_{1\times n} \\ 0_{n\times 1} & 0_{n\times n} \end{pmatrix}\begin{pmatrix} h_1^T \\ \vdots \\ h_{2n}^T \end{pmatrix} z(0) \\
&= \begin{pmatrix} 1_n \\ 0 \end{pmatrix}\begin{pmatrix} (r-1)l_{11}w_1^T & w_1^T \end{pmatrix}\begin{pmatrix} x(0) \\ v(0) \end{pmatrix},
\end{aligned}$$

which implies that $x_i(t) \to (r-1)l_{11}w_1^T x(0) + w_1^T v(0)$ and $v_i(t) \to 0$ as $t \to \infty$. So the theorem is proved.□

4 Simulation

In this section, several simulation results are presented to illustrate the proposed protocols introduced in Section 3.

Example 1. We consider a system with three agents under the protocol (4) and (20). The corresponding Laplacian matrix and initial value are chosen as

$$L = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, x(0) = \begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix}, \text{ and } \mathrm{v}(0) = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}.$$

Moreover, the values of parameter r are shown in Figure 1. Note that the digraph G has a directed spanning tree. Figure 1 is the state trajectories of multi-agent system under (4) with different parameters r. From Figure 1, we can see the convergence speed is quicker when parameter r increases. In addition, the protocol (4) has an exponential convergence.

Figure 2 is the state trajectories of multi-agent system under consensus protocol (20) with different parameters r. From Figure 2, we can see that the convergence speed of our consensus protocol with parameter $r = 1.5$ is quicker than $r = 1$. That is to say, the convergence speed of our consensus protocol is quicker than the classic one.

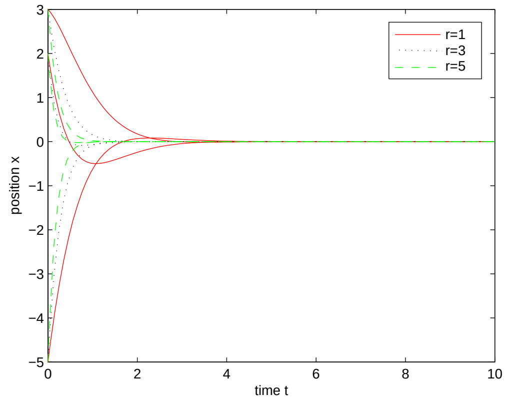
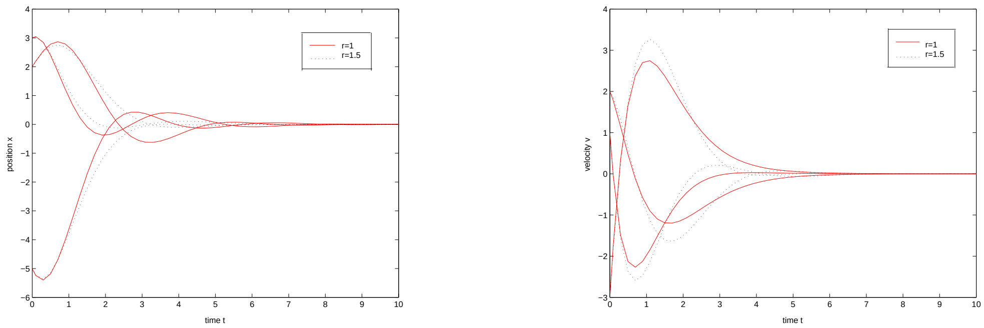
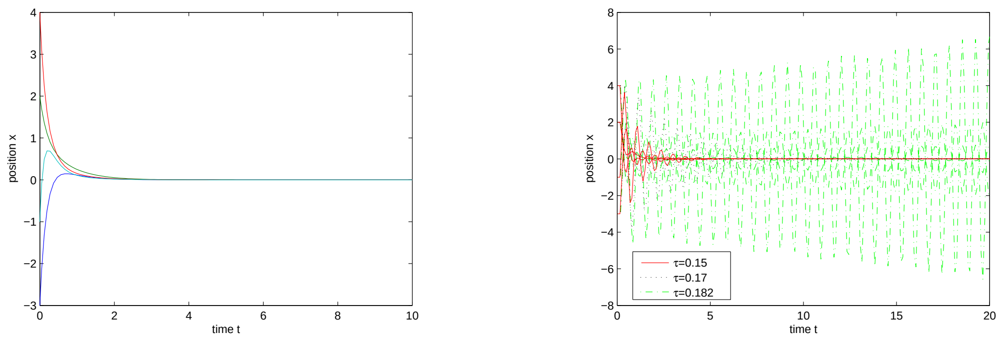


Figure 1: State trajectories of multi-agent system under protocol (4) with different parameter r



(a) Position trajectories (b) Velocity trajectories

Figure 2: State trajectories of multi-agent system under consensus protocol (20)



(a) Under protocol (4) (b) Under protocol (5)

Figure 3: State trajectories of multi-agent system

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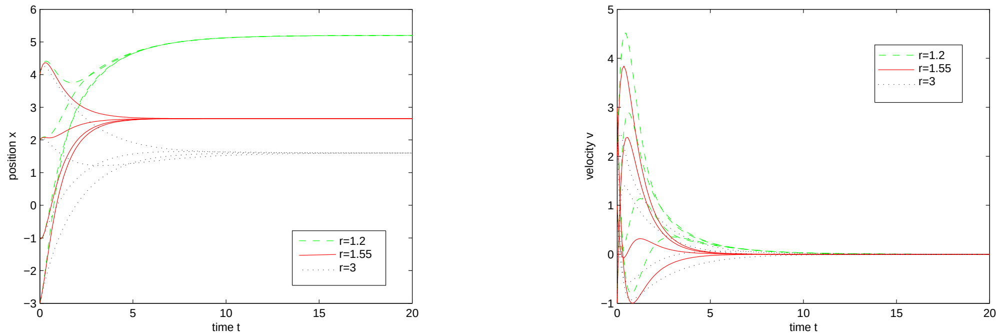

(a) Position trajectories (b) Velocity trajectories

Figure 4: State trajectories of multi-agent system under consensus protocol (20)

Example 2. We consider a system with four agents. In Figure 3, we consider the protocol (4) and (5). The corresponding Laplacian matrix and initial values are chosen as

$$L = \begin{pmatrix} 2 & 0 & -2 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -2 & 3 & -1 \\ 0 & 0 & -4 & 4 \end{pmatrix}, x(0) = \begin{pmatrix} -3 \\ 2 \\ 4 \\ -1 \end{pmatrix}.$$

Then $D = \text{diag}(-2, -1, -3, -4)$. Note that the digraph G has a directed spanning tree, then the conditions of Theorem 1 are satisfied. Figure 3 (a) is state trajectories of multi-agent system under protocol (4). By (12), if $\tau < 0.1706$, then global convergence is achieved. Figure 3 (b) is state trajectories of multi-agent system under protocol (5) with different parameters τ. In Figure 3 (b), it can be seen that multi-agent system is divergent with $\tau = 0.182$. It is clear from Figure 3 that the convergence can be asymptotically achieved under the conditions given by Theorem 1 and 2.

Example 3. We consider a system with four agents. In Figure 4, we consider the consensus protocol (20). The corresponding Laplacian matrix and initial values are chosen as

$$L = \begin{pmatrix} 2 & 0 & -2 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -2 & 2 \end{pmatrix}, x(0) = \begin{pmatrix} -3 \\ 2 \\ 4 \\ -1 \end{pmatrix} \text{and } \mathrm{v}(0) = \begin{pmatrix} 2 \\ 1 \\ 3 \\ -1 \end{pmatrix}.$$

The values of parameter r are shown in Figure 4. Easy to verify, $\Gamma_i(i = 1, \cdots, 3,$ namely, when $r_1 = 1.2, r_2 = 1.55$, and $r_3 = 3$, respectively.) has a zero eigenvalue

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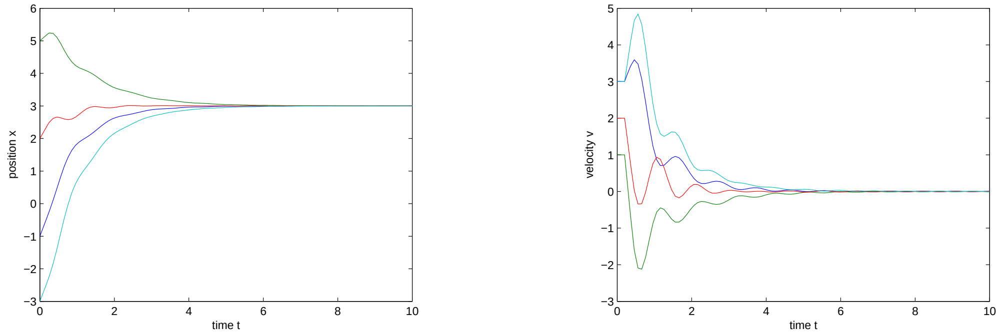

(a) Position trajectories (b) Velocity trajectories

Figure 5: State trajectories of agents under consensus protocol(21)

and three nonzero eigenvalues which have negative real parts. Accordingly, the left eigenvectors of zero eigenvalue of L are taken $w_{11} = \begin{pmatrix} 0.5 & 0.5 & 1 & 0.5 \end{pmatrix}^T, w_{12} = \begin{pmatrix} 0.1818 & 0.1818 & 0.3636 & 0.1818 \end{pmatrix}^T$ and $w_{13} = \begin{pmatrix} 0.05 & 0.05 & 0.1 & 0.05 \end{pmatrix}^T$, respectively. The position consensus convergence values $x_i = (r-1)l_{11}w_{1i}, i = 1, \cdots, 3$, of agents are 5.2, 2.6545, and 1.6. It is clear from Figure 4 that consensus can be asymptotically achieved under the conditions given by Theorem 3.

In Figure 4, note that the consensus protocol (20) is not convergent when $r = 1$. According to Figure 4, we also obtain that the convergence speed is not quicker when parameter r increases. In this example, by plentiful simulations, we find that the convergence speed is quicker when parameter r is equal to 1.5 or so. Therefore, we can choose appropriate parameter r in order to accelerate convergence of consensus protocol. As to how to choose r to obtain the maximal consensus speed, it will be investigated in our future work.

Example 4. We consider a system with four agents. In Figure 5, we consider the consensus protocol (21). The corresponding Laplacian matrix and initial values are chosen as

$$L = \begin{pmatrix} 3 & -2 & 0 & -1 \\ -2 & 3 & -1 & 0 \\ 0 & -1 & 3 & -2 \\ -1 & 0 & -2 & 3 \end{pmatrix}, x(0) = \begin{pmatrix} -1 \\ 5 \\ 2 \\ -3 \end{pmatrix} \text{ and } v(0) = \begin{pmatrix} 3 \\ 1 \\ 2 \\ 3 \end{pmatrix}.$$

The values of parameter are $r = 4/3$ and $\tau = 0.2$. We get $\max_i \frac{\sqrt{\lambda_i^2+\omega_0^2(\lambda_i+(r-1)l_{11})^2}}{\omega_0^2} = 0.9643 < 1$ when $\lambda_i = 6$. Easy to verify, the conditions of Theorem 4 can be satisfied. It is clear from Figure 5 that consensus can be asymptotically achieved under the conditions given by Theorem 4.

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5 Conclusion

In this paper, a kind of new consensus protocol of the multi-agent systems is presented. We study two protocols of multi-agent systems with single-integrator kinematics, and consider the consensus protocols of multi-agent systems with double-integrator dynamics. We derive some sufficient conditions on consensus. Also we find parameter r play an important role in convergence speed. Especially, for consensus protocol (20), we can choose appropriate parameter r in order to accelerate its convergence. How to choose r to obtain the maximal consensus speed is interesting and challenging, which will be further studied in our future work. Finally, numerical examples illustrate our theoretical results.

Acknowledgement. *This research is supported by NSFC (60973015), A Project supported by Scientific Research Fund of Sichuan Provincial Education Department (13ZB0107), Sichuan Province Sci. & Tech. Research Project (2011JY0002), Research Project of Leshan Normal University(Z1255).*

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ALMOST DIFFERENCE SEQUENCE SPACES DERIVED BY USING A GENERALIZED WEIGHTED MEAN

ALİ KARAİSA AND FARUK ÖZGER

Abstract. In this study, we define the spaces $fs(u, v, \triangle)$, $f(u, v, \triangle)$ and $f_0(u, v, \triangle)$ that consist of all sequence whose $G(u, v, \triangle)-$ transforms are in the set of almost convergent sequence and series spaces. Some topological properties of the new almost convergent sets have been investigated as well as γ- and β-duals of the spaces $f(u, v, \triangle)$ and $fs(u, v, \triangle)$. Also, the characterization of certain matrix classes on/into the mentioned spaces has exhaustively been examined. Finally, some identities and inclusion relations related to core theorems are established.

1. Introduction

To investigate the topological properties of the almost convergent sequence and series spaces and the studies connected with their duals are very recent (see [1, 16, 25, 29]) since the background information about these spaces have not known properly; nevertheless, the role played by the algebraical, geometrical and topological properties of the new Banach spaces which are the matrix domains of triangle matrices in sequence spaces is very well-known (see [7, 8, 9, 10, 11, 12, 13, 14]).

The sets $f(G)$ and $f_0(G)$ which are derived by the generalized weighted mean have recently been introduced and studied in [29]. Matrix domains of the double band $B(r, s)$ and triple band $B(r, s, t)$ matrices in the sets of almost null f_0 and almost convergent f sequence spaces have been investigated by Başar and Kirişçi [16] and Sönmez [2], respectively. They have examined some algebraical and topological properties of certain almost null and almost convergent spaces. Following these authors, Kayaduman and Şengönül have subsequently introduced some almost convergent spaces which are the matrix domains of the Riesz matrix [26] and Cesàro matrix of order 1 [25] in the sets of almost null f_0 and almost convergent f sequences. Recently, Karaisa and Karabıyık studied some new almost convergent spaces which are the matrix domains of A^r matrix [1].

The general aim of this study is to fill a gap in literature by extending certain topological spaces and to investigate some topological properties in addition to some core theorems.

The paper is divided into six sections. The next section contains a foreknowledge on the main argument of this study. Section 3 introduces the almost convergent sequence and series spaces which are the matrix domains of the $G(u, v, \triangle)$ matrix in the almost convergent sequence and series spaces fs and f, respectively, in addition to give certain theorems related to behavior of some sequences. Section 4 determines the beta- and gamma-duals of the spaces $fs(u, v, \triangle)$ and $f(u, v, \triangle)$ and contains some stated and proved theorems on the characterization of the matrix mappings on/into the sequence spaces $fs(u, v, \triangle)$, $f(u, v, \triangle)$ and $f_0(u, v, \triangle)$. The final section examines and compares the present results in the context of Banach sequence spaces as studied by other authors.

2. Preliminaries and notations

In this section, we start with recalling the most important notations, definitions and make a few remarks on notions which are needed in this study.

By w, we shall denote the space of all real or complex valued sequences. We shall write ℓ_∞, c and c_0 for the spaces of all bounded, convergent, and null sequences respectively. Also by bs and cs, we denote the spaces of all bounded and convergent series. Let μ and γ be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then, we say that A defines a matrix mapping from μ into γ, and we denote it by writing $A : \mu \to \gamma$, if for every sequence $x = (x_k) \in \mu$ the sequence $Ax = (Ax)_n$, the $A-$transform of x, is in γ; where

$$(Ax)_n = \sum_k a_{nk} x_k \quad (n \in \mathbb{N}). \tag{2.1}$$

The notation $(\mu : \lambda)$ denotes the class of all matrices A such that $A : \mu \to \lambda$. Thus, $A \in (\mu : \lambda)$ if and only if the series on the right hand side of (2.1) converges for each $n \in \mathbb{N}$ and every $x \in \mu$, and we have

2010 *Mathematics Subject Classification.* 46A45,40J05,40C05.
Key words and phrases. almost convergence; Schauder basis; β- and γ -duals; matrix mappings; core theorems.

$Ax = \{(Ax)_n\}_{n\in\mathbb{N}} \in \lambda$ for all $x \in \mu$. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. We write $e = (1, 1, \cdots)$ and U for the set of all sequences $u = (u_k)$ such that $u_k \neq 0$ for all $k \in \mathbb{N}$. For $u \in U$, let $\frac{1}{u} = (\frac{1}{u_k})$. Let $u, v \in U$ and define the matrix $G(u, v) = (g_{nk})$ by

$$g_{nk} = \begin{cases} u_n v_k, & (k < n), \\ u_n v_n, & (k = n), \\ 0, & (k > n) \end{cases}$$

for all $k \in \mathbb{N}$, where u_n depends only on n and v_k only on k. The matrix $G(u, v)$, defined above, is called as generalized weighted mean or factorable matrix.

The matrix domain μ_A of an infinite matrix A in a sequence space μ is defined by

$$\mu_A = \{x = (x_k) \in \omega : Ax \in \mu\} \tag{2.2}$$

which is a sequence space. Although in the most cases the new sequence space μ_A generated by the limitation matrix A from a sequence space μ is the expansion or the contraction of the original space μ.

A sequence (b_k) in a normed space X is called a *Schauder basis* for X if and only if for each $x \in X$, there exists a unique sequence (α_k) of scalars such that $x = \sum_{k=0}^{\infty} \alpha_k b_k$.

Let K be a subset of $\mathbb{N}$. The natural density $\delta(K)$ of $K \subseteq \mathbb{N}$ is $\lim_n n^{-1}|\{\{k \leqslant n : k \in K\}\}|$ provided it exists, where $|E|$ denotes the cardinality of a set E. A sequence $x = (x_k)$ is called statistically convergent ($st-$convergent) to the number l, denoted $st - \lim x$, if every $\epsilon > 0$, $\delta(\{k : |x_k - l| \geqslant \epsilon\}) = 0$, [32]. We write S and S_0 to denote the sets of all statistically convergent sequences and statistically null sequences. The statistically convergent sequences were studied by several authors (see [32, 6, 5]). The concepts of the statistical boundedness, statistical limit superior (or briefly $st - \limsup$) and statistical limit inferior (or briefly $st - \liminf$) have been introduced by Fridy and Orhan in [32]. They have also studied on the notions of the statistical core (or briefly $st-$core) of a statistically bounded sequence as the closed interval $[st - \liminf, st - \limsup]$.

We list the following functionals on ℓ_∞:

$$\begin{aligned} l(x) &= \liminf_{k\to\infty} x_k, \ L(x) = \limsup_{k\to\infty} x_k, \\ q_\sigma(x) &= \limsup_{m\to\infty} \sup_{n\in\mathbb{N}} \sum_{i=0}^{m} x_{\sigma^i(n)}, \\ L^*(x) &= \limsup_{m\to\infty} \sup_{n\in\mathbb{N}} \sum_{i=0}^{m} x_{n+i}. \end{aligned}$$

The $\sigma-$core of a real bounded sequence x is defined as the closed interval $[-q_\sigma(-x), q_\sigma(x)]$ and also the inequality $q_\sigma(Ax) \leqslant q_\sigma(x)$ holds for all bounded sequences x. The Knopp-core (shortly $K-$core) of x is the interval $[l(x), L(x)]$ while the Banach core (in short $B-$core) of x defined by the interval $[-L^*(-x), L^*(x)]$. In particular, when $\sigma(n) = n + 1$ because of the equality $q_\sigma(x) = L^*(x)$, $\sigma-$core of x is reduced to the $B-$core of x (see [21, 27]). The necessary and sufficient conditions for an infinite matrix matrix A to satisfy the inclusion $K - core(Ax) \subseteq B - core(x)$ for each bounded sequences x obtained in [4].

We now focus on the sets of almost convergent sequences. A continuous linear functional ϕ on ℓ_∞ is called a Banach limit if (i) $\phi(x) \geqslant 0$ for $x = (x_k), x_k \geqslant 0$ for every k, (ii) $\phi(x_{\sigma(k)}) = \phi(x_k)$ where σ is shift operator which is defined on ω by $\sigma(k) = k + 1$ and (iii) $\phi(e) = 1$ where $e = (1, 1, 1, \ldots)$. A sequence $x = (x_k) \in \ell_\infty$ is said to be almost convergent to the generalized limit α if all Banach limits of x are α [22] and denoted by $f - \lim x = \alpha$. In other words, $f - \lim x_k = \alpha$ uniformly in n if and only if

$$\lim_{m\to\infty} \frac{1}{m+1} \sum_{k=0}^{m} x_{k+n} \text{ uniformly in } n.$$

The characterization given above was proved by Lorentz in [22]. We denote the sets of all almost convergent sequences f and series fs by

$$f = \left\{ x = (x_k) \in \omega : \lim_{m\to\infty} t_{mn}(x) = \alpha \text{ uniformly in } n \right\},$$

where

$$t_{mn}(x) = \sum_{k=0}^{m} \frac{1}{m+1} x_{k+n}, \ t_{-1,n} = 0$$

and

$$fs = \left\{ x = (x_k) \in \omega : \exists l \in \mathbb{C} \ni \lim_{m\to\infty} \sum_{k=0}^{m} \sum_{j=0}^{n+k} \frac{x_j}{m+1} = l \text{ uniformly in } n \right\}.$$

We know that the inclusions $c \subset f \subset \ell_\infty$ strictly hold. Because of these inclusions, norms $\|.\|_f$ and $\|.\|_\infty$ of the spaces f and ℓ_∞ are equivalent. So the sets f and f_0 are BK-spaces with the norm $\|x\|_f = \sup_{m,n} |t_{mn}(x)|$.

Remark that the γ- and β-duals of the set fs were found by Başar and Kirişçi. Now, we give a theorem about non-existence of Schauder basis of the space fs.

Theorem 2.1. *The set fs has no Schauder basis.*

Proof. Let us define the matrix $S = (s_{nk})$ by $s_{nk} = 1$ $(0 \leqslant k \leqslant n)$, $s_{nk} = 0$ $(n > k)$. Then $x \in fs$ if and only if $(Sx)_n \in f$ for all n. As a result, since the set f has no basis the set fs has no basis too. □

3. The sequence space $fs(u, v, \triangle)$, $f(u, v, \triangle)$ and their topological properties

For a sequence $x = (x_k)$, we denote the difference sequence space by $\triangle x = (x_k - x_{k-1})$. Let $u = (u_k)_{k=0}^\infty$ and $v = (v_k)_{k=0}^\infty$ be sequence of real numbers such that $u_k \neq 0$ and $v_k \neq 0$ for all $k \in \mathbb{N}$. Recently, the difference sequence space of weighted mean $\mu(u, v, \triangle)$ have been introduced in [23], where $\mu = c_0, c$, and ℓ_∞. These sequence space defined as the matrix domains of the triangle W in the space c_0, c, and ℓ_∞, respectively. The matrix $G(u, v, \triangle) = W = (w_{nk})$ is defined by

$$w_{nk} = \begin{cases} u_n(v_k - v_{k+1}), & (k < n), \\ u_n v_n, & (k = n), \\ 0, & (k > n). \end{cases}$$

In this section, we define the new sequence space $fs(u, v, \triangle)$ and $f(u, v, \triangle)$ derived by using the generalized weighted mean and topological properties of the sets. Much of the study given in this section is, by now, classical and not very difficult, but it is necessary to give. Now, we introduce the new spaces $f(u, v, \triangle)$, $f_0(u, v, \triangle)$ and $fs(u, v, \triangle)$ as the sets of all sequences such that their $G(u, v, \triangle)-$ transforms are in the spaces f, f_0 and fs, respectively, that is

$$f(u, v, \triangle) = \left\{ x \in \omega : \exists \alpha \in \mathbb{C} \ni \lim_{m\to\infty} \frac{1}{m+1} \sum_{k=0}^{m} \sum_{i=0}^{k+n} u_{k+n} v_i (x_i - x_{i-1}) = \alpha \text{ uniformly in } n \right\},$$

$$f_0(u, v, \triangle) = \left\{ x \in \omega : \lim_{m\to\infty} \frac{1}{m+1} \sum_{k=0}^{m} \sum_{i=0}^{k+n} u_{k+n} v_i (x_i - x_{i-1}) = 0 \text{ uniformly in } n \right\}$$

and

$$fs(u, v, \triangle) = \left\{ x = (x_k) \in \omega : \exists l \in \mathbb{C} \ni \lim_{m\to\infty} \frac{1}{m+1} \sum_{k=0}^{m} \sum_{j=0}^{n+k} \sum_{i=0}^{j} u_j v_i (x_i - x_{i-1}) = l \text{ uniformly in } n \right\}.$$

We can redefine the spaces $fs(u, v, \triangle)$, $f(u, v, \triangle)$ and $f_0(u, v, \triangle)$ by the notation of (2.2)

$$f_0(u, v, \triangle) = (f_0)_{G(u,v,\triangle)},\ f(u, v, \triangle) = f_{G(u,v,\triangle)} \text{ and } fs(u, v, \triangle) = (fs)_{G(u,v,\triangle)}.$$

Define the sequence $y = (y_k)$, which will be frequently used, as the $G(u, v, \triangle)$-transform of a sequence $x = (x_k)$, i.e.

$$y_k = \sum_{i=0}^{k} u_k v_i \triangle x_i = \sum_{i=0}^{k} u_k \nabla v_i x_i,\ (\nabla v = v_i - v_{i+1})\ (k \in \mathbb{N}). \tag{3.1}$$

Theorem 3.1. *The spaces $f(u, v, \triangle)$ and $fs(u, v, \triangle)$ has no Schauder basis.*

Proof. Since, it is known that the matrix domain μ_A of a normed sequence space μ has a basis if and only if μ has a basis whenever $A = (a_{nk})$ is a triangle [7, Remark 2.4] and the space f has no Schauder basis by [28, Corollary 3.3] we have $f(u, v, \triangle)$ has no Schauder basis. Since the set fs has no basis in theorem 2.1, $fs(u, v, \triangle)$ has no Schauder basis. □

Theorem 3.2. *The following statements hold.*

(i): *The sets $f(u, v, \triangle)$ and $f_0(u, v, \triangle)$ are linear spaces with the co-ordinatewise addition and scalar multiplication which are BK-spaces with the norm*

$$\|x\|_{f(u,v,\triangle)} = \sup_m \left| \frac{1}{m+1} \sum_{k=0}^{m} \sum_{i=0}^{k} u_k \nabla v_i x_{n+i} \right|. \tag{3.2}$$

(ii): *The set $fs(u,v,\triangle)$ is a linear space with the co-ordinatewise addition and scalar multiplication which is a BK-space with the norm*

$$\|x\|_{fs(u,v,\triangle)} = \sup_m \left| \sum_{k=0}^{m} \frac{1}{m+1} \sum_{k=0}^{m} \sum_{j=0}^{n+k} \sum_{i=0}^{j} u_j \nabla v_i x_i \right|.$$

Proof. Since the second part can be similarly proved, we only focus on the first part. Since the sets f and f_0 endowed with the norm $\|.\|_\infty$ are BK-spaces (see[30, Example 7.3.2(b)]) and the matrix $W = (w_{nk})$ is normal, Theorem 4.3.2 of Wilansky [3, p.61] gives the fact that the spaces $f(u,v,\triangle)$ and $f_0(u,v,\triangle)$ are BK-spaces with the norm in (3.2). □

Now, we may give the following theorem concerning the isomorphism between our spaces and the sets f, f_0 and fs.

Theorem 3.3. *The sets $f(u,v,\triangle), f_0(u,v,\triangle)$ and $fs(u,v,\triangle)$ are linearly isomorphic to the sets f, f_0 and fs respectively; i.e. $f(u,v,\triangle) \cong f$, $f_0(u,v,\triangle) \cong f_0$ and $fs(u,v,\triangle) \cong fs$.*

Proof. To prove the fact that $f(u,v,\triangle) \cong f$ we should show the existence of a linear bijection between the spaces $f(u,v,\triangle)$ and f. Consider the transformation T defined with the notation of (2.2) from $f(u,v,\triangle)$ to f by $x \mapsto y = Tx = G(u,v,\triangle)x$. The linearity of T is clear. Further, it is clear that $x = \theta$ whenever $Tx = \theta$ and hence, T is injective.

Let $y = (y_k) \in f(u,v,\triangle)$ and define the sequence $x = (x_k)$ by

$$x_k = \sum_{i=0}^{k} \frac{1}{v_i} \left[\frac{y_i}{u_i} - \frac{y_{i-1}}{u_{i-1}} \right] = \sum_{i=0}^{k-1} \frac{1}{u_i} \left(\frac{1}{v_i} - \frac{1}{v_{i+1}} \right) y_i + \frac{1}{u_k v_k} y_k \quad \text{for each } k \in \mathbb{N}. \tag{3.3}$$

Then, since

$$(\triangle x)_{i+n} = \frac{1}{v_i} \left[\frac{y_{n+i}}{u_i} - \frac{y_{n+i-1}}{u_{i-1}} \right] \quad \text{for each } i \in \mathbb{N} \tag{3.4}$$

we have

$$\begin{aligned} f_{G(u,v,\triangle)} - \lim x &= \lim_{m\to\infty} \sum_{k=0}^{m} \frac{1}{m+1} \sum_{i=0}^{k} u_k v_i (\triangle x)_{n+i} \text{ uniformly in } n \\ &= \lim_{m\to\infty} \sum_{k=0}^{m} \frac{1}{m+1} \sum_{i=0}^{k} u_k v_i \frac{1}{v_i} \left[\frac{y_{n+i}}{u_i} - \frac{y_{n+i-1}}{u_{i-1}} \right] \text{ uniformly in } n \\ &= \lim_{m\to\infty} \frac{1}{m+1} \sum_{k=0}^{m} y_{k+n} \text{ uniformly in } n \\ &= f - \lim y \end{aligned}$$

which implies that $x \in f(u,v,\triangle)$. As a result, T is surjective. Hence, T is a linear bijection which implies that the spaces $f(u,v,\triangle)$ and f are linearly isomorphic, as desired. In the same way, it can be shown that $f_0(u,v,\triangle)$ and $fs(u,v,\triangle)$ are linearly isomorphic to f_0 and fs, respectively and so we omit the details. □

Theorem 3.4. *Let the spaces $f_0(u,v,\triangle)$ and $f(u,v,\triangle)$ be given. Then,*

(i): *$f_0(u,v,\triangle) \subset f(u,v,\triangle)$ strictly hold.*

(ii): *If $(u_n) \in c$ and $(v_k - v_{k+1}) \in \ell_1$, then $\ell_\infty \subset f(u,v,\triangle)$ holds.*

Proof. (i) Let $x = (x_k) \in f_0(u,v,\triangle)$ which means that $G(u,v,\triangle) \in f_0$. Since $f_0 \subset f$, $G(u,v,\triangle) \in f$. This implies that $x \in f(u,v,\triangle)$. Thus, we have $f_0(u,v,\triangle) \subset f(u,v,\triangle)$.

Now, we show that this inclusion is strict. Let us consider the sequence $z = \{z_k\}$ define by,

$$z_k = \sum_{i=0}^{k-1} \frac{1}{u_i} \left(\frac{1}{v_i} - \frac{1}{v_{i+1}} \right) + \frac{1}{u_k v_k} \quad \text{for each } k \in \mathbb{N}$$

we have that

$$f_{G(u,v,\triangle)} - \lim z = \lim_{m\to\infty} \sum_{k=0}^{m} \frac{1}{m+1} \sum_{i=0}^{k} u_k v_i (\triangle z)_{n+i} = \lim_{m\to\infty} \sum_{k=0}^{m} \frac{1}{m+1} = e = (1,1,\cdots)$$

which means that $z \in f(u,v,\triangle) \setminus f_0(u,v,\triangle)$ that is to say that the inclusion is strict.
(ii) Let $(u_n) \in c$ and $(v_k - v_{k+1}) \in \ell_1$, then we have $G(u,v,\triangle) \in c$ for all $x \in \ell_\infty$. Hence, since $c \subset f$, $G(u,v,\triangle) \in f$ which implies that $x = (x_k) \in f(u,v,\triangle)$. This completes the proof. □

4. Certain matrix mappings on the sets $fs(u,v,\triangle)$ and $f(u,v,\triangle)$ and some duals

In this section, we shall characterize some matrix transformations between the spaces of generalized difference almost convergent sequence and almost convergent series in addition to paranormed and classical sequence spaces after giving $\beta-$ and γ duals of the spaces $fs(u,v,\triangle)$ and $f(u,v,\triangle)$. We start with the definition of the beta and gamma duals.

If x and y are sequences and X and Y are subsets of ω, then we write $x \cdot y = (x_k y_k)_{k=0}^{\infty}$, $x^{-1} * Y = \{a \in \omega : a \cdot x \in Y\}$ and

$$M(X,Y) = \bigcap_{x \in X} x^{-1} * Y = \{a : a \cdot x \in Y \text{ for all } x \in X\}$$

for the multiplier space of X and Y; in particular, we use the notations $X^{\beta} = M(X, cs)$ and $X^{\gamma} = M(X, bs)$ for the $\beta-$and $\gamma-$duals of X.

Lemma 4.1. [31] *$A = (a_{nk}) \in (f : \ell_\infty)$ if and only if*

$$\sup_n \sum_k |a_{nk}| < \infty. \tag{4.1}$$

Lemma 4.2. [31] *$A = (a_{nk}) \in (f : c)$ if and only if (4.1) holds and there are $\alpha, \alpha_k \in \mathbb{C}$ such that*

$$\lim_{n\to\infty} a_{nk} = \alpha_k \textit{ for all } k \in \mathbb{N},$$

$$\lim_{n\to\infty} \sum_k a_{nk} = \alpha,$$

$$\lim_{n\to\infty} \sum_k \Delta(a_{nk} - \alpha_k) = 0.$$

Theorem 4.3. *Let $u, v \in U$, $a = (a_k) \in w$.*
The $\gamma-$ dual of the space $f(u,v,\triangle)$ is the intersection of the sets

$$\begin{aligned} b_1 &= \left\{ a = (a_k) \in \omega : \sup_n \sum_n \left| \frac{1}{u_k} \left[\frac{a_k}{v_k} + \left(\frac{1}{v_i} - \frac{1}{v_{i+1}} \right) \sum_{i=k+1}^{n} a_i \right] \right| < \infty \right\}, \\ b_2 &= \left\{ a = (a_k) \in \omega : \sup_n \left| \frac{a_n}{u_n v_n} \right| < \infty \right\}. \end{aligned}$$

Proof. Take any sequence $a = (a_k) \in \omega$ and consider the following equality

$$\begin{aligned} \sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n} a_k \left[\sum_{i=0}^{k-1} \frac{1}{u_k} \left(\frac{1}{v_i} - \frac{1}{v_{i+1}} \right) y_i + \frac{1}{u_k v_k} y_k \right] \\ &= \sum_{k=0}^{n-1} \frac{1}{u_k} \left[\frac{a_k}{v_k} + \left(\frac{1}{v_i} - \frac{1}{v_{i+1}} \right) \sum_{i=k+1}^{n} a_i \right] y_k + \frac{a_n}{u_n v_n} y_n \\ &= (Ey)_n \end{aligned} \tag{4.2}$$

$$e_{nk} = \begin{cases} \frac{1}{u_k} \left[\frac{a_k}{v_k} + \left(\frac{1}{v_i} - \frac{1}{v_{i+1}} \right) \sum_{i=k+1}^{n} a_i \right], & (0 \leqslant k \leqslant n-1), \\ \frac{1}{u_n v_n} a_n, & (k = n), \\ 0, & (k > n) \end{cases} \tag{4.3}$$

for all $k, n \in \mathbb{N}$. Thus, we deduce from (4.2) that $ax = (a_k x_k) \in bs$ whenever $x = (x_k) \in f(u,v,\triangle)$ if and only if $Ey \in \ell_\infty$ whenever $y = (y_k) \in f$ where $E = (e_{nk})$ is defined in (4.3). Therefore with the help of Lemma 4.1 $\{f(u,v,\triangle)\}^{\gamma} = b_2 \cap b_1$. □

Theorem 4.4. *The $\beta-$ dual of the space $f(u,v,\triangle)$ is the intersection of the sets*

$$\begin{aligned} b_3 &= \left\{ a = (a_k) \in \omega : \lim_{n\to\infty} e_{nk} \textit{ exists} \right\}, \\ b_4 &= \left\{ a = (a_k) \in \omega : \lim_{n\to\infty} \sum_{k=0}^{n} e_{nk} \textit{ exists} \right\}, \\ b_5 &= \left\{ a = (a_k) \in \omega : \lim_{n\to\infty} \sum_k \triangle \left[e_{nk} - a_k \right] < \infty \right\}, \end{aligned}$$

where $a_k = \lim_{n\to\infty} e_{nk}$. Then, $\{f(u,v,\triangle)\}^{\beta} = \cap_{k=1}^{5} b_k$.

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Proof. Let us take any sequence $a \in \omega$. By (4.2) $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in f(u, v, \triangle))$ if and only if $Ey \in c$ whenever $y = (y_k) \in f$ where $E = (e_{nk})$ defined in (4.3), we derive the consequence by Lemma 4.2 that $\{f(u, v, \triangle)\}^{\beta} = \cap_{k=1}^{5} b_k$. □

Theorem 4.5. *The $\gamma-$ dual of the space $fs(u, v, \triangle)$ is the intersection of the sets*

$$\begin{aligned} b_6 &= \left\{ a = (a_k) \in \omega : \sup_n \sum_k |\triangle e_{nk}| < \infty \right\}, \\ b_7 &= \left\{ a = (a_k) \in \omega : \lim_{k\to\infty} e_{nk} = 0 \right\}. \end{aligned}$$

in other words we have $\{fs(u, v, \triangle)\}^{\gamma} = b_6 \cap b_7$.

Proof. This may be obtained in the similar way as mentioned in the proof of Theorem 4.3 with Lemma 4.1 instead of Lemma 4.12(viii). So we omit details. □

Theorem 4.6. *Defined the set b_8 by*

$$b_8 = \left\{ a = (a_k) \in \omega : \lim_{n\to\infty} \sum_k \left|\triangle^2 e_{nk}\right| < \infty \right\}.$$

Then, $\{fs(u, v, \triangle)\}^{\beta} = b_3 \cap b_6 \cap b_7 \cap b_8$.

Proof. This may be obtained in the similar way as mentioned in the proof of Theorem 4.4 with Lemma 4.2 instead of Lemma 4.12(vii). So, we omit details. □

For the sake of brevity the following notations will be used:

$$\begin{aligned} \widetilde{a}(n, k, m) &= \frac{1}{m+1} \sum_{i=0}^{m} \widetilde{a}_{n+i,k}, \qquad \tilde{a}(n, k) = \sum_{i=0}^{n} \tilde{a}_{ik}, \\ c(n, k, m) &= \frac{1}{m+1} \sum_{i=0}^{m} c_{n+i,k}, \qquad c(n, k) = \sum_{i=0}^{n} c_{ik}, \end{aligned}$$

where c_{nk} is defined in (4.9) and $\widetilde{a}_{nk} = \sum_{j=k}^{\infty} \frac{a_{n,j+1}}{u_{j+1}} (\frac{1}{v_k} - \frac{1}{v_{k-1}})$ for all $k, m, n \in \mathbb{N}$.

Assume that the infinite matrices $E = (e_{nk})$ and $A = (a_{nk})$ map the sequences $x = (x_k)$ and $y = (y_k)$ which are connected with the relation (3.1) to the sequences $\xi = (\xi_n)$ and $\sigma = (\sigma_n)$, respectively, i.e.,

$$\xi_n = \big(Ex\big)_n = \sum_k e_{nk} x_k \ \text{ for all } \ n \in \mathbb{N}, \tag{4.4}$$

$$\sigma_n = \big(Ay\big)_n = \sum_k a_{nk} y_k \ \text{ for all } \ n \in \mathbb{N}. \tag{4.5}$$

One can easily conclude here that the method E is directly applied to the terms of the sequence $x = (x_k)$ while the method A is applied to the $G(u, v, \Delta)$-transform of the sequence $x = (x_k)$. So, the methods E and A are essentially different.

Now, suppose that the matrix product $AG(u, v, \Delta)$ exists which is a much weaker assumption than the conditions on the matrix A belonging to any matrix class, in general. It is not difficult to see that the sequence in (4.5) reduces to the sequence in (4.4) as follows:

$$\begin{aligned} \sigma_n &= \sum_k \sum_{i=0}^{k} u_k \nabla v_i x_i a_{nk} \\ &= \sum_k \sum_{i=k} u_i a_{ni} \nabla v_k x_k = \xi_n, \end{aligned}$$

where $\nabla v_k = v_k - v_{k-1}$. Hence, the matrices $E = (e_{nk})$ and $A = (a_{nk})$ are connected with the relation

$$a_{nk} = \sum_{i=k} (\frac{1}{v_k} - \frac{1}{v_{k-1}}) \frac{e_{n,i+1}}{u_{i+1}} \ \text{or} \tag{4.6}$$

$$e_{nk} = \sum_{i=k} u_i a_{ni} \nabla v_k x_k \ \text{for all } k, n \in \mathbb{N}. \tag{4.7}$$

Note that the methods A and E are not necessarily equivalent since the order of summation may not be reversed.

We now give the following fundamental theorems connected with the matrix mappings on/into the almost convergent spaces $f(u,v,\Delta)$ and $fs(u,v,\Delta)$:

Theorem 4.7. *Let μ be any given sequence space and the matrices $E=(e_{nk})$ and $A=(a_{nk})$ are connected with the relation (4.6). Then, $E\in(f(u,v,\Delta):\mu)$ if and only if*

$$A\in(f:\mu)$$

and

$$(e_{nk})_{k\in\mathbb{N}}\in[f(u,v,\Delta)]^{\beta} \text{ for all } n\in\mathbb{N}. \tag{4.8}$$

Proof. Suppose that $E=(e_{nk})$ and $A=(a_{nk})$ are connected with the relation (4.6) and let μ be any given sequence space and keep in mind that the spaces $f(u,v,\Delta)$ and f are norm isomorphic.

Let $E\in(f(u,v,\Delta):\mu)$ and take any sequence $x\in f(u,v,\Delta)$ and keep in mind that $y=G(u,v,\Delta)x$. Then $(e_{nk})_{k\in\mathbb{N}}\in[f(u,v,\Delta)]^{\beta}$ that is, (4.8) holds for all $n\in\mathbb{N}$ and $AG(u,v,\Delta)$ exists which implies that $(a_{nk})_{k\in\mathbb{N}}\in\ell_1=f^{\beta}$ for each $n\in\mathbb{N}$. Thus, Ay exists for all $y\in f$. Hence, by the equality (4.6) we have $A\in(f:\mu)$.

On the other hand, assume that (4.8) holds and $A\in(f:\mu)$. Then, we have $(a_{nk})_{k\in\mathbb{N}}\in\ell_1$ for all $n\in\mathbb{N}$ which gives together with $(e_{nk})_{k\in\mathbb{N}}\in[f(u,v,\Delta)]^{\beta}$ for each $n\in\mathbb{N}$ that Ex exists. Then, the equality $Ex=Ay$ in (4.6) again holds. Hence $Ex\in\mu$ for all $x\in f(u,v,\Delta)$, that is $E\in(f(u,v,\Delta):\mu)$. □

Theorem 4.8. *Let μ be any given sequence space and the elements of the infinite matrices $F=(f_{nk})$ and $C=(c_{nk})$ are connected with the relation*

$$c_{nk}=\sum_{i=k}^{n}u_n v_{i-k}(f_{i-k,k}-f_{i-k-1,k}) \text{ for all } k,n\in\mathbb{N}. \tag{4.9}$$

Then, $F=(f_{nk})\in(\mu:f(u,v,\Delta))$ if and only if $C\in(\mu:f)$.

Proof. Let $t=(t_k)\in\mu$ and consider the following equality

$$\begin{aligned}\{G(u,v,\Delta)(Ft)\}_n &= \sum_{i=0}^{n}u_n v_i\Delta(Ft)_i\\ &= \sum_{i=0}^{n}u_n v_i\sum_k(f_{i,k}-f_{i-1,k})t_k\\ &= \sum_k\sum_{i=k}^{n}u_n v_{i-k}(f_{i-k,k}-f_{i-k-1,k})t_k\end{aligned}$$

for all $n\in\mathbb{N}$. Whence, it can be seen from here that $Ft\in f(u,v,\Delta)$ whenever $t\in\mu$ if and only if $Ct\in f$ whenever $t\in\mu$. This completes the proof. □

Theorem 4.9. *Let μ be any given sequence space and the matrices $E=(e_{nk})$ and $A=(a_{nk})$ are connected with the relation (4.6). Then, $E\in(fs(u,v,\Delta):\mu)$ if and only if $A\in(fs:\mu)$ and $(e_{nk})_{k\in\mathbb{N}}\in[fs(u,v,\Delta)]^{\beta}$ for all $n\in\mathbb{N}$.*

Proof. The proof is based on the proof the Theorem 4.7. □

Theorem 4.10. *Let μ be any given sequence space and the elements of the infinite matrices $F=(f_{nk})$ and $C=(c_{nk})$ are connected with the relation (4.9). Then, $F=(f_{nk})\in(\mu:fs(u,v,\Delta))$ if and only if $C\in(\mu:fs)$.*

Proof. The proof is based on the proof the Theorem 4.8. □

By Theorem 4.7, Theorem 4.8, Theorem 4.9 and Theorem 4.10 we have quite a few outcomes depending on the choice of the space μ to characterize certain matrix mappings. Hence, by the help of these theorems the necessary and sufficient conditions for the classes $(f(u,v,\Delta):\mu)$, $(\mu:f(u,v,\Delta))$, $(fs(u,v,\Delta):\mu)$ and $(\mu:fs(u,v,\Delta))$ may be derived by replacing the entries of E and A by those of the entries of $A=EG^{-1}(u,v,\Delta)$ and $C=G(u,v,\Delta)F$, respectively; where the necessary and sufficient conditions on the matrices F and C are read from the concerning results in the existing literature.

Lemma 4.11. *Let $A=(a_{nk})$ be an infinite matrix. Then, the following statements hold:*

i: $A\in(c_0(p):f)$ if and only if

$$\exists N>1\ni\sup_{m\in\mathbb{N}}\sum_k|a(n,k,m)|\,N^{-1/p_k}<\infty \text{ for all } n\in\mathbb{N}, \tag{4.10}$$

$$\exists\alpha_k\in\mathbb{C} \text{ for all } k\in\mathbb{N}\ni\lim_{m\to\infty}a(n,k,m)=\alpha_k \text{ uniformly in } n. \tag{4.11}$$

ii: $A \in (c(p) : f)$ if and only if (4.10) and (4.11) hold and

$$\exists \alpha \in \mathbb{C} \ni \lim_{m\to\infty} \sum_k a(n,k,m) = \alpha \text{ uniformly in } n. \tag{4.12}$$

iii: $A \in (\ell_\infty(p) : f)$ if and only if (4.10) and (4.11) hold and

$$\exists N > 1 \ni \lim_{m\to\infty} \sum_k |a(n,k,m) - \alpha_k| \, N^{1/p_k} = 0 \text{ uniformly in } n. \tag{4.13}$$

Lemma 4.12. *Let $A = (a_{nk})$ be an infinite matrix. Then, the following statements hold.*

i: *[Duran, [33]] $A \in (\ell_\infty : f)$ if and only if (4.1) holds and*

$$f - \lim a_{nk} = \alpha_k \textit{ exists for each fixed } k, \tag{4.14}$$

$$\lim_{m\to\infty} \sum_k |a(n,k,m) - \alpha_k| = 0 \textit{ uniformly in } n, \tag{4.15}$$

ii: *[King, [34]] $A \in (c : f)$ if and only if (4.1), (4.14) hold and*

$$f - \lim \sum_k a_{nk} = \alpha \tag{4.16}$$

iii: *[Başar and Çolak, [19]] $A \in (cs : f)$ if and only if (4.14) holds and*

$$\sup_{n\in\mathbb{N}} \sum_k |\Delta a_{nk}| < \infty \ , \tag{4.17}$$

iv: *[Başar and Çolak, [19]] $A \in (bs : f)$ if and only if (4.14), (4.17) hold and*

$$\lim_k a_{nk} = 0 \textit{ exists for each fixed } n, \tag{4.18}$$

$$\lim_{q\to\infty} \sum_k \frac{1}{q+1} \sum_{i=0}^{q} |\Delta\left[a(n+i,k) - \alpha_k\right]| = 0 \textit{ uniformly in } n, \tag{4.19}$$

v: *[Duran, [33]] $A \in (f : f)$ if and only if (4.1), (4.14), (4.16) hold and*

$$\lim_{m\to\infty} \sum_k |\Delta\left[a(n,k,m) - \alpha_k\right]| = 0 \textit{ uniformly in } n, \tag{4.20}$$

vi: *[Başar, [18]] $A \in (fs : f)$ if and only if (4.14), (4.18), (4.20) and (4.19) hold.*

vii: *[Duran, [15]] $A \in (fs : c)$ if and only if (4.2), (4.17), (4.18) hold and*

$$\lim_{n\to\infty} \sum_k \left|\Delta^2 a_{nk}\right| = \alpha, \tag{4.21}$$

viii: *$A \in (fs : \ell_\infty)$ if and only if (4.17) and (4.18) hold.*

ix: *[Başar and Solak, [17]] $A \in (bs : fs)$ if and only if (4.18), (4.19) hold and*

$$\sup_{n\in\mathbb{N}} \sum_k |\Delta a(n,k)| < \infty \ , \tag{4.22}$$

$$f - \lim a(n,k) = \alpha_k \textit{ exists for each fixed } k, \tag{4.23}$$

x: *[Başar, [18]] $A \in (fs : fs)$ if and only if (4.19), (4.22), (4.23) hold and*

$$\lim_{q\to\infty} \sum_k \frac{1}{q+1} \sum_{i=0}^{q} \left|\Delta^2\left[a(n+i,k) - \alpha_k\right]\right| = 0 \textit{ uniformly in } n, \tag{4.24}$$

xi: *[Başar and Çolak, [19]] $A \in (cs : fs)$ if and only if (4.22) and (4.23) hold.*

xii: *[Başar, [20]] $A \in (f : cs)$ if and only if*

$$\sup_{n\in\mathbb{N}} \sum_k |a(n,k)| < \infty \ , \tag{4.25}$$

$$\sum_n a_{nk} = \alpha_k \textit{ exists for each fixed } k, \tag{4.26}$$

$$\sum_n \sum_k a_{nk} = \alpha, \tag{4.27}$$

$$\lim_{m\to\infty} \sum_k |\Delta\left[a(n,k) - \alpha_k\right]| = 0. \tag{4.28}$$

Now, we give our main results related to matrix mappings on/into the spaces of almost convergent series $fs(\Delta^{(m)})$ and sequences $f(\Delta^{(m)})$.

Corollary 4.13. *Let $A = (a_{nk})$ be an infinite matrix. Then, the following statements hold:*

- **i:** $A \in (fs(u, v, \Delta) : f)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in [f(u, v, \Delta)]^\beta$ for all $n \in \mathbb{N}$ and (4.14), (4.18) hold with $\widetilde{a}_{nk}$ instead of a_{nk}, (4.20) holds with $\widetilde{a}(n, k, m)$ instead of $a(n, k, m)$ and (4.19) holds with $\tilde{a}(n, k)$ instead of $a(n, k)$.
- **ii:** $A \in (fs(u, v, \Delta) : c)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in [f(u, v, \Delta)]^\beta$ for all $n \in \mathbb{N}$ and (4.2), (4.17), (4.18) and (4.21) hold with $\widetilde{a}_{nk}$ instead of a_{nk}.
- **iii:** $A \in (fs(\Delta^{(m)}) : \ell_\infty)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in [f(u, v, \Delta)]^\beta$ for all $n \in \mathbb{N}$ and (4.17) and (4.18) hold with $\widetilde{a}_{nk}$ instead of a_{nk}.
- **iv:** $A \in (fs(u, v, \Delta) : fs)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in [f(u, v, \Delta)]^\beta$ for all $n \in \mathbb{N}$ and (4.19), (4.22), (4.23) and (4.24) hold with $\tilde{a}(n, k)$ instead of $a(n, k)$.
- **v:** $A \in (cs : fs(u, v, \Delta))$ if and only if (4.22) and (4.23) hold with $c(n, k)$ instead of $a(n, k)$.
- **vi:** $A \in (bs : fs(u, v, \Delta))$ if and only if (4.18) holds with c_{nk} instead of a_{nk}, (4.19), (4.22) and (4.23) hold with $c(n, k)$ instead of $a(n, k)$.
- **vii:** $A \in (fs : fs(u, v, \Delta))$ if and only if (4.19), (4.22), (4.23) and (4.24) hold with $c(n, k)$ instead of $a(n, k)$.

Corollary 4.14. *Let $A = (a_{nk})$ be an infinite matrix. Then, the following statements hold:*

- **i:** $A \in (c(p) : f(u, v, \Delta))$ if and only if (4.10), (4.11) and (4.12) hold with $c(n, k, m)$ instead of $a(n, k, m)$.
- **ii:** $A \in (c_0(p) : f(u, v, \Delta))$ if and only if (4.10) and (4.11) hold with $c(n, k, m)$ instead of $a(n, k, m)$.
- **iii:** $A \in (\ell_\infty(p) : f(u, v, \Delta))$ if and only if (4.10), (4.11) and (4.13) hold with $c(n, k, m)$ instead of $a(n, k, m)$.

Corollary 4.15. *Let $A = (a_{nk})$ be an infinite matrix. Then, the following statements hold:*

- **i:** $A \in (f(u, v, \Delta) : \ell_\infty)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in [f(u, v, \Delta)]^\beta$ *for all* $n \in \mathbb{N}$ *and (4.1) holds with* $\widetilde{a}_{nk}$ *instead of* a_{nk}.
- **ii:** $A \in (f(u, v, \Delta) : c)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in [f(u, v, \Delta)]^\beta$ *for all* $n \in \mathbb{N}$ *and (4.1), (4.2), (4.2) and (4.2) hold with* $\widetilde{a}_{nk}$ *instead of* a_{nk}.
- **iii:** $A \in (f(u, v, \Delta) : bs)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in [f(u, v, \Delta)]^\beta$ *for all* $n \in \mathbb{N}$ *and (4.22) holds with* $\tilde{a}(n, k)$ *instead of* $a(n, k)$.
- **iv:** $A \in (f(u, v, \Delta) : cs)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in [f(u, v, \Delta)]^\beta$ *for all* $n \in \mathbb{N}$ *and (4.2)-(4.25) hold with* $\tilde{a}(n, k)$ *instead of* $a(n, k)$.

Corollary 4.16. *Let $A = (a_{nk})$ be an infinite matrix. Then, the following statements hold:*

- **i:** $A \in (\ell_\infty : f(u, v, \Delta))$ *if and only if (4.1), (4.14) hold with* c_{nk} *instead of* a_{nk} *and (4.15) holds with* $c(n, k, m)$ *instead of* $a(n, k, m)$.
- **ii:** $A \in (f : f(u, v, \Delta))$ *if and only if (4.1), (4.14), (4.20) hold with* $c(n, k, m)$ *instead of* $a(n, k, m)$ *and (4.16) hold with* c_{nk} *instead of* a_{nk}.
- **iii:** $A \in (c : f(u, v, \Delta))$ *if and only if (4.1), (4.14) and (4.16) hold with* c_{nk} *instead of* a_{nk}.
- **iv:** $A \in (bs : f(u, v, \Delta))$ *if and only if (4.14), (4.17), (4.18) hold with* c_{nk} *instead of* a_{nk} *and (4.19) holds with* $c(n, k)$ *instead of* $a(n, k)$.
- **v:** $A \in (fs : f(u, v, \Delta))$ *if and only if (4.14), (4.18) hold with* c_{nk} *instead of* a_{nk}*, (4.20) holds with* $c(n, k, m)$ *instead of* $a(n, k, m)$ *and (4.19) holds with* $c(n, k)$ *instead of* $a(n, k)$.
- **vi:** $A \in (cs : f(u, v, \Delta))$ *if and only if (4.17) and (4.18) hold with* c_{nk} *instead of* a_{nk}.

Characterization of the classes $(f(u, v, \Delta) : f_\infty)$, $(f_\infty : f(u, v, \Delta))$, $(fs(u, v, \Delta) : f_\infty)$ and $(f_\infty : fs(u, v, \Delta))$ is redundant since the spaces of almost bounded sequences f_∞ and ℓ_∞ are equal.

5. Some core theorems

Let us start with the definition of $G(u, v, \triangle)-$core of x. Let $x \in \ell_\infty$ then $G(u, v, \triangle)-$core of x defined by the closed interval $[G(x), g(x)]$, where

$$\begin{aligned} G(x) &= \limsup_{m\to\infty} \sup_{n\in\mathbb{N}} \sum_{k=0}^{m} \frac{1}{m+1} \sum_{i=0}^{k} u_k \nabla v_i x_{n+i}, \\ g(x) &= \liminf_{n\to\infty} \sup_{n\in\mathbb{N}} \sum_{k=0}^{m} \frac{1}{m+1} \sum_{i=0}^{k} u_k \nabla v_i x_{n+i}. \end{aligned}$$

Hence, it is easy to see that $G(u, v, \triangle)-$core of x is α if and only if $f(u, v, \triangle) - \lim x = \alpha$.

Theorem 5.1. *$G(u,v,\triangle)-core(Ax)\subseteq K-core(x),(G(Ax)\leqslant L(x))$ for all $x\in\ell_\infty$ if and only if $A\in(c:f(u,v,\triangle))_{reg}$ and*

$$\lim_{m\to\infty}\sup_{n\in\mathbb{N}}\sum_j\frac{1}{m+1}\left|\sum_{k=0}^{m}u_k\sum_{i=0}^{k}\nabla v_i a_{n+i,j}\right|=1. \tag{5.1}$$

Proof. Assume that $G(u,v,\triangle)-core(Ax)$. Let $x=(x_k)$ be a convergent sequences, so we have $L(x)=l(x)$. By this assumption, we have

$$l(x)\leqslant g(Ax)\leqslant G(Ax)\leqslant L(x)$$

Hence, we obtain $G(Ax)=g(Ax)=\lim x$ which implies that $A\in(c:f(\triangle^{(m)}))_{reg}$. Now, let us consider the sequences $B=b_{ij}(n)$ of infinite matrices defined by

$$b_{ij}(n)=\frac{1}{m+1}\sum_{j=0}^{m}u_k\sum_{i=0}^{k}\nabla v_i a_{n+i,j}.$$

Since $A\in(c:f(u,v,\triangle))_{reg}$, it is easy to see that the conditions of Lemma 2 (see Das, [21]) are satisfied for the matrix sequence A. Hence, there exists $y\in\ell_\infty$ such that $\|y\|\leqslant 1$ and

$$G(Ay)=\limsup_{m\to\infty}\sup_{n}\sum_j|b_{ij}(n)|. \tag{5.2}$$

Hence, $x=e=(1,1,1\ldots)$, by using the hypothesis, we can write

$$\begin{aligned}1 &= g(Ae)\leqslant\liminf_{m\to\infty}\sup_{n\in\mathbb{N}}\sum_j|b_{ij}(n)|\leqslant\limsup_{m\to\infty}\sup_{n\in\mathbb{N}}\sum_j|b_{ij}(n)|\\ &= G(Ay)\leqslant L(y)\leqslant\|y\|\leqslant 1.\end{aligned}$$

Which proves necessity of (5.1).

On the other hand, let $A\in(c:f(u,v,\triangle))_{reg}$ and (5.1) hold for all $x\in\ell_\infty$. We define any real number μ we write $\mu^+=\max\{0,\mu\}$ and $\mu^-=\max\{-\mu,0\}$ then, $|\mu|=\mu^++\mu^-$ and $\mu=\mu^+-\mu^-$. Hence, for any given $\epsilon>0$, there exists a $j_0\in\mathbb{N}$ such that $x_j<L(x)+\epsilon$ for all $j>j_0$. Then, we can write

$$\begin{aligned}\sum_j b_{ij}(n) &= \sum_{j<j_0}(b_{ij}(n)x_j+\sum_{j\geqslant j_0}(b_{ij}(n)^+x_j-\sum_{j\geqslant j_0}(b_{ij}(n))^-x_j\\ &\leqslant \|x\|_\infty\sum_{j<j_0}|b_{ij}(n)|+[L(x)+\epsilon]\sum_{j\geqslant j_0}|b_{ij}(n|+\|x\|_\infty\sum_{j\geqslant j_0}[|b_{ij}(n)|-b_{ij}(n)].\end{aligned}$$

Thus, by applying the $\limsup_m\sup_{n\in\mathbb{N}}$ above equation and using hypothesis, we have $G(x)\leqslant L(x)+\epsilon$ This completes the proof, since ϵ is arbitrary and $x\in\ell_\infty$. □

Theorem 5.2. *$A\in(S\cap\ell_\infty:f(u,v,\triangle))_{reg}$ if and only if $A\in(c:f(u,v,\triangle))_{reg}$ and*

$$\lim_{m\to\infty}\sum_{j\in E}\frac{1}{m+1}\left|\sum_{k=0}^{m}u_k\sum_{i=0}^{k}\nabla v_i a_{n+i,j}\right|=0. \tag{5.3}$$

for every $E\subseteq\mathbb{N}$ with $\delta(E)=0$.

Proof. Firstly, suppose that $A\in(S\cap\ell_\infty:f(u,v,\triangle))_{reg}$. Then $A\in(c:f(u,v,\triangle))_{reg}$ immediately follows from the fact that $c\subseteq S\cap\ell_\infty$. Now, define sequence $s=(s_k)$ for all $x\in\ell_\infty$ as

$$s_k=\begin{cases}x_k, & k\in E,\\ 0, & k\notin E,\end{cases}$$

where E any subset of $\mathbb{N}$ with $\delta(E)=0$. By our assumption, since $s\in S_0$, we have $As\in f(u,v,\triangle)$. On the other hand, since $As=\sum_{k\in E}a_{nk}s_k$, the matrix $C=(c_{nk})$ defined by

$$c_{nk}=\begin{cases}a_{nk}, & k\in E,\\ 0, & k\notin E\end{cases}$$

for all n, must belong to the class $(\ell_\infty:f(u,v,\triangle))$. Hence, the necessity (5.3) follows from Corollary (4.16)(i). Conversely, let $A\in(c:f(u,v,\triangle))_{reg}$ and (5.3) hold for all $x\in\ell_\infty$. Let x any sequence in $S\cap\ell_\infty$ with

$st-\lim x = s$ and write $E=\{i : |x_i - p| \geqslant \epsilon\}$ for an given $\epsilon > 0$, so that $\delta(E)=0$. Since $A \in (c : f(u,v,\triangle))_{reg}$ and $f(u,v,\triangle)-\lim\sum_k a_{nk}=1$, we have

$$\begin{aligned} f(u,v,\triangle)-\lim(Ax) &= f(u,v,\triangle)-\lim\left(\sum_k a_{nk}(x_k-p)+p\sum_k a_{nk}\right) \\ &= f(u,v,\triangle)-\lim\sum_k a_{nk}(x_k-p)+p \\ &= \lim_{m\to\infty}\sup_{n\in\mathbb{N}}\sum_j \frac{1}{m+1}\sum_{k=0}^{m}u_k\sum_{i=0}^{k}\nabla v_i a_{n+i,j}(x_j-p)+p \end{aligned}$$

On the other hand, since we have

$$\left|\sum_j \frac{1}{m+1}\sum_{k=0}^{m}u_k\sum_{i=0}^{k}\nabla v_i a_{n+i,j}(x_j-p)\right| \leqslant \|x\|_\infty \sum_{j\in E}\frac{1}{m+1}\left|\sum_{k=0}^{m}u_k\sum_{i=0}^{k}\nabla v_i a_{n+i,j}(x_j-p)\right| + \epsilon\|A\|$$

the condition (5.3) implies that

$$\lim_{m\to\infty}\sum_j \frac{1}{m+1}\sum_{k=0}^{m}u_k\sum_{i=0}^{k}\nabla v_i a_{n+i,j}(x_j-p)=0 \text{ uniformly } n.$$

Therefore, $f(u,v,\triangle)-\lim(Ax)=st-\lim x$ that is $A\in(S\cap\ell_\infty : f(u,v,\triangle))_{reg}$ which completes the proof. □

Theorem 5.3. *$G(u,v,\triangle)-core(Ax)\subseteq st-core(x)$ for all $x\in\ell_\infty$ if and only if $A\in(S\cap\ell_\infty : f(u,v,\triangle))_{reg}$ and* (5.1) *holds.*

Proof. Assume that the inclusion $G(u,v,\triangle)-core(Ax)\subseteq st-core(x)$ holds for each bounded sequence x. Then, $G(Ax)\leqslant st-\sup(x)$ for all $x\in\ell_\infty$. Hence one may easily see that the following inequalities hold:

$$st-\inf(x)\leqslant g(Ax)\leqslant G(Ax)\leqslant st-\sup(x).$$

If $x\in(S\cap\ell_\infty)$, then we have $st-\inf(x)=st-\sup(x)=st-\lim x$. Which implies that $st-\sup(x)=g(x)=G(x)=f(u,v,\triangle)-\lim(Ax)$ that is $A\in(c : f(u,v,\triangle))_{reg}$.

On the other hand, let $A\in(S\cap\ell_\infty : f(u,v,\triangle))_{reg}$ and (5.1) hold for all $x\in\ell_\infty$. Then $st-\sup(x)$ is finite. Let E be a subset of $\mathbb{N}$ defined by $E=\{k : x_k > st-\sup(x)+\epsilon\}$ for a given $\epsilon>0$. Then obvious that $\delta(E)=0$ and $x_k\leqslant st-\sup(x)+\epsilon$, if $k\notin E$.

We define any real number μ we write $\mu^+=\max\{0,\mu\}$ and $\mu^-=\max\{-\mu,0\}$ then $|\mu|=\mu^++\mu^-$ and $\mu=\mu^+-\mu^-$, then for fixed positive integer m we can write

$$\begin{aligned} \sum_j b_{ij}(n) &= \sum_{j<j_0} b_{ij}(n)x_j+\sum_{\substack{j\geqslant j_0\\ j\in E}}(b_{ij}(n))^+x_j \\ &+ \sum_{\substack{j>j_0\\ k\notin E}}(b_{ij}(n))^-x_j-\sum_{j\geqslant j_0}(b_{ij}(n))^+x_j \\ &\leqslant \|x\|_\infty\sum_{j<j_0}|b_{ij}(n)|+[st-\sup(x)+\epsilon]\sum_{\substack{j\geqslant j_0\\ j\notin E}}|b_{ij}(n)| \\ &+ \|x\|_\infty\sum_{\substack{j\geqslant j_0\\ j\in E}}|b_{ij}(n)|+\|x\|_\infty\sum_{j\geqslant j_0}[|b_{ij}(n)|-b_{ij}(n)]. \end{aligned}$$

Whence, applying $\limsup_m \sup_{n\in\mathbb{N}}$ to the above equation and using the hypothesis, we obtain $G(x)\leqslant st-\sup(x)+\epsilon$. This completes the proof since ϵ is arbitrary and $x\in\ell_\infty$. □

6. Conclusion

The general aim of this study is to fill a gap in literature by extending certain topological spaces and to investigate some topological properties in addition to some core theorems.

Domain of the generalized difference matrix $B(r,s)$ in the almost convergent spaces f and f_0 was introduced by Başar and Kirişçi in [16]. One of the nice part of their paper was to find beta- and gamma duals of the set of almost convergent series fs. As a generalization of the spaces Başar and Kirişçi, the sequence space $f(B)$ which is matrix domain of the triple band matrix $B(r,s,t)$ in the almost convergent space f has been examined by Sönmez.

In the present paper, we study the domains of the triangle matrix $G(u, v, \triangle)$ in the almost convergent sequence spaces f and f_0 and series space fs. Nevertheless, the present results does not compare with the results obtained by Sönmez [2] and Başar and Kirişçi [16]. But our results are more general and more comprehensive than the corresponding results of Kayaduman and Şengönül [25, 26], since the space $f(u, v, \triangle)$ and $f_0(u, v, \triangle)$ reduce in the cases $v_k - v_{k-1} = r_k, u_n = \frac{1}{R_n}$ and $v_k - v_{k-1} = 1, u_n = \frac{1}{n}$ to the Riesz sequence space $\widehat{f}, \widehat{f_0}$ and to the Cesàro sequence space $\widetilde{f}, \widetilde{f_0}$, respectively.

Acknowledgement

We thank the referees for their careful reading of the original manuscript and for the valuable comments.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 19, NO.1, 2015, COPYRIGHT 2015 EUDOXUS PRESS, LLC

Ground state solutions for discrete nonlinear Schrödinger equations with potentials

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Abstract

In this paper, we consider discrete nonlinear Schrödinger equations with periodic potentials and bounded potentials. Under a more general super-quadratic condition instead of the classical Ambrosetti-Rabinowitz condition, we prove the existence of ground state solutions of the equations by using the critical point theory in combination with the Nehari manifold approach.

Key words Solitons; Discrete nonlinear Schrödinger equations; Nehari manifold; Ground state solutions; Critical point theory.

2000 Mathematics Subject Classification 39A10, 39A12.

1 Introduction

In this paper, we study discrete solitons for the discrete nonlinear Schrödinger (DNLS) equation

$$i\dot{\psi}_n = -\Delta\psi_n + v_n\psi_n - f(n, \psi_n), \quad n \in \mathbb{Z}, \tag{1.1}$$

where

$$\Delta\psi_n = \psi_{n+1} + \psi_{n-1} - 2\psi_n$$

is the discrete Laplacian in one spatial dimension, potential $V = \{v_n\}$ is a sequence of real number.

We assume that $f(n, 0) = 0$ and the nonlinearity $f(n, u)$ is gauge invariant, i.e.,

$$f(n, e^{i\theta}u) = e^{i\theta}f(n, u), \quad \theta \in \mathbb{R}.$$

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Since solitons are spatially localized time-periodic solutions and decay to zero at infinity. Thus ψ_n has the form

$$\psi_n = u_n e^{-i\omega t},$$

and

$$\lim_{|n|\to\infty} \psi_n = 0,$$

where $\{u_n\}$ is a real valued sequence and $\omega \in \mathbb{R}$ is the temporal frequency. Then (1.1) becomes

$$-\Delta u_n + v_n u_n - \omega u_n = f(n, u_n), \quad n \in \mathbb{Z}, \tag{1.2}$$

and

$$\lim_{|n|\to\infty} u_n = 0 \tag{1.3}$$

holds. Naturally, if we look for discrete solitons of equation (1.1), we just need to get the solutions of equation (1.2) satisfying (1.3).

DNLS equation is one of the most important inherently discrete models, having a crucial role in the modeling of a great variety of phenomena, ranging from solid state and condensed matter physics to biology (see [1–4, 11] and reference therein). For example, Davydov ([1]) studied the discrete nonlinear Schrödinger in molecular biology and Su et al. ([11]) considered the equation in condensed matter physics.

In the past decade, the existence of discrete solitons in DNLS equations has been considered in a number of studies. If the potential is unbounded, the existence of nontrivial solitons of DNLS equations was obtained in [12, 16, 17] by Nehari manifold method and in [5] by the mountain pass theorem and the fountain theorem, respectively. If the potential is periodic, the existence of discrete solitons for the periodic DNLS equations has been studied, see [6–10, 15] and the reference therein. In our paper, we consider two cases of the potentials, one is periodic; the other is nonperiodic corresponding to a potential well in the sense that $\lim_{|n|\to\infty} v_n$ exists and is equal to $\sup_{\mathbb{Z}} V$.

In some paper (see, e.g. [14]), the following classical Ambrosetti-Rabinowitz superlinear condition is assumed:

$$0 < \mu F(n, u) \leq f(n, u)u, \quad \text{for some} \ \ \mu > 2 \ \ \text{and} \ u \neq 0, \tag{1.4}$$

where $F(n, u)$ is the primitive function of $f(n, u)$, *i.e.*,

$$F(n, u) = \int_0^u f(n, s)ds.$$

It is easy to see that (1.4) implies that $F(n, u) \geq C|u|^\mu$, for some constant $C > 0$ and $|u| \geq 1$.

In this paper, instead of (1.4) we assume the following super-quadratic condition

$$\lim_{|u|\to\infty} F(n, u)/u^2 = +\infty \ \ \text{uniformly for} \ \ n \in \mathbb{Z}. \tag{1.5}$$

It is well known that (1.4) implies (1.5). It is also well known that many nonlinearities such as

$$f(n, u) = u \ln(1 + |u|),$$

do not satisfy (1.4). A crucial role that (1.4) plays is to ensure the boundedness of Palais-Smale sequences. We obtain the existence of ground state solutions without Palais-Smale condition.

This paper is organized as follows: In Section 2, we establish the variational framework associated with (1.2). We consider two cases of the potentials in Section 3 and in Section 4, respectively.

2 Preliminaries

In this section, we establish the variational framework associated with (1.2). Let

$$A = -\triangle + V - \omega \text{ and } E = l^2(\mathbb{Z}),$$

and

$$l^p \equiv l^p(\mathbb{Z}) = \left\{ u = \{u_n\}_{n\in\mathbb{Z}} : \forall\, n \in \mathbb{Z}, u_n \in \mathbb{R}, \|u\|_{l^p} = \left(\sum_{n\in\mathbb{Z}} |u_n|^p\right)^{\frac{1}{p}} < \infty \right\}. \tag{2.1}$$

Then the following embedding between l^p spaces holds,

$$l^q \subset l^p, \|u\|_{l^p} \leq \|u\|_{l^q}, 1 \leq q \leq p \leq \infty. \tag{2.2}$$

Consider the functional J defined on E by

$$J(u) = \frac{1}{2}\ (Au, u) - \sum_{n\in\mathbb{Z}} F(n, u_n). \tag{2.3}$$

where $(\cdot, \cdot)$ is the inner product in l^2. Then $J \in C^1(E, \mathbb{R})$ and the derivative of J we have the following formula,

$$(J'(u), v) = (Au, v) - \sum_{n\in\mathbb{Z}} f(n, u_n) v_n, \quad \forall\, v \in E. \tag{2.4}$$

Equation (2.4) implies that (1.2) is the corresponding Euler-Lagrange equation for J. Thus, we have reduced the problem of finding a nontrivial solution of (1.2) to that of seeking a nonzero critical point of the functional J on E.

We define the Nehari manifold

$$N = \{u \in E \setminus \{0\} : J'(u)u = 0\}, \tag{2.5}$$

and

$$c = \inf_{u\in N} J(u). \tag{2.6}$$

We are concern with the existence of ground state solutions, *i.e.*, solutions corresponding to the least positive critical value of the variational functional.

3 The periodic potential case

Now, we present the following basic hypotheses in order to establish the main results in this Section:

(V_1) The discrete potential $V = \{v_n\}$ is assumed to be $T-$periodic in n, *i.e.*, $v_{n+T} = v_n$ and $v_n - \omega > 0$ for all n.

(f_1) $f \in C(\mathbb{Z} \times \mathbb{R}, \mathbb{R})$ and $f(n,u) = f(n+T,u)$, and there exist $a > 0$, $p \in (2,\infty)$ such that

$$|f(n,u)| \leq a(1+|u|^{p-1}), \quad \text{for all } n \in \mathbb{Z} \text{ and } u \in \mathbb{R}.$$

(f_2) $\lim\limits_{|u|\to 0} f(n,u)/u = 0$ uniformly for $n \in \mathbb{Z}$.

(f_3) $\lim\limits_{|u|\to \infty} F(n,u)/u^2 = +\infty$ uniformly for $n \in \mathbb{Z}$, where $F(n,u)$ is the primitive function of $f(n,u)$, i.e.,

$$F(n,u) = \int_0^u f(n,s)ds.$$

(f_4) $u \mapsto f(n,u)/|u|$ is strictly increasing on $(-\infty,0)$ and $(0,\infty)$ for each $n \in \mathbb{Z}$.

Under the above hypotheses, our results can be stated as follows.

Theorem 3.1. *Assume that conditions $(V_1), (f_1) - (f_4)$ hold. Then equation (1.2) has a ground state solution.*

By the condition (V_1), we may introduce an equivalent norm in E by setting

$$\|u\|^2 := (Au,u),$$

then the functional J can be rewritten as

$$J(u) = \frac{1}{2}\ \|u\|^2 - I(u),$$

where $I(u) = \sum\limits_{n\in\mathbb{Z}} F(n,u_n)$.

We assume that (V_1) and $(f_1) - (f_4)$ are satisfied from now on.

Lemma 3.1. *(a)* $F(n,u) > 0$ *and* $\frac{1}{2}f(n,u)u > F(n,u)$ *for all* $u \neq 0$.

(b) $J(u) > 0$, *for all* $u \in N$.

Proof. *(a)* From (f_2) and (f_4), it is easy to get that

$$F(n,u) > 0, \quad \text{for all } u \neq 0. \tag{3.1}$$

Set $H(n,u) = \frac{1}{2}f(n,u)u - F(n,u)$, by (f_4), we have

$$H(n,u) = \frac{u}{2}f(n,u) - \int_0^u f(n,s)ds > \frac{u}{2}f(n,u) - \frac{f(n,u)}{u}\int_0^u sds = 0.$$

So $\frac{1}{2}f(n,u)u > F(n,u)$ for all $u \neq 0$.

(b) For all $u \in N$, by (a), we have

$$J(u) = J(u) - \frac{1}{2}J'(u)u = \sum_{n\in\mathbb{Z}} \left(\frac{1}{2}f(n,u)u - F(n,u)\right) > 0. \qquad \square$$

Lemma 3.2. (a) *$I'(u) = o(\|u\|)$ as $u \to 0$.*

(b) *$s \mapsto I'(su)u/s$ is strictly increasing for all $u \neq 0$ and $s > 0$.*

(c) *$I(su)/s^2 \to \infty$ uniformly for u on the weakly compact subsets of $E \setminus \{0\}$, as $s \to \infty$.*

Proof. (a) and (b) are easy to be shown from (f_2) and (f_4), respectively. Next we verify (c). Let $W \subset E \setminus \{0\}$ be weakly compact and let $\{u^{(k)}\} \subset W$. It suffices to show that if $s^{(k)} \to \infty$ as $k \to \infty$, then so does a subsequence of $I(s^{(k)}u^{(k)})/(s^{(k)})^2$. Passing to a subsequence if necessary, $u^{(k)} \rightharpoonup u \in E \setminus \{0\}$ and $u_n^{(k)} \to u_n$ for every n, as $k \to \infty$.

Since $|s^{(k)}u_n^{(k)}| \to \infty$ and $u^{(k)} \neq 0$, by (f_3) and (3.1), we have

$$\frac{I(s^{(k)}u^{(k)})}{(s^{(k)})^2} = \sum_{n\in\mathbb{Z}} \frac{F(n,u_n^k)}{(s^{(k)}u_n^{(k)})^2}(u_n^{(k)})^2 \to \infty \quad \text{as } k \to \infty. \qquad \square$$

Lemma 3.3. *For each $w \in E \setminus \{0\}$, there exists a unique $s_w > 0$ such that $s_w w \in N$.*

Proof. Let $g(s) := J(sw),\ s > 0$. Since

$$g'(s) = J'(sw)w = s(\|w\|^2 - s^{-1}I'(sw)w),$$

from (b) of Lemma 3.2, then there exists a unique s_w, such that $g'(s) > 0$ whenever $0 < s < s_w$, $g'(s) < 0$ whenever $s > s_w$ and $g'(s_w) = J'(s_w w)w = 0$. So $s_w w \in N$. $\square$

Remark 3.1. By (a) and (c) of Lemma 3.2, $g(s) > 0$ for $s > 0$ small and $g(s) < 0$ for $s > 0$ large. Together with Lemma 3.3, we have s_w is a unique maximum of $g(s)$ and $s_w w$ is the unique point on the ray $s \mapsto sw$ $(s > 0)$ which intersects N. That is, $u \in N$ is the unique maximum of J on the ray. Therefore, we may define the mapping $\hat{m} : E \setminus \{0\} \to N$ and $m : S \to N$ by setting

$$\hat{m}(w) := s_w w \quad \text{and} \quad m := \hat{m}|_S,$$

where $S = \{u \in E : \|u\| = 1\}$.

Lemma 3.4. *For each compact subset $\mathcal{V} \subset S$, there exists a constant $C_{\mathcal{V}}$ such that $s_w \leq C_{\mathcal{V}}$ for all $w \in \mathcal{V}$.*

Proof. Suppose by contradiction, $s_w^{(k)} \to \infty$ as $k \to \infty$. By Lemma 3.1 and (f_3), we have

$$0 < \frac{J(s_w^{(k)}w)}{(s_w^{(k)})^2} = \frac{1}{2}\|w\|^2 - \sum_{n\in\mathbb{Z}} \frac{F(n,s_w^{(k)}w_n)}{(s_w^{(k)})^2 w_n^2} w_n^2 \to -\infty, \quad \text{as } k \to \infty,$$

this is a contradiction. $\square$

Lemma 3.5. (a) *The mapping* $\hat{m}$ *is continuous.*

(b) *The mapping* m *is a homeomorphism between* S *and* N*, and the inverse of* m *is given by* $m^{-1}(u) = \frac{u}{\|u\|}$.

Proof. (a) Suppose $w_n \to w \neq 0$. Since $\hat{m}(tu) = \hat{m}(u)$ for each $t > 0$, we may assume $w_n \in S$ for all n. Write $\hat{m}(w_n) = s_{w_n} w_n$. By Lemma 3.3 and Lemma 3.4, $\{s_{w_n}\}$ is bounded, hence $s_{w_n} \to s > 0$ after passing to a subsequence if needed. Since N is closed and $\hat{m}(w_n) = s_{w_n} w_n \to sw, sw \in N$. Hence $sw = s_w w = \hat{m}(w)$ by the uniqueness of s_w of Lemma 3.3.

(b) This is an immediate consequence of (a). $\square$

Lemma 3.6. J *is coercive on* N*, i.e.,* $J(u) \to \infty$ *as* $\|u\| \to \infty$*,* $u \in N$*.*

Proof. Suppose by contradiction, there exists a sequence $\{u^{(k)}\} \subset N$ such that $\|u^{(k)}\| \to \infty$ and $J(u^{(k)}) \leq d$, for some $d \in [c, \infty)$. Let $v^{(k)} = \frac{u^{(k)}}{\|u^{(k)}\|}$, then there exists a subsequence, still denoted by the same notation, such that $v^{(k)} \rightharpoonup v$ and $v_n^{(k)} \to v_n$ for every n, as $k \to \infty$.

First we know that there exist $\delta > 0$ and $n_k \in \mathbb{Z}$ such that

$$|v_{n_k}^{(k)}| \geq \delta. \tag{3.2}$$

Indeed, if not, then $v^{(k)} \to 0$ in l^∞ as $k \to \infty$. For $q > 2$,

$$\|v^{(k)}\|_{l^q}^q \leq \|v^{(k)}\|_{l^\infty}^{q-2} \|v^{(k)}\|_{l^2}^2$$

we have $v^{(k)} \to 0$ in all l^q, $q > 2$.

Note that by (f_1) and (f_2), for any $\varepsilon > 0$, there exists $c_\varepsilon > 0$ such that

$$|f(n,u)| \leq \varepsilon|u| + c_\varepsilon|u|^{p-1} \quad \text{and} \quad |F(n,u)| \leq \varepsilon|u|^2 + c_\varepsilon|u|^p. \tag{3.3}$$

Then for each $s > 0$, we have

$$\sum_{n\in\mathbb{Z}} F(n, sv_n^{(k)}) \leq \varepsilon s^2 \|v^{(k)}\|_{l^2}^2 + c_\varepsilon s^p \|v^{(k)}\|_{l^p}^p$$

which implies that $\sum\limits_{n\in\mathbb{Z}} F(n, sv_n^{(k)}) \to 0$ as $k \to \infty$. Therefore

$$d \geq J(u^{(k)}) \geq J(sv^{(k)}) = \frac{s^2}{2}\|v^{(k)}\|^2 - \sum_{n\in\mathbb{Z}} F(n, sv_n^{(k)}) \to \frac{s^2}{2}, \tag{3.4}$$

as $k \to \infty$. This is a contradiction if $s > \sqrt{2d}$.

Due to periodicity of coefficients, we know J and N are both invariant under T-translation. Making such shifts, we can assume that $1 \leq n_k \leq T-1$ in (3.2). Moreover, passing to a subsequence, we can assume that $n_k = n_0$ is independent of k.

Next we may extract a subsequence, still denoted by $\{v^{(k)}\}$, such that $v_n^{(k)} \to v_n$ for all $n \in \mathbb{Z}$. Specially, for $n = n_0$, inequality (3.2) shows that $|v_{n_0}| \geq \delta$, so $v \neq 0$. Since $|u_n^{(k)}| \to \infty$ as $k \to \infty$, it follows again from (f_3) and Lemma 3.1 that

$$0 \leq \frac{J(u^{(k)})}{\|u^{(k)}\|^2} = \frac{1}{2} - \sum_{n\in\mathbb{Z}} \frac{F(n, u_n^{(k)})}{(u_n^{(k)})^2}(v_n^{(k)})^2 \to -\infty \qquad as\ k \to \infty,$$

a contradiction again. □

We shall consider the functional $\hat{\Psi} : E \setminus \{0\} \to \mathbb{R}$ and $\Psi : S \to \mathbb{R}$ defined by

$$\hat{\Psi} := J(\hat{m}(w)) \quad \text{and} \quad \Psi := \hat{\Psi}|_S. \tag{3.5}$$

Lemma 3.7. *(a)* $\hat{\Psi} \in C^1(E \setminus \{0\}, \mathbb{R})$ *and*

$$\hat{\Psi}'(w)z = \frac{\|\hat{m}(w)\|}{\|w\|} J'(\hat{m}(w))z \quad \text{for all } w, z \in E, w \neq 0.$$

(b) $\Psi \in C^1(S, \mathbb{R})$ *and*

$$\Psi'(w)z = \|m(w)\| J'(m(w))z \quad \text{for all } z \in T_w(S) = \{v \in E : (w, v) = 0\}.$$

(c) If $\{w_n\}$ *is a Palais-Smale sequence for* Ψ *if and only if* $\{m(w_n)\}$ *is a Palais-Smale sequence for* J.

(d) w *is a critical point of* Ψ *if and only if* $m(w)$ *is a nontrivial critical point of* J*. Moreover, the corresponding values of* Ψ *and* J *coincide and* $\inf_S \Psi = \inf_N J$.

Proof.(a) Let $w \in E \setminus \{0\}$ and $z \in E$. By Remark 3.1 and the mean value theorem, we obtain

$$\begin{aligned} \hat{\Psi}(w+tz) - \hat{\Psi}(w) &= J(s_{w+tz}(w+tz)) - J(s_w w) \\ &\leq J(s_{w+tz}(w+tz)) - J(s_{w+tz}(w)) \\ &= J'(s_{w+tz}(w+\tau_t tz))s_{w+tz}tz, \end{aligned}$$

where $|t|$ is small enough and $\tau_t \in (0, 1)$. Similarly,

$$\begin{aligned} \hat{\Psi}(w+tz) - \hat{\Psi}(w) &= J(s_{w+tz}(w+tz)) - J(s_w w) \\ &\geq J(s_w(w+tz)) - J(s_w(w)) \\ &= J'(s_w(w+\eta_t tz))s_w tz, \end{aligned}$$

where $\eta_t \in (0, 1)$. From the proof of Lemma 3.5, the function $w \mapsto s_w$ is continuous, combining these two inequalities that

$$\lim_{t\to 0} \frac{\hat{\Psi}(w+tz) - \hat{\Psi}(w)}{t} = s_w J'(s_w w)z = \frac{\|\hat{m}(w)\|}{\|w\|} J'(\hat{m}(w))z.$$

Hence the Gâteaux derivative of $\hat{\Psi}$ is bounded linear in z and continuous in w. It follows that $\hat{\Psi}$ is a class of C^1 (see [14], Proposition1.3).

(b) follows from (a). Note only that since $w \in S$, $m(w) = \hat{m}(w)$.

(c) Let $\{w_n\}$ be a Palais-Smale sequence for Ψ, and let $u_n = m(w_n) \in N$. Since for every $w_n \in S$ we have an orthogonal splitting $E = T_{w_n}S \oplus \mathbb{R}w_n$. Using (b) we have

$$\|\Psi'(w_n)\| = \sup_{\substack{z\in T_{w_n}S\\ \|z\|=1}} \Psi'(w_n)z = \|m(w_n)\| \sup_{\substack{z\in T_{w_n}S\\ \|z\|=1}} J'(m(w_n))z = \|u_n\| \sup_{\substack{z\in T_{w_n}S\\ \|z\|=1}} J'(u_n)z,$$

Using (b) again, then

$$\begin{aligned} \|\Psi'(w_n)\| &\leq \|u_n\|\|J'(u_n)\| = \|u_n\| \sup_{\substack{z\in T_{w_n}S,\ t\in\mathbb{R}\\ z+tw\neq 0}} \frac{J'(u_n)(z+tw)}{\|z+tw\|} \\ &\leq \|u_n\| \sup_{z\in T_{w_n}S\setminus\{0\}} \frac{J'(u_n)(z)}{\|z\|} = \|\Psi'(w_n)\|, \end{aligned}$$

Therefore

$$\|\Psi'(w_n)\| = \|u_n\|\|J'(u_n)\|. \tag{3.6}$$

Noticing that $c \leq J(u_n) = \frac{1}{2}\ \|u_n\|^2 - I(u_n) \leq \frac{1}{2}\ \|u_n\|^2$, then $\|u_n\| \geq \sqrt{2c}$. Together with Lemma 3.6, $\sqrt{2c} \leq \|u_n\| \leq \sup_n \|u_n\| < \infty$. Hence $\{w_n\}$ is a Palais-Smale sequence for Ψ if and only if $\{u_n\}$ is a Palais-Smale sequence for J.

(d) By (3.6), $\Psi'(w) = 0$ if and only if $J'(m(w)) = 0$. The other part is clear. □

Proof of Theorem 3.1. If $u_0 \in N$ satisfies $J(u_0) = c$, then $m^{-1}(u_0) \in S$ is a minimizer of Ψ and therefore a critical point of Ψ, so u_0 is a critical point of J by Lemma 3.7. It remains to show that there exists a minimizer $u \in N$ of $J|_N$. Let $\{w^{(k)}\} \subset S$ be a minimizing sequence for Ψ. By Ekeland's variational principle we may assume $\Psi(w^{(k)}) \to c,\ \Psi'(w^{(k)}) \to 0$ as $k \to \infty$, hence $J(u^{(k)}) \to c$, $J'(u^{(k)}) \to 0$ as $k \to \infty$, where $u^{(k)} := m(w^{(k)}) \in N$.

First, we know that $\{u^{(k)}\}$ is bounded in N by Lemma 3.6, then there exists a subsequence, still denoted by the same notation, such that $u^{(k)}$ weakly converges to some $u \in E$. We claim that there exist $\delta > 0$ and $n_k \in \mathbb{Z}$ such that

$$|u^{(k)}_{n_k}| \geq \delta. \tag{3.7}$$

Indeed, if not, then $u^{(k)} \to 0$ in l^∞ as $k \to \infty$. From the simple fact that, for $q > 2$,

$$\|u^{(k)}\|^q_{l^q} \leq \|u^{(k)}\|^{q-2}_{l^\infty}\|u^{(k)}\|^2_{l^2}$$

we have $u^{(k)} \to 0$ in all $l^q,\ q > 2$. By (4.6), we know

$$\begin{aligned} \sum_{n\in\mathbb{Z}} f(n, u^{(k)}_n)u^{(k)}_n &\leq \varepsilon \sum_{n\in\mathbb{Z}} |u^{(k)}_n| \cdot |u^{(k)}_n| + c_\varepsilon \sum_{n\in\mathbb{Z}} |u^{(k)}_n|^{p-1} \cdot |u^{(k)}_n| \\ &\leq \varepsilon\|u^{(k)}\|_{l^2} \cdot \|u^{(k)}\|_{l^2} + c_\varepsilon\|u^{(k)}\|^{p-1}_{l^p} \cdot \|u^{(k)}\|_{l^p} \\ &\leq \varepsilon\|u^{(k)}\|_{l^2} \cdot \|u^{(k)}\| + c_\varepsilon\|u^{(k)}\|^{p-1}_{l^p} \cdot \|u^{(k)}\|, \end{aligned}$$

which implies that $\sum_{n\in\mathbb{Z}} f(n,u_n^{(k)})u_n^{(k)} = o(\|u^{(k)}\|)$ as $k\to\infty$. Therefore, we have

$$o(\|u^{(k)}\|) = \big(J'(u^{(k)}), u^{(k)}\big) = \|u^{(k)}\|^2 - \sum_{n\in\mathbb{Z}} f(n,u_n^{(k)})u_n^{(k)} = \|u^{(k)}\|^2 - o(\|u^{(k)}\|).$$

So $\|u^{(k)}\|^2 \to 0$, as $k\to\infty$, contrary to $u^{(k)}\in N$.

Since J and J' are both invariant under T-translation. Making such shifts, we can assume that $1\le n_k\le T-1$ in (3.7). Moreover passing to a subsequence, we can assume that $n_k = n_0$ is independent of k. Extract a subsequence, still denoted by $\{u^{(k)}\}$, we have $u^{(k)} \rightharpoonup u$ and $u_n^{(k)} \to u_n$ for all $n\in\mathbb{Z}$. Specially, for $n=n_0$, inequality (3.7) shows that $|u_{n_0}|\ge\delta$, so $u\ne 0$. Hence $u\in N$.

Finally we need to show that $J(u)=c$. By Lemma 3.1 and Fatou's lemma, we have

$$\begin{aligned} c &= \lim_{k\to\infty}\left(J(u^{(k)}) - \frac{1}{2}J'(u^{(k)})u^{(k)}\right) = \lim_{k\to\infty}\sum_{n\in\mathbb{Z}}\left(\frac{1}{2}f(n,u_n^{(k)})u_n^{(k)} - F(n,u_n^{(k)})\right) \\ &\ge \sum_{n\in\mathbb{Z}}\left(\frac{1}{2}f(n,u_n)u_n - F(n,u_n)\right) = J(u) - \frac{1}{2}J'(u)u = J(u)\ge c. \end{aligned}$$

Hence $J(u)=c$. Theorem 3.1 is complete. $\square$

4 The potential well cases

Now we consider

$$-\Delta u_n + v_n u_n - \omega u_n = f(u_n), \quad n\in\mathbb{Z}, \tag{4.1}$$

for the case where $V=\{v_n\}$ is nonperiodic corresponding to a potential well. Since the nonlinearity is autonomous, the conditions on f needs modified slightly. More precisely, we make the following assumptions.

(V_2) $0<\inf_{\mathbb{Z}} V \le \sup_{\mathbb{Z}} V = V_\infty < \infty$ with $V_\infty := \lim_{|n|\to\infty} v_n$ and $v_n - \omega > 0$, for all n.

(f_1') $f\in C(\mathbb{R},\mathbb{R})$ and there exist $a>0$, $p\in(2,\infty)$ such that

$$|f(u)| \le a(1+|u|^{p-1}), \quad \text{for all} \quad u\in\mathbb{R}.$$

(f_2') $\lim_{|u|\to 0} f(u)/u = 0$.

(f_3') $\lim_{|u|\to\infty} F(u)/u^2 = +\infty$, where $F(u)$ is the primitive function of $f(u)$, i.e.,

$$F(u) = \int_0^u f(s)ds.$$

(f_4') $u\mapsto f(u)/|u|$ is strictly increasing on $(-\infty,0)$ and $(0,\infty)$.

Theorem 4.1. *Assume that conditions* $(V_2), (f_1') - (f_4')$ *hold. Then equation (4.1) has a ground state solution.*

Consider the functional $\tilde{J}$ defined on E by

$$\tilde{J}(u) = \frac{1}{2}\ (Au, u) - \sum_{n\in\mathbb{Z}} F(u_n). \tag{4.2}$$

By the condition (V_2), we may introduce an equivalent norm in E by setting

$$\|u\|^2 := (Au, u),$$

then the functional $\tilde{J}$ can be rewritten as

$$\tilde{J}(u) = \frac{1}{2}\ \|u\|^2 - \tilde{I}(u),$$

where $\tilde{I}(u) = \sum\limits_{n\in\mathbb{Z}} F(u_n)$.

We define the Nehari manifold

$$\tilde{N} = \{u \in E \setminus \{0\} : \tilde{J}'(u)u = 0\}, \tag{4.3}$$

and

$$\tilde{c} = \inf_{u\in\tilde{N}} \tilde{J}(u). \tag{4.4}$$

We assume that (V_2) and $(f_1') - (f_4')$ are satisfied from now on. The argument is the same as in Lemma 3.1-3.7 with $f(u)$ which does not depend on n, except that the proof of Lemma 3.6 needs to be slightly modified.

Lemma 4.1. $\tilde{J}$ *is coercive on* $\tilde{N}$, *i.e.,* $\tilde{J}(u) \to \infty$ *as* $\|u\| \to \infty$, $u \in \tilde{N}$.

Proof. Suppose by contradiction, there exists a sequence $\{u^{(k)}\} \subset \tilde{N}$ such that $\|u^{(k)}\| \to \infty$ and $\tilde{J}(u^{(k)}) \le d$, for some $d \in [c, \infty)$. Let $v^{(k)} = \frac{u^{(k)}}{\|u^{(k)}\|}$, then there exists a subsequence, still denoted by the same notation, such that $v^{(k)} \rightharpoonup v$ and $v_n^{(k)} \to v_n$ for every n, as $k \to \infty$.

First we claim that there exist $\delta > 0$ and $n_k \in \mathbb{Z}$ such that

$$|v_{n_k}^{(k)}| \ge \delta. \tag{4.5}$$

Indeed, if not, then $v^{(k)} \to 0$ in l^∞ as $k \to \infty$. For $q > 2$,

$$\|v^{(k)}\|_{l^q}^q \le \|v^{(k)}\|_{l^\infty}^{q-2} \|v^{(k)}\|_{l^2}^2$$

we have $v^{(k)} \to 0$ in all l^q, $q > 2$.

Note that by (f_1') and (f_2'), for any $\varepsilon > 0$, there exists $c_\varepsilon > 0$ such that

$$|f(u)| \le \varepsilon|u| + c_\varepsilon|u|^{p-1} \quad \text{and} \quad |F(u)| \le \varepsilon|u|^2 + c_\varepsilon|u|^p. \tag{4.6}$$

Then for each $s > 0$, we have

$$\sum_{n\in\mathbb{Z}} F(sv_n^{(k)}) \leq \varepsilon s^2 \|v^{(k)}\|_{l^2}^2 + c_\varepsilon s^p \|v^{(k)}\|_{l^p}^p$$

which implies that $\sum_{n\in\mathbb{Z}} F(sv_n^{(k)}) \to 0$ as $k \to \infty$. Therefore

$$d \geq \tilde{J}(u^{(k)}) \geq \tilde{J}(sv^{(k)}) = \frac{s^2}{2}\|v^{(k)}\|^2 - \sum_{n\in\mathbb{Z}} F(sv_n^{(k)}) \to \frac{s^2}{2}, \tag{4.7}$$

as $k \to \infty$. This is a contradiction if $s > \sqrt{2d}$.

Making shifts $\bar{v}^{(k)} = \{\bar{v}_n^{(k)}\} = \{v_{n+z_k}^{(k)}\}$ such that $\bar{v}^{(k)} \rightharpoonup \bar{v} \neq 0$ and $\bar{v}_n^{(k)} \to \bar{v}_n$ for all $n \in \mathbb{Z}$. Since $|u_n^{(k)}| \to \infty$, as $k \to \infty$, it follows again from (f_3') and Lemma 3.1 (b) that

$$0 \leq \frac{\tilde{J}(u^{(k)})}{\|u^{(k)}\|^2} = \frac{1}{2} - \sum_{n\in\mathbb{Z}} \frac{F(u_n^{(k)})}{(u_n^{(k)})^2}(v_n^{(k)})^2 = \frac{1}{2} - \sum_{n\in\mathbb{Z}} \frac{F(\bar{u}_n^{(k)})}{(\bar{u}_n^{(k)})^2}(\bar{v}_n^{(k)})^2 \to -\infty,$$

as $k \to \infty$ with $\bar{u}^{(k)} := \{\bar{u}_n^{(k)}\} = \{u_{n+z_k}^{(k)}\}$. This is a contradiction again. □

Proof of Theorem 4.1. We consider an associated equation

$$-\Delta u_n + V_\infty u_n = f(u_n), \quad n \in \mathbb{Z}, \tag{4.8}$$

we define the energy functional $\tilde{J}_\infty$ by replacing V with V_∞, $c_\infty = \inf_{N_\infty} \tilde{J}_\infty(u)$, here $\tilde{N}_\infty = \{u \in E \setminus \{0\} : \tilde{J}_\infty'(u)u = 0\}$. Since V_∞ is constant, by Theorem 3.1, $c_\infty > 0$ is attained at some $u_\infty \in \tilde{N}_\infty$.

If $V < V_\infty$, for all $t > 0$, we have

$$\tilde{c} \leq \tilde{J}(tu_\infty) < \tilde{J}_\infty(tu_\infty) \leq \tilde{J}_\infty(u_\infty) = c_\infty.$$

So we may assume that $\tilde{c} < c_\infty$, since otherwise $\tilde{c} = c_\infty$ and the assertion follows from Theorem 3.1.

Let $\{w^{(k)}\} \subset S$ be a minimizing sequence for Ψ. By Ekeland's variational principle we may assume $\Psi(w^{(k)}) \to \tilde{c}$, $\Psi'(w^{(k)}) \to 0$ as $k \to \infty$, hence $\tilde{J}(u^{(k)}) \to \tilde{c}, \tilde{J}'(u^{(k)}) \to 0$ as $k \to \infty$, where $u^{(k)} := m(w^{(k)})$.

First, we know $\{u^{(k)}\}$ is bounded by Lemma 4.1, making shifts $\bar{u}^{(k)} = \{\bar{u}_n^{(k)}\} = \{u_{n+z_k}^{(k)}\}$ such that $\bar{u}^{(k)} \rightharpoonup u \neq 0$, then $\{z_k\}$ is bounded. If not, up to a subsequence if needed, $z_k \to \infty$, then u is a critical point of $\tilde{J}_\infty$ and

$$\begin{aligned}\tilde{c} &= \lim_{k\to\infty}\left(\tilde{J}(u^{(k)}) - \frac{1}{2}\tilde{J}'(u^{(k)})u^{(k)}\right) = \lim_{k\to\infty}\sum_{n\in\mathbb{Z}}\left(\frac{1}{2}f(u_n^{(k)})u_n^{(k)} - F(u_n^{(k)})\right) \\ &\geq \sum_{n\in\mathbb{Z}}\left(\frac{1}{2}f(u_n)u_n - F(u_n)\right) = \tilde{J}_\infty(u) - \frac{1}{2}\tilde{J}_\infty'(u)u = \tilde{J}_\infty(u) \geq c_\infty,\end{aligned}$$

this contradicts $\tilde{c} < c_\infty$. Hence the sequence $\{z_k\}$ is bounded, without loss of generality, we may assume that $z_k = 0$ and therefore $\bar{u}^{(k)} = u^{(k)}$ for all k. Then $u \neq 0$ and $u \in \tilde{N}$.

Finally we need to show that $\tilde{J}(u) = \tilde{c}$. By Lemma 3.1 and Fatou's lemma, we have

$$\begin{aligned}\tilde{c} &= \lim_{k\to\infty}\left(\tilde{J}(u^{(k)}) - \frac{1}{2}\tilde{J}'(u^{(k)})u^{(k)}\right) = \lim_{k\to\infty}\sum_{n\in\mathbb{Z}}\left(\frac{1}{2}f(u_n^{(k)})u_n^{(k)} - F(u_n^{(k)})\right)\\ &\geq \sum_{n\in\mathbb{Z}}\left(\frac{1}{2}f(u_n)u_n - F(u_n)\right) = \tilde{J}(u) - \frac{1}{2}\tilde{J}'(u)u = \tilde{J}(u) \geq \tilde{c}.\end{aligned}$$

Hence $\tilde{J}(u) = \tilde{c}$. Theorem 4.1 is complete. □

Acknowledgments

This work is supported by Program for the National Natural Science Foundation of China (No. 11371313) and the Foundation of Yuncheng University(YQ-2014011,SWSX201402).

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On fuzzy approximating spaces *

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February 14, 2014

Abstract: In this paper, we introduce the concept of fuzzy approximating spaces and obtain a decision condition that fuzzy topological spaces are fuzzy approximating spaces.

Keywords: Fuzzy set; Fuzzy approximation space; Fuzzy topology; Fuzzy approximating space; (CC) axiom.

1 Introduction

Rough set theory, proposed by Pawlak [5, 6], is a new mathematical tool for data reasoning. It may be seen as an extension of classical set theory and has been successfully applied to machine learning, intelligent systems, inductive reasoning, pattern recognition, mereology, image processing, signal analysis, knowledge discovery, decision analysis, expert systems and many other fields.

The basic structure of rough set theory is an approximation space. Based on it, lower and upper approximations can be induced. Using these approximations, knowledge hidden in information systems may be revealed and expressed in the form of decision rules (see [6]).

Topological structure is an important base for knowledge extraction and processing. Therefore, an interesting and natural research topic in rough set theory is to study the relationship between rough sets (resp. fuzzy rough sets) and topologies (resp. topologies).

The purpose of this paper is to investigate fuzzy approximating space, i.e., a particular type of fuzzy topological spaces where the given fuzzy topology coincides with the fuzzy topology induced by some reflexive fuzzy relation.

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*This work is supported by the National Natural Science Foundation of China (No. 11061004) and Guangxi University Science and Technology Research Project (No. 2013ZD020).

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2 Preliminaries

Throughout this paper, U denotes a finite and nonempty set called the universe. I denotes $[0,1]$. $F(U)$ denotes the set of all fuzzy sets in U. For $a \in I$, $\bar{a}$ denotes the constant fuzzy set in U.

We recall some basic operations on $F(U)$ as follows [10]: for any $A, B \in F(U)$, $\{A_j : j \in J\} \subseteq F(U)$ and $\lambda \in I$,

(1) $A = B \iff A(x) = B(x)$ for each $x \in U$.

(2) $A \subseteq B \iff A(x) \leq B(x)$ for each $x \in U$.

(3) $A = B^c \iff A(x) = 1 - B(x)$ for each $x \in U$.

(4) $(\bigcap\limits_{j\in J} A)(x) = \bigwedge\limits_{j\in J} A(x)$ for each $x \in U$.

(5) $(\bigcup\limits_{j\in J} A)(x) = \bigvee\limits_{j\in J} A(x)$ for each $x \in U$.

(6) $(\lambda A)(x) = \lambda \wedge A(x)$ for each $x \in U$.

A fuzzy set is called a fuzzy point in U, if it takes the value 0 for each $y \in U$ except one, say, $x \in U$. If its value at x is λ $(0 < \lambda \leq 1)$, we denote this fuzzy point by x_λ, where the point x is called its support and λ is called its height (see [3]).

Remark 2.1. $A = \bigcup\limits_{x\in U} (A(x)x_1) \quad (A \in F(U))$.

Definition 2.2 ([2])**.** *$\tau \subseteq F(U)$ is called a fuzzy topology on U, if*

(i) For each $a \in I$, $\bar{a} \in \tau$.

(ii) $A \in \tau,\ B \in \tau$ imply $A \cap B \in \tau$.

(iii) $\{A_j : j \in J\} \subseteq \tau$ implies $\bigcup\limits_{j\in J} A_j \in \tau$.

In this case the pair (U, τ) is called a fuzzy topological space. Every member of τ is called a fuzzy open set in U. Its complement is called a fuzzy closed set in U.

We denote $\tau^c = \{A \in F(U) : A^c \in \tau\}$.

It should be pointed out that if (i) in Definition 2.2 is replaced by

$(i)'$ $\bar{0}, \bar{1} \in \tau$,

then τ is a fuzzy topology in the sense of Chang [1]. We can see that a fuzzy topology in the sense of Lowen must be a fuzzy topology in the sense of Chang. In this paper, we always consider the fuzzy topology in the sense of Lowen.

The interior and closure of A denoted respectively by $int(A)$ and $cl(A)$ for each $A \in F(U)$, are defined as follows:

$$int_\tau(A) = \bigcup\{B \in \tau : B \subseteq A\}, \quad cl_\tau(A) = \bigcap\{B \in \tau^c : B \supseteq A\}.$$

3 Fuzzy approximation spaces and fuzzy rough sets

Recall that R is a fuzzy relation on U if $R \in F(U \times U)$.

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Let R be a fuzzy relation on U. R is called reflexive if $R(x,x)=1$ for each $x \in U$. R is called symmetric if $R(x,y)=R(y,x)$ for any $x,y \in U$. R is called transitive if $R(x,z) \geq R(x,y) \wedge R(y,z)$ for any $x,y,z \in U$. R is called preorder if R is reflexive and transitive.

Definition 3.1 ([7]). *Let R be a fuzzy relation on U. The pair (U,R) is called a fuzzy approximation space. For each $A \in F(U)$, the fuzzy lower and the fuzzy upper approximation of A with respect to (U,R), denoted by $\underline{R}(A)$ and $\overline{R}(A)$ are respectively, defined as follows:*

$$\underline{R}(A)(x) = \bigwedge_{y \in U} (A(y) \vee (1-R(x,y))) \ \ (x \in U) \quad and \quad \overline{R}(A)(x) = \bigvee_{y \in U} (A(y) \wedge R(x,y)) \ (x \in U).$$

The pair $(\underline{R}(A), \overline{R}(A))$ is called the fuzzy rough set of A with respect to (U,R).

Remark 3.2. (1) $\overline{R}(x_1)(y) = R(y,x)$, $\underline{R}((x_1)^c)(y) = 1 - R(y,x)$ $(x,y \in U)$.
(2) *For each $a \in I$, $\overline{R}(\overline{a}) \subseteq \ \bar{a} \ \subseteq \underline{R}(\bar{a})$;*
(3) *If R is reflexive, then for each $a \in I$, $\underline{R}(\bar{a}) = \bar{a} = \overline{R}(\bar{a})$.*

Let (U,R) be a fuzzy approximation space. (U,R) is call reflexive (resp. preorder) if R is reflexive (resp. preorder).

Theorem 3.3 ([7, 8]). *Let (U,R) be a fuzzy approximation space. Then for any $A,B \in F(U)$, $\{A_j : j \in J\} \subseteq F(U)$ and $\lambda \in I$,*
(1) $A \subseteq B \Longrightarrow \underline{R}(A) \subseteq \underline{R}(B)$, $\overline{R}(A) \subseteq \overline{R}(B)$.
(2) $\underline{R}(A^c) = (\overline{R}(A))^c$, $\overline{R}(A^c) = (\underline{R}(A))^c$.
(3) $\underline{R}(A \cap B) = \underline{R}(A) \cap \underline{R}(B)$, $\overline{R}(A \cup B) = \overline{R}(A) \cup \overline{R}(B)$.
(4) $\underline{R}(\bigcap_{j \in J} A_j) = \bigcap_{j \in J} (\underline{R}(A_j))$, $\overline{R}(\bigcup_{j \in J} A_j) = \bigcup_{j \in J} (\overline{R}(A_j))$.
(5) $\underline{R}(\bar{\lambda} \cup A) = \bar{\lambda} \cup \underline{R}(A)$, $\overline{R}(\lambda A) = \lambda \overline{R}(A)$.

Theorem 3.4 ([4, 7, 9]). *Let (U,R) be a fuzzy approximation space. Then*

$$\begin{array}{lcl} (1)\ R\ is\ reflexive & \Longleftrightarrow & \forall A \in F(U), \underline{R}(A) \subseteq A. \\ & \Longleftrightarrow & \forall A \in F(U), A \subseteq \overline{R}(A). \\ (2)\ R\ is\ preorder & \Longrightarrow & \forall A \in F(U),\ \underline{R}(\underline{R}(A)) = \underline{R}(A), \quad \overline{R}(\overline{R}(A)) = \overline{R}(A). \end{array}$$

4 Fuzzy approximation spaces and fuzzy topologies

In this paper, we denote

$$\tau_R = Fix(\underline{R}) \ (i.e., \{A \in F(U) : A = \underline{R}(A)\}), \quad \theta_R = \{\underline{R}(A) : A \in F(U)\}.$$

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4.1 Fuzzy topologies induced by approximation spaces

Proposition 4.1. *Let (U, R) be a fuzzy approximation space. If R is preorder, then $\tau_R = \theta_R$.*

Proof. Obviously, $\tau_R \subseteq \theta_R$. By Theorem 3.4(2), $\tau_R \supseteq \theta_R$. So $\tau_R = \theta_R$. □

Theorem 4.2 ([7])**.** *Let (U, R) be a preorder fuzzy approximation space. Then*
(1) *θ_R is a fuzzy topology on U.*
(2) *$\underline{R}$ is the interior operator of θ_R.*
(3) *$\overline{R}$ is the closure operator of θ_R.*

Theorem 4.3. *Let (U, R) be a reflexive fuzzy approximation space. Then the following properties hold.*
(1) *τ_R is a fuzzy topology on U.*
(2)
$$int_{\tau_R}(A) \subseteq \underline{R}(A) \subseteq A \subseteq \overline{R}(A) \subseteq cl_{\tau_R}(A) \quad (A \in F(U)).$$
(3) *$A \in (\tau_R)^c \iff A = \overline{R}(A)$.*
(4) *For each $a \in I$, $\bar{a} \in (\tau_R)^c$.*

Proof. (1) This holds by Remark 3.2, Theorem 3.3 and Theorem 3.4(1).
(2) For each $A \in F(U)$, by Theorem 3.3(1),

$$\begin{aligned} int_{\tau_R}(A) &= \bigcup\{B \in \tau_R : B \subseteq A\} \subseteq \bigcup\{B \in \tau_R : \underline{R}(B) \subseteq \underline{R}(A)\} \\ &= \bigcup\{B \in F(U) : B = \underline{R}(B) \subseteq \underline{R}(A)\} \subseteq \underline{R}(A). \end{aligned}$$

By Theorem 3.3(2), $cl_{\tau_R}(A) = (int_{\tau_R}(A^c))^c \supseteq (\underline{R}(A^c))^c = \overline{R}(A)$. By Theorem 3.4(1), $\underline{R}(A) \subseteq A \subseteq \overline{R}(A)$.

Thus
$$int_{\tau_R}(A) \subseteq \underline{R}(A) \subseteq A \subseteq \overline{R}(A) \subseteq cl_{\tau_R}(A).$$
(3) This holds by Theorem 3.3(2).
(4) This holds by (3) and Remark 3.2. □

Definition 4.4. *Let (U, R) be a reflexive fuzzy approximation space. τ_R is called the fuzzy topology induced by (U, R) or R on U.*

4.2 Fuzzy approximation spaces induced by fuzzy topologies

Definition 4.5. *Let τ be a fuzzy topology on U. Define a fuzzy relation R_τ on U by*
$$R_\tau(x, y) = cl_\tau(y_1)(x) \quad ((x, y) \in U \times U).$$
Then R_τ is called the fuzzy relation induced by τ on U and (U, R_τ) is called the fuzzy approximation space induced by τ on U.

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Theorem 4.6. *Let τ be a fuzzy topology on U and let R_τ be the fuzzy relation induced by τ on U. Then the following properties hold.*

(1) *R_τ is reflexive.*

(2)

$$\underline{R_\tau}(A) \subseteq int_\tau(A) \subseteq A \subseteq cl_\tau(A) \subseteq \overline{R_\tau}(A) \quad (A \in F(U)).$$

Proof. (1) For each $x \in U$,

$$R_\tau(x,x) = cl_\tau(x_1)(x) \geq (x_1)(x) = 1.$$

Then R_τ is reflexive.

(2) For each $A \in F(U)$, by Remark 2.1,

$$\begin{aligned} cl_\tau(A) &= cl_\tau(\bigcup_{y\in U}(A(y)y_1)) = \bigcup_{y\in U} cl_\tau(A(y)y_1) = \bigcup_{y\in U} cl_\tau(\overline{A(y)} \cap y_1) \\ &\subseteq \bigcup_{y\in U}(cl_\tau(\overline{A(y)}) \cap cl_\tau(y_1)) = \bigcup_{y\in U}(\overline{A(y)} \cap cl_\tau(y_1)) = \bigcup_{y\in U}(A(y)cl_\tau(y_1)). \end{aligned}$$

Then for each $x \in U$,

$$cl_\tau(A)(x) \leq \bigvee_{y\in U}(A(y)cl_\tau(y_1))(x) = \bigvee_{y\in U}(A(y)\wedge cl_\tau(y_1)(x)) = \bigvee_{y\in U}(A(y)\wedge R_\tau(x,y)) = \overline{R_\tau}(A)(x).$$

Thus $cl_\tau(A) \subseteq \overline{R_\tau}(A)$. By Theorem 3.3(2),

$$int_\tau(A) = (cl_\tau(A^c))^c \supseteq (\overline{R_\tau}(A^c))^c = \underline{R_\tau}(A).$$

Hence

$$\underline{R_\tau}(A) \subseteq int_\tau(A) \subseteq A \subseteq cl_\tau(A) \subseteq \overline{R_\tau}(A).$$

□

5 Fuzzy approximating spaces

As can be seen from Section 4, a reflexive fuzzy relation yields a fuzzy topology. In this section, we consider a interested problem, that is, under which conditions can the given fuzzy topology coincides with the fuzzy topology induced by some reflexive fuzzy relation?

Definition 5.1. *Let (U, σ) be a fuzzy topological space. If there exists a reflexive fuzzy relation R on U such that $\tau_R = \sigma$, then (U, σ) is called a fuzzy approximating space.*

The following condition for a fuzzy topology τ on U is called (CC) axiom,

(CC) axiom: for any $\lambda \in I$ and $A \in F(U)$, $cl_\tau(\lambda A) = \lambda cl_\tau(A)$.

Proposition 5.2. *Let τ be a fuzzy topology on U. If τ satisfies (CC) axiom, then $\overline{R_\tau}$ is the closure operator of τ.*

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Proof. For each $A \in F(U)$, by Remark 2.1 and (CC) axiom,

$$cl_\tau(A) = cl_\tau(\bigcup_{y\in U}(A(y)y_1)) = \bigcup_{y\in U} cl_\tau(A(y)y_1) = \bigcup_{y\in U}(A(y)cl_\tau(y_1)).$$

Then for each $x \in U$,

$$cl_\tau(A)(x) = \bigvee_{y\in U}(A(y)cl_\tau(y_1))(x) = \bigvee_{y\in U}(A(y)\wedge cl_\tau(y_1)(x)) = \bigvee_{y\in U}(A(y)\wedge R_\tau(x,y)) = \overline{R_\tau}(A)(x).$$

Thus $\overline{R_\tau}(A) = cl_\tau(A)$. Hence $\overline{R_\tau}$ is the closure operator of τ. □

Proposition 5.3. *Let R be a preorder fuzzy relation on U. Then τ_R satisfies (CC) axiom.*

Proof. For any $\lambda \in I$ and $A \in F(U)$, by Theorem 3.3(5), Proposition 4.1 and Theorem 4.2(3),

$$cl_{\tau_R}(\lambda A) = cl_{\theta_R}(\lambda A) = \overline{R}(\lambda A) = \lambda\overline{R}(A) = \lambda cl_{\theta_R}(A) = \lambda cl_{\tau_R}(A).$$

Thus τ_R satisfies (CC) axiom. □

Proposition 5.4. *Let τ be a fuzzy topology on U, let R_τ be the fuzzy relation induced by τ on U and let τ_{R_τ} be the fuzzy topology induced by R_τ on U. Then*

$$\tau_{R_\tau} = \tau \quad \textit{if and only if} \quad \tau \quad \textit{satisfies} \quad \textit{(CC)} \quad \textit{axiom.}$$

Proof. Necessity. For each $A \in F(U)$, by Theorem 4.6(2), $cl_\tau(A) \subseteq \overline{R_\tau}(A)$. By Theorems 4.6(1) and 4.3(2),

$$cl_\tau(A) = cl_{\tau_{R_\tau}}(A) \supseteq \overline{R_\tau}(A).$$

Then $cl_\tau(A) = \overline{R_\tau}(A)$. So $\overline{R}$ is the closure operator of τ. For any $a \in I$ and $A \in F(U)$, by Theorem 3.3(5),

$$cl_\tau(\lambda A) = \overline{R_\tau}(\lambda A) = \lambda\overline{R_\tau}(A) = \lambda cl_\tau(A).$$

Thus τ satisfies (CC) axiom.

Sufficiency. By Theorem 4.6(1), R_τ is reflexive. For any $x, y, z \in U$, put $cl_\tau(z_1)(y) = \lambda$. By Remark 2.1,

$$\begin{aligned} \lambda cl_\tau(y_1) &= cl_\tau(\lambda y_1) = cl_\tau(cl_\tau(z_1)(y)y_1) \\ &\subseteq cl_\tau(\bigcup_{t\in U}(cl_\tau(z_1)(t)t_1)) = cl_\tau(cl_\tau(z_1)) = cl_\tau(z_1). \end{aligned}$$

Then

$$\begin{aligned} R_\tau(x,y) \wedge R_\tau(y,z) &= cl_\tau(y_1)(x) \wedge cl_\tau(z_1)(y) = cl_\tau(y_1)(x) \wedge \lambda \\ &= \lambda \wedge cl_\tau(y_1)(x) = (\lambda cl_\tau(y_1))(x) \\ &\leq cl_\tau(z_1)(x) = R_\tau(x,z). \end{aligned}$$

Then R_τ is transitive.

So R_τ is preorder. For each $A \in F(U)$, by Propositions 4.1 and 4.2(3), $cl_{\tau_{R_\tau}}(A) = cl_{\theta_{R_\tau}}(A) = \overline{R_\tau}(A)$. By (CC) axiom and Proposition 5.2, $\overline{R_\tau}(A) = cl_\tau(A)$. So $cl_{\tau_{R_\tau}}(A) = cl_\tau(A)$. Thus $\tau_{R_\tau} = \tau$. □

Theorem 5.5. *Let (U, τ) be a fuzzy topological space. If τ satisfies (CC) axiom, then (U, τ) is a fuzzy approximating space.*

Proof. This holds by Proposition 5.4. □

Example 5.6. *$\{\bar{a} : a \in I\}$ is a fuzzy approximating space.*

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On some self-adjoint fractional finite difference equations

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Abstract. Recently, the existence of solution for the fractional self-adjoint equation $\Delta^{\nu}_{\nu-1}(p\Delta y)(t) = h(t)$ for order $0 < \nu \leq 1$ was reported in [9]. In this paper, we investigated the self-adjoint fractional finite difference equation $\Delta^{\nu}_{\nu-2}(p\Delta y)(t) = h(t, p(t+\nu-2)\Delta y(t+\nu-2))$ via the boundary conditions $y(\nu-2) = 0$, such that $\Delta y(\nu-2) = 0$ and $\Delta y(\nu+b) = 0$. Also, we analyzed the self-adjoint fractional finite difference equation $\Delta^{\nu}_{\nu-2}(p\Delta^2 y)(t) = h(t, p(t+\nu-3)\Delta^2 y(t+\nu-3))$ via the boundary conditions $y(\nu-2) = 0$, $\Delta y(\nu-2) = 0$, $\Delta^2 y(\nu-2) = 0$ and $\Delta^2 y(\nu+b) = 0$. Finally, we conclude a result about the existence of solution for the general equation $\Delta^{\nu}_{\nu-2}(p\Delta^m y)(t) = h(t, p(t+\nu-m-1)\Delta^m y(t+\nu-m-1))$ via the boundary conditions $y(\nu-2) = \Delta y(\nu-2) = \Delta^2 y(\nu-2) = \cdots = \Delta^m y(\nu-2) = 0$ and $\Delta^m y(\nu+b) = 0$ for order $1 < \nu \leq 2$.

1 Introduction and Preliminaries

One of the trends in fractional calculus [1, 2, 3, 4] is the discrete version of it (see for example Refs.[5, 6, 7, 8, 7, 19, 9, 11, 12, 13, 14, 15, 16, 17]and the references therein). Some recent applications of the discrete fractional calculus revealed its huge potential applications to solve real world problems (see for example Refs.[19, 20, 21, 22] and the references therein).

Very recently, Awasthi provided some results about the fractional self-adjoint equation $\Delta^{\nu}_{\nu-1}(p\Delta y)(t) = h(t)$ for order $0 < \nu \leq 1$ ([9]). In this paper we investigate the fractional finite difference equation $\Delta^{\nu}_{\nu-2}(p\Delta y)(t) = h(t, p(t+\nu-2)\Delta y(t+\nu-2))$ in the presence of the boundary conditions $y(\nu-2) = 0$, $\Delta y(\nu-2) = 0$ and $\Delta y(\nu+b) = 0$, such that $b \in \mathbb{N}_0$, $t \in \mathbb{N}_0^{b+2}$, $1 < \nu \leq 2$ and $h : \mathbb{N}_{\nu-2}^{b+\nu} \times \mathbb{R} \to \mathbb{R}$ is a map. In addition, we discuss the existence of the solution corresponding to the fractional finite difference equation $\Delta^{\nu}_{\nu-2}(p\Delta^2 y)(t) = h(t, p(t+\nu-3)\Delta^2 y(t+\nu-3))$ with the boundary conditions $y(\nu-2) = 0$, $\Delta y(\nu-2) = 0$, $\Delta^2 y(\nu-2) = 0$ and $\Delta^2 y(\nu+b) = 0$, such that $1 < \nu \leq 2$, $b \in \mathbb{N}_0$, $t \in \mathbb{N}_0^{b+3}$ and $h : \mathbb{N}_{\nu-2}^{\nu+b} \times \mathbb{R} \to \mathbb{R}$ represents a map. For the second equation we provided an example.

We recall that $t^{\underline{\nu}} := \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}$ for all $t, \nu \in \mathbb{R}$ provided that the right-hand side is defined (see for example Ref.[15] and the references therein). In the following we use the notations $\mathbb{N}_a = \{a, a+1, a+2, \ldots\}$ for all $a \in \mathbb{R}$ and $\mathbb{N}_a^b = \{a, a+1, a+2, \ldots, b\}$ for all real numbers a and b such that $b - a$ denotes a natural number. Let $\nu > 0$ with $m - 1 < \nu < m$ for some natural number m. Then the ν-th fractional sum of f based at a is

defined by $\Delta_a^{-\nu}f(t) = \frac{1}{\Gamma(\nu)}\sum_{k=a}^{t-\nu}(t-\sigma(k))^{\underline{\nu-1}}f(k)$ for all $t \in \mathbb{N}_{a+\nu}$, where $\sigma(k) = k+1$ is the forward jump operator (see [15]). Similarly, we define $\Delta_a^{\nu}f(t) = \frac{1}{\Gamma(-\nu)}\sum_{k=a}^{t+\nu}(t-\sigma(k))^{\underline{-\nu-1}}f(k)$ for all $t \in \mathbb{N}_{a+N-\nu}$.

Lemma 1.1. *[9] Let $h : \mathbb{N}_a \to \mathbb{R}$ be a mapping and m a natural number. Then, the general solution of the equation $\Delta_{a+\nu-m}^{\nu}y(t) = h(t)$ is given by $y(t) = \sum_{i=1}^{m} C_i(t-a)^{\underline{\nu-i}} + \Delta_a^{-\nu}h(t)$ for all $t \in \mathbb{N}_{a+\nu-m}$, where $C_1, \cdots, C_m$ are arbitrary constants.*

Let $h : \mathbb{N}_{\nu-m} \times \mathbb{R} \to \mathbb{R}$ be a mapping and m a natural number. By using a similar proof, one can check that the general solution of the equation $\Delta_{\nu-m}^{\nu}y(t) = h(t, y(t+\nu-m+1))$ is given by

$$y(t) = \sum_{i=1}^{m} C_i t^{\underline{\nu-i}} + \Delta^{-\nu}h(t, y(t+\nu-m+1))$$

for all $t \in \mathbb{N}_{\nu-m}$. In particular, the general solution has the following representation

$$y(t) = \sum_{i=1}^{m} C_i t^{\underline{\nu-i}} + \frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-\sigma(s))^{\underline{\nu-1}}h(s, y(s+\nu-m+1)) \tag{1. 1}$$

for all $t \in \mathbb{N}_{\nu-m}$. By considering the details, note that $\sum_{k=a}^{b}(t-\sigma(k))^{\underline{\nu-1}}f(k) = 0$ whenever $b < a$. Also for $\nu, \mu > 0$ with $m-1 < \nu \leq m$ and $n-1 < \mu \leq n$, the domain of the operator Δ is given by $\mathcal{D}\{\Delta_a^{-\nu}f\} = \mathbb{N}_{a+\nu}$, $\mathcal{D}\{\Delta_a^{\nu}f\} = \mathbb{N}_{a+m-\nu}$, $\mathcal{D}\{\Delta_{a+n-\mu}^{-\nu}\Delta_a^{\mu}f\} = \mathbb{N}_{a+n+\nu-\mu}$, $\mathcal{D}\{\Delta_{a+\mu}^{\nu}\Delta_a^{-\mu}f\} = \mathbb{N}_{a+\mu+m-\nu}$, $\mathcal{D}\{\Delta_{a+n-\mu}^{\nu}\Delta_a^{\mu}f\} = \mathbb{N}_{a+n-\mu+m-\nu}$ and $\mathcal{D}\{\Delta_{a+\mu}^{-\nu}\Delta_a^{-\mu}f\} = \mathbb{N}_{a+\nu+\mu}$ (for more details see [9], [14] and [15]). One can find next result about composing a difference with a sum in [19].

Lemma 1.2. *Let $f : \mathbb{N}_a \to \mathbb{R}$ be a map, $k \in \mathbb{N}_0$ and $\mu > k$ with $n-1 < \mu \leq n$. Then $\Delta^k\Delta_a^{-\mu}f(t) = \Delta_a^{k-\mu}f(t)$ for all $t \in \mathbb{N}_{a+\mu}$ and $\Delta^k\Delta_a^{\mu}f(t) = \Delta_a^{k+\mu}f(t)$ for all $t \in \mathbb{N}_{a+n-\mu}$.*

By making use of (1. 1) and the last Lemma for $k = 1$, we finally report that

$$\Delta y(t) = \sum_{i=1}^{m} C_i(\nu-i)t^{\underline{\nu-i-1}} + \frac{1}{\Gamma(\nu-1)}\sum_{s=0}^{t-\nu+1}(t-\sigma(s))^{\underline{\nu-2}}h(t, y(t+\nu-m+1)),$$

which will be used in proving the main results. Let (X, d) be a metric space, $T : X \to X$ be a self-map and Ψ be the family of nondecreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that $\sum_{n=1}^{\infty}\psi^n(t) < \infty$ for all $t > 0$. We recall that T is an α-ψ-contractive mapping whenever there exists $\psi \in \Psi$ and $\alpha : X \times X \to [0,\infty)$ such that $\alpha(x,y)d(Tx,Ty) \leq \psi(d(x,y))$ for all $x, y \in X$. Also, T is called α-admissible whenever $\alpha(x,y) \geq 1$ implies that $\alpha(Tx,Ty) \geq 1$. In our proofs we used the next result which have been proved in [18].

Theorem 1.3. *[18] Let (X, d) be a complete metric space, $T : X \to X$ an α-admissible α-ψ-contractive map such that $\alpha(x_0, Tx_0) \geq 1$ for some $x_0 \in X$. If x_n is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$, for all n and $x_n \to x$ for some $x \in X$, then $\alpha(x_n, x) \geq 1$ for all n. Then T has a fixed point.*

2 Main results

In this section we present the main results of this manuscript.

Lemma 2.1. *Let $b \in \mathbb{N}_0$, $t \in \mathbb{N}_0^{b+2}$, $1 < \nu \leq 2$ and $h : \mathbb{N}_{\nu-2}^{b+\nu} \times \mathbb{R} \to \mathbb{R}$ be a map. Then y_0 is a solution of the problem $\Delta_{\nu-2}^{\nu}(p\Delta y)(t) = h(t, p(t+\nu-2)\Delta y(t+\nu-2))$ via the boundary conditions $y(\nu-2) = 0$, $\Delta y(\nu-2) = 0$ and $\Delta y(\nu+b) = 0$ if and only if y_0 is a solution of the fractional sum equation*

$$y(t) = \sum_{s=0}^{b} G(t,s)h(t, p(t+\nu-2)\Delta y(t+\nu-2)),$$

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where

$$G(t,s)=\frac{1}{\Gamma(\nu)}\begin{cases}\displaystyle\sum_{k=\nu-2}^{t-1}\frac{-k^{\underline{\nu-1}}(\nu+b-\sigma(s))^{\underline{\nu-1}}}{p(k)(\nu+b)^{\underline{\nu-1}}}+\sum_{k=s+\nu}^{t-1}\frac{(k-\sigma(s))^{\underline{\nu-1}}}{p(k)} & s\le t-\nu-1,\\[2ex] \displaystyle\sum_{k=\nu-2}^{t-1}\frac{-k^{\underline{\nu-1}}(\nu+b-\sigma(s))^{\underline{\nu-1}}}{p(k)(\nu+b)^{\underline{\nu-1}}} & t-\nu\le s.\end{cases}$$

Proof. Let y_0 be a solution of the problem $\Delta^{\nu}_{\nu-2}(p\Delta y)(t)=h(t,p(t+\nu-2)\Delta y(t+\nu-2))$ via the boundary conditions $y(\nu-2)=0$, $\Delta y(\nu-2)=0$ and $\Delta y(\nu+b)=0$. If $x(t)=(p\Delta y)(t)$, then $x_0=p\Delta y_0$ is a solution of the equation $\Delta^{\nu}_{\nu-2}x(t)=h(t,x(t+\nu-2))$. By using Lemma 1.1, we get

$$x_0(t)=C_1t^{\underline{\nu-1}}+C_2t^{\underline{\nu-2}}+\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-\sigma(s))^{\underline{\nu-1}}h(s,x_0(s+\nu-2))$$

and so $\Delta y_0(t)=\frac{1}{p(t)}\big[C_1t^{\underline{\nu-1}}+C_2t^{\underline{\nu-2}}+\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-\sigma(s))^{\underline{\nu-1}}h(s,p(s+\nu-2)\Delta y_0(s+\nu-2))\big]$. Since $\Delta y_0(\nu-2)=0$, we have

$$0=\frac{1}{p(\nu-2)}\big[C_1(\nu-2)^{\underline{\nu-1}}+C_2(\nu-2)^{\underline{\nu-2}}+\frac{1}{\Gamma(\nu)}\sum_{s=0}^{(\nu-2)-\nu}((\nu-2)-\sigma(s))^{\underline{\nu-1}}h(s,p(s+\nu-2)\Delta y_0(s+\nu-2))\big].$$

Since $(\nu-2)^{\underline{\nu-1}}=0$ and $\sum_{s=0}^{-2}((\nu-2)-\sigma(s))^{\underline{\nu-1}}h(s,p(s+\nu-2)\Delta y_0(s+\nu-2))=0$, $C_2=0$. Hence,

$$\Delta y_0(t)=\frac{1}{p(t)}\big[C_1t^{\underline{\nu-1}}+\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-\sigma(s))^{\underline{\nu-1}}h(s,p(s+\nu-2)\Delta y_0(s+\nu-2))\big].$$

Since $\Delta y_0(\nu+b)=0$, $C_1=\frac{-1}{(\nu+b)^{\underline{\nu-1}}\Gamma(\nu)}\sum_{s=0}^{b}(\nu+b-\sigma(s))^{\underline{\nu-1}}h(s,p(s+\nu-2)\Delta y_0(s+\nu-2))$ and so

$$\Delta y_0(t)=\frac{1}{p(t)}\big[\frac{-t^{\underline{\nu-1}}}{(\nu+b)^{\underline{\nu-1}}\Gamma(\nu)}\sum_{s=0}^{b}(\nu+b-\sigma(s))^{\underline{\nu-1}}h(s,p(s+\nu-2)\Delta y_0(s+\nu-2))$$

$$+\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-\sigma(s))^{\underline{\nu-1}}h(s,p(s+\nu-2)\Delta y_0(s+\nu-2))\big].$$

Thus, we obtain

$$y_0(t)-y_0(\nu-2)=\sum_{k=\nu-2}^{t-1}\Delta y_0(k)=\frac{1}{\Gamma(\nu)}\sum_{k=\nu-2}^{t-1}\sum_{s=0}^{b}\frac{-k^{\underline{\nu-1}}(\nu+b-\sigma(s))^{\underline{\nu-1}}}{p(k)(\nu+b)^{\underline{\nu-1}}}h(s,p(s+\nu-2)\Delta y_0(s+\nu-2))$$

$$+\frac{1}{\Gamma(\nu)}\sum_{k=\nu-2}^{t-1}\sum_{s=0}^{k-\nu}\frac{(k-\sigma(s))^{\underline{\nu-1}}}{p(k)}h(s,p(s+\nu-2)\Delta y_0(s+\nu-2)).$$

But, we have $y_0(\nu-2)=0$. Thus by interchanging the order of summations, we get

$$y_0(t)=\frac{1}{\Gamma(\nu)}\sum_{s=0}^{b}\sum_{k=\nu-2}^{t-1}\frac{-k^{\underline{\nu-1}}(\nu+b-\sigma(s))^{\underline{\nu-1}}}{p(k)(\nu+b)^{\underline{\nu-1}}}h(s,p(s+\nu-2)\Delta y_0(s+\nu-2))$$

$$+\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-1-\nu}\sum_{k=s+\nu}^{t-1}\frac{(k-\sigma(s))^{\underline{\nu-1}}}{p(k)}h(s,p(s+\nu-2)\Delta y_0(s+\nu-2))$$

$$= \sum_{s=0}^{b} G(t,s)h(s, p(s+\nu-2)\Delta y_0(s+\nu-2)).$$

Now, let y_0 be a solution of the fractional sum equation $y(t) = \sum_{s=0}^{b} G(t,s)h(s, p(s+\nu-2)\Delta y(s+\nu-2))$. Then, we have

$$y_0(t) = \frac{1}{\Gamma(\nu)} \sum_{k=\nu-2}^{t-1} \sum_{s=0}^{b} \frac{-k^{\underline{\nu-1}}(\nu+b-\sigma(s))^{\underline{\nu-1}}}{p(k)(\nu+b)^{\underline{\nu-1}}} h(s, p(s+\nu-2)\Delta y_0(s+\nu-2))$$

$$+\frac{1}{\Gamma(\nu)} \sum_{k=\nu-2}^{t-1} \sum_{s=0}^{k-\nu} \frac{(k-\sigma(s))^{\underline{\nu-1}}}{p(k)} h(s, p(s+\nu-2)\Delta y_0(s+\nu-2)).$$

By using the mentioned details, we get $y_0(\nu-2) = 0$. Since

$$\Delta\Big[\frac{1}{\Gamma(\nu)} \sum_{k=\nu-2}^{t-1} \sum_{s=0}^{k-\nu} \frac{(k-\sigma(s))^{\underline{\nu-1}}}{p(k)} h(s, p(s+\nu-2)\Delta y_0(s+\nu-2))\Big]$$

$$= \frac{1}{\Gamma(\nu-1)} \sum_{k=\nu-2}^{t-2} \sum_{s=0}^{k-\nu} \frac{(k-\sigma(s))^{\underline{\nu-2}}}{p(k)} h(s, p(s+\nu-2)\Delta y_0(s+\nu-2)),$$

we obtain $\Delta y_0(\nu-2) = 0$. By using a similar calculation, one can get that $\Delta y_0(\nu+b) = 0$. Moreover,

$$\Delta^{\nu}(p\Delta y_0)(t) = \Delta^{\nu}\Bigg[p(t)\Delta\Bigg(\frac{1}{\Gamma(\nu)} \sum_{k=\nu-2}^{t-1} \sum_{s=0}^{b} \frac{-k^{\underline{\nu-1}}(\nu+b-\sigma(s))^{\underline{\nu-1}}}{p(k)(\nu+b)^{\underline{\nu-1}}} h(s, p(s+\nu-2)\Delta y_0(s+\nu-2))$$

$$+\frac{1}{\Gamma(\nu)} \sum_{k=\nu-2}^{t-1} \sum_{s=0}^{k-\nu} \frac{(k-\sigma(s))^{\underline{\nu-1}}}{p(k)} h(s, p(s+\nu-2)\Delta y_0(s+\nu-2))\Bigg)\Bigg]$$

$$= \Delta^{\nu}\Bigg[\frac{p(t)}{\Gamma(\nu-1)} \sum_{k=\nu-2}^{t-2} \sum_{s=0}^{b} \frac{-(\nu-1)k^{\underline{\nu-2}}(\nu+b-\sigma(s))^{\underline{\nu-1}}}{p(k)(\nu+b)^{\underline{\nu-1}}} h(s, p(s+\nu-2)\Delta y_0(s+\nu-2))\Bigg]$$

$$+\Delta^{\nu}\Bigg[p(t) \sum_{k=\nu-2}^{t-2} \frac{1}{p(k)} \Delta^{-\nu} h(k, p(k+\nu-2)\Delta y_0(k+\nu-2))\Bigg].$$

Since $\Delta^{\nu} t^{\underline{\nu-2}} = \frac{\Gamma(\nu-1)t^{\underline{\nu-2-\nu}}}{\Gamma(\nu-\nu-1)} = 0$ and

$$\Delta^{\nu}\Bigg[p(t) \sum_{k=\nu-2}^{t-2} \frac{1}{p(k)} \Delta^{-\nu} h(k, p(k+\nu-2)\Delta y_0(k+\nu-2))\Bigg] = h(t, p(t+\nu-2)\Delta y_0(t+\nu-2)),$$

a simple calculation shows us that $\Delta^{\nu}(p\Delta y_0)(t) = h(t, p(t+\nu-2)\Delta y_0(t+\nu-2))$. This completes the proof. □

It is important to note that the sum $\sum_{s=0}^{k} G(t,s)$ is bounded, where G is the Green function in last result. Denote the upper bound of $\sum_{s=0}^{b} |G(t,s)|$ by M_G.

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Theorem 2.2. *Suppose that $\psi \in \Psi$ has this property that $\psi(2t) \leq 2\psi(t)$ for all $t \geq 0$ and $p : \mathbb{N}_{\nu-2}^{b+\nu} \to \mathbb{R}^+$ is a bounded function such that $p(t) \leq C$ for all $t \in \mathbb{N}_{\nu-2}^{b+\nu}$. Let $h : \mathbb{N}_{\nu-2}^{b+\nu} \times \mathbb{R} \to \mathbb{R}$ be a map such that*

$$|h(t,a) - h(t,b)| \leq \frac{1}{2CM_G}\psi(|a-b|)$$

for all $a, b \in \mathbb{R}$ and $t \in \mathbb{N}_{\nu-2}^{b+\nu}$. Then the problem $\Delta_{\nu-2}^{\nu}(p\Delta y)(t) = h(t, p(t+\nu-2)\Delta y(t+\nu-2))$ via the boundary conditions $y(\nu-2) = 0$, $\Delta y(\nu-2) = 0$ and $\Delta y(\nu+b) = 0$ has at least one solution.

Proof. Suppose that $\mathcal{X}$ is the space of real valued functions on $\mathbb{N}_0$ via the norm $\|y\| = max\{|y(t)| : t \in \mathbb{N}_0\}$. It is known that $\mathcal{X}$ is a Banach space. Define $T : \mathcal{X} \to \mathcal{X}$ by $Tx(t) = \sum_{s=0}^{b} G(t,s)h(s, p(s+\nu-2)\Delta x(s+\nu-2))$. By using Lemma 2.1, we know that $y_0 \in \mathcal{X}$ is a solution of the problem if and only if y_0 is a fixed point of T. Let $x, y \in \mathcal{X}$ and $t \in \mathbb{N}_0$. Hence,

$$|Tx(t) - Ty(t)| = |\sum_{s=0}^{b} G(t,s)h(s, p(s+\nu-2)\Delta x(s+\nu-2)) - \sum_{s=0}^{b} G(t,s)h(s, p(s+\nu-2)\Delta y(s+\nu-2))|$$

$$\begin{aligned}
&\leq \sum_{s=0}^{b} |G(t,s)||h(t, p(s+\nu-2)\Delta x(s+\nu-2)) - h(t, p(s+\nu-2)\Delta y(s+\nu-2))| \\
&\leq M_G \times \frac{1}{2CM_G}\psi(|p(s+\nu-2)\Delta x(s+\nu-2) - p(s+\nu-2)\Delta y(s+\nu-2)|) \\
&= \frac{p(s+\nu-2)}{2C}\psi(|x(s+\nu-1) - x(s+\nu-2) - y(s+\nu-1) + y(s+\nu-2)|) \\
&\leq \frac{1}{2}\psi(\|x-y\| + \|x-y\|) \leq \psi(\|x-y\|)
\end{aligned}$$

for all $x, y \in \mathcal{X}$. If we define the function $\alpha : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ by $\alpha(x,y) = 1$ for all $x, y \in \mathcal{X}$, then we have $\alpha(x,y)d(Tx,Ty) \leq \psi d(x,y)$ for all $x, y \in \mathcal{X}$. Thus, T is an α-ψ-contractive mapping. It is clear that $\alpha(x,y) \geq 1$ implies that $\alpha(Tx,Ty) \geq 1$ for all $x, y \in \mathcal{X}$ and so T is α-admissible. Also, $\alpha(x_0, Tx_0) \geq 1$ for all $x_0 \in \mathcal{X}$. Also, for each sequence $\{x_n\}$ in $\mathcal{X}$ with $\alpha(x_n, x_{n+1}) \geq 1$ and $x_n \to x$, we have $\alpha(x_n, x) \geq 1$ for all n. By using Theorem 1.3, T has a fixed point $y_0 \in \mathcal{X}$ and so y_0 is a solution of the problem. □

Now, we consider the fractional finite difference equation $\Delta_{\nu-2}^{\nu}(p\Delta^2 y)(t) = h(t, p(t+\nu-3)\Delta^2 y(t+\nu-3))$ via the boundary conditions $y(\nu-2) = 0$, $\Delta y(\nu-2) = 0$, $\Delta^2 y(\nu-2) = 0$ and $\Delta^2 y(\nu+b) = 0$, where $b \in \mathbb{N}_0$, $t \in \mathbb{N}_0^{b+3}$, $1 < \nu \leq 2$ and $h : \mathbb{N}_{\nu-2}^{b+\nu} \times \mathbb{R} \to \mathbb{R}$ is a map. First, we give next key result.

Lemma 2.3. *Let $h : \mathbb{N}_{\nu-2}^{b+\nu} \times \mathbb{R} \to \mathbb{R}$ be a mapping, $b \in \mathbb{N}_0$, $t \in \mathbb{N}_0^{b+3}$ and $1 < \nu \leq 2$. Then y_0 is a solution of the problem $\Delta_{\nu-2}^{\nu}(p\Delta^2 y)(t) = h(t, p(t+\nu-3)\Delta^2 y(t+\nu-3))$ via the boundary conditions $y(\nu-2) = 0$, $\Delta y(\nu-2) = 0$, $\Delta^2 y(\nu-2) = 0$ and $\Delta^2 y(\nu+b) = 0$ if and only if y_0 is a solution of the fractional sum equation $y(t) = \sum_{s=0}^{b} G(t,s)h(t, p(t+\nu-3)\Delta^2 y(t+\nu-3))$, where*

$$G(t,s) = \frac{1}{\Gamma(\nu)} \begin{cases} \displaystyle\sum_{\tau=\nu-2}^{t-1}\sum_{k=\nu-2}^{\tau-1} \frac{-k^{\underline{\nu-1}}(\nu+b-\sigma(s))^{\underline{\nu-1}}}{p(k)(\nu+b)^{\underline{\nu-1}}} + \sum_{\tau=s+1+\nu}^{t-1}\sum_{k=s+\nu}^{\tau-1} \frac{(k-\sigma(s))^{\underline{\nu-1}}}{p(k)} & s \leq t-\nu-2, \\ \displaystyle\sum_{\tau=\nu-2}^{t-1}\sum_{k=\nu-2}^{\tau-1} \frac{-k^{\underline{\nu-1}}(\nu+b-\sigma(s))^{\underline{\nu-1}}}{p(k)(\nu+b)^{\underline{\nu-1}}} & t-\nu-1 \leq s. \end{cases}$$

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Proof. Let y_0 be a solution of the problem $\Delta^{\nu}_{\nu-2}(p\Delta^2 y)(t) = h(t, p(t+\nu-3)\Delta^2 y(t+\nu-3))$ via the boundary conditions $y(\nu-2) = 0$, $\Delta y(\nu-2) = 0$, $\Delta^2 y(\nu-2) = 0$ and $\Delta^2 y(\nu+b) = 0$. If $x(t) = (p\Delta^2 y)(t)$, then $x_0 = p\Delta^2 y_0$ is a solution of the equation $\Delta^{\nu}_{\nu-2} x(t) = h(t, x(t+\nu-3))$. By using Lemma 1.1, we get

$$x_0(t) = C_1 t^{\underline{\nu-1}} + C_2 t^{\underline{\nu-2}} + \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t-\sigma(s))^{\underline{\nu-1}} h(s, x_0(s, \nu-3)).$$

By using the conditions $\Delta^2 y_0(\nu-2) = 0$ and $\Delta^2 y_0(\nu+b) = 0$, we conclude that $C_2 = 0$ and

$$C_1 = \frac{-1}{(\nu+b)^{\underline{\nu-1}}\Gamma(\nu)} \sum_{s=0}^{b} (\nu+b-\sigma(s))^{\underline{\nu-1}} h(s, p(s+\nu-3)\Delta^2 y_0(s+\nu-3)).$$

Thus, we obtain

$$\Delta^2 y_0(t) = \frac{1}{p(t)} \Big[\frac{-t^{\underline{\nu-1}}}{(\nu+b)^{\underline{\nu-1}}\Gamma(\nu)} \sum_{s=0}^{b} (\nu+b-\sigma(s))^{\underline{\nu-1}} h(s, p(s+\nu-3)\Delta^2 y_0(s+\nu-3))$$
$$+ \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t-\sigma(s))^{\underline{\nu-1}} h(s, p(s+\nu-3)\Delta^2 y_0(s+\nu-3)) \Big]$$

and so by applying the condition $\Delta y_0(\nu-2) = 0$ and interchanging the order of summations, we get

$$\Delta y_0(t) = \frac{1}{\Gamma(\nu)} \sum_{s=0}^{b} \sum_{k=\nu-2}^{t-1} \frac{-k^{\underline{\nu-1}}(\nu+b-\sigma(s))^{\underline{\nu-1}}}{p(k)(\nu+b)^{\underline{\nu-1}}} h(s, p(s+\nu-3)\Delta^2 y_0(s+\nu-3))$$
$$+ \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-1-\nu} \sum_{k=s+\nu}^{t-1} \frac{(k-\sigma(s))^{\underline{\nu-1}}}{p(k)} h(s, p(s+\nu-3)\Delta^2 y_0(s+\nu-3)).$$

Again, by applying the condition $y_0(\nu-2) = 0$ and interchanging the order of summations, we get

$$y_0(t) = \frac{1}{\Gamma(\nu)} \sum_{s=0}^{b} \sum_{\tau=\nu-2}^{t-1} \sum_{k=\nu-2}^{\tau-1} \frac{-k^{\underline{\nu-1}}(\nu+b-\sigma(s))^{\underline{\nu-1}}}{p(k)(\nu+b)^{\underline{\nu-1}}} h(s, p(s+\nu-3)\Delta^2 y_0(s+\nu-3))$$
$$+ \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-2-\nu} \sum_{\tau=s+1+\nu}^{t-1} \sum_{k=s+\nu}^{\tau-1} \frac{(k-\sigma(s))^{\underline{\nu-1}}}{p(k)} h(s, p(s+\nu-3)\Delta^2 y_0(s+\nu-3))$$
$$= \sum_{s=0}^{b} G(t,s) h(s, p(s+\nu-3)\Delta^2 y_0(s+\nu-3)).$$

Now, let y_0 be a solution of the fractional sum equation $y(t) = \sum_{s=0}^{b} G(t,s) h(s, p(s+\nu-3)\Delta^2 y(s+\nu-3))$. Then, we have

$$y_0(t) = \frac{1}{\Gamma(\nu)} \sum_{s=0}^{b} \sum_{\tau=\nu-2}^{t-1} \sum_{k=\nu-2}^{\tau-1} \frac{-k^{\underline{\nu-1}}(\nu+b-\sigma(s))^{\underline{\nu-1}}}{p(k)(\nu+b)^{\underline{\nu-1}}} h(s, p(s+\nu-3)\Delta^2 y_0(s+\nu-3))$$
$$+ \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-2-\nu} \sum_{\tau=s+1+\nu}^{t-1} \sum_{k=s+\nu}^{\tau-1} \frac{(k-\sigma(s))^{\underline{\nu-1}}}{p(k)} h(s, p(s+\nu-3)\Delta^2 y_0(s+\nu-3)).$$

Similar to proof of Lemma 2.1, one can conclude that $y(\nu-2) = 0$, $\Delta y(\nu-2) = 0$, $\Delta^2 y(\nu-2) = 0$ and $\Delta^2 y(\nu+b) = 0$. Also $\Delta^{\nu}_{\nu-2}(p\Delta^2 y_0)(t) = h(t, p(t+\nu-3)\Delta^2 y_0(t+\nu-3))$. This completes the proof. □

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Similar to above results one can get that $\sum_{s=0}^{b} G(t,s)$ is bounded, where G is the Green function in last result. Denote the bound of the Green function by M'_G. Also, we can provide next result.

Theorem 2.4. *Suppose that $\psi \in \Psi$ has this property that $\psi(4t) \leq 4\psi(t)$ for all $t \geq 0$ and $p : \mathbb{N}_{\nu-2}^{b+\nu} \to \mathbb{R}^+$ is a bounded function such that $p(t) \leq C$ for all $t \in \mathbb{N}_{\nu-2}^{b+\nu}$. Let $h : \mathbb{N}_{\nu-2}^{b+\nu} \times \mathbb{R} \to \mathbb{R}$ be a map such that*

$$|h(t,a) - h(t,b)| \leq \frac{1}{4CM'_G}\psi(|a-b|)$$

for all $a, b \in \mathbb{R}$ and $t \in \mathbb{N}_{\nu-2}^{b+\nu}$. Then the problem $\Delta_{\nu-2}^{\nu}(p\Delta^2 y)(t) = h(t, p(t+\nu-3)\Delta^2 y(t+\nu-3))$ via the boundary conditions $y(\nu-2) = 0$, $\Delta y(\nu-2) = 0$, $\Delta^2 y(\nu-2) = 0$ and $\Delta^2 y(\nu+b) = 0$ has at least one solution.

Here, we present an example for the second problem.

Example 2.1. *Consider the problem*

$$\Delta_{-0.5}^{1.5}(e^{-t}\Delta^2 y(t)) = \tan t + \frac{\sin(e^{1.5-t}\Delta^2 y(t-1.5))}{\cosh(t+11)} - \frac{4}{4-t^2}$$

via the boundary conditions $y(-0.5) = 0$, $\Delta y(-0.5) = 0$, $\Delta^2 y(-0.5) = 0$ and $\Delta^2 y(4.5) = 0$. Let $\nu = 1.5$, $p(z) = e^{-z}$, $h(t,a) = \tan t + \frac{\sin a}{\cosh(t+11)} - \frac{4}{4-t^2}$ and $b = 3$. Note that, $0 < p(z) \leq e^{0.5}$ for all $z \in \mathbb{N}_{-0.5}^{4.5}$. Put $C = e^{0.5}$. Now, we show that $M'_G = 527.9447$. In fact, the Green function is given by

$$G(t,s) = \frac{1}{\Gamma(1.5)} \begin{cases} \displaystyle\sum_{\tau=-0.5}^{t-1}\sum_{k=-0.5}^{\tau-1} \frac{-k^{\underline{0.5}}(4.5-\sigma(s))^{\underline{0.5}}}{p(k)(4.5)^{\underline{0.5}}} + \sum_{\tau=s+2.5}^{t-1}\sum_{k=s+1.5}^{\tau-1} \frac{(k-\sigma(s))^{\underline{0.5}}}{p(k)} & s \leq t-3.5, \\ \displaystyle\sum_{\tau=-0.5}^{t-1}\sum_{k=-0.5}^{\tau-1} \frac{-k^{\underline{0.5}}(4.5-\sigma(s))^{\underline{0.5}}}{p(k)(4.5)^{\underline{0.5}}} & t-2.5 \leq s. \end{cases}$$

Thus, $G(2.5,0) = \sum_{\tau=-0.5}^{1.5}\sum_{k=-0.5}^{\tau-1} \frac{-k^{\underline{0.5}}3.5^{\underline{0.5}}}{\Gamma(1.5)p(k)(4.5)^{\underline{0.5}}} = 2 \times \frac{-(-0.5)^{\underline{0.5}}(3.5)^{\underline{0.5}}}{\Gamma(1.5)p(-0.5)(4.5)^{\underline{0.5}}} + \frac{-(0.5)^{\underline{0.5}}(3.5)^{\underline{0.5}}}{\Gamma(1.5)p(0.5)(4.5)^{\underline{0.5}}} = -1.4655$. By some calculations, one can get all values of the Green function which we have arranged those in next table.

t	2.5	3.5	4.5	5.5	6.5
$G(t,0)$	-1.4655	-4.4239	-9.4107	-16.6908	-23.9512
$G(t,1)$	-1.2560	-7.6338	-19.2329	-36.3505	-53.467
$G(t,2)$	-1.0048	-6.1071	-25.1322	-55.1956	-85.2591
$G(t,3)$	-0.6699	-4.0714	-16.7548	-58.8741	-100.9934

Table 3.1: Values of the Green function

Thus, $M'_G = \sum_{s=0}^{3}|G(t,s)| = 527.9447$. Now, define the non-decreasing function $\psi : [0,\infty) \to [0,\infty)$ by $\psi(t) = e^{-1}t$. Then $\psi \in \Psi$ and

$$|h(t,a) - h(t,b)| = \frac{1}{\cosh(t+11)}|\sin a - \sin b| \leq \frac{2}{e^{t+11} + e^{-t-11}}|a-b| \leq \frac{2}{e^{t+11}}|a-b|.$$

Since $-0.5 \leq t \leq 4.5$, $e^{10.5} \leq e^{t+11} \leq e^{15.5}$ and so

$$|h(t,a) - h(t,b)| \leq \frac{2}{e^{10.5}}|a-b| \leq \frac{e^{-1}}{4e^{0.5} \times 527.9447}|a-b| = \frac{1}{4CM'_G}\psi(|a-b|).$$

Hence by using Theorem 2.4, the problem has at least one solution.

By using the provided technique, one can get the next general result.
Suppose that $\psi \in \Psi$ has this property that $\psi(2^m t) \leq 2^m \psi(t)$ for all $t \geq 0$ and $p : \mathbb{N}_{\nu-2}^{b+\nu} \to \mathbb{R}^+$ is a bounded function such that $p(t) \leq C$ for all $t \in \mathbb{N}_{\nu-2}^{b+\nu}$. Let $h : \mathbb{N}_{\nu-2}^{b+\nu} \times \mathbb{R} \to \mathbb{R}$ be a map such that

$$|h(t,a) - h(t,b)| \leq \frac{1}{2^m C M_G^{(m-1)}} \psi(|a-b|)$$

for all $a, b \in \mathbb{R}$ and $t \in \mathbb{N}_{\nu-2}^{b+\nu}$. Then the problem $\Delta_{\nu-2}^{\nu}(p\Delta^m y)(t) = h(t, p(t+\nu-m-1)\Delta^m y(t+\nu-m-1))$ via the boundary conditions $y(\nu-2) = \Delta y(\nu-2) = \Delta^2 y(\nu-2) = \cdots = \Delta^m y(\nu-2) = 0$ and $\Delta^m y(\nu+b) = 0$ has at least one solution. Here, $M_G^{(m-1)}$ is the bound of the related Green function.

Acknowledgments

Research of the second and third authors was supported by Azarbaijan Shahid Madani University.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 19, NO.1, 2015, COPYRIGHT 2015 EUDOXUS PRESS, LLC

Local shape control of a weighted interpolation surface

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Abstract

This paper proposes a new weighted bivariate blending rational interpolator only based on function values. The interpolation function with some free parameters has simple and explicit mathematical representation, and it is C^1-continuous for any positive parameters and weighted coefficient. More important, the shape of the interpolating surfaces can be modified by selecting suitable parameters and weighted coefficient. In order to meet the needs of practical design, a local shape control method is employed to control the shape of interpolating surfaces. In the special case, the method of "Central Point Value Control" is discussed.

Keywords: rational spline, bivariate blending interpolation, weighted interpolation, shape control, computer-aided geometric design

1 Introduction

In various applications such as industrial design and manufacture, atmospheric analysis, geology and medical imaging, etc., it is often necessary to generate a smooth surface that interpolates a prescribed set of data. For most applications, C^1 smoothness is generally sufficient. There are many ways to tackle this problem [2, 3, 4, 9, 12, 13, 18], such as the polynomial spline method, the NURBS method and the Bèzier method. These methods are effective and applied widely in shape design of industrial products. However, one of the disadvantages of the polynomial spline method is that curve and surface can not be adjusted locally for unchanged given data. The NURBS and Bézier methods are the so-called "no-interpolating type" methods; this means that the constructed curve and surface do not fit with the given data, so the given points play the role of the control points.

Here, we are concerned with the C^1 bivariate rational spline interpolations with a simple and explicit mathematical representation, which can be modified by using new parameters. In fact, in recent years, motivated by the univariate rational spline interpolation [1, 5, 10, 11, 14, 15, 17], the C^1 bivariate rational spline, which has a simple and explicit mathematical representation with parameters, has been studied [6, 7, 8, 19]. Since the parameters in the interpolation function are selective according to the control constrains, the constrained control of the shape becomes possible. However, in order to maintain smoothness of these interpolating surfaces, the parameters of y-direction must be restricted, these parameters can not be selected freely for different interpolating

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subregion. To overcome this problem, in this paper, a new weighted bivariate blending rational spline is constructed, and a local shape control technique of the interpolating surfaces is also developed.

This paper is arranged as follows. In Section 2, a weighted bivariate blending rational interpolator only based on the function values is constructed. Section 3 gives the basis and matric representation of this interpolation. In Section 4, the error estimate formula of the interpolation is derived. Section 5 deal with the local shape control of the interpolating surface, in the special case, the "Central Point Value Control" technique is developed, and numerical examples are presented to show the performance of the method.

2 Interpolation

Let $\Omega : [a, b; c, d]$ be the plane region, and $\{(x_i, y_i, f_{i,j}, i = 1, 2, \cdots, n+1; j = 1, 2, \cdots, m+1\}$ be a given set of data points, where $a = x_1 < x_2 < \cdots < x_{n+1} = b$ and $c = y_1 < y_2 < \cdots < y_{m+1} = d$ are the knot spacings, $f_{i,j}$ represent $f(x_i, y_j)$ at the point (x_i, y_j). Let $h_i = x_{i+1} - x_i$, and $l_j = y_{j+1} - y_j$, and for any point $(x, y) \in [x_i, x_{i+1}; y_j, y_{j+1}]$ in the xy-plane, let $\theta = (x - x_i)/h_i$ and $\eta = (y - y_j)/l_j$. Denote $\Delta^{(x)}_{i,j} = (f_{i+1,j} - f_{i,j})/h_i, \Delta^{(y)}_{i,j} = (f_{i,j+1} - f_{i,j})/l_j$. First, for each $y = y_j, j = 1, 2, \cdots, m+1$, construct the x-direction interpolant curve in $[x_i, x_{i+1}]$; this is given by

$$p_{i,j}(x) = \frac{(1-\theta)^3\alpha_{i,j}f_{i,j} + \theta(1-\theta)^2V_{i,j} + \theta^2(1-\theta)W_{i,j} + \theta^3 f_{i+1,j}\beta_{i,j}}{(1-\theta)^2\alpha_{i,j} + 2\theta(1-\theta) + \theta^2\beta_{i,j}}, \ i = 1, 2, \cdots, n, \tag{1}$$

where

$$\begin{aligned} V_{i,j} &= 2f_{i,j} + \alpha_{i,j}f_{i+1,j}, \\ W_{i,j} &= (2 + \beta_{i,j})f_{i+1,j} - \beta_{i,j}h_i\Delta^{(x)}_{i+1,j}, \end{aligned}$$

with $\alpha_{i,j} > 0, \beta_{i,j} > 0$. This interpolation is called the rational cubic interpolation based only on function values which satisfies

$$p_{i,j}(x_i) = f_{i,j}, \quad p_{i,j}(x_{i+1}) = f_{i+1,j}, \quad p'_{i,j}(x_i) = \Delta^{(x)}_{i,j}, \quad p'_{i,j}(x_{i+1}) = \Delta^{(x)}_{i+1,j}.$$

For each pair of $(i, j), i = 1, 2, \cdots, n-1$ and $j = 1, 2, \cdots, m-1$, using the x-direction interpolation function $p_{i,j}(x)$, define the interpolation function $P^1(x, y)$ on each rectangular region $[x_i, x_{i+1}; y_j, y_{j+1}]$ as follows:

$$\begin{aligned} P^1_{i,j}(x, y) = (1-\eta)^3 p_{i,j}(x) + \eta(1-\eta)^2 V^1_{i,j} + \eta^2(1-\eta)W^1_{i,j} + \eta^3 p_{i,j+1}(x), \\ i = 1, 2, \cdots, n-1; \ j = 1, 2, \cdots, m-1, \end{aligned} \tag{2}$$

where $V^1_{i,j} = 2p_{i,j}(x) + p_{i,j+1}(x)$, $W^1_{i,j} = 3p_{i,j+1}(x) - l_j(p_{i,j+2}(x) - p_{i,j+1}(x))/l_{j+1}$. It is easy to prove that $P^1_{i,j}(x, y)$ is C^1-continuous in the interpolating region $[x_1, x_n; y_1, y_m]$, no matter what the parameters $\alpha_{i,s}, \beta_{i,s}$ $(s = j, j+1, j+2)$ might be, and which satisfies

$$P^1_{i,j}(x_r, y_s) = f_{r,s}, \ \frac{\partial P^1_{i,j}(x_r, y_s)}{\partial x} = \Delta^{(x)}_{r,s}, \ \frac{\partial P^1_{i,j}(x_r, y_s)}{\partial y} = \Delta^{(y)}_{r,s}, \ r = i, i+1, \ s = j, j+1.$$

The interpolating scheme above begins in x-direction first. Now, let the interpolation function begins with y-direction first. For each $x = x_i, i = 1, 2, \cdots, n+1$, denote the y-direction interpolation in $[y_j, y_{j+1}]$ by

$$p^*_{i,j}(y) = \frac{(1-\eta)^3\gamma_{i,j}f_{i,j} + \eta(1-\eta)^2V^*_{i,j} + \eta^2(1-\eta)W^*_{i,j} + \eta^3 f_{i,j+1}\tau_{i,j}}{(1-\eta)^2\gamma_{i,j} + 2\eta(1-\eta) + \eta^2\tau_{i,j}}, \ j = 1, 2, \cdots, m, \tag{3}$$

where

$$V^*_{i,j} = 2f_{i,j} + \gamma_{i,j}f_{i,j+1},$$
$$W^*_{i,j} = (2 + \tau_{i,j})f_{i,j+1} - l_j\tau_{i,j}\Delta^{(y)}_{i,j+1},$$

with $\gamma_{i,j} > 0, \tau_{i,j} > 0$.

For each pair $(i, j), i = 1, 2, \cdots, n-1$ and $j = 1, 2, \cdots, m-1$, define the bivariate blending rational interpolation function $P^2_{i,j}(x, y)$ on $[x_i, x_{i+1}; y_j, y_{j+1}]$ as follows:

$$\begin{aligned} P^2_{i,j}(x,y) = (1-\theta)^3 p^*_{i,j}(y) + \theta(1-\theta)^2 V^{2,*}_{i,j} + \theta^2(1-\theta)W^{2,*}_{i,j} + \theta^3 p^*_{i+1,j}(y), \\ i = 1, 2, \cdots, n-1; \quad j = 1, 2, \cdots, m-1, \end{aligned} \tag{4}$$

where $V^{2,*}_{i,j} = 2p^*_{i,j}(y) + p^*_{i+1,j}(y)$, $W^{2,*}_{i,j} = 3p^*_{i+1,j}(y) - h_i(p^*_{i+2,j}(y) - p^*_{i+1,j}(y))/h_{i+1}$.

Similarly, the interpolation function $P^2_{i,j}(x, y)$ is C^1-continuous in the interpolating region $[x_1, x_n; y_1, y_m]$, no matter what the parameters $\gamma_{r,j}, \tau_{r,j}$ $(r = i, i+1, i+2)$ might be, and which satisfies

$$P^2_{i,j}(x_r, y_s) = f_{r,s}, \ \frac{\partial P^2_{i,j}(x_r, y_s)}{\partial x} = \Delta^{(x)}_{r,s}, \ \frac{\partial P^2_{i,j}(x_r, y_s)}{\partial y} = \Delta^{(y)}_{r,s}, \ r = i, i+1, \ s = j, j+1.$$

Where, we let

$$P_{i,j}(x, y) = \lambda_{i,j}P^1_{i,j}(x, y) + (1 - \lambda_{i,j})P^2_{i,j}(x, y), \tag{5}$$

with the weighted coefficient $\lambda_{i,j} \in [0, 1]$, then $P_{i,j}(x, y)$ is called the weighted bivariate blending rational interpolation on $[x_i, x_{i+1}; y_j, y_{j+1}]$. It is obvious that $P_{i,j}(x, y)$ defined by (5) is C^1-continuous in the interpolating region $[x_1, x_n; y_1, y_m]$, no matter what the parameters $\alpha_{i,s}, \beta_{i,s}$ $(s = j, j+1, j+2)$ and $\gamma_{r,j}, \tau_{r,j}$ $(r = i, i+1, i+2)$ might be, and which satisfies

$$P_{i,j}(x_r, y_s) = f_{r,s}, \ \frac{\partial P_{i,j}(x_r, y_s)}{\partial x} = \Delta^{(x)}_{r,s}, \ \frac{\partial P_{i,j}(x_r, y_s)}{\partial y} = \Delta^{(y)}_{r,s}, \ r = i, i+1, \ s = j, j+1.$$

Example 1. Let the interpolated function be $f(x, y) = \sin(x^2 - xy + y)$, $(x, y) \in [0.2, 1.2; 0.2, 1.2]$. Thus, the interpolation function of $f(x, y)$ can be defined in $[0.2, 1; 0.2, 1]$ by (5). Let $h_i = l_j = 0.2$, $\alpha_{i,j} = \alpha_{i,j+1} = \alpha_{i,j+2} = 0.4$, $\beta_{i,j} = \beta_{i,j+1} = \beta_{i,j+2} = 0.8$, $\gamma_{i,j} = \gamma_{i+1,j} = \gamma_{i+2,j} = 0.2$, $\tau_{i,j} = \tau_{i+1,j} = \tau_{i+2,j} = 0.6$, $\lambda_{i,j} = 0.5$, and $P(x, y)$ is the interpolation function defined in (5). Figure 1 and Figure 2 show the graphs of the surfaces $f(x, y)$ and $P(x, y)$, respectively. Figure 3 is the graph of error function $f(x, y) - P(x, y)$ with $\alpha_{i,s} = 0.4, \beta_{i,s} = 0.8, \gamma_{r,j} = 0.2, \tau_{r,j} = 0.6, \lambda_{i,j} = 0.5$. Figure 4 shows the graph of error function $f(x, y) - P(x, y)$ with $\alpha_{i,s} = 1250, \beta_{i,s} = 325, \gamma_{r,j} = 202, \tau_{r,j} = 186, \lambda_{i,j} = 0.5$. Figure 3 shows that the interpolator defined by (5) has a good approximation to the interpolated function $f(x, y)$. From Figure 3 and Figure 4, it is easy to see that the errors of the interpolation have only a small floating for a big change of the parameters, it means that the weighted bivariate blending rational interpolation defined by (5) is stable for parameters. Thus, we can finely adjust the surface shape by selecting suitable parameters.

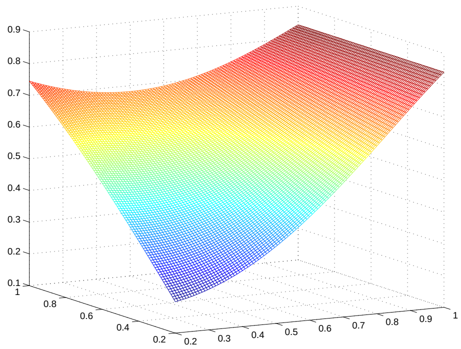

Figure 1: Graph of $f(x, y)$.

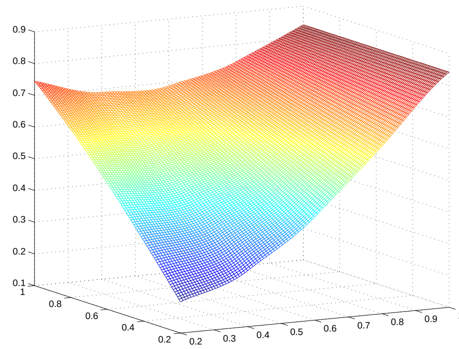

Figure 2: Graph of $P(x, y)$.

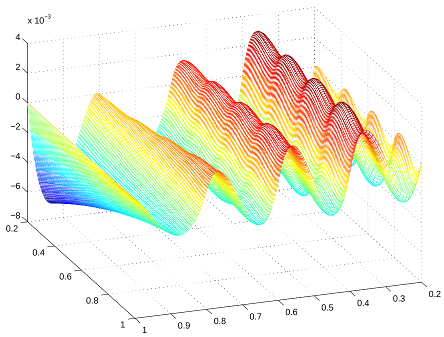

Figure 3: Graph of $f(x, y) - P(x, y)$ with $\alpha_{i,s} = 0.4$, $\beta_{i,s} = 0.8, \gamma_{r,j} = 0.2, \tau_{r,j} = 0.6, \lambda_{i,j} = 0.5$.

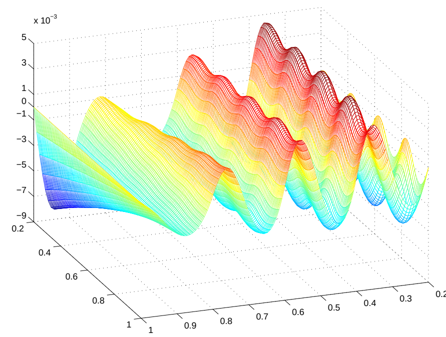

Figure 4: Graph of $f(x, y) - P(x, y)$ with $\alpha_{i,s} = 1250$, $\beta_{i,s} = 325$, $\gamma_{r,j} = 202$, $\tau_{r,j} = 186$, $\lambda_{i,j} = 0.5$.

3 Basis and matrix representation of the interpolation

In the following of this paper, for the interpolation defined by (5), consider the equally spaced knots case, namely, for all $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$, $h_i = h_j$ and $l_i = l_j$. From (1)-(5), the interpolation function defined in (5) can be written as follows:

$$P_{i,j}(x, y) = \sum_{r=0}^{2} \sum_{s=0}^{2} \omega_{r,s}(\theta, \eta) f_{i+r,j+s}, \tag{6}$$

where

$$\begin{aligned}
\omega_{0,0}(\theta, \eta) &= \lambda_{i,j} a_{i,j}^{(x,1)} b_{i,j}^{(x,1)} + (1 - \lambda_{i,j}) a_{i,j}^{(y,1)} b_{i,j}^{(y,1)}, \\
\omega_{1,0}(\theta, \eta) &= \lambda_{i,j} a_{i,j}^{(x,2)} b_{i,j}^{(x,1)} + (1 - \lambda_{i,j}) a_{i+1,j}^{(y,1)} b_{i,j}^{(y,2)}, \\
\omega_{2,0}(\theta, \eta) &= \lambda_{i,j} a_{i,j}^{(x,3)} b_{i,j}^{(x,1)} + (1 - \lambda_{i,j}) a_{i+2,j}^{(y,1)} b_{i,j}^{(y,3)}, \\
\omega_{0,1}(\theta, \eta) &= \lambda_{i,j} a_{i,j+1}^{(x,1)} b_{i,j}^{(x,2)} + (1 - \lambda_{i,j}) a_{i,j}^{(y,2)} b_{i,j}^{(y,1)}, \\
\omega_{1,1}(\theta, \eta) &= \lambda_{i,j} a_{i,j+1}^{(x,2)} b_{i,j}^{(x,2)} + (1 - \lambda_{i,j}) a_{i+1,j}^{(y,2)} b_{i,j}^{(y,2)}, \\
\omega_{2,1}(\theta, \eta) &= \lambda_{i,j} a_{i,j+1}^{(x,3)} b_{i,j}^{(x,2)} + (1 - \lambda_{i,j}) a_{i+2,j}^{(y,2)} b_{i,j}^{(y,3)},
\end{aligned}$$

$$\omega_{0,2}(\theta,\eta) = \lambda_{i,j} a_{i,j+2}^{(x,1)} b_{i,j}^{(x,3)} + (1-\lambda_{i,j}) a_{i,j}^{(y,3)} b_{i,j}^{(y,1)},$$
$$\omega_{1,2}(\theta,\eta) = \lambda_{i,j} a_{i,j+2}^{(x,2)} b_{i,j}^{(x,3)} + (1-\lambda_{i,j}) a_{i+1,j}^{(y,3)} b_{i,j}^{(y,2)},$$
$$\omega_{2,2}(\theta,\eta) = \lambda_{i,j} a_{i,j+2}^{(x,3)} b_{i,j}^{(x,3)} + (1-\lambda_{i,j}) a_{i+2,j}^{(y,3)} b_{i,j}^{(y,3)},$$

with

$$a_{i,j}^{(x,1)} = \frac{(1-\theta)^2(2\theta+(1-\theta)\alpha_{i,j})}{(1-\theta)^2\alpha_{i,j}+2\theta(1-\theta)+\theta^2\beta_{i,j}},$$
$$a_{i,j}^{(x,2)} = \frac{\theta(1-\theta)^2\alpha_{i,j}+\theta^2(2-\theta)\beta_{i,j}+2\theta^2(1-\theta)}{(1-\theta)^2\alpha_{i,j}+2\theta(1-\theta)+\theta^2\beta_{i,j}},$$
$$a_{i,j}^{(x,3)} = -\frac{\theta^2(1-\theta)\beta_{i,j}}{(1-\theta)^2\alpha_{i,j}+2\theta(1-\theta)+\theta^2\beta_{i,j}},$$
$$a_{i,j}^{(y,1)} = \frac{(1-\eta)^2(2\eta+(1-\eta)\gamma_{i,j})}{(1-\eta)^2\gamma_{i,j}+2\eta(1-\eta)+\eta^2\tau_{i,j}},$$
$$a_{i,j}^{(y,2)} = \frac{\eta(1-\eta)^2\gamma_{i,j}+\eta^2(2-\eta)\tau_{i,j}+2\eta^2(1-\eta)}{(1-\eta)^2\gamma_{i,j}+2\eta(1-\eta)+\eta^2\tau_{i,j}},$$
$$a_{i,j}^{(y,3)} = -\frac{\eta^2(1-\eta)\tau_{i,j}}{(1-\eta)^2\gamma_{i,j}+2\eta(1-\eta)+\eta^2\tau_{i,j}},$$
$$b_{i,j}^{(x,1)} = b_{i,j}^{(x,1)}(\eta) = (1-\eta)^2(1+\eta),$$
$$b_{i,j}^{(x,2)} = b_{i,j}^{(x,2)}(\eta) = \eta(1+2\eta-2\eta^2),$$
$$b_{i,j}^{(x,3)} = b_{i,j}^{(x,3)}(\eta) = -\eta^2(1-\eta),$$
$$b_{i,j}^{(y,1)} = b_{i,j}^{(y,1)}(\theta) = (1-\theta)^2(1+\theta),$$
$$b_{i,j}^{(y,2)} = b_{i,j}^{(y,2)}(\theta) = \theta(1+2\theta-2\theta^2),$$
$$b_{i,j}^{(y,3)} = b_{i,j}^{(y,3)}(\theta) = -\theta^2(1-\theta).$$

The terms $\omega_{r,s}(\theta,\eta), (r=0,1,2; s=0,1,2)$ are called the basis of the bivariate interpolation defined by (5), which satisfy

$$\sum_{r=0}^{2}\sum_{s=0}^{2}\omega_{r,s}(\theta,\eta) = 1. \tag{7}$$

Denote

$$a_{i,j}^{(x)} = \begin{pmatrix} a_{i,j}^{(x,1)} & a_{i,j}^{(x,2)} & a_{i,j}^{(x,3)} \end{pmatrix}, \quad a_{i,j+1}^{(x)} = \begin{pmatrix} a_{i,j+1}^{(x,1)} & a_{i,j+1}^{(x,2)} & a_{i,j+1}^{(x,3)} \end{pmatrix},$$
$$a_{i,j+2}^{(x)} = \begin{pmatrix} a_{i,j+2}^{(x,1)} & a_{i,j+2}^{(x,2)} & a_{i,j+2}^{(x,3)} \end{pmatrix}, \quad F_1^{(x)} = \begin{pmatrix} f_{i,j} & f_{i+1,j} & f_{i+2,j} \end{pmatrix}^T,$$
$$F_2^{(x)} = \begin{pmatrix} f_{i,j+1} & f_{i+1,j+1} & f_{i+2,j+1} \end{pmatrix}^T, \quad F_3^{(x)} = \begin{pmatrix} f_{i,j+2} & f_{i+1,j+2} & f_{i+2,j+2} \end{pmatrix}^T,$$
$$a_{i,j}^{(y)} = \begin{pmatrix} a_{i,j}^{(y,1)} & a_{i,j}^{(y,2)} & a_{i,j}^{(y,3)} \end{pmatrix}, \quad a_{i+1,j}^{(y)} = \begin{pmatrix} a_{i+1,j}^{(y,1)} & a_{i+1,j}^{(y,2)} & a_{i+1,j}^{(y,3)} \end{pmatrix},$$
$$a_{i+2,j}^{(y)} = \begin{pmatrix} a_{i+2,j}^{(y,1)} & a_{i+2,j}^{(y,2)} & a_{i+2,j}^{(y,3)} \end{pmatrix}, \quad F_1^{(y)} = \begin{pmatrix} f_{i,j} & f_{i,j+1} & f_{i,j+2} \end{pmatrix}^T,$$
$$F_2^{(y)} = \begin{pmatrix} f_{i+1,j} & f_{i+1,j+1} & f_{i+1,j+2} \end{pmatrix}^T, \quad F_3^{(y)} = \begin{pmatrix} f_{i+2,j} & f_{i+2,j+1} & f_{i+2,j+2} \end{pmatrix}^T,$$
$$B_{i,j}^{(x)} = \begin{pmatrix} b_{i,j}^{(x,1)} & b_{i,j}^{(x,2)} & b_{i,j}^{(x,3)} \end{pmatrix}^T, \quad B_{i,j}^{(y)} = \begin{pmatrix} b_{i,j}^{(y,1)} & b_{i,j}^{(y,2)} & b_{i,j}^{(y,3)}) \end{pmatrix}^T,$$

and let

$$A_{i,j}^{(x)} = \begin{pmatrix} a_{i,j}^{(x)} & a_{i,j+1}^{(x)} & a_{i,j+2}^{(x)} \end{pmatrix}, \quad A_{i,j}^{(y)} = \begin{pmatrix} a_{i,j}^{(y)} & a_{i+1,j}^{(y)} & a_{i+2,j}^{(y)} \end{pmatrix},$$
$$F_{i,j}^{(x)} = \begin{pmatrix} F_1^{(x)} & 0 & 0 \\ 0 & F_2^{(x)} & 0 \\ 0 & 0 & F_3^{(x)} \end{pmatrix}, \quad F_{i,j}^{(y)} = \begin{pmatrix} F_1^{(y)} & 0 & 0 \\ 0 & F_2^{(y)} & 0 \\ 0 & 0 & F_3^{(y)} \end{pmatrix},$$

then the interpolation function $P_{i,j}(x,y)$ defined by (5) can be represented by matrices as follows:

$$P_{i,j}(x,y) = \lambda_{i,j} A_{i,j}^{(x)} F_{i,j}^{(x)} B_{i,j}^{(x)} + (1-\lambda_{i,j}) A_{i,j}^{(y)} F_{i,j}^{(y)} B_{i,j}^{(y)}.$$

Furthermore, Denoting

$$M = \max\{|f_{i+r,j+s}|, r = 0,1,2; j = 0,1,2\}.$$

From (6), it follows that

$$\begin{aligned} |P_{i,j}(x,y)| &\le \sum_{r=0}^{2}\sum_{s=0}^{2} |\omega_{r,s}(\theta,\eta) f_{i+r,j+s}| \le M \sum_{r=0}^{2}\sum_{s=0}^{2} |\omega_{r,s}(\theta,\eta)| \\ &\le M|\lambda_{i,j}|(|b_{i,j}^{(x,1)}| \sum_{k=0}^{2} |a_{i,j}^{(x,k)}| + |b_{i,j}^{(x,2)}| \sum_{k=0}^{2} |a_{i,j+1}^{(x,k)}| + |b_{i,j}^{(x,3)}| \sum_{k=0}^{2} |a_{i,j+2}^{(x,k)}|) \\ &\quad + M|(1-\lambda_{i,j})|(|b_{i,j}^{(y,1)}| \sum_{k=0}^{2} |a_{i,j}^{(y,k)}| + |b_{i,j}^{(y,2)}| \sum_{k=0}^{2} |a_{i+1,j}^{(y,k)}| + |b_{i,j}^{(y,3)}| \sum_{k=0}^{2} |a_{i+2,j}^{(x,k)}|). \end{aligned} \tag{8}$$

It is easy to obtain that

$$\begin{aligned} &\sum_{k=0}^{2} |a_{i,j+s}^{(x,k)}| = 1 + \frac{2\theta^2(1-\theta)\beta_{i,j}}{(1-\theta)^2\alpha_{i,j+s} + 2\theta(1-\theta) + \theta^2\beta_{i,j+s}} \le 3,\ s = 0,1,2, \\ &\sum_{k=0}^{2} |a_{i+r,j}^{(y,k)}| = 1 + \frac{2\eta^2(1-\eta)\tau_{i,j}}{(1-\eta)^2\gamma_{i+r,j} + 2\eta(1-\eta) + \eta^2\tau_{i+r,j}} \le 3,\ r = 0,1,2, \\ &|b_{i,j}^{(\xi,1)}| = (1-\xi)^2(1+\xi) \le 1,\ \xi = x,y, \\ &|b_{i,j}^{(\xi,2)}| = \xi(1+2\xi-2\xi^2) \le 1,\ \xi = x,y, \\ &|b_{i,j}^{(\xi,3)}| = \xi^2(1-\xi) \le \frac{4}{27},\ \xi = x,y. \end{aligned}$$

Thus, the following bounded theorem can be obtained.

Theorem 1. *For the bivariate blending interpolation $P_{i,j}(x,y)$ defined in (6), no matter what positive values the parameters $\alpha_{i,s}, \beta_{i,s}$ and $\gamma_{r,j}, \tau_{r,j}$ take, the values of $P_{i,j}(x,y)$ satisfy*

$$|P_{i,j}(x,y)| \le \frac{58}{9} M.$$

4 Error estimates of the interpolation

For the error estimation of the interpolator defined in (6), note that the interpolator is local, without loss of generality, it is only necessary to consider the interpolating region $[x_i, x_{i+1}; y_j, y_{j+1}]$ in order to process its error estimates. Let $f(x, y) \in C^2$ be the interpolated function, and $P_{i,j}(x, y)$ is the interpolation function defined by (6) over $[x_i, x_{i+1}; y_j, y_{j+1}]$.

Denoting

$$\|\frac{\partial f}{\partial y}\| = \max_{(x,y)\in D} |\frac{\partial f(x,y)}{\partial y}|, \quad \|\frac{\partial P}{\partial y}\| = \max_{(x,y)\in D} |\frac{\partial P_{i,j}(x,y)}{\partial y}|,$$

where $D = [x_i, x_{i+1}; y_j, y_{j+1}]$. By the Taylor expansion and the Peano-Kernel Theorem [16] gives the following:

$$\begin{aligned} |f(x,y) - P_{i,j}(x,y)| &\le |f(x,y) - f(x,y_j)| + |P_{i,j}(x,y_j) - P_{i,j}(x,y)| + |f(x,y_j) - P_{i,j}(x,y_j)| \\ &\le l_j(\|\frac{\partial f}{\partial y}\| + \|\frac{\partial P}{\partial y}\|) + |\int_{x_i}^{x_{i+2}} \frac{\partial^2 f(\tau, y_j)}{\partial x^2} R_x[(x-\tau)_+] d\tau| \\ &\le l_j(\|\frac{\partial f}{\partial y}\| + \|\frac{\partial P}{\partial y}\|) + \|\frac{\partial^2 f(x,y_j)}{\partial x^2}\| \int_{x_i}^{x_{i+2}} |R_x[(x-\tau)_+]| d\tau, \end{aligned} \tag{9}$$

where $\|\frac{\partial^2 f(x,y_j)}{\partial x^2}\| = \max_{x\in[x_i,x_{i+1}]} |\frac{\partial^2 f(x,y_j)}{\partial x^2}|$, $R_x[(x-\tau)_+] = \lambda_{i,j} R_x^1[(x-\tau)_+] + (1-\lambda_{i,j}) R_x^2[(x-\tau)_+]$, and

$$\begin{aligned} R_x^1[(x-\tau)_+] &= \begin{cases} (x-\tau) - a_{i,j}^{(x,2)}(x_{i+1}-\tau) - a_{i,j}^{(x,3)}(x_{i+2}-\tau), & x_i < \tau < x, \\ -a_{i,j}^{(x,2)}(x_{i+1}-\tau) - a_{i,j}^{(x,3)}(x_{i+2}-\tau), & x < \tau < x_{i+1}, \\ -a_{i,j}^{(x,3)}(x_{i+2}-\tau), & x_{i+1} < \tau < x_{i+2}, \end{cases} \\ &= \begin{cases} r_x^1(\tau), & x_i < \tau < x, \\ s_x^1(\tau), & x < \tau < x_{i+1}, \\ t_x^1(\tau), & x_{i+1} < \tau < x_{i+2}, \end{cases} \end{aligned}$$

$$\begin{aligned} R_x^2[(x-\tau)_+] &= \begin{cases} (x-\tau) - b_{i,j}^{(y,2)}(x_{i+1}-\tau) - b_{i,j}^{(y,3)}(x_{i+2}-\tau), & x_i < \tau < x, \\ -b_{i,j}^{(y,2)}(x_{i+1}-\tau) - b_{i,j}^{(y,3)}(x_{i+2}-\tau), & x < \tau < x_{i+1}, \\ -b_{i,j}^{(y,3)}(x_{i+2}-\tau), & x_{i+1} < \tau < x_{i+2}, \end{cases} \\ &= \begin{cases} r_x^2(\tau), & x_i < \tau < x, \\ s_x^2(\tau), & x < \tau < x_{i+1}, \\ t_x^2(\tau), & x_{i+1} < \tau < x_{i+2}, \end{cases} \end{aligned}$$

Thus, by simple integral calculation, it can be derived that

$$\int_{x_i}^{x_{i+2}} |R_x^1[(x-\tau)_+]| d\tau = h_i^2 B_1(\alpha_{i,j}, \beta_{i,j}, \theta),$$

where

$$B_1(\alpha_{i,j}, \beta_{i,j}, \theta) = \frac{1}{2}\theta(1-\theta) + \frac{\beta_{i,j}^2 \theta^3 (1-\theta)^2}{U(\alpha_{i,j}, \beta_{i,j}, \theta)}, \tag{10}$$

with

$$U(\alpha_{i,j},\beta_{i,j},\theta)=((1-\theta)^2\alpha_{i,j}+2\theta(1-\theta)+\theta^2\beta_{i,j})((1-\theta)^2\alpha_{i,j}+2\theta(1-\theta)+\theta\beta_{i,j});$$

$$\int_{x_i}^{x_{i+2}}|R_x^2[(x-\tau)_+]|d\tau=h_i^2B_2(\theta),$$

where

$$B_2(\theta)=\frac{\theta(1-3\theta^3+2\theta^4)}{2(1+\theta-\theta^2)}. \tag{11}$$

Let

$$B_{i,j}^{(x)}=\max_{\theta\in[0,1]}\{\lambda_{i,j}B_1(\alpha_{i,j},\beta_{i,j},\theta)+(1-\lambda_{i,j})B_2(\theta)\}. \tag{12}$$

This leads to the following theorem.

Theorem 2. *Let $f(x,y)\in C^2$ be the interpolated function, and $P_{i,j}(x,y)$ be its interpolator defined by (6) in $[x_i,x_{i+1};y_j,y_{j+1}]$. Whatever the positive values of the parameters $\alpha_{i,s},\beta_{i,s}$ and $\gamma_{r,j},\tau_{r,j}$ might be, the error of the interpolation satisfies*

$$|f(x,y)-P_{i,j}(x,y)|\le l_j(\|\frac{\partial f}{\partial y}\|+\|\frac{\partial P}{\partial y}\|)+h_i^2\|\frac{\partial^2 f(x,y_j)}{\partial x^2}\|B_{i,j}^{(x)},$$

where $B_{i,j}^{(x)}$ defined by (12).

It is easy to derive that

$$B_1(\alpha_{i,j},\beta_{i,j},\theta)\le 1;\quad B_2(\theta)\le 0.150902.$$

Thus, we have the following theorem.

Theorem 3. *For any positive parameters $\alpha_{i,s},\beta_{i,s},\gamma_{r,j},\tau_{r,j}$ and weight coefficient $\lambda_{i,j}$, $B_{i,s}^{(x)}$ are bounded, and*

$$0\le B_{i,j}^{(x)}\le 1.$$

Similarly, denoting $\|\frac{\partial^2 f(x,y_{j+1})}{\partial x^2}\|=\max_{x\in[x_i,x_{i+1}]}|\frac{\partial^2 f(x,y_{j+1})}{\partial x^2}|$, then the following theorem holds.

Theorem 4. *Let $f(x,y)\in C^2$ be the interpolated function, and $P_{i,j}(x,y)$ be its interpolator defined by (6) in $[x_i,x_{i+1};y_j,y_{j+1}]$. Whatever the positive values of the parameters $\alpha_{i,s},\beta_{i,s}$ and $\gamma_{r,j},\tau_{r,j}$ might be, the error of the interpolation satisfies*

$$|f(x,y)-P_{i,j}(x,y)|\le l_j(\|\frac{\partial f}{\partial y}\|+\|\frac{\partial P}{\partial y}\|)+h_i^2\|\frac{\partial^2 f(x,y_{j+1})}{\partial x^2}\|B_{i,j+1}^{(x)},$$

where $B_{i,j+1}^{(x)}=\max_{\theta\in[0,1]}\{\lambda_{i,j}B_1(\alpha_{i,j+1},\beta_{i,j+1},\theta)+(1-\lambda_{i,j})B_2(\theta)\}$, $B_1(\alpha_{i,j},\beta_{i,j},\theta)$ defined by (10).

Note that the interpolator defined by (6) is symmetric about two variables x and y, we denote $\|\frac{\partial f}{\partial x}\|=\max_{(x,y)\in D}|\frac{\partial f(x,y)}{\partial x}|$, $\|\frac{\partial P}{\partial x}\|=\max_{(x,y)\in D}|\frac{\partial P_{i,j}(x,y)}{\partial x}|$, and $\|\frac{\partial^2 f(x_r,y)}{\partial y^2}\|=\max_{y\in[y_j,y_{j+1}]}|\frac{\partial^2 f(x_r,y)}{\partial y^2}|$, $r=1,2$, then the following theorem can be obtained.

Theorem 5. *Let $f(x,y) \in C^2$ be the interpolated function, and $P_{i,j}(x,y)$ be its interpolator defined by (6) in $[x_i, x_{i+1}; y_j, y_{j+1}]$. Whatever the positive values of the parameters $\alpha_{i,s}, \beta_{i,s}$ and $\gamma_{r,j}, \tau_{r,j}$ might be, the error of the interpolation satisfies*

$$|f(x,y) - P_{i,j}(x,y)| \leq h_i(\|\frac{\partial f}{\partial x}\| + \|\frac{\partial P}{\partial x}\|) + l_j^2\|\frac{\partial^2 f(x_r,y)}{\partial y^2}\|B_{r,j}^{(y)},\ r = 1,2,$$

where $B_{r,j}^{(y)} = \max_{\eta\in[0,1]}\{\lambda_{i,j}B_1(\gamma_{r,j},\tau_{r,j},\eta) + (1-\lambda_{i,j})B_2(\eta)\}$, *$B_1(\alpha_{i,j},\beta_{i,j},\theta)$ is defined by (10), $B_2(\theta)$ is defined by (11).*

5 Local shape control of the interpolation

The shape of the interpolating surface on the interpolating region depends on the interpolating data. Generally speaking, when the interpolating data are given, the shape of interpolating surface is fixed. However, for the interpolation defined by (6), since there are parameters and weighted coefficient, the shape of the interpolating surface can be modified.

For any point (x,y) in the interpolating region $[x_i, x_{i+1}; y_j, y_{j+1}]$, let (θ,η) be its local coordinate. We consider the following local point control issue: the practical design requires the value of the interpolation function at the point (x,y) be equal to a real number M, where $M \in (\min\{f_{i+r,j+s}, r = 0,1; j = 0,1\}, \max\{f_{i+r,j+s}, r = 0,1; j = 0,1\})$. Thus

$$\sum_{r=0}^{2}\sum_{s=0}^{2}\omega_{r,s}(\theta,\eta)f_{i+r,j+s} = M, \tag{13}$$

where the local coordinate (θ,η) is given, for the given weighted coefficient $\lambda_{i,j}$, only the parameters $\alpha_{i,s}, \beta_{i,s}$ $(s = j, j+1, j+2)$ and $\gamma_{r,j}, \tau_{r,j}$ $(r = i, i+1, i+2)$ are unknown. If there exist parameters $\alpha_{i,s}, \beta_{i,s}$ and $\gamma_{r,j}, \tau_{r,j}$ which satisfy Eq. (13), then the problem is solved.

For example, the central point is more important than other points for shape control of the interpolating surface, we consider the "Central Point Value Control" problem. Assume $\alpha_{i,j} = \alpha_{i,j+1} = \alpha_{i,j+2}, \beta_{i,j} = \beta_{i,j+1} = \beta_{i,j+2}$, and denote by α_i and β_i respectively; assume $\gamma_{i,j} = \gamma_{i+1,j} = \gamma_{i+2,j}, \tau_{i,j} = \tau_{i+1,j} = \tau_{i+2,j}$, and denote by γ_j and τ_j respectively. Let the weighted coefficient $\lambda_{i,j} = \frac{1}{2}$. Since (x,y) is the central point in $[x_i, x_{i+1}; y_j, y_{j+1}]$, $\theta = \frac{1}{2}, \eta = \frac{1}{2}$. In this case, Eq. (13) becomes

$$\begin{aligned} &2(k_1 - 32M)\alpha_i + 2(k_2 - 32M)\beta_i + 2(k_1 - 32M)\gamma_j + 2(k_3 - 32M)\tau_j \\ &+k_1\alpha_i\gamma_j + k_3\alpha_i\tau_j + k_2\beta_i\gamma_j + k_4\beta_i\tau_j + k_0 = 0, \end{aligned} \tag{14}$$

where

$$\begin{aligned} k_1 &= 6f_{i,j} + 9f_{i,j+1} - f_{i,j+2} + 9f_{i+1,j} + 12f_{i+1,j+1} - f_{i+1,j+2} - f_{i+2,j} - f_{i+2,j+1}, \\ k_2 &= 3f_{i,j} + 3f_{i,j+1} + 15f_{i+1,j} + 24f_{i+1,j+1} - 3f_{i+1,j+2} - 4f_{i+2,j} - 7f_{i+2,j+1} + f_{i+2,j+2}, \\ k_3 &= (3f_{i,j} + 15f_{i,j+1} - 4f_{i,j+2} + 3f_{i+1,j} + 24f_{i+1,j+1} - 7f_{i+1,j+2} - 3f_{i+2,j+1} + f_{i+2,j+2}), \\ k_4 &= (9f_{i,j+1} - 3f_{i,j+2} + 9f_{i+1,j} + 36f_{i+1,j+1} - 9f_{i+1,j+2} - 3f_{i+2,j} - 9f_{i+2,j+1} + 2f_{i+2,j+2}), \\ k_0 &= 24f_{i,j} + 36f_{i,j+1} - 4f_{i,j+2} + 36f_{i+1,j} + 48f_{i+1,j+1} - 4f_{i+1,j+2} - 4f_{i+2,j} - 4f_{i+2,j+1} - 128M. \end{aligned}$$

If there exist positive solutions α_i, β_i and γ_j, τ_j which satisfy Eq. (14), then the interpolation $P_{i,j}(x,y)$ defined in (6) with these parameters is the solution of "Central Point Value Control" problem. This can be stated in the following theorem.

Theorem 6. *The sufficient condition for "Central Point Value Control" problem having solution is that the number of the variation sign of the sequence* $\{k_i, i = 0, 1, \cdots, 4; k_j - 32M, j = 1, 2, 3\}$ *are not equal to zero.*

Example 2. Let $\Omega : [0, 2; 0, 2]$ be plane region, and the interpolating data are given in Table 1.

Table 1. The interpolating data

(x_i, y_j)	$(0,0)$	$(0,1)$	$(0,2)$	$(1,0)$	$(1,1)$	$(1,2)$	$(2,0)$	$(2,1)$	$(2,2)$
$f_{i,j}$	2.5	1.5	1	2	1	0.5	1	1.5	0.5

Then the interpolation function $P_{i,j}(x, y)$ defined by (5) can be constructed in $[0, 1; 0, 1]$ for the given positive parameters $\alpha_{i,s}, \beta_{i,s}, \gamma_{r,j}, \tau_{r,j}$ and the weighted coefficient $\lambda_{i,j}$. In general case, one can select any positive real numbers as the values of $\alpha_{i,s}, \beta_{i,s}$ and $\gamma_{r,j}, \tau_{r,j}$.

Let $\lambda_{i,j} = \frac{1}{2}$ and $\alpha_i = 0.5, \beta_i = 0.8, \gamma_j = 0.4, \tau_j = 0.6$, for the given interpolation data, denote the interpolation function by $P_1(x, y)$ which is defined over $[0, 1; 0, 1]$. Figure 5 shows the graph of the weighted bivariate blending rational interpolation surface $P_1(x, y)$. In this case, the function value of central point is that $P_1(\frac{1}{2}, \frac{1}{2}) = \frac{17269}{10560} = 1.63532\cdots$.

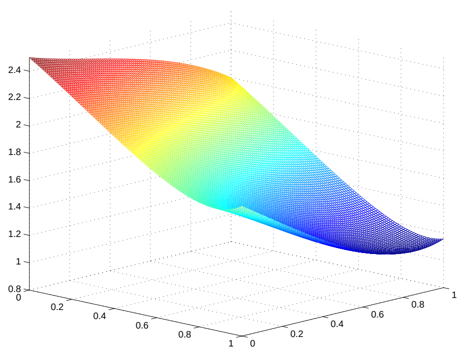

Figure 5: Graph of $P_1(x, y)$.

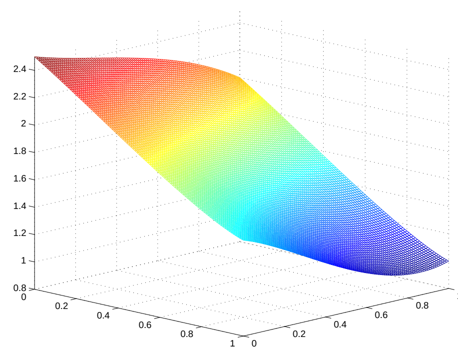

Figure 6: Graph of $P_2(x, y)$.

From the "Central Point Value Control" equation (13), the value of central point can be controlled through modifying the parameters. For example, if the practical design requires $P(\frac{1}{2}, \frac{1}{2}) = 1.65$. It is easy to test that $\alpha_i = 0.5, \beta_i = \frac{55}{58}, \gamma_j = 0.25$ and $\tau_j = 0.25$ satisfy Eq. (14). Denote the interpolation by $P_2(x, y)$. Figure 6 shows the graph of the surface $P_2(x, y)$.

Furthermore, if the practical design requires $P(\frac{1}{2}, \frac{1}{2}) = 1.6$. It is easy to examine that $\alpha_i = 0.5, \beta_i = \frac{45}{62}, \gamma_j = 0.25$ and $\tau_j = 1.5$ satisfy Eq. (14). Denote the interpolation by $P_3(x, y)$. Figure 7 shows the graph of the surface $P_3(x, y)$.

6 Concluding remarks

An explicit representation of a weighted bivariate blending rational interpolator is presented. The interpolation function is C^1-continuous in the interpolating region $[x_1, x_n; y_1, y_m]$. In this interpolation, there are some parameters $\alpha_{i,j}, \beta_{i,j}, \gamma_{i,j}, \tau_{i,j}$ and weighted coefficient $\lambda_{i,j}$. These positive parameters and weighted coefficient can be freely selected according to the needs of practical design.

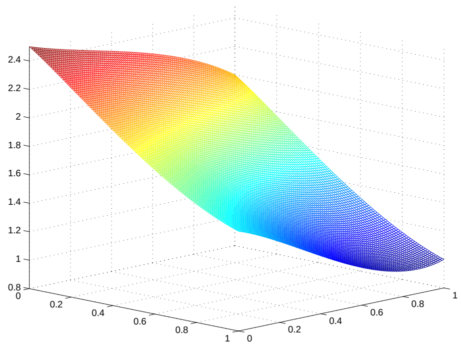

Figure 7: Graph of $P_3(x, y)$.

The local shape control method of the interpolating surfaces is also developed in this paper. And from (13), there are varied expressions, so manifold control scheme can be derived similar to the process for "Central Point Value Control" problem.

Acknowledgment

This research were supported by the National Nature Science Foundation of China (No.61070096) and the Natural Science Foundation of Shandong Province (No.ZR2012FL05).

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GENERALIZED ADDITIVE FUNCTIONAL INEQUALITIES IN C^*-ALGEBRAS

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ABSTRACT. In this paper, we investigate isomorphisms and derivations in real C^*-algebras associated with the following generalized additive functional inequality

$$\|af(x) + bf(y) + cf(z)\| \le \|f(\alpha x + \beta y + \gamma z)\|. \tag{0.1}$$

We moreover prove the Hyers-Ulam stability of homomorphisms and derivations in real C^*-algebras associated with the generalized additive functional inequality (0.1).

1. INTRODUCTION AND PRELIMINARIES

The stability problem of functional equations originated from a question of Ulam [50] concerning the stability of group homomorphisms. Hyers [12] gave a first affirmative answer to the question of Ulam for Banach spaces. Th.M. Rassias [38] provided a generalization of Hyers' Theorem which allows the *Cauchy difference to be unbounded.*

Theorem 1.1. [38] *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x + y) - f(x) - f(y)\| \le \theta(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E$, where θ and p are positive real numbers with $p < 1$. Then there exists an unique additive mapping $L : E \to E'$ satisfying

$$\|f(x) - L(x)\| \le \frac{2\theta}{2 - 2^p}\|x\|^p$$

for all $x \in E$. Also, if for each $x \in E$ the mapping $f(tx)$ is continuous in $t \in \mathbb{R}$, then L is $\mathbb{R}$-linear.

Th.M. Rassias [39] during the 27^{th} International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \ge 1$. Gajda [8] following the same approach as in Th.M. Rassias [38], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [8], as well as by Th.M. Rassias and Šemrl [45] that one cannot prove a Th.M. Rassias' type theorem when $p = 1$. The counterexamples of Gajda [8], as well as of Th.M. Rassias and Šemrl [45] have stimulated several mathematicians to invent new definitions of *approximately additive* or *approximately linear* mappings, cf. Găvruta [9], who among others studied the

2010 *Mathematics Subject Classification.* Primary 39B72; Secondary 46L05.

Key words and phrases. Hyers-Ulam stability, generalized additive functional inequality, C^*-algebra homomorphism, derivation.

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Hyers-Ulam stability of functional equations (cf. the books of Czerwik [3, 4], Hyers, Isac and Th.M. Rassias [13]).

Găvruta [9] provided a further generalization of Th.M. Rassias' Theorem. Isac and Th.M. Rassias [16] applied the Hyers-Ulam stability theory to prove fixed point theorems and study some new applications in Nonlinear Analysis. In [14], Hyers, Isac and Th.M. Rassias studied the asymptoticity aspect of Hyers-Ulam stability of mappings. Several papers have been published on various generalizations and applications of Hyers-Ulam stability to a number of functional equations and mappings (see [1, 5, 6, 18, 19, 20, 21, 23, 24, 26, 28, 29, 30, 32, 34, 40, 41, 42, 43, 47, 48, 49]).

Beginning around the year 1980, the topic of approximate homomorphisms and their stability theory in the field of functional equations and inequalities were taken up by several mathematicians (cf. Hyers and Th.M. Rassias [15], Th.M. Rassias [42] and the references therein).

Definition 1.2. Let A and B be real C^*-algebras.

(i) An $\mathbb{R}$-linear mapping $H : A \to B$ is called a *C^*-algebra homomorphism* if $H(xy) = H(x)H(y)$ and $H(x^*) = H(x)^*$ for all $x, y \in A$. If a C^*-algebra homomorphism $H : A \to B$ is bijective, then the C^*-algebra homomorphism $H : A \to B$ is called a *C^*-algebra isomorphism*.

(ii) An $\mathbb{R}$-linear mapping $D : A \to A$ is called a *derivation* if $D(xy) = D(x)y + xD(y)$ for all $x, y \in A$.

A real C^*-algebra A, endowed with the Lie product $[x, y] := \frac{xy-yx}{2}$ on A, is called a *Lie C^*-algebra* (see [25, 27, 33]).

Definition 1.3. Let A and B be Lie C^*-algebras.

(i) An $\mathbb{R}$-linear mapping $H : A \to B$ is called a *Lie $*$-homomorphism* if $H([x, y]) = [H(x), H(y)]$ and $H(x^*) = H(x)^*$ for all $x, y \in A$. If a Lie $*$-homomorphism $H : A \to B$ is bijective, then the Lie $*$-homomorphism $H : A \to B$ is called a *Lie $*$-isomorphism*.

(ii) An $\mathbb{R}$-linear mapping $D : A \to A$ is called a *Lie derivation* if $D([x, y]) = [x, D(y)] + [D(x), y]$ for all $x, y \in A$.

Gilányi [10] showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(x - y)\| \le \|f(x + y)\| \tag{1.2}$$

then f satisfies the quadratic functional equation

$$2f(x) + 2f(y) = f(x + y) + f(x - y).$$

See also [46]. Fechner [7] and Gilányi [11] proved the Hyers-Ulam stability of the functional inequality (1.2). Park, Cho and Han [31] investigated the Jordan-von Neumann type Cauchy-Jensen additive mappings and prove their stability, and Cho and Kim [2] proved the Hyers-Ulam stability of the Jordan-von Neumann type Cauchy-Jensen additive mappings.

The purpose of this paper is to prove the Hyers-Ulam stability of generalized additive functional inequalities associated with homomorphisms and derivations in C^*-algebras.

In Section 2, we investigate isomorphisms in C^*-algebras associated with the functional inequality (0.1).

In Section 3, we investigate derivations on C^*-algebras associated with the functional inequality (0.1).

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In Section 4, we prove the Hyers-Ulam stability of homomorphisms in C^*-algebras associated with the functional inequality (0.1).

In Section 5, we prove the Hyers-Ulam stability of derivations on C^*-algebras associated with the functional inequality (0.1).

Throughout this paper, we assume that $a, b, c, \alpha, \beta, \gamma$ are nonzero real numbers. Let A be a real C^*-algebra with C^*-norm $\|\cdot\|_A$, and let B be a real C^*-algebra with C^*-norm $\|\cdot\|_B$.

2. Isomorphisms in C^*-algebras

Consider a mapping $f : A \to B$ satisfying the following functional inequality

$$\|af(x) + bf(y) + cf(z)\|_B \leq \|f(\alpha x + \beta y + \gamma z)\|_B \tag{2.1}$$

for all $x, y, z \in A$.

In this section, we investigate isomorphisms in C^*-algebras associated with the functional inequality (2.1).

Theorem 2.1. *Let $p \neq 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a nonzero bijective mapping satisfying* (2.1) *and* $\lim_{t\in\mathbb{R}, t\to 0} f(tx) = 0$ *for all $x \in A$ such that*

$$\begin{aligned} \|f(xy) - f(x)f(y)\|_B &\leq \theta(\|x\|_A^{2p} + \|y\|_A^{2p}), & (2.2)\\ \|f(x^*) - f(x)^*\|_B &\leq 2\theta\|x\|_A^p & (2.3)\end{aligned}$$

for all $x, y \in A$. Then the bijective mapping $f : A \to B$ is a C^*-algebra isomorphism.

Proof. By Theorem 2.7 of [22], the mapping $f : A \to B$ is $\mathbb{R}$-linear.

(i) Assume that $p < 1$. By (2.2),

$$\begin{aligned} \|f(xy) - f(x)f(y)\|_B &= \lim_{n\to\infty} \frac{1}{4^n}\|f(4^n xy) - f(2^n x)f(2^n y)\|_B \\ &\leq \lim_{n\to\infty} \frac{4^{np}}{4^n}\theta(\|x\|_A^{2p} + \|y\|_A^{2p}) = 0 \end{aligned}$$

for all $x, y \in A$. So

$$f(xy) = f(x)f(y)$$

for all $x, y \in A$.

By (2.3),

$$\|f(x^*) - f(x)^*\|_B = \lim_{n\to\infty} \frac{1}{2^n}\|f(2^n x^*) - f(2^n x)^*\|_B \leq \lim_{n\to\infty} \frac{2\cdot 2^{np}}{2^n}\theta\|x\|_A^{2p} = 0$$

for all $x \in A$. So

$$f(x^*) = f(x)^*$$

for all $x \in A$.

(ii) Assume that $p > 1$. By a similar method to the proof of the case (i), one can prove that the mapping $f : A \to B$ satisfies

$$\begin{aligned} f(xy) &= f(x)f(y), \\ f(x^*) &= f(x)^* \end{aligned}$$

for all $x, y \in A$.

Therefore, the bijective mapping $f : A \to B$ is a C^*-algebra isomorphism. □

Theorem 2.2. *Let $p \neq 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a nonzero bijective mapping satisfying* (2.1), (2.3) *and* $\lim_{t\in\mathbb{R},t\to 0} f(tx) = 0$ *for all $x \in A$ such that*

$$\|f(xy) - f(x)f(y)\|_B \leq \theta \cdot \|x\|_A^p \cdot \|y\|_A^p \tag{2.4}$$

for all $x, y \in A$. Then the bijective mapping $f : A \to B$ is a C^*-algebra isomorphism.

Proof. By Theorem 2.7 of [22], the mapping $f : A \to B$ is $\mathbb{R}$-linear.

(i) Assume that $p < 1$. By (2.4),

$$\begin{aligned}\|f(xy) - f(x)f(y)\|_B &= \lim_{n\to\infty} \frac{1}{4^n}\|f(4^n xy) - f(2^n x)f(2^n y)\|_B \\ &\leq \lim_{n\to\infty} \frac{4^{np}}{4^n}\theta \cdot \|x\|_A^p \cdot \|y\|_A^p = 0\end{aligned}$$

for all $x, y \in A$. So

$$f(xy) = f(x)f(y)$$

for all $x, y \in A$.

(ii) Assume that $p > 1$. By a similar method to the proof of the case (i), one can prove that the mapping $f : A \to B$ satisfies

$$f(xy) = f(x)f(y)$$

for all $x, y \in A$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Remark 2.3. If we replace $\|f(xy) - f(x)f(y)\|_B$ by $\|f([x, y]) - [f(x), f(y)]\|_B$ in the statements of Theorem 2.1 and Theorem 2.2, then we can show that $f : A \to B$ is a Lie $*$-isomorphism instead of a C^*-algebra isomorphism, respectively.

3. Derivations on C^*-algebras

Consider a mapping $f : A \to A$ satisfying the following functional inequality

$$\|af(x) + bf(y) + cf(z)\|_A \leq \|f(\alpha x + \beta y + \gamma z)\|_A \tag{3.1}$$

for all $x, y, z \in A$.

In this section, we investigate derivations on C^*-algebras associated with the functional inequality (3.1).

Theorem 3.1. *Let $p \neq 1$ and θ be nonnegative real numbers, and let $f : A \to A$ be a nonzero mapping satisfying* (3.1) *and* $\lim_{t\in\mathbb{R},t\to 0} f(tx) = 0$ *for all $x \in A$ such that*

$$\|f(xy) - f(x)y - xf(y)\|_A \leq \theta(\|x\|_A^{2p} + \|y\|_A^{2p}) \tag{3.2}$$

for all $x, y \in A$. Then the mapping $f : A \to A$ is a derivation on a C^*-algebra.

Proof. By Theorem 2.7 of [22], the mapping $f : A \to A$ is $\mathbb{R}$-linear.

(i) Assume that $p < 1$. By (3.2),

$$\begin{aligned}\|f(xy) - f(x)y - xf(y)\|_A &= \lim_{n\to\infty} \frac{1}{4^n}\|f(4^n xy) - f(2^n x) \cdot 2^n y - 2^n x \cdot f(2^n y)\|_A \\ &\leq \lim_{n\to\infty} \frac{4^{np}}{4^n}\theta(\|x\|_A^{2p} + \|y\|_A^{2p}) = 0\end{aligned}$$

for all $x, y \in A$. So

$$f(xy) = f(x)y + xf(y)$$

for all $x, y \in A$.

(ii) Assume that $p > 1$. By a similar method to the proof of the case (i), one can prove that the mapping $f : A \to A$ satisfies

$$f(xy) = f(x)y + xf(y)$$

for all $x, y \in A$.

Therefore, the mapping $f : A \to A$ is a derivation on a C^*-algebra, as desired. □

Theorem 3.2. *Let $p \neq 1$ and θ be nonnegative real numbers, and let $f : A \to A$ be a nonzero mapping satisfying (3.1) and $\lim_{t\in\mathbb{R},t\to 0} f(tx) = 0$ for all $x \in A$ such that*

$$\|f(xy) - f(x)y - xf(y)\|_A \leq \theta \cdot \|x\|_A^p \cdot \|y\|_A^p \tag{3.3}$$

for all $x, y \in A$. Then the mapping $f : A \to A$ is a derivation on a C^*-algebra.

Proof. By Theorem 2.7 of [22], the mapping $f : A \to A$ is $\mathbb{R}$-linear.

(i) Assume that $p < 1$. By (3.3),

$$\begin{aligned}\|f(xy) - f(x)y - xf(y)\|_A &= \lim_{n\to\infty} \frac{1}{4^n}\|f(4^n xy) - f(2^n x)\cdot 2^n y - 2^n x \cdot f(2^n y)\|_A \\ &\leq \lim_{n\to\infty} \frac{4^{np}}{4^n}\theta \cdot \|x\|_A^p \cdot \|y\|_A^p = 0\end{aligned}$$

for all $x, y \in A$. So

$$f(xy) = f(x)y + xf(y)$$

for all $x, y \in A$.

(ii) Assume that $p > 1$. By a similar method to the proof of the case (i), one can prove that the mapping $f : A \to A$ satisfies

$$f(xy) = f(x)y + xf(y)$$

for all $x, y \in A$.

Therefore, the mapping $f : A \to A$ is a derivation on a C^*-algebra, as desired. □

Remark 3.3. If we replace $\|f(xy) - f(x)y - xf(y)\|_A$ by $\|f([x, y]) - [f(x), y] - [x, f(y)]\|_A$ in the statements of Theorem 3.1 and Theorem 3.2, then we can show that $f : A \to B$ is a Lie derivation instead of a derivation on a C^*-algebra, respectively.

4. Stability of homomorphisms in C^*-algebras

In this section, we prove the Hyers-Ulam stability of homomorphisms in C^*-algebras.

Theorem 4.1. *Let $f : A \to B$ be a mapping satisfying $\lim_{t\in\mathbb{R},t\to 0} f(tx) = 0$ for all $x \in A$. When $|\alpha| > |\beta|$ and $0 < p < 1$, or $|\alpha| < |\beta|$ and $p > 1$, if there exists a $\theta \geq 0$ satisfying (2.2) and (2.3) such that*

$$\begin{aligned}\|\alpha f(x) + \beta f(y) + \gamma f(z)\|_B &\leq \|f(\alpha x + \beta y + \gamma z)\|_B \\ &+ \theta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p)\end{aligned} \tag{4.1}$$

for all $x, y, z \in A$, then there exists a unique C^*-algebra homomorphism $H : A \to B$ satisfying

$$\|f(x) - H(x)\|_B \le \frac{\theta(|\alpha|^p + |\beta|^p)}{|\alpha||\beta|(|\beta|^{p-1} - |\alpha|^{p-1})}\|x\|_A^p \tag{4.2}$$

for all $x \in A$.

Proof. Let $\xi = -\frac{\alpha}{\beta}$. By Corollary 3.7 of [22], there exists a unique $\mathbb{R}$-linear mapping $H : A \to B$ satisfying (4.2). The mapping $H : A \to B$ is defined by $H(x) := \lim_{n\to\infty} \frac{f(\xi^n x)}{\xi^n}$ for all $x \in A$.

By (2.2),

$$\begin{aligned} \|H(xy) - H(x)H(y)\|_B &= \lim_{n\to\infty} \frac{1}{\xi^{2n}}\|f(\xi^{2n}xy) - f(\xi^n x)f(\xi^n y)\|_B \\ &\le \lim_{n\to\infty} \frac{\xi^{2np}}{\xi^{2n}}\theta(\|x\|_A^{2p} + \|y\|_A^{2p}) = 0 \end{aligned}$$

for all $x, y \in A$. So

$$H(xy) = H(x)H(y)$$

for all $x, y \in A$.

By (2.3),

$$\|H(x^*) - H(x)^*\|_B = \lim_{n\to\infty} \frac{1}{\xi^n}\|f(\xi^n x^*) - f(\xi^n x)^*\|_B \le \lim_{n\to\infty} \frac{\xi^{np}}{\xi^n}2\theta \cdot \|x\|_A^p = 0$$

for all $x \in A$. So

$$H(x^*) = H(x)^*$$

for all $x \in A$.

Therefore, the mapping $H : A \to B$ is a unique C^*-algebra homomorphism, as desired. □

We prove another stability of generalized additive functional inequalities.

Theorem 4.2. *Let $f : A \to B$ be a mapping satisfying $\lim_{t\in\mathbb{R},t\to 0} f(tx) = 0$ for all $x \in A$. When $|\alpha| > |\beta|$ and $p > 1$, or $|\alpha| < |\beta|$ and $0 < p < 1$, if there exists a $\theta \ge 0$ satisfying (2.2), (2.3) and (4.1), then there exists a unique C^*-algebra homomorphism $H : A \to B$ satisfying*

$$\|f(x) - H(x)\|_B \le \frac{\theta(|\alpha|^p + |\beta|^p)}{|\alpha||\beta|(|\alpha|^{p-1} - |\beta|^{p-1})}\|x\|_A^p \tag{4.3}$$

for all $x \in A$.

Proof. Let $\xi = -\frac{\alpha}{\beta}$. By Corollary 3.10 of [22], there exists a unique $\mathbb{R}$-linear mapping $H : A \to B$ satisfying (4.3). The mapping $H : A \to B$ is defined by $H(x) := \lim_{n\to\infty} \xi^n f(\frac{x}{\xi^n})$ for all $x \in A$.

By (2.2),

$$\begin{aligned}\|H(xy) - H(x)H(y)\|_B &= \lim_{n\to\infty} \xi^{2n} \left\| f\left(\frac{xy}{\xi^{2n}}\right) - f\left(\frac{x}{\xi^n}\right) f\left(\frac{y}{\xi^n}\right)\right\|_B \\ &\le \lim_{n\to\infty} \frac{\xi^{2n}}{\xi^{2np}} \theta(\|x\|_A^{2p} + \|y\|_A^{2p}) = 0\end{aligned}$$

for all $x, y \in A$. So

$$H(xy) = H(x)H(y)$$

for all $x, y \in A$.

By (2.3),

$$\|H(x^*) - H(x)^*\|_B = \lim_{n\to\infty} \xi^n \left\| f\left(\frac{x^*}{\xi^n}\right) - f\left(\frac{x}{\xi^n}\right)^*\right\|_B \le \lim_{n\to\infty} \frac{\xi^n}{\xi^{np}} 2\theta \cdot \|x\|_A^p = 0$$

for all $x \in A$. So

$$H(x^*) = H(x)^*$$

for all $x \in A$.

Therefore, the mapping $H : A \to B$ is a unique C^*-algebra homomorphism, as desired. □

Remark 4.3. If we replace $\|f(xy) - f(x)f(y)\|_B$ by $\|f([x, y]) - [f(x), f(y)]\|_B$ in the statements of Theorem 4.1 and Theorem 4.2, then we can show that there is a unique Lie $*$-homomorphism instead of a unique C^*-algebra homomorphism, respectively.

5. Stability of derivations on C^*-algebras

In this section, we prove the Hyers-Ulam stability of derivations on C^*-algebras.

Theorem 5.1. *Let $f : A \to A$ be a mapping satisfying $\lim_{t\in\mathbb{R}, t\to 0} f(tx) = 0$ for all $x \in A$. When $|\alpha| > |\beta|$ and $0 < p < 1$, or $|\alpha| < |\beta|$ and $p > 1$, if there exists a $\theta \ge 0$ satisfying (3.2) such that*

$$\begin{aligned}\|\alpha f(x) + \beta f(y) + \gamma f(z)\|_A &\le \|f(\alpha x + \beta y + \gamma z)\|_A \\ &+ \theta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p) \qquad (5.1)\end{aligned}$$

for all $x, y, z \in A$, then there exists a unique derivation $D : A \to A$ satisfying

$$\|f(x) - D(x)\|_A \le \frac{\theta(|\alpha|^p + |\beta|^p)}{|\alpha||\beta|(|\beta|^{p-1} - |\alpha|^{p-1})} \|x\|_A^p \qquad (5.2)$$

for all $x \in A$.

Proof. Let $\xi = -\frac{\alpha}{\beta}$. By Corollary 3.7 of [22], there exists a unique $\mathbb{R}$-linear mapping $D : A \to A$ satisfying (5.2). The mapping $D : A \to A$ is defined by $D(x) := \lim_{n\to\infty} \frac{f(\xi^n x)}{\xi^n}$ for all $x \in A$.

By (3.2),

$$\begin{aligned}\|D(xy) - D(x)y - xD(y)\|_A &= \lim_{n\to\infty} \frac{1}{\xi^{2n}} \|f(\xi^{2n}xy) - f(\xi^n x) \cdot \xi^n y - \xi^n x \cdot f(\xi^n y)\|_A \\ &\le \lim_{n\to\infty} \frac{\xi^{2np}}{\xi^{2n}} \theta(\|x\|_A^{2p} + \|y\|_A^{2p}) = 0\end{aligned}$$

for all $x, y \in A$. So

$$D(xy) = D(x)y + xD(y)$$

for all $x, y \in A$.

Therefore, the mapping $D : A \to B$ is a unique derivation, as desired. □

We prove another stability of generalized additive functional inequalities.

Theorem 5.2. *Let $f : A \to B$ be a mapping satisfying $\lim_{t\in\mathbb{R},t\to 0} f(tx) = 0$ for all $x \in A$. When $|\alpha| > |\beta|$ and $p > 1$, or $|\alpha| < |\beta|$ and $0 < p < 1$, if there exists a $\theta \geq 0$ satisfying* (3.2) *and* (5.1)*, then there exists a unique derivation $D : A \to A$ satisfying*

$$\|f(x) - D(x)\|_A \leq \frac{\theta(|\alpha|^p + |\beta|^p)}{|\alpha||\beta|(|\alpha|^{p-1} - |\beta|^{p-1})}\|x\|_A^p \tag{5.3}$$

for all $x \in A$.

Proof. Let $\xi = -\frac{\alpha}{\beta}$. By Corollary 3.10 of [22], there exists a unique $\mathbb{R}$-linear mapping $D : A \to A$ satisfying (5.3). The mapping $D : A \to A$ is defined by $D(x) := \lim_{n\to\infty} \xi^n f(\frac{x}{\xi^n})$ for all $x \in A$.

By (3.2),

$$\begin{aligned} \|D(xy) - D(x)y - xD(y)\|_A &= \lim_{n\to\infty} \xi^{2n} \left\| f\left(\frac{xy}{\xi^{2n}}\right) - f\left(\frac{x}{\xi^n}\right) \cdot \frac{y}{\xi^n} - \frac{x}{\xi^n} \cdot f\left(\frac{y}{\xi^n}\right)\right\|_A \\ &\leq \lim_{n\to\infty} \frac{\xi^{2n}}{\xi^{2np}}\theta(\|x\|_A^{2p} + \|y\|_A^{2p}) = 0 \end{aligned}$$

for all $x, y \in A$. So

$$D(xy) = D(x)y + xD(y)$$

for all $x, y \in A$.

Therefore, the mapping $D : A \to A$ is a unique derivation, as desired. □

Remark 5.3. If we replace $\|f(xy) - f(x)y - xf(y)\|_A$ by $\|f([x, y]) - [f(x), y] - [x, f(y)]\|_A$ in the statements of Theorem 5.1 and Theorem 5.2, then we can show that there is a unique Lie derivation instead of a unique derivation on a C^*-algebra, respectively.

Acknowledgments

D. Y. Shin was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2010-0021792).

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Approximation reductions in IVF decision information systems *

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February 14, 2014

Abstract: Rough set theory is a mathematical tool to deal with imprecision and uncertainty in data analysis. The lower and upper approximations of a set respectively characterize the non-numeric and numeric uncertain aspects of the available information. In this paper, we study upper and lower approximation reductions in IVF decision information systems by using rough set theory.

Keywords: IVF; Decision information system; Similarity relation; Lower approximation reduction; Upper approximation reduction.

1 Introduction

The concept of rough set was originally proposed by Pawlak [1] as a mathematical tool to handle imprecision, vagueness and uncertainty in data analysis. This theory has been amply demonstrated to have usefulness and versatility by successful application in machine learning, intelligent systems, inductive reasoning, pattern recognition, mereology, image processing, signal analysis, knowledge discovery, decision analysis, expert systems and many other fields [2, 3, 4, 5]. Rough set theory of deals with the approximation of an arbitrary subset of a universe by two definable or observable subsets called lower and upper approximations. By using the concept of lower and upper approximation in rough sets theory, knowledge hidden in information systems may be unravelled and expressed in the form of decision rules.

In traditional rough set approach, the values of attributes are assumed to be nominal data, i.e. symbols. In many applications, however, the decision attribute-values can be linguistic terms (i.e. interval value fuzzy sets). The traditional rough set approach would treat these values as symbols, thereby some important information included in these values such as the partial ordering and membership degrees is ignored, which means that the traditional rough set

*This work is supported by the National Social Science Foundation of China (No. 12BJL087).

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approach cannot effectively deal with interval value fuzzy initial data. Thus a new rough set model is needed to deal with such data.

Interval-valued fuzzy information systems are the combination of traditional rough sets theory and interval-valued fuzzy sets theory. The purpose of this paper is to investigate attribute reductions in interval-valued fuzzy decision information systems.

2 Preliminaries

In this section, we briefly recall some basic concepts about IVF sets and IVF decision information systems.

Throughout this paper, " interval-valued fuzzy " denote briefly by " IVF ". U denotes a finite and nonempty set called the universe. 2^U denotes the family of all subsets of U. $F(U)$ denotes the set of all fuzzy sets in U. I denotes $[0,1]$ and $[I]$ denotes $\{[a,b] : a,b \in I \ and \ a \leq b\}$. For $a \in I$, denote $\overline{a} = [a,a]$.

2.1 IVF sets

Definition 2.1 ([7]). *$\forall\ a,b \in [I]$, define*

(1) $a = b \Longleftrightarrow a^- = b^-,\ a^+ = b^+.$
(2) $a \leq b \Longleftrightarrow a^- \leq b^-,\ a^+ \leq b^+; \quad a < b \Longleftrightarrow a \leq b,\ a \neq b.$
(3) $a^c = \overline{1} - a = [1 - a^+, 1 - a^-].$

Definition 2.2 ([7]). *$\forall\ \{a_i : i \in J\} \subseteq [I]$, define*
(1) $\bigvee\limits_{i\in J} a_i = [\bigvee\limits_{i\in J} a_i^-,\ \bigvee\limits_{i\in J} a_i^+].$
(2) $\bigwedge\limits_{i\in J} a_i = [\bigwedge\limits_{i\in J} a_i^-,\ \bigwedge\limits_{i\in J} a_i^+].$

Definition 2.3 ([7]). *A mapping $A : U \to [I]$ is called an IVF set on U. Denote*

$$A(x) = [A^-(x), A^+(x)] \quad (x \in U).$$

Then $A^-(x)$ (resp. $A^+(x)$) is called the lower (resp. upper) degree to which x belongs to A. A^- (resp. A^+) is called the lower (resp. upper) IVF set of A.

The set of all IVF sets in U is denoted by $F^{(i)}(U)$.

$\widetilde{U}$ represents the IVF set which satisfies $\widetilde{U}(x) = \overline{1}$ for each $x \in U$.

Similar to fuzzy sets, some basic operations on $F^{(i)}(U)$ as follows: $\forall\ A, B \in F^{(i)}(U)$,
(1) $A = B \iff A(x) = B(x)$ for each $x \in U$.
(2) $A \subseteq B \iff A(x) \leq B(x)$ for each $x \in U$.
(3) (3) $B = A^c$ *or* $\widetilde{U} - A \iff B(x) = A(x)^c$ *or* $\widetilde{U}(x) - A(x)$ for each $x \in U$.
(4) $(A \cap B)(x) = A(x) \wedge B(x)$ for each $x \in U$.
(5) $(A \cup B)(x) = A(x) \vee B(x)$ for each $x \in U$.
Obviously,

$$A = B \iff A^- = B^- \ and \ A^+ = B^+.$$

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2.2 IVF decision information systems

Definition 2.4 ([8]). *$(U, A \cup D)$ is called an IVF decision information system, where $U = \{x_0, x_1, \cdots, x_{n-1}\}$ is the universe, A is a condition attribute set and $D = \{d_k \in F^{(i)}(U) : k = 1, 2, \cdots, r\}$ is a decision attribute set.*

Denote

$$d_k(x_i) = D_{ik} \ (i = 0, 1, \cdots, n-1, \ k = 1, 2, \cdots, r),$$

$$D(x_i) = \frac{D_{i1}}{d_1} + \frac{D_{i2}}{d_2} + \cdots + \frac{D_{ir}}{d_r}.$$

Example 2.5 ([8]). *Table 1 gives an IVF decision information system where $U = \{x_0, x_1, \cdots, x_9\}$, $A = \{a_1, a_2, a_3\}$, $D = \{d_1, d_2, d_3\}$.*

Table 1: An IVF decision information system

	a_1	a_2	a_3	d_1	d_2	d_3
x_0	2	1	3	[0.7,0.9]	[0.15,0.2]	[0.4,0.5]
x_1	3	2	1	[0.3,0.5]	[0.5,0.7]	[0.35,0.4]
x_2	2	1	3	[0.7,0.8]	[0.3,0.4]	[0.1,0.2]
x_3	2	2	3	[0.15,0.2]	[0.5,0.8]	[0.2,0.3]
x_4	1	1	4	[0.05,0.1]	[0.2,0.3]	[0.65,0.9]
x_5	1	1	2	[0.1,0.2]	[0.35,0.5]	[1.0,1.0]
x_6	3	2	1	[0.25,0.4]	[1.0,1.0]	[0.3,0.4]
x_7	1	1	4	[0.1,0.2]	[0.25,0.4]	[0.5,0.6]
x_8	2	1	3	[0.45,0.6]	[0.25,0.3]	[0.2,0.3]
x_9	3	2	1	[0.05,0.1]	[0.8,0.9]	[0.05,0.2]

Obviously,

$$D(x_0) = \frac{D_{01}}{d_1} + \frac{D_{02}}{d_2} + \frac{D_{03}}{d_3} = \frac{[0.7, 0.9]}{d_1} + \frac{[0.15, 0.2]}{d_2} + \frac{[0.4, 0.5]}{d_3}.$$

Definition 2.6. *Let $(U, A \cup D)$ be an IVF decision information system. Then $B \subseteq A$ determines an equivalence relation as follows:*

$$R_B = \{(x, y) \in U \times U : a(x) = a(y) \ (a \in B)\}.$$

R_B forms a partition $U/R_B = \{[x]_B : x \in U\}$ of U where $[x]_B = \{y \in U : (x, y) \in R_B\}$.

The lower and upper approximations of $X \in F^{(i)}(U)$ with regard to (U, R_B) as follows:

$$\underline{R_B}(X)(x) = \bigwedge_{y \in [x]_B} X(y), \quad \overline{R_B}(X)(x) = \bigvee_{y \in [x]_B} X(y) \ (x \in U).$$

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Remark 2.7. *If $y \in [x]_B$, then $[y]_B = [x]_B$. So $\underline{R_B}(X)(y) = \underline{R_B}(X)(x)$ and $\overline{R_B}(X)(y) = \overline{R_B}(X)(x)$.*

Denote $\underline{R_B}(X)([x]_B) = \underline{R_B}(X)(x)$, $\overline{R_B}(X)([x]_B) = \overline{R_B}(X)(x)$ $\quad (*)$

Proposition 2.8. *Let $(U, A \cup D)$ be an IVF decision information system and let $C \subseteq B \subseteq A$. Then $\forall\, X \in F^{(i)}(U)$,*

(1) $\underline{R_B}(\widetilde{U} - X) = \widetilde{U} - \overline{R_B}(X)$.

(2) $\underline{R_C}(X) \subseteq \underline{R_B}(X) \subseteq X \subseteq \overline{R_B}(X) \subseteq \overline{R_C}(X)$.

Proof. (1) $\forall\, x \in U$,

$$\begin{aligned}\underline{R_B}(\widetilde{U} - X)(x) &= \bigwedge_{y\in[x]_B} (\widetilde{U} - X)(y) = \bigwedge_{y\in[x]_B} (\widetilde{U}(y) - X(y))\\ &= \bigwedge_{y\in[x]_B} \widetilde{U}(y) - \bigvee_{y\in[x]_B} X(y)\\ &= \widetilde{U}(x) - \bigvee_{y\in[x]_B} X(y) = (\widetilde{U} - \overline{R_B}(X))(x).\end{aligned}$$

Then $\underline{R_B}(\widetilde{U} - X) = \widetilde{U} - \overline{R_B}(X)$.

(2) Since $C \subseteq B$, $\forall\, x \in U$, $[x]_C \supseteq [x]_B$. Then

$$\underline{R_C}(X)(x) = \bigwedge_{y\in[x]_C} X(y) \le \bigwedge_{y\in[x]_B} X(y) = \underline{R_B}(X)(x).$$

So $\underline{R_C}(X) \subseteq \underline{R_B}(X)$.

Similarly, $\overline{R_B}(X) \subseteq \overline{R_C}(X)$.

$\forall\, x \in U,\ x \in [x]_B$. Then $\underline{R_B}(X)(x) = \bigwedge_{y\in[x]_B} X(y) \le X(x)$. So $\underline{R_B}(X) \subseteq X$.

Similarly, $X \subseteq \overline{R_B}(X)$.

Hence

$$\underline{R_C}(X) \subseteq \underline{R_B}(X) \subseteq X \subseteq \overline{R_B}(X) \subseteq \overline{R_C}(X).$$

□

3 Approximation reductions in IVF decision information systems

3.1 The similarity relation R_D

Definition 3.1. *Let $S = (U, A \cup D)$ be a IVF decision information system where $U = \{x_0, x_1, \cdots, x_n\}$, , A is a condition attribute set $D = \{d_1, d_2, \cdots, d_r\}$.*

Denote

$$d_k(x_i) = D_{ik}\ (i = 0, 1, \cdots, n-1,\ k = 1, 2, \cdots, r),$$

For $i, j \in \{0, 1, \cdots, n-1\}$, define

$$R_D(x_i, x_j) = \bigwedge\{\bar{1} - D_{ik} \wedge D_{jk}| : k = 1, 2, \cdots, r\}.$$

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Obviously, $R_D(x_i, x_i) = \overline{1}$, $R_D(x_i, x_j) = R_D(x_j, x_i)$. Then R_D is a similarity relation on U. We can obtain the following similar decision class $S_D(x)$:

$$S_D(x)(y) = R_D(x, y)\ (y \in U).$$

Denote

$$S_D(x_i) = \frac{x_0}{S_D(x_i)(x_0)} + \frac{x_1}{S_D(x_i)(x_1)} + \frac{x_2}{S_D(x_i)(x_2)} + \frac{x_3}{S_D(x_i)(x_3)} + \frac{x_4}{S_D(x_i)(x_4)} + \frac{x_5}{S_D(x_i)(x_5)} + \frac{x_6}{S_D(x_i)(x_6)}$$

$$+ \frac{x_7}{S_D(x_i)(x_7)} + \frac{x_8}{S_D(x_i)(x_8)} + \frac{x_9}{S_D(x_i)(x_9)}\quad (i = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9),$$

$$U/R_D = \{S_D(x) : x \in U\}.$$

Example 3.2. *In Example 2.5,*

$$\begin{aligned} S_D(x_0)(x_1) = R_D(x_0, x_1) \\ &= (\overline{1} - D_{01} \wedge D_{11}) \bigwedge (\overline{1} - D_{02} \wedge D_{12}) \bigwedge (\overline{1} - D_{03} \wedge D_{13}) \\ &= (\overline{1} - [0.7, 0.9] \wedge [0.3, 0.5]) \bigwedge (\overline{1} - [0.15, 0.2] \wedge [0.5, 0.7]) \\ &\quad \bigwedge (\overline{1} - [0.4, 0.5] \wedge [0.35, 0.4]) \\ &= (\overline{1} - [0.3, 0.5]) \bigwedge (\overline{1} - [0.15, 0.2]) \bigwedge (\overline{1} - [0.35, 0.4]) \\ &= [0.5, 0.7] \bigwedge [0.8, 0.85]) \bigwedge [0.6, 0.65] \\ &= [0.5, 0.65]. \end{aligned}$$

Similarly,

$$S_D(x_0)(x_0) = \overline{1},\quad S_D(x_0)(x_2) = [0.2, 0.3],\quad S_D(x_0)(x_3) = [0.7, 0.8],$$

$$S_D(x_0)(x_4) = [0.5, 0.6],\quad S_D(x_0)(x_5) = [0.5, 0.6],\quad S_D(x_0)(x_6) = [0.6, 0.7],$$

$$S_D(x_0)(x_7) = [0.5, 0.6],\quad S_D(x_0)(x_8) = [0.4, 0.55],\quad S_D(x_0)(x_9) = [0.8, 0.85].$$

Thus

$$S_D(x_0) = \frac{x_0}{\overline{1}} + \frac{x_1}{[0.5, 0.65]} + \frac{x_2}{[0.2, 0.3]} + \frac{x_3}{[0.7, 0.8]} + \frac{x_4}{[0.5, 0.6]} + \frac{x_5}{[0.5, 0.6]} + \frac{x_6}{[0.6, 0.7]} + \frac{x_7}{[0.5, 0.6]} + \frac{x_8}{[0.4, 0.55]} + \frac{x_9}{[0.8, 0.85]}.$$

We can also calculate $S_D(x_i)$ $(i = 1, 2, 3, 4, 5, 6, 7, 8, 9)$. They record in Table 2.

3.2 Upper and lower approximation reductions

Definition 3.3. *Let $S = (U, A \cup D)$ be a IVF decision information system. Denote $U/R_D = \{S_D(x) : x \in U\}$. $\forall\ B \subseteq A$,*

(1) If $\forall\ x \in U$, $\underline{R_B}(S_D(x)) = \underline{R_A}(S_D(x))$, then B is called a lower approximation consistent set of S; If B is a lower approximation consistent set of S and $\forall\ C \subsetneq B, x \in U$, $\underline{R_C}(S_D(x)) \neq \underline{R_A}(S_D(x))$, then B is called a lower approximation reduction of S.

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Table 2: $S_D(x_i)$

	x_0	x_1	x_2	x_3	x_4
$S_D(x_0)$	$\bar{1}$	[0.5,0.65]	[0.2,0.3]	[0.7,0.8]	[0.5,0.6]
$S_D(x_1)$	[0.5,0.65]	$\bar{1}$	[0.5,0.7]	[0.3,0.5]	[0.6,0.65]
$S_D(x_2)$	[0.2,0.3]	[0.5,0.7]	$\bar{1}$	[0.6,0.7]	[0.7,0.8]
$S_D(x_3)$	[0.7,0.8]	[0.3,0.5]	[0.6,0.7]	$\bar{1}$	[0.7,0.8]
$S_D(x_4)$	[0.5,0.6]	[0.6,0.65]	[0.7,0.8]	[0.7,0.8]	$\bar{1}$
$S_D(x_5)$	[0.5,0.6]	[0.5,0.65]]	[0.6,0.7]	[0.5,0.65]	[0.1,0.35]
$S_D(x_6)$	[0.6,0.7]	[0.3,0.5]	[0.6,0.7]	[0.2,0.5]	[0.6,0.7]
$S_D(x_7)$	[0.5,0.6]	[0.6,0.65]	[0.4,0.75]	[0.6,0.75]	[0.4,0.5]
$S_D(x_8)$	[0.4,0.55]	[0.5,0.7]	[0.4,0.55]	[0.7,0.75]	[0.7,0.8]
$S_D(x_9)$	[0.8,0.85]	[0.3,0.5]	[0.6,0.7]	[0.2,0.5]	[0.7,0.8]
−−	−−	−−	−−	−−	−−
	x_5	x_6	x_7	x_8	x_9
$S_D(x_0)$	[0.5,0.6]	[0.6,0.7]	[0.5,0.6]	[0.4,0.55]	[0.8,0.85]
$S_D(x_1)$	[0.5,0.65]	[0.3,0.5]	[0.6,0.65]	[0.5,0.7]	[0.3,0.5]
$S_D(x_2)$	[0.6,0.7]	[0.6,0.7]	[0.4,0.75]	[0.4,0.55]	[0.6,0.7]
$S_D(x_3)$	[0.5,0.65]	[0.2,0.5]	[0.6,0.75]	[0.7,0.75]	[0.2,0.5]
$S_D(x_4)$	[0.1,0.35]	[0.6,0.7]	[0.4,0.5]	[0.7,0.8]	[0.7,0.8]
$S_D(x_5)$	$\bar{1}$	[0.5,0.65]	[0.4,0.5]	[0.7,0.75]	[0.5,0.65]
$S_D(x_6)$	[0.5,0.65]	$\bar{1}$	[0.6,0.7]	[0.6,0.75]	[0.1,0.2]
$S_D(x_7)$	[0.4,0.5]	[0.6,0.7]	$\bar{1}$	[0.7,0.75]	[0.6,0.75]
$S_D(x_8)$	[0.7,0.75]	[0.6,0.75]	[0.7,0.75]	$\bar{1}$	[0.7,0.75]
$S_D(x_9)$	[0.5,0.65]	[0.6,0.75]	[0.6,0.75]	[0.7,0.75]	$\bar{1}$

(2) *If* $\forall\ x \in U$, $\overline{R_B}(S_D(x)) = \overline{R_A}(S_D(x))$, *then* B *is called a upper approximation consistent set of* S*; If* B *is a upper approximation consistent set of* S *and* $\forall\ C \subsetneq B, x \in U$, $\overline{R_C}(S_D(x)) \neq \overline{R_A}(S_D(x))$, *then* B *is called a upper approximation reduction of* S.

Theorem 3.4. *Let* $S = (U, A \cup D)$ *be a IVF decision information system. Denote* $U/R_D = \{S_D(x) : x \in U\}$.

(1) $B \subseteq A$ *is a lower approximation consistent set of* $S \Longleftrightarrow$ *If* $\forall\ x, y, z \in U$, $\underline{R_A}(S_D(x))(y) \neq \underline{R_A}(S_D(x))(z)$, *then* $[y]_B \cap [z]_B = \emptyset$.

(2) $B \subseteq A$ *is a upper approximation consistent set of* $S \Longleftrightarrow$ *If* $\forall\ x, y, z \in U$, $\overline{R_A}(S_D(x))(y) \neq \overline{R_A}(S_D(x))(z)$, *then* $[y]_B \cap [z]_B = \emptyset$.

Proof. (1) " $\Longrightarrow$ " Let B be a lower approximation consistent set of S. Then $\forall\ x \in U$, $\underline{R_B}(S_D(x)) = \underline{R_A}(S_D(x))$. Thus $\forall\ x, y, z \in U$,

$$\underline{R_B}(S_D(x))(y) = \underline{R_A}(S_D(x))(y),\quad \underline{R_B}(S_D(x))(z) = \underline{R_A}(S_D(x))(z).$$

Since $\underline{R_A}(S_D(x))(y) \neq \underline{R_A}(S_D(x))(z)$, we have $\underline{R_B}(S_D(x))(y) \neq \underline{R_B}(S_D(x))(z)$,

Then

$$\bigwedge_{u\in[y]_B} S_D(x)(u) \neq \bigwedge_{v\in[z]_B} S_D(x)(v).$$

Thus $[y]_B \cap [z]_B = \emptyset$

" $\Longleftarrow$ " Since $[y]_B = \cup\{[z]_A : z \in [y]_B\}$, we have

$$\begin{aligned}\underline{R_B}(S_D(x))(y) &= \bigwedge_{u\in[y]_B} S_D(x)(u) = \bigwedge\{S_D(x)(u) : u \in \bigcup_{z\in[y]_B}[z]_A\}\\ &= \bigwedge_{z\in[y]_B}\{S_D(x)(u) : u \in [z]_A\} = \bigwedge_{z\in[y]_B}\bigwedge_{u\in[z]_A} S_D(x)(u)\\ &= \bigwedge_{z\in[y]_B} \underline{R_A}(S_D(x))(z).\end{aligned}$$

If $z \in [y]_B$, then $[y]_B \cap [z]_B \neq \emptyset$. By the Hypothesis, $\underline{R_A}(S_D(x))(y) = \underline{R_A}(S_D(x))(z)$. So

$$\underline{R_B}(S_D(x))(y) = \underline{R_A}(S_D(x))(y).$$

Thus $\forall\ x \in U$, $\underline{R_B}(S_D(x)) = \underline{R_A}(S_D(x))$.

Hence B is a lower approximation consistent set of S.

(2) The proof is similar to (1). □

Definition 3.5. *Let $S = (U, A \cup D)$ be a IVF decision information system. Denote $U/R_D = \{S_D(x) : x \in U\}$.*

$\forall\ x, y, z \in U$, denote

$$\underline{D}(y,z) = \begin{cases} \{a_i \in A : a_i(y) \neq a_i(z)\}, & \underline{R_A}(S_D(x))(y) \neq \underline{R_A}(S_D(x))(z), \\ A, & \underline{R_A}(S_D(x))(y) = \underline{R_A}(S_D(x))(z). \end{cases}$$

$$\overline{D}(y,z) = \begin{cases} \{a_i \in A : a_i(y) \neq a_i(z)\}, & \overline{R_A}(S_D(x))(y) \neq \overline{R_A}(S_D(x))(z), \\ A, & \overline{R_A}(S_D(x))(y) = \overline{R_A}(S_D(x))(z). \end{cases}$$

$\underline{D}(y,z)$ and $\overline{D}(y,z)$ are called the lower and upper approximation attribute set of y and z in S, respectively.

Theorem 3.6. *Let $S = (U, A \cup D)$ be a IVF decision information system. Denote $U/R_D = \{S_D(x) : x \in U\}$. Then $\forall\ B \subseteq A$,*

(1) B is a lower approximation consistent set of $S \Longleftrightarrow \forall\ y, z \in U$, $B \cap \underline{D}(y,z) \neq \emptyset$.

(2) B is a upper approximation consistent set of $S \Longleftrightarrow \forall\ y, z \in U$, $B \cap \overline{D}(y,z) \neq \emptyset$.

Proof. (1) " $\Longrightarrow$ " Suppose B is a lower approximation consistent set of S.

If $\underline{R_A}(S_D(x))(y) \neq \underline{R_A}(S_D(x))(z)$, then by Theorem 3.4(1), $[y]_B \cap [z]_B = \emptyset$. So there exists $a_{i_0} \in B$ such that $a_{i_0}(y) \neq a_{i_0}(z)$. Then

$$a_{i_0} \in B \cap \{a_i \in A : a_i(y) \neq a_i(z)\} = B \cap \underline{D}(y,z).$$

Thus $B \cap \underline{D}(y,z) \neq \emptyset$.

If $\underline{R_A}(S_D(x))(y) = \underline{R_A}(S_D(x))(z)$, then $B \cap \underline{D}(y,z) = B \cap A = B \neq \emptyset$.

" $\Longleftarrow$ " Suppose $B \cap \underline{D}(y,z) \neq \emptyset$. If $\underline{R_A}(S_D(x))(y) \neq \underline{R_A}(S_D(x))(z)$, then

$$B \cap \underline{D}(y,z) = B \cap \{a_i \in A : a_i(y) \neq a_i(z)\} \neq \emptyset.$$

So $[y]_B \cap [z]_B = \emptyset$.

By Theorem 3.4(1), B is a lower approximation consistent set of S.

(2) The proof is similar to (1). $\square$

Example 3.7. *Consider Example 3.2.*

(1) $U/R_A = \{X_1, X_2, X_3, X_4, X_5\}$
$= \{\{x_0, x_2, x_8\}, \{x_1, x_6, x_9\}, \{x_3\}, \{x_4, x_7\}, \{x_5\}\}$.

We need to calculate $\underline{R_A}(S_D(x_0))(X_i)$.

$$\begin{aligned}\underline{R_A}(S_D(x_0))(X_1) &= \bigwedge_{y \in X_1} S_D(x_0)(y) = S_D(x_0)(x_0) \wedge S_D(x_0)(x_2) \wedge S_D(x_0)(x_8) \\ &= \bar{1} \wedge [0.2, 0.3] \wedge [0.4, 0.55] = [0.2, 0.3].\end{aligned}$$

Similarly, $\underline{R_A}(S_D(x_0))(X_2) = [0.5, 0.65]$, $\underline{R_A}(S_D(x_0))(X_3) = [0.7, 0.8]$, $\underline{R_A}(S_D(x_0))(X_4) = [0.5, 0.5]$, $\underline{R_A}(S_D(x_0))(X_5) = [0.5, 0.6]$.

Denote

$$\underline{R_A}(S_D(x_0)) = \frac{X_1}{[0.2, 0.3]} + \frac{X_2}{[0.5, 0.65]} + \frac{X_3}{[0.7, 0.8]} + \frac{X_4}{[0.5, 0.5]} + \frac{X_5}{[0.5, 0.6]}.$$

We can also calculate $\underline{R_A}(S_D(x_i))$ $(i = 1, 2, 3, 4, 5, 6, 7, 8, 9)$. *They record in Table 3.*

(2) *Put* $B = \{a_1, a_2\}$. *Obviously,*

$$U/R_B = \{X_1, X_2, X_3, X_4 \cup X_5\} = \{\{x_0, x_2, x_8\}, \{x_1, x_6, x_9\}, \{x_3\}, \{x_4, x_5, x_7\}\}.$$

We can calculate $\underline{R_B}(S_D(x_i))$ $(i = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9)$. *They record in Table 4.*

By table 3 and table 4, $B = \{a_1, a_2\}$ *is not a lower approximation consistent set of* S.

Thus B *is not a lower approximation reduction of* S.

4 Conclusions

In this paper, we have researched lower approximation reducts and upper approximation reducts in IVF decision information systems. In future work, we will investigate knowledge acquisition in IVF decision information systems..

Table 3: $\underline{R_A}(S_D(x_i))$

	X_1	X_2	X_3	X_4	X_5
$\underline{R_A}(S_D(x_0))$	[0.2,0.3]	[0.5,0.65]	[0.7,0.8]	[0.5,0.5]	[0.5,0.6]
$\underline{R_A}(S_D(x_1))$	[0.5,0.65]	[0.3,0.5]	[0.3,0.5]	[0.6,0.65]	[0.5,0.65]
$\underline{R_A}(S_D(x_2))$	[0.2,0.3]	[0.5,0.7]	[0.6,0.7]	[0.4,0.75]	[0.6.0.7]
$\underline{R_A}(S_D(x_3))$	[0.6,0.7]	[0.2,0.5]	$\bar{1}$	[0.6,0.75]	[0.5,0.65]
$\underline{R_A}(S_D(x_4))$	[0.5,0.6]	[0.6,0.65]	[0.7,0.8]	[0.4,0.5]	[0.1,0.35]
$\underline{R_A}(S_D(x_5))$	[0.5,0.6]]	[0.5,0.65]	[0.5,0.65]	[0.1,0.35]	$\bar{1}$
$\underline{R_A}(S_D(x_6))$	[0.6,0.7]	[0.1,0.2]	[0.2,0.5]	[0.6,0.7]	[0.5,0.65]
$\underline{R_A}(S_D(x_7))$	[0.4,0.6]	[0.6,0.65]	[0.6,0.75]	[0.4,0.5]	[0.4,0.5]
$\underline{R_A}(S_D(x_8))$	[0.4,0.55]	[0.5,0.7]	[0.7,0.75]	[0.7,0.75]	[0.7,0.75]
$\underline{R_A}(S_D(x_9))$	[0.6,0.7]	[0.3,0.5]	[0.2,0.5]	[0.6,0.75]	[0.5,0.65]

Table 4: $\underline{R_B}(S_D(x_1))$

	X_1	X_2	X_3	$X_4 \cup X_5$
$\underline{R_B}(S_D(x_0))$	[0.2,0.3]	[0.5,0.65]	[0.7,0.8]	[0.5,0.5]
$\underline{R_B}(S_D(x_1))$	[0.5,0.65]	[0.3,0.5]	[0.3,0.5]	[0.5,0.65]
$\underline{R_B}(S_D(x_2))$	[0.2,0.3]	[0.5,0.7]	[0.6,0.7]	[0.4,0.7]
$\underline{R_B}(S_D(x_3))$	[0.6,0.7]	[0.2,0.5]	$\bar{1}$	[0.5,0.65]
$\underline{R_B}(S_D(x_4))$	[0.5,0.6]	[0.6,0.65]	[0.7,0.8]	[0.1,0.35]
$\underline{R_B}(S_D(x_5))$	[0.5,0.6]]	[0.5,0.65]	[0.5,0.65]	[0.1,0.35]]
$\underline{R_B}(S_D(x_6))$	[0.6,0.7]	[0.1,0.2]	[0.2,0.5]	[0.5,0.65]
$\underline{R_B}(S_D(x_7))$	[0.4,0.6]	[0.6,0.65]	[0.6,0.75]	[0.4,0.5]]
$\underline{R_B}(S_D(x_8))$	[0.4,0.55]	[0.5,0.7]	[0.7,0.75]	[0.7,0.75]]
$\underline{R_B}(S_D(x_9))$	[0.6,0.7]	[0.3,0.5]	[0.2,0.5]	[0.5,0.65]

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The covering method for attribute reductions of concept lattices *

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February 14, 2014

Abstract: This paper investigates methods for attribute reductions of concept lattices. We establish the relationship between attribute reductions of the object oriented concept lattice generated by a formal context and covering reductions of the covering approximation space induced by the same formal context, and then present the covering method to compute all attribute reductions of this object oriented concept lattices. Moreover, we give an example to compute all attribute or object reductions of the object oriented concept lattice and all reductions of a formal context by using this method, which illustrates that this method is effective and simple.

Keywords: Formal context; Reduction; Concept lattice; Covering method; Covering approximation space.

1 Introduction

Formal concept analysis (FCA), proposed by Wille [7, 12], is an important theory for data analysis and knowledge discovery. Two basic notions in FCA are formal contexts and formal concepts. The set of all formal concepts of a formal context forms a complete lattice, called the concept lattice of the formal context, to reflect the relationship of generalization and specialization among the concepts. Up to now, FCA has been applied to information retrieval, database management systems, software engineering and other aspects (see [2, 4, 6, 9, 13]).

Attribute reductions of concept lattices are one of the key issues in FCA or concept lattice theory and there have been many studies on this topic. For example, Ganter et al. [7] discussed the knowledge reduction issue by removing the reducible objects and attributes of a formal context. Zhang et al. [18] proposed a knowledge reduction method in formal contexts from the viewpoint of lattice isomorphism. Liu et al. [10] put forward reduction method for concept

*This work is supported by the National Natural Science Foundation of China (No. 11226085, 11061004).

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lattices based on rough set theory. Wei et al. [15] investigated the problem of knowledge reduction in decision formal contexts by defining two partial orders between the conditional and the decision concept lattices. Based on granular computing, Wu et al. [14] presented a novel reduction method for decision formal contexts.

The purpose of this paper is to give the covering method for attribute reductions of concept lattices.

2 Preliminaries

In this section, we recall some basic concepts of formal contexts, concept lattices and covering approximation spaces.

Definition 2.1 ([7]). *Let U be a finite set of objects, A be a finite set of features or attributes, and R be a binary relation on $U \times A$ (i.e., $R \subseteq U \times A$). The triple (U, A, R) is called a formal context.*

Remark 2.2. *Let (U, A, R) be a formal context. In this paper, we use the following notations.*

(1) $(u, \alpha) \in R$ is also written as $uR\alpha$, which means that the object u possesses the feature or attribute α.

(2) Let $u \in U$ and $\alpha \in A$, $uR = \{\alpha \in A : uR\alpha\} \subseteq A$ and $R\alpha = \{u \in U : uR\alpha\} \subseteq U$. Thus, $uR\alpha \Longleftrightarrow \alpha \in uR \Longleftrightarrow u \in R\alpha$.

(3) Let $X \in 2^U$ and $B \subseteq A$. $XR = \bigcup\{uR : u \in X\}$ and $RB = \bigcup\{R\alpha : \alpha \in B\}$.

(4) Let $B \subseteq A$. (U, B, R') denotes a formal subcontext of (U, A, R), where $R' = R \bigcap (U \times B)$.

In this paper, all formal context are assumed to be regular, where a formal context (U, A, R) is regular if for each $u \in U$ and each $\alpha \in A$, $uR \neq \emptyset$, $uR \neq A$, $R\alpha \neq \emptyset$ and $R\alpha \neq U$.

Definition 2.3. *Let (U, A, R) be a formal context. Then a pair of approximation operators $\Box : 2^U \longrightarrow 2^A$ and $\Diamond : 2^A \longrightarrow 2^U$ are respectively defined as follows, where $X \in 2^U$ and $B \subset A$.*

$X^{\Box} = \{\alpha \in A : \forall u \in U(uR\alpha \Longrightarrow u \in X)\} = \{\alpha \in A : R\alpha \subset X\}$.

$B^{\Diamond} = \{u \in U : \exists \alpha \in A(uR\alpha \bigwedge \alpha \in B)\} = \{u \in U : uR \bigcap B \neq \emptyset\} = \bigcup\{R\alpha : \alpha \in B\} = RB$.

Definition 2.4 ([16, 17]). *Let (U, A, R) be a formal context, $X \in 2^U$ and $B \subset A$. The ordered pair (X, B) is called an object oriented concept , if $X = B^{\Diamond}$ and $B = X^{\Box}$. X and B are called the extension and the intension of the object oriented concept (X, B), respectively.*

In other words, an object oriented concept is an ordered pair (X, B), where X is the set of objects, each of which possesses at least one attribute in B, and B is the set of attributes, each of which is possessed by one or some objects only in X.

Remark 2.5. *Let (U, A, R) be a formal context and $X \in 2^U$. It is clear that X is the extension of an object oriented concept of (U, A, R) if and only if $X^{\Box\Diamond} = X$.*

Definition 2.6 ([16])**.** *Let (U, A, R) be a formal context. Put*

$$\mathcal{D}(U, A, R) = \{(X, B) : (X, B) \text{ is an object oriented concept of } (U, A, R)\}.$$

Then $\mathcal{D}(U, A, R)$ forms a complete lattice and is called an object oriented concept lattice generated by (U, A, R), in which the meet $\bigvee$ and the join $\bigwedge$ of any two object oriented concepts are respectively defined as follows:

$$(X_1, B_1) \bigvee (X_2, B_2) = (X_1 \bigcup X_2, (X_1 \bigcup X_2)^{\Box}) = (X_1 \bigcup X_2, (B_1 \bigcup B_2)^{\Diamond\Box}),$$

$$(X_1, B_1) \bigwedge (X_2, B_2) = ((B_1 \bigcap B_2)^{\Diamond}, B_1 \bigcap B_2) = ((X_1 \bigcap X_2)^{\Box\Diamond}, B_1 \bigcap B_2).$$

Definition 2.7 ([10])**.** *Let (X_1, B_1) and (X_2, B_2) be two object oriented concepts of a formal context (U, A, R). If $X_1 \subseteq X_2$, which is equivalent to $B_1 \subseteq B_2$, (X_1, B_1) is called a sub-concept of (X_2, B_2), and (X_2, B_2) is called a super-concept of (X_1, B_1), i.e., $(X_1, B_1) \leq (X_2, B_2)$. The relation $\leq$ is the hierarchical order of the concepts. Furthermore, an object oriented concept $(X, B) \in \mathcal{D}(U, A, R)$ is said to be minimal , if $X = X'$ and $B = B'$ whenever $(X', B') \in \mathcal{D}(U, A, R)$ and $(X', B') \leq (X, B)$.*

Definition 2.8 ([10])**.** *Let $\mathcal{D}(U, A_1, R_1)$ and $\mathcal{D}(U, A_2, R_2)$ be two object oriented concept lattices. If there exists a bijective function $h : \mathcal{D}(U, A_1, R_1) \longrightarrow \mathcal{D}(U, A_2, R_2)$ such that $h((X_1, B_1) \bigvee (X_2, B_2)) = h((X_1, B_1)) \bigvee h((X_2, B_2))$ and $h((X_1, B_1) \bigwedge (X_2, B_2)) = h((X_1, B_1)) \bigwedge h((X_2, B_2))$ for whenever (X_1, B_1), $(X_2, B_2) \in \mathcal{D}(U, A_1, R_1)$, then $\mathcal{D}(U, A_1, R_1)$ and $\mathcal{D}(U, A_2, R_2)$ are said to be isomorphic, which is denoted by $\mathcal{D}(U, A_1, R_1) \cong \mathcal{D}(U, A_2, R_2)$.*

Remark 2.9 ([10])**.** *According to Definition 2.8, we have that the structures and the hierarchical orders of two isomorphic concept lattices are identical.*

Definition 2.10 ([10])**.** *Let $\mathcal{D}(U, A, R)$ be an object oriented concept lattice.*

(1) Let $\beta \in A$. Then attribute β is said to be dispensable in $\mathcal{D}(U, A, R)$, if $\mathcal{D}(U, A, R) \cong \mathcal{D}(U, A - \{\beta\}, R \bigcap (U \times (A - \{\beta\})))$; otherwise, attribute α is indispensable in $\mathcal{D}(U, A, R)$.

(2) Let attributes set $B \subseteq A$. Then B is called a consistent set of $\mathcal{D}(U, A, R)$, if $\mathcal{D}(U, A, R) \cong \mathcal{D}(U, B, R \bigcap (U \times B))$.

(3) Let attributes set $B \subseteq A$ be a consistent set of $\mathcal{D}(U, A, R)$. Then B is called an attribute reduction of $\mathcal{D}(U, A, R)$, if for whenever $\beta \in B$, $\mathcal{D}(U, A, R) \ncong \mathcal{D}(U, B - \{\beta\}, R \bigcap (U \times (B - \{\beta\})))$.

Definition 2.11 ([10])**.** *Let $\mathcal{D}(U, A, R)$ be an object oriented concept lattice.*

(1) Let $u \in U$. Then object v is said to be dispensable in $\mathcal{D}(U, A, R)$, if $\mathcal{D}(U, A, R) \cong \mathcal{D}(U - \{v\}, A, R \bigcap ((U - \{v\}) \times A))$; otherwise, object v is indispensable in $\mathcal{D}(U, A, R)$.

(2) Let objects set $V \subseteq U$. Then V is called a consistent set of $\mathcal{D}(U, A, R)$, if $\mathcal{D}(U, A, R) \cong \mathcal{D}(V, A, R \bigcap (V \times A))$.

(3) Let objects set $V \subseteq U$ be a consistent set of $\mathcal{D}(U,A,R)$. Then V is called an object reduction of $\mathcal{D}(U,A,R)$, if for whenever $v \in V$, $\mathcal{D}(U,A,R) \ncong \mathcal{D}(V-\{v\}, A, R\bigcap((V-\{v\})\times A))$.

Definition 2.12 ([10]). *Let (U,A,R) be a formal context, $V \subseteq U$ and $B \subseteq A$. then formal context $(V,B,R\bigcap(V\times B))$ is called a reduction of (U,A,R) , if V and B are an object reduction and an attribute reduction of the object oriented concept lattice $\mathcal{D}(U,A,R)$, respectively.*

Remark 2.13. *Note that there is a dual relation between attribute reductions and object reductions in the object oriented concept lattice $\mathcal{D}(U,A,R)$. So this paper only gives some investigations for attribute reductions in $\mathcal{D}(U,A,R)$. All results for object reductions in $\mathcal{D}(U,A,R)$ can be obtained by the same methods.*

Definition 2.14 ([1]). *Let U, the universe of discourse, be a finite set and $\mathcal{C}$ be a family of subsets of U.*

(1) $\mathcal{C}$ is called a covering of U, if U is the union of elements of $\mathcal{C}$.

(2) The pair $(U,\mathcal{C})$ is called a covering approximation space, if $\mathcal{C}$ is a covering of U.

Definition 2.15 ([8]). *Let $(U,\mathcal{C})$ be a covering approximation space. Put $\mathscr{T} = \{\bigcup\mathcal{F} : \mathcal{F} \subseteq \mathcal{C}\}$. Then $\mathscr{T}$ is called the generalized topology on U induced by $\mathcal{C}$ and $\mathcal{C}$ is called a base for $\mathscr{T}$.*

Definition 2.16 ([19, 20]). *Let $(U,\mathcal{C})$ be a covering approximation space.*

(1) Let $K \in \mathcal{C}$. K is called a reducible element of $\mathcal{C}$, if $K = \bigcup\mathcal{F}$ for some $\mathcal{F} \subseteq \mathcal{C} - \{K\}$; otherwise, K is called an irreducible element of $\mathcal{C}$.

(2) $\mathcal{C}$ is called irreducible , if K is an irreducible element of $\mathcal{C}$ for each $K \in \mathcal{C}$; otherwise $\mathcal{C}$ is called reducible.

Let $(U,\mathcal{C})$ be a covering approximation space. It is known that $\mathcal{C} - \{K\}$ is still a covering of U if K is a reducible element of $\mathcal{C}$ [20]. It follows that a reduction of a covering can be computed by deleting one reducible element in a step [19].

Definition 2.17 ([19, 20]). *Let $(U,\mathcal{C})$ be a covering approximation space. If K_1 is a reducible element of $\mathcal{C}$, K_2 is a reducible element of $\mathcal{C} - \{K_1\}$, $\cdots$, K_n is a reducible element of $\mathcal{C} - \{K_1, K_2, \cdots, K_{n-1}\}$ and $\mathcal{C}' = \mathcal{C} - \{K_1, K_2, \cdots, K_n\}$ is irreducible, then $\mathcal{C}'$ is called a reduction (more precisely, covering reduction) of $\mathcal{C}$.*

In particularly, $\mathcal{C}$ is a reduction of $\mathcal{C}$ if $\mathcal{C}$ is irreducible.

Definition 2.18. *(1) Let (U,A,R) be a formal context. Put $\mathcal{C} = \{R\alpha : \alpha \in A\}$. Then $(U,\mathcal{C})$ is a covering approximation space and it is called to be induced by (U,A,R).*

(2) Let $(U,\mathcal{C})$ be a covering approximation space, where $\mathcal{C} = \{K_\alpha : \alpha \in A\}$. Give a binary relation R on $U \times A$: $uR\alpha$ if and only if $u \in K_\alpha$. Then (U,A,R) is a formal context and it is called to be induced by $(U,\mathcal{C})$.

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Remark 2.19. *There is such a case for a formal context (U, A, R): "$R\alpha = R\beta$ for some $\alpha, \beta \in A$". So, for the sake of conveniences, we allow that $K = K'$ for some $K', K' \in \mathcal{C}$ for a covering approximation space $(U, \mathcal{C})$ in the discussion of this paper.*

3 The main results

In this section, based on the relationship between attribute reductions of $\mathcal{D}(U, A, R)$ and covering reductions of $(U, \mathcal{C})$ where (U, A, R) is a formal context, $\mathcal{D}(U, A, R)$ is the object oriented concept lattice generated by (U, A, R), $(U, \mathcal{C})$ is the covering approximation spaces induced by (U, A, R), we present the covering method to compute all attribute reductions of $\mathcal{D}(U, A, R)$.

Let $(U, \mathcal{C})$ be a covering approximation space, where $\mathcal{C} = \{K_\alpha : \alpha \in A\}$. For $B \subseteq A$, we denote

$$\mathcal{C}_B = \{K_\alpha : \alpha \in B\} \textit{ and } \bigcup \mathcal{C}_B = \bigcup \{K_\alpha : \alpha \in B\}.$$

Obviously, $\mathcal{C}_A = \mathcal{C}$ and $\bigcup \mathcal{C}_A = U$.

Proposition 3.1. *Let (U, A, R) be a formal context and let $(U, \mathcal{C})$ be a covering approximation space.*

(1) If (U, A, R) is induced by $(U, \mathcal{C})$, then $(U, \mathcal{C})$ is induced by (U, A, R).

(2) If $(U, \mathcal{C})$ is induced by (U, A, R), then (U, A, R) is induced by $(U, \mathcal{C})$.

Proof. (1) Let (U, A, R) be induced by $(U, \mathcal{C})$, where $\mathcal{C} = \{K_\alpha : \alpha \in A\}$. Then, for each $u \in U$ and each $\alpha \in A$, $uR\alpha$ if and only if $u \in K_\alpha$. Let $(U, \mathcal{C}')$ be a covering approximation space induced by (U, A, R). Then $\mathcal{C}' = \{R\alpha : \alpha \in A\}$. We only need to prove that $\mathcal{C} = \mathcal{C}'$. Let $\alpha \in A$, it suffices to show that $R\alpha = K_\alpha$. In fact, $u \in R\alpha$ if and only if $uR\alpha$ if and only if $u \in K_\alpha$, so $R\alpha = K_\alpha$.

(2) Let $(U, \mathcal{C})$ be induced by (U, A, R). Then $\mathcal{C} = \{R\alpha : \alpha \in A\}$. Let (U, A, R') be a formal context induced by $(U, \mathcal{C})$. It suffices to show that $R = R'$. In fact, for each $u \in U$ and each $\alpha \in A$, $(u, \alpha) \in R$ if and only if $u \in R\alpha$ if and only if $(u, \alpha) \in R'$, so $R = R'$. □

Remark 3.2. *Let (U, A, R) be a formal context, and let $(U, \mathcal{C})$ be a covering approximation space induced by (U, A, R), where $\mathcal{C} = \{K_\alpha : \alpha \in A\}$. Then $K_\alpha = R\alpha$ for each $\alpha \in A$ and $\bigcup \{K_\alpha : \alpha \in B\} = RB$ for each $B \subseteq A$.*

Proposition 3.3. *Let $(U, \mathcal{C})$ be a covering approximation space and (U, A, R) be a formal context induced by $(U, \mathcal{C})$. Then the following are equivalent for $X \in 2^U$.*

(1) X is the extension of an object oriented concept of (U, A, R).

(2) $X \in \mathscr{T}$, where $\mathscr{T}$ is the generalized topology on U induced by $\mathcal{C}$.

Proof. Let $\mathcal{C} = \{K_\alpha : \alpha \in A\}$. Since (U, A, R) is induced by $(U, \mathcal{C})$, $(U, \mathcal{C})$ is induced by (U, A, R) by Proposition 3.1. So $K_\alpha = R\alpha$ for each $\alpha \in A$ by Remark 3.2.

(1) $\Longrightarrow$ (2): Assume that X is the extension of an object oriented concept of (U, A, R). Then $X = X^{\Box\Diamond}$ by Remark 2.8. Note that $X^{\Box} \subseteq A$. So $X = \bigcup\{R\alpha : \alpha \in X^{\Box}\} = \bigcup\{K_\alpha : \alpha \in X^{\Box}\}$. This proves that X is the union of some elements of $\mathcal{C}$, and hence $X \in \mathscr{T}$.

(2) $\Longrightarrow$ (1): Assume that $X \in \mathscr{T}$. Since $X^{\Box} = \{\beta \in A : R\beta \subseteq X\}$, $X^{\Box\Diamond} = \bigcup\{R\beta : \beta \in X^{\Box}\}$. For each $\beta \in X^{\Box}$, $R\beta \subseteq X$, hence $X^{\Box\Diamond} = \bigcup\{R\beta : \beta \in X^{\Box}\} \subseteq X$. Sine $X \in \mathscr{T}$, then there exists a subset B of A such that $X = \bigcup\{K_\alpha : \alpha \in B\}$. For each $\alpha \in B$, $R\alpha = K_\alpha \subseteq X$, so $\alpha \in X^{\Box}$, hence $K_\alpha = R\alpha \subseteq \bigcup\{R\beta : \beta \in X^{\Box}\} = X^{\Box\Diamond}$. It follows that $X = \bigcup\{K_\alpha : \alpha \in B\} \subseteq X^{\Box\Diamond}$. Consequently, $X^{\Box\Diamond} = X$. By Remark 2.8, X is the extension of an object oriented concept of (U, A, R). $\Box$

By the proof of Proposition 3.3, we have the following remark.

Remark 3.4. *Let (U, A, R) be a formal context, and let $(U, \mathcal{C})$ be a covering approximation space induced by (U, A, R). If $X \in 2^U$ and $B \subseteq A$, then $(X, B) \in \mathcal{D}(U, A, R)$ if and only if $X = \bigcup \mathcal{C}_B$*

Lemma 3.5. *Let (U, A, R) be a formal context and (U, B, R') be a formal subcontext of (U, A, R). For $X \in 2^U$ and $D \subset B$, put*

$X_B^{\Box} = \{\alpha \in B : \forall x \in U(xR'\alpha \Longrightarrow x \in X)\} = \{\alpha \in B : R'\alpha \subset X\}$;

$D_B^{\Diamond} = \{x \in U : \exists \alpha \in B(xR'\alpha \bigwedge \alpha \in D)\} = \{x \in U : xR' \bigcap D \neq \emptyset\} = \bigcup\{R'\alpha : \alpha \in D\} = R'D$.

Then the following hold.

(1) $X_B^{\Box} = X^{\Box} \bigcap B$.

(2) $D_B^{\Diamond} = D^{\Diamond}$.

Proof. (1) Let $\alpha \in X_B^{\Box}$. Then $\alpha \in B$. For each $x \in U$, if $(x, \alpha) \in R$, then $(x, \alpha) \in R \bigcap (U \times B) = R'$, i.e., $xR'\alpha$, hence $x \in X$. It follows that $\alpha \in X^{\Box}$. So $\alpha \in X^{\Box} \bigcap B$. Conversely, let $\alpha \in X^{\Box} \bigcap B$, then $\alpha \in B$. Since $\alpha \in X^{\Box}$, for each $x \in U$, if $(x, \alpha) \in R' \subset R$, then $x \in X$. It follows that $\alpha \in X_B^{\Box}$. Consequently, $X_B^{\Box} = X^{\Box} \bigcap B$.

(2) It is suffices to prove that $R'\alpha = R\alpha$ for each $\alpha \in D$. For each $\alpha \in D \subset B$, $(x, \alpha) \in R$ if and only if $(x, \alpha) \in R \bigcap (U \times B)$, so $R'\alpha = \{x : (x, \alpha) \in R \bigcap (U \times B)\} = \{x : (x, \alpha) \in R\} = R\alpha$. $\Box$

Theorem 3.6. *Let (U, A_1, R_1) and (U, A_2, R_2) be two formal contexts, and let $(U, \mathcal{C}_1)$ and $(U, \mathcal{C}_2)$ be two covering approximation spaces induced by (U, A_1, R_1) and (U, A_2, R_2) respectively. Then the following are equivalent.*

(1) $\mathcal{D}(U, A_1, R_1) \cong \mathcal{D}(U, A_2, R_2)$.

(2) $\mathscr{T}_1 = \mathscr{T}_2$, where $\mathscr{T}_1$ and $\mathscr{T}_2$ are the generalized topologies on U induced by $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively.

Proof. By Remark 2.9, we can use the following equivalent definition of isomorphism between two object oriented concept lattices $\mathcal{D}(U, A_1, R_1)$ and $\mathcal{D}(U, A_2, R_2)$ in the following proof, which was given by Zhang et al. in [18]. $\mathcal{D}(U, A_1, R_1)$ and $\mathcal{D}(U, A_2, R_2)$ are said to be isomorphic if whenever $X \in 2^U$, X is the extension

of an object oriented concept of $\mathcal{D}(U, A_1, R_1)$ if and only if X is the extension of an object oriented concept of $\mathcal{D}(U, A_2, R_2)$.

(1) $\Longrightarrow$ (2): Assume that $\mathcal{D}(U, A_1, R_1) \cong \mathcal{D}(U, A_2, R_2)$. Let $X_1 \in \mathscr{T}_1$, then X_1 is the extension of an object oriented concept of (U, A_1, R_1) by Proposition 3.3. Since $\mathcal{D}(U, A_1, R_1) \cong \mathcal{D}(U, A_2, R_2)$, there exists the extension X_2 of an object oriented concept of (U, A_2, R_2) such that $X_1 = X_2$. By Proposition 3.3, $X_2 \in \mathscr{T}_2$, and hence $X_1 \in \mathscr{T}_2$. This proves that $\mathscr{T}_1 \subseteq \mathscr{T}_2$. Similarly, $\mathscr{T}_2 \subseteq \mathscr{T}_1$. It follows that $\mathscr{T}_1 = \mathscr{T}_2$.

(2) $\Longrightarrow$ (1): Assume that $\mathscr{T}_1 = \mathscr{T}_2$. By Proposition 3.3, X is the extension of an object oriented concept of (U, A_1, R_1) if and only if $X \in \mathscr{T}_1$ and X is the extension of an object oriented concept of (U, A_2, R_2) if and only if $X \in \mathscr{T}_2$. It follows that X is the extension of an object oriented concept of (U, A_1, R_1) if and only if X is the extension of an object oriented concept of (U, A_2, R_2). So $\mathcal{D}(U, A_1, R_1) \cong \mathcal{D}(U, A_2, R_2)$. □

Corollary 3.7. *Let (U, A, R) be a formal context and let $(U, \mathcal{C})$ be a covering approximation space induced by the (U, A, R). Then the following are equivalent for $B \subseteq A$.*

(1) B is a consistent set of the object oriented concept lattice $\mathcal{D}(U, A, R)$.

(2) $\mathcal{C}_B$ is a base for $\mathscr{T}$, where $\mathscr{T}$ is the generalized topologies on U induced by $\mathcal{C}$.

Lemma 3.8. *Let $(U, \mathcal{C})$ be a covering approximation space, where $\mathcal{C} = \{K_\alpha : \alpha \in A\}$, and let (U, A, R) be a formal context induced by the $(U, \mathcal{C})$. Then the following are equivalent for $\beta \in A$.*

(1) β is a dispensable attribute in the object oriented concept lattice $\mathcal{D}(U, A, R)$.

(2) K_β is a reducible element of $\mathcal{C}$.

Proof. (1) $\Longrightarrow$ (2): Assume that β is a dispensable attribute in $\mathcal{D}(U, A, R)$. By [10, Property 3.1], there exists $B \subseteq A - \{\beta\}$ such that $R\beta = RB$ It follows that $K_\beta = \bigcup\{K_\alpha : \alpha \in B\}$ from Remark 3.2. That is K_β is a reducible element of $\mathcal{C}$.

(2) $\Longrightarrow$ (1): It is obtained by reverting the proof of the above (1) $\Longrightarrow$ (2). □

Lemma 3.9. *Let $(U, \mathcal{C})$ be a covering approximation space where $\mathcal{C} = \{K_\alpha : \alpha \in A\}$ and let $B \subseteq A$. Then the following are equivalent.*

(1) $\mathcal{C}_B$ is a reduction of $\mathcal{C}$.

(2) $\mathcal{C}_B$ is an irreducible base for $\mathscr{T}$, where $\mathscr{T}$ is the generalized topologies on U induced by $\mathcal{C}$.

Proof. Let $\mathscr{T}$ be the generalized topologies on U induced by $\mathcal{C}$.

(1) $\Longrightarrow$ (2): Assume that $\mathcal{C}_B$ is a reduction of $\mathcal{C}$. Let $\mathcal{C}_B = \mathcal{C} - \{K_1, K_2, \cdots, K_n\}$ described as in Definition 2.14. Then $\mathcal{C}_B$ is irreducible. Clearly, we only need to prove that K_i is the union of some elements of $\mathcal{C}_B$ for each $i \in \{1, 2, \cdots, n\}$. In fact, since K_n is a reducible element of $\mathcal{C} - \{K_1, K_2, \cdots, K_{n-1}\}$, K_n is the union of some elements of $(\mathcal{C} - \{K_1, K_2, \cdots, K_{n-1}\}) - \{K_n\} = \mathcal{C}_B$. Similarly, since K_{n-1} is a reducible element of $\mathcal{C} - \{K_1, K_2, \cdots, K_{n-2}\}$, K_{n-1} is the union of

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some elements of $(\mathcal{C} - \{K_1, K_2, \cdots, K_{n-2}\}) - \{K_{n-1}\} = \mathcal{C}_B \bigcup \{K_n\}$. Note that K_n is the union of some elements of $\mathcal{C}_B$. So K_{n-1} is the union of some elements of $\mathcal{C}_B$. And so on, K_i is the union of some elements of $\mathcal{C}_B$ for $i = n-2, n-3, \cdots, 2, 1$. Thus, the proof is completed.

(2) $\Longrightarrow$ (1): Assume that $\mathcal{C}_B$ is an irreducible base for $\mathscr{T}$. Let $\mathcal{C} - \mathcal{C}_B = \{K_1, K_2, \cdots, K_n\}$. For each $i \in \{1, 2, \cdots, n\}$, since $\mathcal{C} \subseteq \mathscr{T}$ and $\mathcal{C}_B$ is a base for $\mathscr{T}$, K_i is the union of some elements of $\mathcal{C}_B$. It follows that K_i is a reducible element of $\mathcal{C} - \{K_1, K_2, \cdots, K_{i-1}\}$. On the other hand, $\mathcal{C}_B = \mathcal{C} - \{K_1, K_2, \cdots, K_n\}$ is irreducible. So $\mathcal{C}_B$ is a reduction of $\mathcal{C}$. □

The following theorem gives the relationship between attribute reductions and covering reductions.

Theorem 3.10. *Let (U, A, R) be a formal context and let $(U, \mathcal{C})$ be a covering approximation space induced by the (U, A, R), where $\mathcal{C} = \{K_\alpha : \alpha \in A\}$. Then the following are equivalent for $B \subseteq A$.*

(1) B is an attribute reduction of the object oriented concept lattice $\mathcal{D}(U, A, R)$.

(2) $\mathcal{C}_B$ is a reduction of $\mathcal{C}$.

Proof. Let $\mathscr{T}$ be the generalized topologies on U induced by $\mathcal{C}$.

(1) $\Longrightarrow$ (2): Assume that B is an attribute reduction of the object oriented concept lattice $\mathcal{D}(U, A, R)$. By Lemma 3.9, we only need to prove that $\mathcal{C}_B$ is an irreducible base for $\mathscr{T}$. Note that B is a consistent set of $\mathcal{D}(U, A, R)$. By Corollary 3.7, $\mathcal{C}_B$ is a base for $\mathscr{T}$. It suffices to prove that $\mathcal{C}_B$ is irreducible. Whenever $\beta \in B$. Since B is an attribute reduction of $\mathcal{D}(U, A, R)$, $\mathcal{D}(U, A, R) \ncong \mathcal{D}(U, B - \{\beta\}, R \bigcap (U \times (B - \{\beta\})))$, i.e., β is a indispensable attribute in $\mathcal{D}(U, A, R)$. By Lemma 3.8, K_β is a irreducible element of $\mathcal{C}$. Furthermore, K_β is a irreducible element of $\mathcal{C}_B$. So $\mathcal{C}_B$ is irreducible.

(2) $\Longrightarrow$ (1): Assume that $\mathcal{C}_B$ is a reduction of $\mathcal{C}$. By Lemma 3.9, $\mathcal{C}_B$ is an irreducible base for $\mathscr{T}$. By Corollary 3.7, B is a consistent set of the object oriented concept lattice $\mathcal{D}(U, A, R)$ It suffices to prove that $\mathcal{D}(U, A, R) \ncong \mathcal{D}(U, B - \{\beta\}, R \bigcap (U \times (B - \{\beta\})))$ for whenever $\beta \in B$. In fact, if not, then there exists $\beta \in B$ such that $\mathcal{D}(U, A, R) \cong \mathcal{D}(U, B - \{\beta\}, R \bigcap (U \times (B - \{\beta\})))$. Note that $\mathcal{D}(U, A, R) \cong \mathcal{D}(U, B, R \bigcap (U \times B))$. So $\mathcal{D}(U, B, R \bigcap (U \times B)) \cong \mathcal{D}(U, B - \{\beta\}, R \bigcap (U \times (B - \{\beta\})))$. i.e., β is a dispensable attribute in $\mathcal{D}(U, B, R \bigcap (U \times B))$. By Lemma 3.8, K_β is a reducible element of $\mathcal{C}_B$. This contradicts that $\mathcal{C}_B$ is irreducible. □

Lemma 3.11. *Let $(U, \mathcal{C})$ be a covering approximation space. If $\mathcal{C}_1$ and $\mathcal{C}_2$ are two reductions of $\mathcal{C}$, then $\mathcal{C}_1 = \mathcal{C}_2$.*

Proof. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two reductions of $\mathcal{C}$ and $\mathscr{T}$ be the generalized topologies on U induced by $\mathcal{C}$. Then $\mathcal{C}_1$ and $\mathcal{C}_2$ are two irreducible base for $\mathscr{T}$ from Lemma 3.9. Let $K \in \mathcal{C}_1$, then there is $\mathcal{F} \subseteq \mathcal{C}_2\}$ such that $K = \bigcup \mathcal{F}$. Since $\mathcal{C}_1$ is irreducible, K is not a reducible element of $\mathcal{C}_1$, i.e., K is not a union of some elements of $\mathcal{C}_1 - \{K\}$. It follows that there is $K' \in \mathcal{F}$ such that K' is not a union of some elements of $\mathcal{C}_1 - \{K\}$. Note that $\mathcal{C}_1$ is a base. So there is $\mathcal{F}' \subseteq \mathcal{C}_1\}$ such that $K' = \bigcup \mathcal{F}'$ and $K \in \mathcal{F}'$. Thus, we have $K \supseteq K' \supseteq K$, hence

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$K = K' \subseteq \mathcal{F} \subseteq \mathcal{C}_2$. This proves that $\mathcal{C}_1 \subseteq \mathcal{C}_2$. By the same way, we can prove that $\mathcal{C}_2 \subseteq \mathcal{C}_1$. Consequently, $\mathcal{C}_1 = \mathcal{C}_2$. □

Corollary 3.12. *Let (U, A, R) be a formal context. If B_1 and B_2 are two attribute reductions of the object oriented concept lattice $\mathcal{D}(U, A, R)$, then $\mathscr{R}B_1 = \mathscr{R}B_2$, where $\mathscr{R}B_i = \{R\beta : \beta \in B_i\}$ for $i = 1, 2$.*

In particularly, if B_1 and B_2 are two attribute reductions of $\mathcal{D}(U, A, R)$, then $|B_1| = |B_2|$, where $|A|$ denotes the cardinal number of A for a set A.

By Corollary 3.12, we have the following theorem, which give the covering method to compute all attribute reductions of the object oriented concept lattice $\mathcal{D}(U, A, R)$.

Theorem 3.13. *Let (U, A, R) be a formal context and let $\mathcal{E}$ be the family of all attribute reductions of the object oriented concept lattice $\mathcal{D}(U, A, R)$. If $B = \{\beta_1, \beta_2, \cdots, \beta_n\}$ is an attribute reduction of $\mathcal{D}(U, A, R)$, then $\mathcal{E} = \{\{\alpha_1, \alpha_2, \cdots, \alpha_n\} : R\alpha_i = R\beta_i,\ i = 1, 2, \cdots, n\}$.*

Just as stated in Remark 2.13, all results for object reductions of the object oriented concept lattice $\mathcal{D}(U, A, R)$ can be obtained by the same method, and we omit these results.

The following is an example of a formal context (U, A, R), which comes from [10, Table 1]. As an application of results in this paper, we use the covering method to compute easily attribute or object reductions of $\mathcal{D}(U, A, R)$ and reductions of (U, A, R), where $\mathcal{D}(U, A, R)$ is the object oriented concept lattice generated by (U, A, R).

Example 3.14. *A formal context (U, A, R) is described as the following Table 1, where $U = \{1, 2, 3, 4, 5, 6\}$, $A = \{a, b, c, d, e, f\}$, and the number, which lies in the cross of the row labeled by i ($i = 1, 2, 3, 4, 5, 6$) and the column labeled by t ($t = a, b, c, d, e, f$), is 1 or 0 by $(i, t) \in R$ or $(i, t) \notin R$.*

Table 1: A formal context (U, A, R)

	a	b	c	d	e	f
1	1	0	0	1	0	1
2	1	1	0	1	1	1
3	0	0	0	1	0	0
4	0	0	1	0	1	0
5	1	1	0	1	1	1
6	0	0	1	0	1	0

(1) (U, A, R) induces a covering approximation space $(U, \mathcal{C})$, where $K_a = \{1, 2, 5\}$, $K_b = \{2, 5\}$, $K_c = \{4, 6\}$, $K_d = \{1, 2, 3, 5\}$, $K_e = \{2, 4, 5, 6\}$, $K_f = \{1, 2, 5\}$, and $\mathcal{C} = \{K_\alpha : \alpha \in A\}$. It is easy to see that $K_e = K_b \bigcup K_c$, $K_f = K_a$, and $\{K_a, K_b, K_c, K_d\}$ is irreducible. So $\{K_a, K_b, K_c, K_d\}$ is a base for

the generalized topology induced by $\mathcal{C}$. By Theorem 3.9, $\{K_a, K_b, K_c, K_d\}$ is a reduction of $\mathcal{C}$. It follows that $\{a, b, c, d\}$ is an attribute reduction of the object oriented concept lattice $\mathcal{D}(U, A, R)$. Put $\mathcal{E}$ is the family of all attribute reductions of $\mathcal{D}(U, A, R)$. By Theorem 3.13, $\mathcal{E} = \{\{a, b, c, d\}, \{b, c, d, f\}\}$.

(2) (U, A, R) induces a covering approximation space $(A, \mathscr{J})$, where $L_1 = \{a, d, f\}$, $L_2 = \{a, b, d, e, f\}$, $L_3 = \{d\}$, $L_4 = \{c, e\}$, $L_5 = \{a, b, d, e, f\}$, $L_6 = \{c, e\}$, and $\mathscr{J} = \{L_u : u \in U\}$. It is easy to see that $L_2 = L_5$, $L_4 = L_6$, and $\{L_1, L_2, L_3, L_4\}$ is irreducible. So $\{L_1, L_2, L_3, L_4\}$ is a base for the generalized topology induced by $\mathscr{J}$. By Theorem 3.9, $\{L_1, L_2, L_3, L_4\}$ is a reductions of $\mathscr{J}$. It follows that $\{1, 2, 3, 4\}$ is an object reduction of the object oriented concept lattice $\mathcal{D}(U, A, R)$. Put $\mathcal{G}$ is the family of all object reductions of $\mathcal{D}(U, A, R)$. Then $\mathcal{G} = \{\{1, 2, 3, 4\}, \{1, 2, 3, 6\}, \{1, 3, 4, 5\}, \{1, 3, 5, 6\}\}$.

(3) By the above (1) and (2), we can obtain eight reductions of (U, A, R), i.e., we can obtain a reduction $(V, B, R \bigcap (V \times B))$ of (U, A, R) for each $V \in \mathcal{G}$ and each $B \in \mathcal{E}$. For example, if we pick $\{1, 2, 3, 6\} \in \mathcal{G}$ and $\{b, c, d, f\} \in \mathcal{E}$, then the following reductions of (U, A, R) is obtained.

Table 2: A reduction of the formal context (U, A, R)

	b	c	d	f
1	0	0	1	1
2	1	0	1	1
3	0	0	1	0
6	0	1	0	0

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 19, NO.1, 2015, COPYRIGHT 2015 EUDOXUS PRESS, LLC

A new two-step iterative method for solving nonlinear equations

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Abstract

In this paper, a new iterative method for solving nonlinear equations is developed by using modified homotopy perturbation method. The convergence analysis of the proposed method is also given. The validity and efficiency of our method is illustrated by applying this new method along with some other existing methods on various test problems.

Key words: Homotopy perturbation method; nonlinear equations; iterative methods

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1 Introduction

The construction of numerical methods for solving nonlinear equations has been a sweltring subject for many researchers. Consequently, many techniques like modified Adomian decomposition, fixed point, homotopy perturbation etc. have been developed for the purpose. [1–14].

Homotopy perturbation method(HPM) is to continuously transform a simple problem(easy to solve) into the under study problem(difficult to solve). Because of this fact, scientists and engineers have applied HPM extensively to linear and nonlinear problems. HPM was proposed first by He [15] in 1999 and systematical description which is, in fact, a coupling of traditional pertubation method and homotopy in topology [16]. This method then further improved by He and applied to nonlinear oscillator with discontinuities [17], nonlinear wave equation [18], asymptotology[19], boundary value problem [20], limit cycle and bifurcation of nonlinear problems [21] and many other fields. Thus He's method can be applied to solve various types of nonlinear equations. Subsequently, many kinds of linear and nonlinear problems have been dealt with HPM [22–28].

This paper presents a new numerical technique based on modified homotopy perturbation method to solve linear and nonlinear algebraic equations. The proposed method and a few existing methods are applied for comparison purpose to several test problems in order to reveal the accuracy and convergence of our method.

We need the following results in the sequal.

Theorem 1 [29]. Suppose f is continuous on $[a, b]$ and differentiable in (a, b).Then there exists a point $\delta \in (a, b)$ such that

$f(y) = f(x) + (y - x)f'(x) + \frac{1}{2}(y - x)^2 f''(\delta)$.

For the solution of nonlinear equation by fixed point iterative method, the equation $f(x) = 0$ is usually rewritten as

$$x = g(x), \tag{1}$$

where

(i) there exists $[a, b]$ such that $g(x) \in [a, b]$ for all $x \in [a, b]$,

(ii) there exists $L > 0$ such that $|g'(x)| \leq L < 1$ for all $x \in (a, b)$.

Considering the following iterative scheme:

$$x_{n+1} = g(x_n), \quad n = 0, 1, 2, ..., \tag{2}$$

and starting with a suitable initial approximation x_0, we build up a sequence of approximations, say $\{x_n\}$, for the solution of the nonlinear equation, say α. The scheme will converge to the root α, provided that

(i) the initial approximation x_0 is chosen in the interval $[a, b]$,

(ii) g has a continuous derivative on (a, b),

(iii) $|g'(x)| < 1$ for all $x \in [a, b]$,

(iv) $a \leq g(x) \leq b$ for all $x \in [a, b]$. (see [30])

The order of the convergence for the sequence of approximations derived from an iterative method is defined in the literature as:

Definition 1. Let $\{x_n\}$ converges to α. If there exists an integer constant p and real positive constant C such that

$$\lim_{n\to\infty} \left| \frac{x_{n+1} - \alpha}{(x_n - \alpha)^p} \right| = C,$$

then p is called the order and C the constant of convergence.

To determine the order of convergence of sequence $\{x_n\}$, let us consider the Taylor expansion of $g(x_n)$:

$$g(x_n) = g(x) + \frac{g'(x)}{1!}(x_n - x) + \frac{g''(x)}{2!}(x_n - x)^2 + ... + \frac{g^{(k)}(x)}{k!}(x_n - x)^k + ... \tag{3}$$

using Eq.(1) and Eq.(2) in Eq.(3), we have

$x_{n+1} - x = g'(x)(x_n - x) + \frac{g''(x)}{2!}(x_n - x)^2 + ... + \frac{g^{(k)}(x)}{3!}(x_n - x)^k + ...,$

and state the following theorem:

Theorem 2 [4]. Suppose $g \in C^p[a, b]$. If $g^{(k)}(x) = 0$, for $k = 1, 2, ..., p-1$ and $g^{(p)}(x) \neq 0$, then the sequence $\{x_n\}$ is of order p.

Definition 2. If an iterative method has order of convergence k with number of evaluations in a single iteration e, then $(k)^{\frac{1}{e}}$is called the efficiency index of the method. For example, the efficiency index of Newton Raphson methed, Javidi method [31], Noor et al. method [32], Rafiq and Rafiullah method [11] and Algorithm 2 of the present paper are 1.41421, 1.31607, 1.44225, 1.44225 and 1.44225 respectively.

2 Modified homotopy perturbation method

Consider the nonlinear algebraic equation

$$f\left(x\right)=0,\ \ x\in R. \tag{4}$$

We assume that r is a simple zero of Eq.(4) and α is an initial guess sufficiently close to r. Using Theorem 1 around α for Eq.(4), we have

$$f\left(\alpha\right)+\left(x-\alpha\right)f'\left(\alpha\right)+\frac{1}{2}\left(x-\alpha\right)^{2}f''\left(\delta\right)=0. \tag{5}$$

where δ lies between x and α.

We can rewrite Eq.(5) in the following form

$$x=c+N\left(x\right), \tag{6}$$

where

$$c=\alpha-\frac{f\left(\alpha\right)}{f'\left(\alpha\right)} \tag{7}$$

and

$$N\left(x\right)=-\frac{1}{2}\left(x-\alpha\right)\frac{f''\left(\delta\right)}{f'\left(\alpha\right)}. \tag{8}$$

To illustrate the basic idea of modified homotopy perturbation method, we construct a homotopy $\Theta:\left(R\times[0,1]\right)\times R\rightarrow R$ for Eq.(6), which satisfies

$$\Theta(\varpi,\beta,\theta)=\varpi-c-\beta N(\varpi)+\beta^2(1-\beta)\theta=0,\quad \theta,\varpi\in R \quad \text{and} \quad \beta\in[0,1], \tag{9}$$

where θ is unknown real number and β is embedding parameter. It is obvious that

$$\Theta(\varpi,0,\theta)=\varpi-c=0, \tag{10}$$

$$\Theta(\varpi,1,\theta)=\varpi-c-N(\varpi)=0, \tag{11}$$

The embedding parameter β monotonically increases from zero to unity as the trivial problem $\Theta(\varpi,0,\theta)=\varpi-c=0$ is continuously deformed to original problem $\Theta(\varpi,1,\theta)=\varpi-c-N(\varpi)=0$. The modified HPM uses the homotopy parameter β as an expanding parameter to obtain (see [15] and the references there in)

$$\varpi=x_0+\beta x_1+\beta^2 x_2+... \tag{12}$$

The approximate solution of Eq.(4), therefore, can be readily obtained:

$$r=\lim_{\beta\to 1}\varpi=x_0+x_1+x_2+... \tag{13}$$

The convergence of the series (13) has been proved by He [33].

For the application of modified HPM to Eq.(4), we can write Eq.(9) as follows by expanding $N(\varpi)$ into a Taylor series around x_0 :

$$\varpi-c-\beta\{N(x_o)+(\varpi-x_0)\frac{N'(x_0)}{1!}+(\varpi-x_0)^2\frac{N''(x_0)}{2!}+...\}+\beta^2(1-\beta)\theta=0. \tag{14}$$

Substituting Eq.(12) into Eq.(14) yields

$$x_0+\beta x_1+\beta^2 x_2+...-c-\beta\{N(x_0)+\left(x_0+\beta x_1+\beta^2 x_2+...-x_0\right)\frac{N'(x_0)}{1!}+$$
$$\left(x_0+\beta x_1+\beta^2 x_2+...-x_o\right)^2\frac{N''(x_0)}{2!}+...\}+\beta^2(1-\beta)\theta=0. \tag{15}$$

By equating the terms with identical powers of β, we have

$$\beta^0 = x_0 - c = 0, \tag{16}$$

$$\beta^1 = x_1 - N(x_0) = 0, \tag{17}$$

$$\beta^2 = x_2 - x_1 N'(x_0) + \theta = 0, \tag{18}$$

$$\beta^3 = x_3 - x_2 N'(x_0) - \frac{1}{2} x_1^2 N''(x_0) - \theta = 0. \tag{19}$$

We try to find parameter θ, such that

$$x_2 = 0. \tag{20}$$

Hence by substituting $x_1 = N(x_0)$ from Eq.(17) into Eq.(18), we have

$$x_2 - N(x_0) N'(x_0) + \theta = 0. \tag{21}$$

Setting $x_2 = 0$ into Eq.(21), we have

$$\theta = N(x_0) N'(x_0). \tag{22}$$

Using Eq.(22), $x_1 = N(x_0)$, and $x_2 = 0$ into Eq.(19), we obtain

$$x_3 = \frac{1}{2} N^2(x_0) N''(x_0) + N(x_0) N'(x_0). \tag{23}$$

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where

$$N(x_0) = -\frac{1}{2} \frac{[f(\alpha)]^2 f''(\delta)}{[f'(\alpha)]^3}, \tag{24}$$

$$N'(x_0) = \frac{f(\alpha) f''(\delta)}{[f'(\alpha)]^2}, \tag{25}$$

$$N''(x_0) = -\frac{f''(\delta)}{f'(\alpha)}. \tag{26}$$

Combining Eq.(7) and Eq.(16), we get

$$x_0 = c = \alpha - \frac{f(\alpha)}{f'(\alpha)}. \tag{27}$$

Now using equations(24)-(26) in Eq.(17) and Eq.(23), we have

$$x_1 = -\frac{1}{2}\frac{[f(\alpha)]^2 f''(\delta)}{[f'(\alpha)]^3} \tag{28}$$

and

$$x_3 = -\frac{1}{8}\left(\frac{[f(\alpha)]^4 [f''(\delta)]^3 + 4[f(\alpha)]^3 [f'(\alpha)]^2 [f''(\delta)]^2}{[f'(\alpha)]^7}\right). \tag{29}$$

Using equations(20), (27), (28) and (29) into Eq.(13), we obtain the solution of Eq.(4) as follows:

$$r = x_0 + x_1 + x_2 + x_3 + ... = \alpha - \frac{f(\alpha)}{f'(\alpha)} - \frac{1}{2}\frac{[f(\alpha)]^2 f''(\delta)}{[f'(\alpha)]^3} -$$
$$\frac{1}{8}\left(\frac{[f(\alpha)]^4 [f''(\delta)]^3 + 4[f(\alpha)]^3 [f'(\alpha)]^2 [f''(\delta)]^2}{[f'(\alpha)]^7}\right) + ... \tag{30}$$

This formulations allows us to suggest the following iterative method for solving nonlinear Eq.(4).

Algorithm 1 [32] For a given w_0, calculate the approximate solution w_{n+1} by the iterative scheme

$w_{n+1} = w_n - \frac{f(w_n)}{f'(w_n)} - \frac{1}{2}\frac{[f(w_n)]^2 f''(y_n)}{[f'(w_n)]^3}$,

$y_n = w_n - \frac{f(w_n)}{f'(w_n)}; \quad f'(w_n) \neq 0.$

Algorith 1 has sbeen suggested by Norr et al. **[32]** which we derive in this paper by using modified HPM.

Algorithm 2. For a given w_0, calculate the approximate solution w_{n+1} by the iterative scheme

$$w_{n+1} = w_n - \frac{f(w_n)}{f'(w_n)} - \frac{1}{2}\frac{[f(w_n)]^2 f''(y_n)}{[f'(w_n)]^3} - \frac{1}{8}\left(\frac{[f(w_n)]^4[f''(y_n)]^3+4[f(w_n)]^3[f'(w_n)]^2[f''(y_n)]^2}{[f'(w_n)]^7}\right),$$

$$y_n = w_n - \frac{f(w_n)}{f'(w_n)}; \qquad f'(w_n) \neq 0.$$

3 Convergence analysis

We discuss the convergence analysis of algorithm 2.

Theorem 3. Consider the nonlinear equation $f(x) = 0$. Suppose f is sufficently differentiable. Then for the iterative method defined by Algorithm 2, the convergence is at least of order 3.

Proof: For the iteration function defined as in Algorithm 2, which has a fixed point r, let us define

$$y = x - \frac{f(x)}{f'(x)}, \tag{31}$$

and

$$g(x) = x - \frac{f(x)}{f'(x)} - \frac{1}{2}\frac{[f(x)]^2 f''(y)}{[f'(x)]^3} - \frac{4[f(x)]^3[f'(x)]^2\left[f''(y)\right]^2 + [f(x)]^4\left[f''(y)\right]^3}{8[f'(x)]^7}. \tag{32}$$

We obtain the derivatives and the values of derivatives at r by using the software Mathematica.

$$g'(x) = \frac{f(x)f''(x)}{[f'(x)]^2} - \frac{f(x)f''(y)}{[f'(x)]^2} + \frac{3[f(x)]^2 f''(x)f''(y)}{2[f'(x)]^4} +$$

$$\frac{7f''(x)(4[f(x)]^3[f'(x)]^2\left[f''(y)\right]^2 + [f(x)]^4\left[f''(y)\right]^3)}{8[f'(x)]^8} -$$

$$\frac{[f(x)]^3 f''(x)f'''(y)}{2[f'(x)]^5} - \frac{(12[f(x)]^2[f'(x)]^3\left[f''(y)\right]^2 + 8[f(x)]^3 f'(x)f''(x)\left[f''(y)\right]^2}{8[f'(x)]^7} +$$

$$\frac{4\left[f(x)\right]^3 f'(x)\left[f''(y)\right]^3 + 8\left[f(x)\right]^4 f''(x)f''(y)f^{(3)}(y) + \frac{3[f(x)]^5 f''(x)[f''(y)]^2 f^{(3)}(y)}{[f'(x)]^2})}{8\left[f'(x)\right]^7}. \tag{33}$$

$$g''(x) = \frac{f''(x)}{f'(x)} - \frac{(-2\left[f'(x)\right]^2 - 2f(x)f''(x))f''(y)}{2f'^3(x)} +$$

$$\frac{(4\left[f(x)\right]^3\left[f'(x)\right]^2\left[f''(y)\right]^2 + \left[f(x)\right]^4\left[f''(y)\right]^3)(\frac{-56[f''(x)]^2}{[f'(x)]^9} + \frac{7f^{(3)}(x)}{[f'(x)]^8})}{8} -$$

$$f(x)(\frac{2\left[f''(x)\right]^2}{\left[f'(x)\right]^3} - \frac{f^{(3)}(x)}{\left[f'(x)\right]^2}) - 2f(x)f'(x)(\frac{-3f''(x)f''(y)}{\left[f'(x)\right]^4} + \frac{f(x)f''(x)f^{(3)}(y)}{\left[f'(x)\right]^5}) +$$

$$\frac{(7f''(x)(12\left[f(x)\right]^2\left[f'(x)\right]^3\left[f''(y)\right]^2 + 8\left[f(x)\right]^3 f'(x)f''(x)\left[f''(y)\right]^2}{4\left[f'(x)\right]^8} +$$

$$\frac{4\left[f(x)\right]^3 f'(x)\left[f''(y)\right]^3 + 8\left[f(x)\right]^4 f''(x)f''(y)f^{(3)}(y) + \frac{3[f(x)]^5 f''(x)[f''(y)]^2 f^{(3)}(y)}{[f'(x)]^2}))}{4\left[f'(x)\right]^8} -$$

$$\frac{(\left[f(x)\right]^2(f''(y)(\frac{12[f''(x)]^2}{[f'(x)]^5} - \frac{3f^{(3)}(x)}{[f'(x)]^4})}{2} -$$

$$\frac{\frac{\frac{6f(x)\left[f''(x)\right]^2 f^{(3)}(y)}{\left[f'(x)\right]^6} + \frac{f''(x)f^{(3)}(y)}{f'(x)} - \frac{2f(x)\left[f''(x)\right]^2 f^{(3)}(y)}{\left[f'(x)\right]^3} + \frac{f(x)f^{(3)}(x)f^{(3)}(y)}{\left[f'(x)\right]^2} + \frac{\left[f(x)\right]^2\left[f''(x)\right]^2 f^{(4)}(y)}{\left[f'(x)\right]^4}}{\left[f'(x)\right]^3}))}{2} -$$

$$\frac{(4\left[f'(x)\right]^2(6f(x)\left[f'(x)\right]^2 + 3\left[f(x)\right]^2 f''(x))\left[f''(y)\right]^2 + (12\left[f(x)\right]^2\left[f'(x)\right]^2 + 4\left[f(x)\right]^3 f''(x))\left[f''(y)\right]^3}{8f'^7(x)} +$$

$$\frac{\frac{24[f(x)]^4 f''(x)[f''(y)]^2 f^{(3)}(y)}{f'(x)} + 24\,[f(x)]^2\, f'(x)(2f'(x)f''(x)\,[f''(y)]^2 + 2f(x)f''(x)f''(y)f^{(3)}(y))}{8f'^7(x)}+$$

$$\frac{[f(x)]^4\,(\frac{3f''(x)\left[f''(y)\right]^2 f^{(3)}(y)}{f'(x)} - \frac{6f(x)\left[f''(x)\right]^2\left[f''(y)\right]^2 f^{(3)}(y)}{[f'(x)]^3} + \frac{3f(x)\left[f''(y)\right]^2 f^{(3)}(x)f^{(3)}(y)}{[f'(x)]^2}}{8\,[f'(x)]^7}+$$

$$\frac{\frac{6[f(x)]^2\left[f''(x)\right]^2 f''(y)\left[f^{(3)}(y)\right]^2}{[f'(x)]^4} + \frac{3[f(x)]^2[f''(x)]^2\left[f''(y)\right]^2 f^{(4)}(y)}{[f'(x)]^4}}{8\,[f'(x)]^7}+$$

$$\frac{4\,[f(x)]^3\,(\left[f''(y)\right]^2(2\left[f''(x)\right]^2 2f'(x)f^{(3)}(x)) + \frac{8f(x)\left[f''(x)\right]^2 f''(y)f^{(3)}(y)}{f'(x)}}{8\,[f'(x)]^7}+$$

$$\frac{[f'(x)]^2\,(\frac{2f''(x)f''(y)f^{(3)}(y)}{f'(x)} - \frac{4f(x)\left[f''(x)\right]^2 f''(y)f^{(3)}(y)}{[f'(x)]^3}}{8\,[f'(x)]^7}+$$

$$\frac{\frac{2f(x)f''(y)f^{(3)}(x)f^{(3)}(y)}{[f'(x)]^2} + \frac{2[f(x)]^2\left[f''(x)\right]^2\left[f^{(3)}(x)\right]^2}{[f'(x)]^4} + \frac{2[f(x)]^2\left[f''(x)\right]^2 f''(y)f^{(4)}(y)}{[f'(x)]^4}}{8\,[f'(x)]^7}))). \quad (34)$$

$$g'''(x) = \frac{3\,[f''(x)]^2}{[f'(x)]^2} - \frac{f^{(3)}(x)}{f'(x)} + \frac{f''(y)(-6f'(x)f''(x) - 2f(x)f^{(3)}(x))}{2\,[f'(x)]^3} - 3f'(x)(\frac{2\,[f''(x)]^2}{[f'(x)]^3} - \frac{f^{(3)}(x)}{[f'(x)]^2})+$$

$$\frac{3(-2\,[f'(x)]^2 - 2f(x)f''(x))(\frac{-3f''(x)f''(y)}{[f'(x)]^4} + \frac{f(x)f''(x)f^{(3)}(y)}{[f'(x)]^5})}{2}+$$

$$\frac{(3(\frac{-56[f''(x)]^2}{[f'(x)]^9} + \frac{7f^{(3)}(x)}{[f'(x)]^8})(12\,[f(x)]^2\,[f'(x)]^3\left[f''(y)\right]^2 + 8\,[f(x)]^3\, f'(x)f''(x)\left[f''(y)\right]^2}{8}+$$

$$\frac{4\left[f(x)\right]^3 f'(x)\left[f''(y)\right]^3+8\left[f(x)\right]^4 f''(x)f''(y)f^{(3)}(y)+\frac{3[f(x)]^5 f''(x)\left[f''(y)\right]^2 f^{(3)}(y)}{[f'(x)]^2}))}{8}+$$

$$\frac{(4\left[f(x)\right]^3\left[f'(x)\right]^2\left[f''(y)\right]^2+\left[f(x)\right]^4\left[f''(y)\right]^3)(\frac{504[f''(x)]^3}{[f'(x)]^{10}}-\frac{168f''(x)f^{(3)}(x)}{[f'(x)]^9}+\frac{7f^{(4)}(x)}{8})}{8}-$$

$$f(x)(\frac{-6\left[f''(x)\right]^3}{\left[f'(x)\right]^4}+\frac{6f''(x)f^{(3)}(x)}{\left[f'(x)\right]^3}-\frac{f^{(4)}(x)}{\left[f'(x)\right]^2})-3f(x)f'(x)(f''(y)(\frac{12\left[f''(x)\right]^2}{\left[f'(x)\right]^5}-\frac{3f^{(3)}(x)}{\left[f'(x)\right]^4})-$$

$$\frac{6f(x)\left[f''(x)\right]^2 f^{(3)}(y)}{\left[f'(x)\right]^6}+\frac{\frac{f''(x)f^{(3)}(y)}{f'(x)}-\frac{2f(x)\left[f''(x)\right]^2 f^{(3)}(y)}{[f'(x)]^3}+\frac{f(x)f^{(3)}(x)f^{(3)}(y)}{[f'(x)]^2}+\frac{[f(x)]^2\left[f''(x)\right]^2 f^{(4)}(y)}{[f'(x)]^4})}{\left[f'(x)\right]^3})+$$

$$\frac{(21f''(x)(4\left[f'(x)\right]^2(6f(x)\left[f'(x)\right]^2+3\left[f(x)\right]^2 f''(x))\left[f''(y)\right]^2}{8\left[f'(x)\right]^8}+$$

$$\frac{(12\left[f(x)\right]^2\left[f'(x)\right]^2 4\left[f(x)\right]^3 f''(x))\left[f''(y)\right]^3}{8\left[f'(x)\right]^8}+$$

$$\frac{\frac{24[f(x)]^4 f''(x)\left[f''(y)\right]^2 f^{(3)}(y)}{f'(x)}}{8\left[f'(x)\right]^8}+$$

$$\frac{24\left[f(x)\right]^2 f'(x)(2f'(x)f''(x)\left[f''(y)\right]^2+2f(x)f''(y)f^{(3)}(y))}{8\left[f'(x)\right]^8}+$$

$$\frac{\left[f(x)\right]^4(\frac{3f''(x)\left[f''(y)\right]^2 f^{(3)}(y)}{f'(x)}-\frac{6f(x)\left[f''(x)\right]^2\left[f''(y)\right]^2 f^{(3)}(y)}{[f'(x)]^3}+\frac{3f(x)\left[f''(y)\right]^2 f^{(3)}(x)f^{(3)}(y)}{[f'(x)]^2}}{8\left[f'(x)\right]^8}+$$

$$\frac{\frac{6[f(x)]^2\left[f''(x)\right]^2 f''(y)\left[f''(y)\right]^2}{[f'(x)]^4}+\frac{3[f(x)]^2\left[f''(x)\right]^2\left[f''(y)\right]^2 f^{(4)}(y)}{[f'(x)]^4})}{8\left[f'(x)\right]^8}+$$

$$\frac{4\left[f(x)\right]^3(\left[f''(y)\right]^2(2\left[f''(x)\right]^2+2f'(x)f^{(3)}(x))}{8\left[f'(x)\right]^8}+$$

$$\frac{\frac{8f(x)\left[f''(x)\right]^2 f''(y)f^{(3)}(y)}{f'(x)}+[f'(x)]^2(\frac{2f''(x)f''(y)f^{(3)}(y)}{f'(x)}-\frac{4f(x)\left[f''(x)\right]^2 f''(y)f^{(3)}(y)}{[f'(x)]^3}}{8\left[f'(x)\right]^8}+$$

$$\frac{\frac{2f(x)f''(y)f^{(3)}(x)f^{(3)}(y)}{[f'(x)]^2}+\frac{2[f(x)]^2[f''(x)]^2\left[f^{(3)}(y)\right]^2}{[f'(x)]^4}+\frac{2[f(x)]^2[f''(x)]^2 f''(y)f^{(4)}(y)}{[f'(x)]^4}))))}{8\left[f'(x)\right]^8}-$$

$$\frac{([f(x)]^2(\frac{3f(x)f''(x)(\frac{12\left[f''(x)\right]^2}{[f'(x)]^5}-\frac{3f^{(3)}(x)}{[f'(x)]^4})f^{(3)}(x)}{[f'(x)]^2}+f''(y)(\frac{-60\left[f''(x)\right]^3}{[f'(x)]^6}+\frac{36f''(x)f^{(3)}(x)}{[f'(x)]^5}-\frac{3f^{(4)}(x)}{[f'(x)]^4})}{2}-$$

$$\frac{\frac{9f''(x)(\frac{f''(x)f^{(3)}(y)}{f'(x)}-\frac{2f(x)\left[f''(x)\right]^2 f^{(3)}(y)}{[f'(x)]^3}+\frac{f(x)f^{(3)}(x)f^{(3)}(y)}{[f'(x)]^2}+\frac{[f(x)]^2\left[f''(x)\right]^2 f^{(4)}(y)}{[f'(x)]^4}}{[f'(x)]^4})+\frac{(\frac{-3\left[f''(x)\right]^2 f^{(3)}(y)}{[f'(x)]^2}+\frac{6f(x)\left[f''(x)\right]^3 f^{(3)}(y)}{[f'(x)]^4}}{[f'(x)]^3}}{2}+$$

$$\frac{\frac{\frac{2f^{(3)}(x)f^{(3)}(y)}{f'(x)}-\frac{6f(x)f''(x)f^{(3)}(x)f^{(3)}(y)}{[f'(x)]^3}+\frac{f(x)f^{(3)}(y)f^{(4)}(x)}{[f'(x)]^2}+\frac{3f(x)\left[f''(x)\right]^2 f^{(4)}(y)}{[f'(x)]^3}-\frac{6[f(x)]^2\left[f''(x)\right]^3 f^{(4)}(y)}{[f'(x)]^5}}{[f'(x)]^3}}{2}+$$

$$\frac{\frac{\frac{3[f(x)]^2 f''(x)f^{(3)}(x)f^{(4)}(y)}{[f'(x)]^4}+\frac{[f(x)]^3[f''(x)]^3 f^{(5)}(y)}{[f'(x)]^6})}{[f'(x)]^3}))}{2}-(4\left[f'(x)\right]^2([f''(y)]^2(6\left[f'(x)\right]^3+18f(x)f'(x)f''(x)+$$

$$3\left[f(x)\right]^2 f^{(3)}(x))+\left[f''(y)\right]^3 (24f(x)\left[f'(x)\right]^3+36\left[f(x)\right]^2 f'(x)f''(x)+4\left[f(x)\right]^3 f^{(3)}(x))+$$

$$\frac{9f(x)f''(x)(12\left[f(x)\right]^2\left[f'(x)\right]^2+4\left[f(x)\right]^3 f''(x)\left[f''(y)\right]^2 f^{(3)}(y)}{\left[f'(x)\right]^2}+$$

$$12(6f(x)\left[f'(x)\right]^2+3\left[f(x)\right]^2 f''(x))(2f'(x)f''(x)\left[f''(y)\right]^2+2f(x)f''(x)f''(y)f^{(3)}(y))+$$

$$12\left[f(x)\right]^3 f'(x)(\frac{3f''(x)\left[f''(y)\right]^2 f^{(3)}(y)}{f'(x)}-\frac{6f(x)\left[f'(x)\right]^2\left[f''(y)\right]^2 f^{(3)}(y)}{\left[f'(x)\right]^3}+$$

$$\frac{3f(x)\left[f''(y)\right]^2 f^{(3)}(x)f^{(3)}(y)}{\left[f'(x)\right]^2}+\frac{6\left[f(x)\right]^2\left[f''(x)\right]^2 f''(y)\left[f^{(3)}(y)\right]^2}{\left[f'(x)\right]^4}+$$

$$\frac{3\left[f(x)\right]^2\left[f''(x)\right]^2\left[f''(y)\right]^2 f^{(4)}(y)}{\left[f'(x)\right]^4}+36\left[f(x)\right]^2 f'(x)(\left[f''(y)\right]^2(2\left[f''(x)\right]^2+2f'(x)f^{(3)}(x))+$$

$$\frac{8f(x)\left[f''(x)\right]^2 f''(y)f^{(3)}(y)}{f'(x)}+\left[f'(x)\right]^2(\frac{2f''(x)f''(y)f^{(3)}(y)}{f'(x)}-\frac{4f(x)\left[f''(x)\right]^2 f''(y)f^{(3)}(y)}{\left[f'(x)\right]^3}+$$

$$\frac{2f(x)f''(y)f^{(3)}(x)f^{(3)}(y)}{\left[f'(x)\right]^2}+\frac{2[f(x)]^2\left[f''(x)\right]^2\left[f^{(3)}(y)\right]^2}{\left[f'(x)\right]^4}+\frac{2[f(x)]^2\left[f''(x)\right]^2 f''(y)f^{(4)}(y)}{\left[f'(x)\right]^4}))+$$

$$[f(x)]^4(\frac{-9\left[f''(x)\right]^2\left[f''(y)\right]^2 f^{(3)}(y)}{\left[f'(x)\right]^2}+\frac{18f(x)\left[f''(x)\right]^3\left[f''(y)\right]^2 f^{(3)}(y)}{\left[f'(x)\right]^4}+\frac{6\left[f''(y)\right]^2 f^{(3)}(x)f^{(3)}(y)}{f'(x)}-$$

$$\frac{18f(x)f''(x)\left[f''(y)\right]^2 f^{(3)}(x)f^{(3)}(y)}{\left[f'(x)\right]^3}+\frac{18f(x)\left[f''(x)\right]^2 f''(y)\left[f^{(3)}(y)\right]^2}{f'(x)}-$$

$$\frac{36[f(x)]^2\left[f''(x)\right]^3 f''(y)\left[f^{(3)}(y)\right]^2}{\left[f'(x)\right]^5}+\frac{18[f(x)]^2 f''(x)f''(y)f^{(3)}(y)\left[f^{(3)}(y)\right]^2}{\left[f'(x)\right]^4}+$$

$$\frac{6[f(x)]^3\left[f''(x)\right]^3 f''(y)\left[f^{(3)}(y)\right]^3}{\left[f'(x)\right]^6}+\frac{3f(x)\left[f''(y)\right]^2 f^{(3)}(y)f^{(4)}(x)}{\left[f'(x)\right]^5}+$$

$$\frac{9f(x)\left[f''(x)\right]^2\left[f''(y)\right]^2 f^{(4)}(y)}{\left[f'(x)\right]^3}-\frac{18\left[f(x)\right]^2\left[f''(x)\right]^3\left[f''(y)\right]^2 f^{(4)}(y)}{\left[f'(x)\right]^5}+$$

$$\frac{9\left[f(x)\right]^2 f''(x)\left[f''(y)\right]^2 f^{(3)}(x)f^{(4)}(y)}{\left[f'(x)\right]^4}+$$

$$\frac{18\left[f(x)\right]^3\left[f''(x)\right]^3 f''(y)f^{(3)}(x)f^{(4)}(y)}{\left[f'(x)\right]^6}+\frac{3\left[f(x)\right]^3\left[f''(x)\right]^3\left[f''(y)\right]^2 f^{(5)}(y)}{\left[f'(x)\right]^6})+$$

$$\frac{4\left[f(x)\right]^3(\frac{6f(x)f''(x)f''(y)(2[f''(x)]^2+2f'(x)f^{(3)}(y))f^{(3)}(y)}{[f'(x)]^6}+\left[f''(y)\right]^2(6f''(x)f^{(3)}(x)+2f'(x)f^{(4)}(x))}{8\left[f'(x)\right]^7}+$$

$$\frac{6f'(x)f''(x)(\frac{2f''(x)f''(y)f^{(3)}(y)}{f'(x)}-\frac{4f(x)[f''(x)]^2f''(y)f^{(3)}(y)}{[f'(x)]^3}}{8\left[f'(x)\right]^7}+$$

$$\frac{\frac{2f(x)f''(y)f^{(3)}(x)f^{(3)}(y)}{[f'(x)]^2}+\frac{2[f(x)]^2[f''(x)]^2\left[f^{(3)}(y)\right]^2}{[f'(x)]^4}+\frac{2[f(x)]^2[f''(x)]^2f''(y)f^{(4)}(y)}{[f'(x)]^4})}{8\left[f'(x)\right]^7}+$$

$$\frac{[f'(x)]^2(\frac{-6[f''(x)]^2f''(y)f^{(3)}(y)}{[f'(x)]^2}+\frac{12f(x)[f''(x)]^3f''(y)f^{(3)}(y)}{[f'(x)]^4}+\frac{4f''(y)f^{(3)}(x)f^{(3)}(y)}{f'(x)}}{8\left[f'(x)\right]^7}-$$

$$\frac{\frac{12f(x)f''(x)f''(y)f^{(3)}(x)f^{(3)}(y)}{[f'(x)]^3}+\frac{6f(x)[f''(x)]^2f''(y)\left[f^{(3)}(y)\right]^2}{[f'(x)]^3}-\frac{12[f(x)]^2[f''(x)]^3\left[f^{(3)}(y)\right]^2}{[f'(x)]^5}}{8\left[f'(x)\right]^7}+$$

$$\frac{\frac{6[f(x)]^2 f''(x) f^{(3)}(x)\left[f^{(3)}(y)\right]^2}{[f'(x)]^4} + \frac{2f(x)f''(y)f^{(3)}(y)f^{(4)}(x)}{[f'(x)]^2} + \frac{6f(x)[f''(x)]^2 f''(y)f^{(4)}(y)}{[f'(x)]^3} - \frac{12[f(x)]^2[f''(x)]^3 f''(y)f^{(4)}(y)}{[f'(x)]^5}}{8\left[f'(x)\right]^7} +$$

$$\frac{\frac{6[f(x)]^2 f''(x)f''(y)f^{(3)}(x)f^{(4)}(y)}{[f'(x)]^4} + \frac{6[f(x)]^3[f''(x)]^3 f^{(3)}(y)f^{(4)}(y)}{[f'(x)]^6} + \frac{2[f(x)]^3[f''(x)]^3 f''(y)f^{(5)}(y)}{[f'(x)]^6}))}{8\left[f'(x)\right]^7}. \tag{35}$$

Now, one can easily see that

$$g(r) = r, \quad g'(r) = 0 = g''(r), \quad g'''(r) = \frac{6\left[f''(r)\right]^2}{[f'(r)^2]} - \frac{f^{(3)}(r)}{f'(r)} - 3f'(r)\left(\frac{2\left[f''(r)\right]^2}{\left[f'(r)\right]^3} - \frac{\left[f(r)\right]^3}{\left[f'(r)\right]^2}\right). \tag{36}$$

But $g'''(x) \neq 0$ provides that

$\frac{6[f''(r)]^2}{[f'(r)^2]} - \frac{f^{(3)}(r)}{f'(r)} - 3f'(r)(\frac{2[f''(r)]^2}{[f'(r)]^3} - \frac{[f(r)]^3}{[f'(r)]^2}) \neq 0$

and so, in general, the Algorithm 2 is of order 3. Based on result Eq.(36), by using Taylor's series expansion of $g(x_n)$ about r and with the help of Eq.(2), we obtain the error equation as follows:

$$\begin{aligned} x_{n+1} &= g(x_n) \\ &= g(r) + g'(r)(x_x - r) + \frac{g''(r)}{2!}(x_x - r)^2 + \frac{g'''(r)}{3!}(x_x - r)^3 + O(x_x - r)^4 \end{aligned} \tag{37}$$

from Eq.(36) and Eq.(37),we get

$x_{n+1} = r + \frac{g'''(r)}{3!}e_n^3 + O(e_n^4)$

$= r + \frac{1}{6}(\frac{6[f''(r)]^2}{[f'(r)^2]} - \frac{f^{(3)}(r)}{f'(r)} - 3f'(r)(\frac{2[f''(r)]^2}{[f'(r)]^3} - \frac{[f(r)]^3}{[f'(r)]^2}))e_n^3 + O(e_n^4)$

$= r + (\frac{[f''(r)]^2}{[f'(r)^2]} - \frac{f^{(3)}(r)}{6f'(r)} - \frac{1}{2}f'(r)(\frac{2[f''(r)]^2}{[f'(r)]^3} - \frac{[f(r)]^3}{[f'(r)]^2}))e_n^3 + O(e_n^4)$

$= r + c_1 e_n^3 + O(e_n^4)$

where $e_n = x_n - r$ and $c_1 = \frac{[f''(r)]^2}{[f'(r)^2]} - \frac{f^{(3)}(r)}{6f'(r)} - \frac{1}{2}f'(r)(\frac{2[f''(r)]^2}{[f'(r)]^3} - \frac{[f(r)]^3}{[f'(r)]^2})$ Thus,

$e_{n+1} = c_1 e_n^3 + O(e_n^4)$

Which shows that Algorithm 2 has convergence order at least 3.

4 Test problems

In this section, we present some examples to illustrate the efficiency of the Algorithm 2 derived in section 2. For each problem, the comparison of our algorithm with Newton-Raphson method (NR), the method of Javidi (JM) [31], the Noor et al. method(NM) **[32]** and the method of Rafiq and Rafiullah (RR) [11] is given in the form of the tables.

Example 1. $x^5 + x^4 + 4x^2 - 15 = 0$ with $x_0 = 1.5$. The numerical results obtained are given in Table 1. The exact solution prospected is $x = 1.34742809896830498171$.
Table 1

Method	No.of Ite.	x[k]	|f(x[k])|	Diff. of two iterations	Evaluations
NR	5	1.3474280989683050	6.851044e-16	6.616654e-14	10
JM	3	1.3474280989683050	6.851044e-16	1.317722e-10	12
NM	3	1.3474280989683050	6.851044e-16	1.606799e-06	9
RR	3	1.3474280989683050	6.851044e-16	8.567259e-08	9
ALGO2	3	1.3474280989683050	6.851044e-16	1.510379e-06	9

Example 2. $\sin(\ x) - \frac{1}{3}x = 0$ with $x_0 = 1.9$. In Table 2, we present the numerical results.
Table 2

Method	No.of Ite.	x[k]	|f(x[k])|	Diff. of two iterations	Evaluations
NR	5	2.2788626600758283	1.249294e-17	6.302925e-12	10
JM	4	2.2788626600758283	1.249294e-17	2.108617e-11	16
NM	3	2.2788626600758283	1.249294e-17	9.493745e-07	9
RR	3	2.2788626600758283	1.249294e-17	1.286399e-06	9
ALGO2	3	2.2788626600758283	1.249294e-17	2.454444e-06	9

Example 3. $10xe^{-x^2} - 1 = 0$ with $x_0 = 0$.The exact solution prospected is $x = 0.10102584831568519737$ and the numerical solutions obtained are given in Table 3.
Table 3

Method	No.of Ite.	x[k]	|f(x[k])|	Diff. of two iterations	Evaluations
NR	4	0.1010258483156852	2.552517e-17	3.167545e-14	8
JM	3	0.1010258483156852	2.552517e-17	1.387112e-14	12
NM	3	0.1010258483156852	2.552517e-17	1.261299e-08	9
RR	3	0.1010258483156852	2.552517e-17	1.838993e-08	9
ALGO2	3	0.1010258483156852	2.552517e-17	1.266315e-08	9

Example 4. $e^{-x^2+x+2} - 1 = 0$ with $x_0 = 2.1$. The numerical results are given in Table 4 and the exact solution prospected is $x = 2.00000000000000000000$.
Table 4

Method	No.of Ite.	x[k]	|f(x[k])|	Diff. of two iterations	Evaluations
NR	4	1.9999999999999969	9.300000e-15	5.172728e-08	8
JM	3	2.0000000000000001	3.000000e-16	2.520513e-06	12
NM	3	2.0000000000000000	0.000000e+00	8.242000e-07	9
RR	3	2.0000000000000000	0.000000e+00	9.466958e-10	9
ALGO2	3	2.0000000000000000	0.000000e+00	1.093701e-06	9

Example 5. $\ln(x^2 + x + 2) - x + 1 = 0$ with $x_0 = -1$. The exact solution prospected is $x = 4.15259073675715827500$ and the comparison of numerical results is given in Table 5.
Table 5

Method	No.of Ite.	x[k]	|f(x[k])|	Diff. of two iterations	Evaluations
NR	5	4.1525907367571608	1.520771e-15	2.047030e-07	8
JM	-	-	-	-	Diverges
NM	3	4.1525907367571584	7.528743e-17	2.424885e-05	18
RR	4	4.1525907367571583	1.505894e-17	5.115773e-08	9
ALGO2	3	4.1525907367571576	4.065405e-16	4.809359e-05	9

Example 6. $\cos(x) - x = 0$ with $x_0 = 1$. The numerical solutions obtained are given in Table 6 and the exact solution prospected is $x = 0.73908513321516064166$.
Table 6

Method	No.of Ite.	x[k]	|f(x[k])|	Diff. of two iterations	Evaluations
NR	4	0.7390851332151606	2.770350e-18	1.701233e-10	8
JM	3	0.7390851332151606	2.770350e-18	1.634240e-08	12
NM	3	0.7390851332151606	2.770350e-18	4.833743e-11	9
RR	3	0.7390851332151606	2.770350e-18	1.997531e-09	9
ALGO2	3	0.7390851332151606	2.770350e-18	5.244611e-11	9

Example 7. $e^{-x} + \cos(x) = 0$ with $x_0 = 1.75$. In Table 7, we present the numerical results. The exact prospected solution is $x = 1.74613953040801241765$.
Table 7

Method	No.of Ite.	x[k]	|f(x[k])|	Diff. of two iterations	Evaluations
NR	3	1.7461395304080124	2.045916e-17	7.679900e-13	6
JM	2	1.7461395304080124	2.045916e-17	2.016449e-08	8
NM	2	1.7461395304080124	2.045916e-17	1.080398e-08	8
RR	2	1.7461395304080124	2.045916e-17	1.341787e-08	8
ALGO2	2	1.7461395304080124	2.045916e-17	1.080322e-08	8

Example 8. $\sin^{-1}(x^2 - 1) - \frac{1}{2}x + 1 = 0$ with $x_0 = 0.3$. The exact solution prospected is $x = 0.59481096839836917752$. and the numerical results obtained are given in Table 8.
Table 8

Method	No.of Ite.	x[k]	|f(x[k])|	Diff. of two iterations	Evaluations
NR	4	0.5948109683983692	2.622997e-18	9.482505e-10	8
JM	3	0.5948109683983692	2.622997e-18	8.222903e-08	12
NM	3	0.5948109683983692	7.964943e-18	2.068484e-06	9
RR	3	0.5948109683983692	2.622997e-18	4.993354e-07	9
ALGO2	3	0.5948109683983692	7.964943e-18	2.185594e-06	9

Example 9. $x - 2 - \exp(-x) = 0$ with $x_0 = 2.0$. The numerical solutions obtained are given in Table 9 and the exact solution prospected is $x = 2.12002823898764122948$.
Table 9

Method	No.of Ite.	x[k]	|f(x[k])|	Diff. of two iterations	Evaluations
NR	3	2.1200282389876412	3.302368e-17	3.651460e-08	6
JM	3	2.1200282389876412	3.302368e-17	1.106344e-12	12
NM	3	2.1200282389876412	3.302368e-17	1.652310e-14	9
RR	3	2.1200282389876412	3.302368e-17	9.357932e-15	9
ALGO2	3	2.1200282389876412	3.302368e-17	1.650296e-14	9

Example 10. $x^2-(1-x)^5=0$ with $x_0=0.2$. In Table 10, we present the numerical results. The exact solution prospected is $x=0.34595481584824201796$.
Table 10

Method	No.of Ite.	x[k]	|f(x[k])|	Diff. of two iterations	Evaluations
NR	5	0.3459548158482420	3.280895e-18	1.213571e-12	10
JM	4	0.3459548158482420	3.280895e-18	2.047864e-14	16
NM	4	0.3459548158482420	3.280895e-18	2.528477e-13	12
RR	4	0.3459548158482420	3.280895e-18	7.789585e-15	12
ALGO2	4	0.3459548158482420	3.280895e-18	2.329867e-13	12

Example 11. $\ln(x)=0$ with $x_0=0.5$.The exact solution prospected is $x=1.00000000000000000000$ and the numerical results obtained are given in Table 11.
Table 11

Method	No.of Ite.	x[k]	|f(x[k])|	Diff. of two iterations	Evaluations
NR	5	1.0000000000000000	0.000000e+00	3.001887e-09	10
JM	4	1.0000000000000000	0.000000e+00	6.191414e-07	16
NM	4	1.0000000000000000	0.000000e+00	3.906397e-09	12
RR	4	1.0000000000000000	0.000000e+00	1.942042e-10	12
ALGO2	4	1.0000000000000000	0.000000e+00	3.639530e-09	12

5 Conclusion

In order to find the solution of linear and nonlinear algebraic equation, we have introduce a new two-step iterative method based on homotopy perturba-

tion method. The comparison of the proposed method with Newton-Raphson method (NR), the method of Javidi (JM) [31], Noor et al. method(NM) **[32]** and the method of Rafiq and Rafiullah(RR) [11] in the previous section reveals the efficiency of our method.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

Acknowledgements

This study was supported by research funds from Dong-A University.

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Unisoft Filters in R_0-algebras

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Abstract. The notion of uni-soft filters in R_0-algebras is introduced, and some related properties are investigated. Characterizations of a uni-soft filter are established, and a new uni-soft filter from old one is constructed.

1. Introduction

To solve complicated problem in economics, engineering, and environment, we can not successfully use classical methods because of various uncertainties typical for those problems. Uncertainties can not be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, theory of (intuitionistic) fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets. However, all of these theories have their own difficulties which are pointed out in [9]. Maji et al. [8] and Molodtsov [9] suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory.

To overcome these difficulties, Molodtsov [9] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. At present, works on the soft set theory are progressing rapidly. Maji et al. [8] described the application of soft set theory to a decision making problem. Maji et al. [7] also studied several operations on the theory of soft sets. Chen et al. [2] presented a new definition of soft set parametrization reduction, and compared this definition to the related concept of attributes reduction in rough set theory.

R_0-algebras, which are different from BL-algebras, have been introduced by Wang [11] in order to an algebraic proof of the completeness theorem of a formal deductive system [12]. The filter theory in R_0-algebras is discussed in [10].

2010 Mathematics Subject Classification: 06F35, 03G25, 06D72.

Keywords: soft set, uni-soft filter, δ-exclusive set, exclusive filter.

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In this paper, we apply the notion of uni-soft property to the filter theory in R_0-algebras. We introduced the concept of (normal) uni-soft filters in R_0-algebras, and investigate related properties. We establish characterizations of a (normal) uni-soft filter, and make a new uni-soft filter from old one. We provide a condition for an uni-soft filter to be normal, and construct a condensational property of a normal uni-soft filter.

2. Preliminaries

2.1. Basic results on R_0-algebras.

Definition 2.1 ([11]). Let L be a bounded distributive lattice with order-reversing involution $\neg$ and a binary operation $\to$. Then $(L, \wedge, \vee, \neg, \to)$ is called an R_0-*algebra* if it satisfies the following axioms:

(R1) $x \to y = \neg y \to \neg x$,
(R2) $1 \to x = x$,
(R3) $(y \to z) \wedge ((x \to y) \to (x \to z)) = y \to z$,
(R4) $x \to (y \to z) = y \to (x \to z)$,
(R5) $x \to (y \vee z) = (x \to y) \vee (x \to z)$,
(R6) $(x \to y) \vee ((x \to y) \to (\neg x \vee y)) = 1$.

Let L be an R_0-algebra. For any $x, y \in L$, we define $x \odot y = \neg(x \to \neg y)$ and $x \oplus y = \neg x \to y$. It is proven that $\odot$ and $\oplus$ are commutative, associative and $x \oplus y = \neg(\neg x \odot \neg y)$, and $(L, \wedge, \vee, \odot, \to, 0, 1)$ is a residuated lattice. In the following, let x^n denote $x \odot x \odot \cdots \odot x$ where x appears n times for $n \in \mathbb{N}$.

We refer the reader to the book [3] for further information regarding R_0-algebras.

Definition 2.2 ([10]). A nonempty subset F of L is called a *filter* of L if it satisfies

(F1) $1 \in F$,
(F2) $(\forall x \in F)\ (\forall y \in L)\ (x \to y \in F \ \Rightarrow\ y \in F)$.

Lemma 2.3 ([10]). *Let F be a nonempty subset of L. Then F is a filter of L if and only if it satisfies*

(1) $(\forall x \in F)\ (\forall y \in L)\ (x \leq y \ \Rightarrow\ y \in F)$.
(2) $(\forall x, y \in F)\ (x \odot y \in F)$.

Lemma 2.4 ([10]). *Let L be an R_0-algebra. Then the following properties hold:*

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(2.1) $$(\forall x, y \in L)\,(x \le y \;\Leftrightarrow\; x \to y = 1)\,,$$

(2.2) $$(\forall x, y \in L)\,(x \le y \to x)\,,$$

(2.3) $$(\forall x \in L)\,(\neg x = x \to 0)\,,$$

(2.4) $$(\forall x, y \in L)\,((x \to y) \vee (y \to x) = 1)\,,$$

(2.5) $$(\forall x, y \in L)\,(x \le y \;\Rightarrow y \to z \le x \to z,\; z \to x \le z \to y)\,,$$

(2.6) $$(\forall x, y \in L)\,(((x \to y) \to y) \to y = x \to y)\,,$$

(2.7) $$(\forall x, y \in L)\,(x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x))\,,$$

(2.8) $$(\forall x \in L)\,(x \odot \neg x = 0,\; x \oplus \neg x = 1)\,,$$

(2.9) $$(\forall x, y \in L)\,(x \odot y \le x \wedge y,\; x \odot (x \to y) \le x \wedge y)\,,$$

(2.10) $$(\forall x, y, z \in L)\,((x \odot y) \to z = x \to (y \to z))\,,$$

(2.11) $$(\forall x, y \in L)\,(x \le y \to (x \odot y))\,,$$

(2.12) $$(\forall x, y, z \in L)\,(x \odot y \le z \;\Leftrightarrow\; x \le y \to z)\,,$$

(2.13) $$(\forall x, y, z \in L)\,(x \le y \;\Rightarrow\; x \odot z \le y \odot z)\,,$$

(2.14) $$(\forall x, y, z \in L)\,(x \to y \le (y \to z) \to (x \to z))\,,$$

(2.15) $$(\forall x, y, z \in L)\,((x \to y) \odot (y \to z) \le x \to z)\,.$$

2.2. **Basic results on soft set theory.** Soft set theory was introduced by Molodtsov [9] and Çağman et al. [1].

In what follows, let U be an initial universe set and E be a set of parameters. We say that the pair (U, E) is a *soft universe*. Let $\mathscr{P}(U)$ (resp. $\mathscr{P}(E)$) denotes the power set of U (resp. E).

By analogy with fuzzy set theory, the notion of soft set is defined as follows:

Definition 2.5 ([1, 9])**.** A soft set of E over U (a soft set of E for short) is:

any function $f_A : E \to \mathscr{P}(U)$, such that $f_A(x) = \emptyset$ if $x \notin A$, for $A \in \mathscr{P}(E)$,

or, equivalently, any set

$$\mathscr{F}_A := \{(x, f_A(x)) \mid x \in E,\; f_A(x) \in \mathscr{P}(U),\; f_A(x) = \emptyset \text{ if } x \notin A\}\,,$$

for $A \in \mathscr{P}(E)$.

Definition 2.6 ([1])**.** Let $\mathscr{F}_A$ and $\mathscr{F}_B$ be soft sets of E. We say that $\mathscr{F}_A$ is a *soft subset* of $\mathscr{F}_B$, denoted by $\mathscr{F}_A \widetilde{\subseteq} \mathscr{F}_B$, if $f_A(x) \subseteq f_B(x)$ for all $x \in E$.

Definition 2.7 ([5]). For any non-empty subset A of E, a soft set $\mathscr{F}_A$ of E is said to satisfy the *uni-soft property* (it is also called a *uni-soft set* in [5]) if it satisfies:

$$(\forall x, y \in A)\,(x \hookrightarrow y \in A \;\Rightarrow\; f_A(x \hookrightarrow y) \subseteq f_A(x) \cup f_A(y))$$

where $\hookrightarrow$ is a binary operation in E.

3. Uni-soft filters

In what follows, denote by $S(U, L)$ the set of all soft sets of L over U where L is an R_0-algebra unless otherwise specified.

Definition 3.1. A soft set $\mathscr{F}_L \in S(U, L)$ is called a *filter of L based on the uni-soft property* (briefly, *uni-soft filter* of L) if it satisfies:

$$(\forall x \in L)\,(f_L(1) \subseteq f_L(x))\,, \tag{3.1}$$

$$(\forall x, y \in L)\,(f_L(y) \subseteq f_L(x \to y) \cup f_L(x))\,. \tag{3.2}$$

Example 3.2. Let $L = \{0, a, b, c, 1\}$ be a set with the order $0 < a < b < c < 1$, and the following Cayley tables:

x	$\neg x$
0	1
a	c
b	b
c	a
1	0

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	b	b	1	1	1
c	a	a	b	1	1
1	0	a	b	c	1

Then $(L, \wedge, \vee, \neg, \to)$ is an R_0-algebra (see [4]) where $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$. Let $\mathscr{F}_L \in S(U, L)$ be given as follows:

$$\mathscr{F}_L = \{(0, \delta_1), (a, \delta_1), (b, \delta_1), (c, \delta_2), (1, \delta_2)\}$$

where δ_1 and δ_2 are subsets of U with $\delta_2 \subsetneq \delta_1$. Then $\mathscr{F}_L$ is an uni-soft filter of L.

Proposition 3.3. *Let $\mathscr{F}_L \in S(U, L)$ be a uni-soft filter of L. Then the following properties are valid:*

(1) *$\mathscr{F}_L$ is order-reversing, that is,*

$$(\forall x, y \in L)\ (x \leq y \;\Rightarrow\; f_L(y) \subseteq f_L(x)).$$

(2) $(\forall x, y \in L)\,(f_L(x \to y) = f_L(1) \;\Rightarrow\; f_L(y) \subseteq f_L(x))\,.$

(3) $(\forall x, y \in L)\,(f_L(x \odot y) = f_L(x) \cup f_L(y) = f_L(x \wedge y))\,.$

(4) $(\forall x \in L)(\forall n \in \mathbb{N})\,(f_L(x^n) = f_L(x))\,.$

(5) $(\forall x \in L)\,(f_L(0) = f_L(x) \cup f_L(\neg x))\,.$

(6) $(\forall x, y \in L)\,(f_L(x \to z) \subseteq f_L(x \to y) \cup f_L(y \to z))\,.$

(7) $(\forall x, y, z \in L)\,(x \odot y \leq z \;\Rightarrow\; f_L(z) \subseteq f_L(x) \cup f_L(y))\,.$

(8) $(\forall x, y \in L)\,(f_L(x) \cup f_L(x \to y) = f_L(y) \cup f_L(y \to x) = f_L(x) \cup f_L(y))\,.$

(9) $(\forall x, y \in L)\,(f_L(x \odot (x \to y)) = f_L(y \odot (y \to x)) = f_L(x) \cup f_L(y))$.

(10) $(\forall x, y, z \in L)\,(f_L(x \to (\neg z \to z)) \subseteq f_L(x \to (\neg z \to y)) \cup f_L(y \to z))$.

(11) $(\forall x, y, z \in L)\,(f_L(x \to (x \to z)) \subseteq f_L(x \to (y \to z)) \cup f_L(x \to y))$.

Proof. (1) Let $x, y \in L$ be such that $x \leq y$. Then $x \to y = 1$, and so

$$f_L(y) \subseteq f_L(x) \cup f_L(x \to y) = f_L(x) \cup f_L(1) = f_L(x)$$

by (3.1) and (3.2).

(2) Let $x, y \in L$ be such that $f_L(x \to y) = f_L(1)$. Then

$$f_L(y) \subseteq f_L(x) \cup f_L(x \to y) = f_L(x) \cup f_L(1) = f_L(x)$$

by (3.1) and (3.2).

(3) Since $x \odot y \leq x \wedge y$ for all $x, y \in L$, it follows from (1) that $f_L(x) \cup f_L(y) \subseteq f_L(x \odot y)$. Using (2.11) and (1), we have $f_L(y \to (x \odot y)) \subseteq f_L(x)$. It follows from (3.2) that

$$f_L(x \odot y) \subseteq f_L(y \to (x \odot y)) \cup f_L(y) \subseteq f_L(x) \cup f_L(y).$$

Therefore $f_L(x \odot y) = f_L(x) \cup f_L(y)$. Since $y \leq x \to y$ and $x \odot (x \to y) \leq x \wedge y$ for all $x, y \in L$, we have $f_L(x \to y) \subseteq f_L(y)$ and

$$\begin{aligned} f_L(x) \cup f_L(y) &\subseteq f_L(x \wedge y) \subseteq f_L(x \odot (x \to y)) \\ &= f_L(x) \cup f_L(x \to y) \\ &\subseteq f_L(x) \cup f_L(y) \end{aligned}$$

by (1). Hence $f_L(x \wedge y) = f_L(x) \cup f_L(y)$ for all $x, y \in L$.

(4) It follows from (3).

(5) Note that $x \odot \neg x = 0$ for all $x \in L$. Using (3), we have

$$f_L(0) = f_L(x \odot \neg x) = f_L(x) \cup f_L(\neg x)$$

for all $x, y \in L$.

(6) Combining (2.15), (1) and (3), we have the desired result.

(7) It follows from (1) and (3).

(8) Since $y \leq x \to y$ for all $x, y \in L$, it follows from (1) that

$$f_L(x) \cup f_L(x \to y) \subseteq f_L(x) \cup f_L(y). \tag{3.3}$$

Since $x \odot (x \to y) \leq x \wedge y$ for all $x, y \in L$, we have

$$f_L(x) \cup f_L(y) = f_L(x \wedge y) \subseteq f_L(x \odot (x \to y)) = f_L(x) \cup f_L(x \to y) \tag{3.4}$$

by (3) and (1). Hence $f_L(x) \cup f_L(y) = f_L(x) \cup f_L(x \to y)$. Similarly,

$$f_L(y) \cup f_L(y \to x) = f_L(x) \cup f_L(y)$$

for all $x, y \in L$.

(9) Using (3), we have

$$f_L(x \odot (x \to y)) = f_L(x) \cup f_L(x \to y)$$

and

$$f_L(y \odot (y \to x)) = f_L(y) \cup f_L(y \to x)$$

for all $x, y \in L$. It follows from (8) that

$$f_L(x \odot (x \to y)) = f_L(y \odot (y \to x)) = f_L(x) \cup f_L(y).$$

(10) Note that

$$\begin{aligned}(x \to (\neg z \to y)) \odot (y \to z) &= ((x \odot \neg z) \to y) \odot (y \to z) \\ &\leq (x \odot \neg z) \to z = x \to (\neg z \to z)\end{aligned}$$

for all $x, y, z \in L$. Using (1) and (3), we have

$$\begin{aligned}f_L(x \to (\neg z \to z)) &\subseteq f_L((x \to (\neg z \to y)) \odot (y \to z)) \\ &= f_L(x \to (\neg z \to y)) \cup f_L(y \to z)\end{aligned}$$

for all $x, y, z \in L$.

(11) Note that

$$(x \to (y \to z)) \odot (x \to y) = (y \to (x \to z)) \odot (x \to y) \leq x \to (x \to z)$$

for all $x, y, z \in L$. It follows from (1) and (3) that

$$\begin{aligned}f_L(x \to (x \to z)) &\subseteq f_L\left((x \to (y \to z)) \odot (x \to y)\right) \\ &= f_L(x \to (y \to z)) \cup f_L(x \to y)\end{aligned}$$

for all $x, y, z \in L$. □

Definition 3.4. Let $\mathscr{F}_L \in S(U, L)$ be a unisoft filter of L. Then for any $\delta \in \mathscr{P}(U)$, the *$\delta$-exclusive set* of $\mathscr{F}_L$ is defined by

$$e(\mathscr{F}_L; \delta) = \{x \in L \mid f_L(x) \subseteq \delta\}.$$

If $\mathscr{F}_L$ is a uni-soft filter of L, every δ-exclusive set $e(\mathscr{F}_L; \delta)$ is called an *exclusive filter* of L.

We provide characterizations of a uni-soft filter.

Theorem 3.5. *Let $\mathscr{F}_L \in S(U, L)$. Then $\mathscr{F}_L$ is a uni-soft filter of L if and only if the following assertion is valid:*

$$(\forall \delta \in \mathscr{P}(U))\,(e(\mathscr{F}_L; \delta) \neq \emptyset \;\Rightarrow\; e(\mathscr{F}_L; \delta) \text{ is a filter of } L)\,. \tag{3.5}$$

Proof. Assume that $\mathscr{F}_L \in S(U, L)$ satisfies (3.5). For any $x \in L$, let $f_L(x) = \delta$. Then $x \in e(\mathscr{F}_L; \delta)$. Since $e(\mathscr{F}_L; \delta)$ is a filter of L, we have $1 \in e(\mathscr{F}_L; \delta)$ and so $f_L(1) \subseteq \delta = f_L(x)$. For any $x, y \in L$, let $f_L(x \to y) \cup f_L(x) = \delta$. Then $x \to y \in e(\mathscr{F}_L; \delta)$ and $x \in e(\mathscr{F}_L; \delta)$. Since $e(\mathscr{F}_L; \delta)$ is a filter of L, it follows that $y \in e(\mathscr{F}_L; \delta)$. Hence

$$f_L(y) \subseteq \delta = f_L(x \to y) \cup f_L(x).$$

Conversely, suppose that $\mathscr{F}_L$ is a uni-soft filter of L. Let $\delta \in \mathscr{P}(U)$ be such that $e(\mathscr{F}_L; \delta) \neq \emptyset$. Then there exists $a \in e(\mathscr{F}_L; \delta)$, and so $f_L(a) \subseteq \delta$. It follows from (3.1) that $f_L(1) \subseteq f_L(a) \subseteq \delta$. Thus $1 \in e(\mathscr{F}_L; \delta)$. Let $x, y \in L$ be such that $x \to y \in e(\mathscr{F}_L; \delta)$ and $x \in e(\mathscr{F}_L; \delta)$. Then $f_L(x \to y) \subseteq \delta$ and $f_L(x) \subseteq \delta$. It follows from (3.2) that

$$f_L(y) \subseteq f_L(x \to y) \cup f_L(x) \subseteq \delta,$$

that is, $y \in e(\mathscr{F}_L; \delta)$. Thus $\mathscr{F}_L$ is a uni-soft filter of L. □

Theorem 3.6. *Let $\mathscr{F}_L \in S(U, L)$. Then $\mathscr{F}_L$ is a uni-soft filter of L if and only if the following assertions are valid:*

(1) *$\mathscr{F}_L$ is order-reversing,*
(2) $(\forall x, y \in L)\,(f_L(x \odot y) = f_L(x) \cup f_L(y))$.

Proof. The necessity follows from (1) and (3) of Proposition 3.3.

Conversely, suppose that $\mathscr{F}_L$ satisfies two conditions (1) and (2). Let $x, y \in L$. Since $x \leq 1$, we have $f_L(1) \subseteq f_L(x)$ by (1). Note that $x \odot (x \to y) \leq y$ for all $x, y \in L$. It follows from (2) and (1) that

$$f_L(y) \subseteq f_L(x \odot (x \to y)) = f_L(x) \cup f_L(x \to y)$$

for all $x, y \in L$. Therefore $\mathscr{F}_L$ is a uni-soft filter of L. □

Theorem 3.7. *Let $\mathscr{F}_L \in S(U, L)$. Then $\mathscr{F}_L$ is a uni-soft filter of L if and only if the following assertion is valid:*

$$(\forall x, y, z \in L)\,(f_L(x \to z) \subseteq f_L(y) \cup f_L((x \to y) \to z)). \tag{3.6}$$

Proof. Assume that $\mathscr{F}_L$ is a uni-soft filter of L and let $x, y, z \in L$. Since $y \leq x \to y$, it follows from Proposition 3.3(1) and (3.2) that

$$\begin{aligned} f_L(x \to z) \subseteq f_L(z) &\subseteq f_L(x \to y) \cup f_L((x \to y) \to z) \\ &\subseteq f_L(y) \cup f_L((x \to y) \to z) \end{aligned}$$

for all $x, y, z \in L$.

Conversely, suppose that $\mathscr{F}_L$ satisfies the inclusion (3.6). If we take $x = 0$ and $y = z$ in (3.6), then

$$\begin{aligned} f_L(1) = f_L(0 \to z) &\subseteq f_L(z) \cup f_L((0 \to z) \to z) \\ &= f_L(z) \cup f_L(1 \to z) \\ &= f_L(z) \cup f_L(z) = f_L(z) \end{aligned}$$

for all $z \in L$, and if we put $x = 1$, $y = x$ and $z = y$ in (3.6), then

$$f_L(y) = f_L(1 \to y) \subseteq f_L(x) \cup f_L((1 \to x) \to y) = f_L(x) \cup f_L(x \to y)$$

for all $x, y \in L$. Therefore $\mathscr{F}_L$ is a uni-soft filter of L. □

Theorem 3.8. *Let $\mathscr{F}_L \in S(U, L)$. Then $\mathscr{F}_L$ is a uni-soft filter of L if and only if the following assertion is valid:*

$$(\forall x, y, z \in L)\,(x \leq y \to z \;\Rightarrow\; f_L(z) \subseteq f_L(x) \cup f_L(y))\,. \tag{3.7}$$

Proof. Suppose that $\mathscr{F}_L$ is a uni-soft filter of L. Let $x, y, z \in L$ be such that $x \leq y \to z$. Then $f_L(y \to z) \subseteq f_L(x)$ by Proposition 3.3(1), and so

$$f_L(z) \subseteq f_L(y \to z) \cup f_L(y) \subseteq f_L(x) \cup f_L(y)$$

by (3.2).

Conversely, assume that $\mathscr{F}_L$ satisfies the condition (3.7). Let $x, y \in L$. Since $x \leq 1 = x \to 1$, we have $f_L(1) \subseteq f_L(x)$ by (3.7). Note that $x \to y \leq x \to y$. It follows from (3.7) that $f_L(y) \subseteq f_L(x \to y) \cup f_L(x)$. Therefore $\mathscr{F}_L$ is a uni-soft filter of L. □

Proposition 3.9. *Every uni-soft filter $\mathscr{F}_L$ of L satisfies:*

$$(\forall x, y, z \in L)\,(f_L(x \to (y \to z)) \subseteq f_L((x \to y) \to z))\,. \tag{3.8}$$

Proof. Let $x, y, z \in L$. Since $1 = y \to (x \to y) \leq ((x \to y) \to z) \to (y \to z)$, we have

$$f_L(((x \to y) \to z) \to (y \to z)) \subseteq f_L(1)$$

by Proposition 3.3(1). It follows from (2.2), Proposition 3.3(1), (3.2) and (3.1) that

$$\begin{aligned} f_L(x \to (y \to z)) &\subseteq f_L(y \to z) \\ &\subseteq f_L((x \to y) \to z) \cup f_L(((x \to y) \to z) \to (y \to z)) \\ &\subseteq f_L((x \to y) \to z) \cup f_L(1) \\ &= f_L((x \to y) \to z) \end{aligned}$$

for all $x, y, z \in L$. This completes the proof. □

The following example shows that the converse of Proposition 3.9 may not be true in general.

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Example 3.10. Let $L = \{0, a, b, c, d, 1\}$ be a set with the order $0 < a < b < c < d < 1$, and the following Cayley tables:

x	$\neg x$
0	1
a	d
b	c
c	b
d	a
1	0

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	1	1	1
b	c	c	1	1	1	1
c	b	b	b	1	1	1
d	a	a	b	c	1	1
1	0	a	b	c	d	1

Then $(L, \wedge, \vee, \neg, \to)$ is an R_0-algebra (see [4]) where $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$. Let $\mathscr{F}_L \in S(U, L)$ be given as follows:

$$\mathscr{F}_L = \{(0, \delta_1), (a, \delta_2), (b, \delta_2), (c, \delta_2), (d, \delta_1), (1, \delta_1)\}$$

where δ_1 and δ_2 are subsets of U with $\delta_2 \subsetneq \delta_1$. Then $\mathscr{F}_L$ satisfies the condition (3.8), but $\mathscr{F}_L$ is not a uni-soft filter of L since $f_L(1) = \delta_1 \nsubseteq \delta_2 = f_L(a)$.

Proposition 3.11. *For a uni-soft filter $\mathscr{F}_L$ of L, the following are equivalent:*

$$f_L(y \to x) \subseteq f_L(y \to (y \to x)) \quad \textit{for all} \quad x, y \in L, \tag{3.9}$$

$$f_L((z \to y) \to (z \to x)) \subseteq f_L(z \to (y \to x)) \quad \textit{for all} \quad x, y, z \in L. \tag{3.10}$$

Proof. Assume that (3.9) is valid and let $x, y, z \in L$. Using (R4), (2.5) and (2.14), we have

$$z \to (y \to x) \leq z \to (z \to ((z \to y) \to x)).$$

It follows from (R4), (3.9) and Proposition 3.3(1) that

$$\begin{aligned} f_L((z \to y) \to (z \to x)) = \ & f_L(z \to ((z \to y) \to x)) \\ \subseteq \ & f_L(z \to (z \to ((z \to y) \to x))) \\ \subseteq \ & f_L(z \to (y \to x)). \end{aligned}$$

Conversely, suppose that (3.10) holds. If we use z instead of y in (3.10), then

$$\begin{aligned} f_L(z \to x) &= f_L(1 \to (z \to x)) \\ &= f_L((z \to z) \to (z \to x)) \\ &\subseteq f_L(z \to (z \to x)), \end{aligned}$$

which proves (3.9). □

We make a new uni-soft filter from old one.

Theorem 3.12. *Let $\mathscr{F}_L \in S(U, L)$. For a subset δ of U, define a soft set $\mathscr{F}_L^*$ of L by*

$$f_L^* : L \to \mathscr{P}(U), \ x \mapsto \begin{cases} f_L(x) & \text{if } x \in e(\mathscr{F}_L; \delta), \\ \tau & \text{otherwise} \end{cases}$$

where τ is a subset of U such that $\tau \supsetneq \bigcup_{x\in e(\mathscr{F}_L;\delta)} f_L(x)$. If $\mathscr{F}_L$ is a uni-soft filter of L, then so is $\mathscr{F}_L^*$.

Proof. Assume that $\mathscr{F}_L$ is a uni-soft filter of L. Then $e(\mathscr{F}_L;\delta)$ $(\neq \emptyset)$ is a filter of L for all $\delta \in \mathscr{P}(U)$. Hence $1 \in e(\mathscr{F}_L;\delta)$, and so $f_L^*(1) = f_L(1) \subseteq f_L(x) \subseteq f_L^*(x)$ for all $x \in L$. Let $x, y \in L$. If $x \in e(\mathscr{F}_L;\delta)$ and $x \to y \in e(\mathscr{F}_L;\delta)$, then $y \in e(\mathscr{F}_L;\delta)$. Hence

$$f_L^*(y) = f_L(y) \subseteq f_L(x) \cup f_L(x \to y) = f_L^*(x) \cup f_L^*(x \to y).$$

If $x \notin e(\mathscr{F}_L;\delta)$ or $x \to y \notin e(\mathscr{F}_L;\delta)$, then $f_L^*(x) = \tau$ or $f_L^*(x \to y) = \tau$. Thus

$$f_L^*(y) \subseteq \tau = f_L^*(x) \cup f_L^*(x \to y).$$

Therefore $\mathscr{F}_L^*$ is a uni-soft filter of L. □

Corollary 3.13. *Let $\mathscr{F}_L \in S(U,L)$. For a subset δ of U, define a soft set $\mathscr{F}_L^*$ of L by*

$$f_L^* : L \to \mathscr{P}(U),\ x \mapsto \begin{cases} f_L(x) & \text{if } x \in e(\mathscr{F}_L;\delta), \\ U & \text{otherwise.} \end{cases}$$

If $\mathscr{F}_L$ is a uni-soft filter of L, then so is $\mathscr{F}_L^*$.

Proof. Straightforward. □

Theorem 3.14. *Any filter of L can be realized as an exclusive filter of some uni-soft filter of L.*

Proof. Let F be a filter of L. For a nonempty subset δ of U, let $\mathscr{F}_L$ be a soft set of L defined by

$$f_L : L \to \mathscr{P}(U),\ x \mapsto \begin{cases} \delta & \text{if } x \in F, \\ U & \text{otherwise.} \end{cases}$$

Obviously $f_L(1) \subseteq f_L(x)$ for all $x \in L$. For any $x, y \in L$, if $x \in F$ and $x \to y \in F$, then $y \in F$. Hence $f_L(x) \cup f_L(x \to y) = \delta = f_L(y)$. If $x \notin F$ or $x \to y \notin F$, then $f_L(x) = U$ or $f_L(x \to y) = U$. Thus $f_L(y) \subseteq U = f_L(x) \cup f_L(x \to y)$. Therefore $\mathscr{F}_L$ is a uni-soft filter of L and clearly $e(\mathscr{F}_L;\delta) = F$. This completes the proof. □

Acknowledgements

The authors wish to thank the anonymous reviewers for their valuable suggestions.

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The admissible function and non-admissible function in the unit disc *

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Abstract

In this paper, we study the multiple values and uniqueness problem of two non-admissible functions in the unit disc sharing some values, and also investigate the problem of admissible function and non-admissible function sharing some values, and obtain the following conclusions:

1) Two non-constant non-admissible functions can not be identical if they share four distinct values IM;

2) An admissible function can share at most three distinct values IM with a non-admissible function.

Moreover, we also obtain some theorems which are analogous version of the uniqueness theorems of meromorphic functions sharing some values on the whole complex plane given by Yi and Cao.

Key words: uniqueness; meromorphic function; admissible; non-admissible.
Mathematical Subject Classification (2010): 30D 35.

*The author was supported by the NSF of China(11301233, 61202313), the Natural Science Foundation of Jiang-Xi Province in China (20132BAB211001), and the Foundation of Education Department of Jiangxi (GJJ14644) of China. The second author was supported by the NSF of China (11371225, 11171013). The third author was supported by the NSF of China (11461042).

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1 Introduction and basic notions in the Nevanlinna theory in the unit disc $\mathbb{D}$

Considering meromorphic function f, we shall assume that the readers are familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions such as the proximity function $m(r, f)$, counting function $N(r, f)$, characteristic function $T(r, f)$, the first and second main theorems, lemma on the logarithmic derivatives etc. of Nevalinna theory, (see Hayman [6], Yang [14] and Yi and Yang [16]). For a meromorphic function f, $S(r, f)$ denotes any quantity satisfying $S(r, f) = o(T(r, f))$ for all r outside a possible exceptional set of finite logarithmic measure.

In 1926, R. Nevanlinna [9] proved his famous five-value theorem: *For two nonconstant meromorphic functions f and g on $\mathbb{C}$, if they have the same inverse images (ignoring multiplicities) for five distinct values, then $f(z) \equiv g(z)$.* After his very work, the uniqueness of meromorphic functions with shared values on $\mathbb{C}$ attracted many investigations (for references, see the book [16] or some recent papers [1, 3, 5, 8, 13, 12]). It is well-known that two nonconstant rational functions f_1, f_2 satisfy $\lim_{r\to+\infty} \frac{T(r,f_i)}{\log r} < +\infty (i = 1, 2)$ that share four distinct values IM must be identical. However, *for two nonconstant meromorphic functions f_1, f_2 in the unit disc satisfy $\lim_{r\to 1^-} \frac{T(r,f_i)}{\log \frac{1}{1-r}} < +\infty (i = 1, 2)$, they may not be identical if f_1, f_2 share four distinct values IM (see Corollary 2.3).* Hence, we will mainly study the uniqueness of meromorphic functions in the unit disc $\mathbb{D}$ in this paper.

To state some uniqueness theorems of meromorphic functions in the unit disc $\mathbb{D}$, we need the following basic notations and definitions of meromorphic functions in $\mathbb{D}$(see [2], [4], [7], [11]).

Definition 1.1 *Let f be a meromorphic function in $\mathbb{D}$ and $\lim_{r\to 1^-} T(r, f) = \infty$. Then*

$$D(f) := \limsup_{r\to 1^-} \frac{T(r, f)}{-\log(1 - r)}$$

is called the (upper) index of inadmissibility of f. If $D(f) = \infty$, f is called admissible.

Definition 1.2 *Let f be a meromorphic function in $\mathbb{D}$ and $\lim_{r\to 1^-} T(r, f) = \infty$. Then*

$$\rho(f) := \limsup_{r\to 1^-} \frac{\log^+ T(r, f)}{-\log(1 - r)}$$

is called the order (of growth) of f.

In 2005, Titzhoff [11] investigated the uniqueness of two admissible functions in the unit disc $\mathbb{D}$ by using the Second Main Theorem for admissible functions (see [11, Theorem 3]) and obtained the five values theorem for admissible functions in the unit disc $\mathbb{D}$ as follows.

Theorem 1.1 (see [11, Theorem 3]). *Let f be an admissible meromorphic function in $\mathbb{D}$, q be a positive integer and $a_1, a_2, \ldots, a_q$ be pairwise distinct complex numbers. Then,*

for $r \to 1^-$, $r \notin E$,

$$(q-2)T(r,f) \le \sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{f-a_j}\right) + S(r,f),$$

where $E \subset (0,1)$ *is a possibly occurring exceptional set with* $\int_E \frac{dr}{1-r} < \infty$. *If the order of* f *is finite, the remainder* $S(r,f)$ *is a* $O\left(\log \frac{1}{1-r}\right)$ *without any exceptional set.*

Theorem 1.2 (see [5, 11]). *If two admissible function* f, g *share five distinct values, then* $f \equiv g$.

For admissible functions, Theorem 1.1 plays a very important role in studies of the uniqueness problems of meromorphic functions in the unit disc. From Theorem 1.1 (the Second Main Theorem for admissible functions (see [11, Theorem 3]), we can get a lot of results similar to meromorphic functions in the complex plane. However, Theorem 1.1 does not hold for non-admissible functions in the unit disc. Thus it is interesting to consider the uniqueness theory of non-admissible functions in the unit disc.

For non-admissible functions, the following theorem also plays a very important role in studying their uniqueness problems.

Theorem 1.3 (see [11, Theorem 2]). *Let* f *be a meromorphic function in* $\mathbb{D}$ *and* $\lim_{r\to 1^-} T(r,f) = \infty$, q *be a positive integer and* $a_1, a_2, \ldots, a_q$ *be pairwise distinct complex numbers. Then, for* $r \to 1^-$, $r \notin E$,

$$(q-2)T(r,f) \le \sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{f-a_j}\right) + \log \frac{1}{1-r} + S(r,f).$$

Remark 1.1 *In contrast to admissible functions, the term* $\log \frac{1}{1-r}$ *in Theorem 1.3 does not necessarily enter the remainder* $S(r,f)$ *because the non-admissible function* f *may have* $T(r,f) = O\left(\log \frac{1}{1-r}\right)$.

Remark 1.2 *We can see that* $S(r,f) = o\left(\log \frac{1}{1-r}\right)$ *holds in Theorem 1.3 without a possible exception set when* $0 < D(f) < \infty$.

From Theorem 1.3 and Lemma 3.4 in [16], we can get the following result for non-admissible functions in the unit disc which is used in this paper.

Lemma 1.1 *Let* $f(z)$ *be a meromorphic function in* $\mathbb{D}$ *and* $\lim_{r\to 1^-} T(r,f) = \infty$, $a_j (j = 1, 2, \ldots, q)$ *be* q *distinct complex numbers, and* $k_j (j = 1, 2, \ldots, q)$ *be positive integers or* ∞. *If* f *is a non-admissible function, then*

$$\begin{aligned}(q-2)T(r,f) <& \sum_{j=1}^{q} \frac{k_j}{k_j+1} \overline{N}_{k_j)}\left(r, \frac{1}{f-a_j}\right) + \sum_{j=1}^{q} \frac{1}{k_j+1} N\left(r, \frac{1}{f-a_j}\right)\\ &+ \log \frac{1}{1-r} + S(r,f),\end{aligned}$$

and

$$\left(q-2-\sum_{j=1}^{q}\frac{1}{k_j+1}\right)T(r,f)\leq\sum_{j=1}^{q}\frac{k_j}{k_j+1}\overline{N}_{k_j)}\left(r,\frac{1}{f-a}\right)+\log\frac{1}{1-r}+S(r,f),$$

where $S(r,f)$ is stated as in Theorem 2.1.

The main purpose of this paper is to investigate the uniqueness problem of non-admissible functions in the unit disc. In section 2, the uniqueness of two non-admissible functions in $\mathbb{D}$ are investigated and some theorems show that the number and weight of sharing values is related to the index of inadmissibility of functions in $\mathbb{D}$. In section 3, the problem of admissible function and non-admissible function sharing some values is studied, and one of those results show that admissible function and non-admissible function can share at most three distinct values IM.

2 Multiple values and uniqueness of non-admissible functions in the unit disc $\mathbb{D}$

We use $\mathbb{C}$ to denote the open complex plane, $\widehat{\mathbb{C}}:=\mathbb{C}\bigcup\{\infty\}$ to denote the extended complex plane, and $\mathbb{D}=\{z:|z|<1\}$ to denote the unit disc. Let S be a set of distinct elements in $\widehat{\mathbb{C}}$ and $\subseteq\mathbb{C}$. Define

$$E(S,\mathbb{D},f)=\bigcup_{a\in S}\{z\in\mathbb{D}|f_a(z)=0,\quad \textit{counting}\quad \textit{multiplicities}\},$$

$$\overline{E}(S,\mathbb{D},f)=\bigcup_{a\in S}\{z\in\mathbb{D}|f_a(z)=0,\quad \textit{ignoring}\quad \textit{multiplicities}\},$$

where $f_a(z)=f(z)-a$ if $a\in\mathbb{C}$ and $f_\infty(z)=1/f(z)$.

Let f,g be two non-constant meromorphic functions in $\mathbb{D}$. If $E(S,\mathbb{D},f)=E(S,\mathbb{D},g)$, we say f and g share the set S CM(counting multiplicities) in $\mathbb{D}$. If $\overline{E}(S,\mathbb{D},f)=\overline{E}(S,\mathbb{D},g)$, we say f and g share the set S IM(ignoring multiplicities) in $\mathbb{D}$. In particular, when $S=\{a\}$, where $a\in\widehat{\mathbb{C}}$, we say f and g share the value a CM in $\mathbb{D}$ if $E(S,\mathbb{D},f)=E(S,\mathbb{D},g)$, and we say f and g share the value a IM in $\mathbb{D}$ if $\overline{E}(S,\mathbb{D},f)=\overline{E}(S,\mathbb{D},g)$. If $\mathbb{D}$ is replaced by $\mathbb{C}$, we give the simple notation as before, $E(S,f),\overline{E}(S,f)$ and so on(see [14]).

Let $f(z)$ be a non-constant meromorphic function in the unit disc, an arbitrary complex number $a\in\widehat{\mathbb{C}}$, and k be a positive integer. We use $\overline{E}_{k)}(a,\mathbb{D},f)$ to denote the set of zeros of $f-a$ in $\mathbb{D}$, with multiplicities no greater than k, in which each zero is counted only once. We say that $f(z)$ and $g(z)$ share the value a in $\mathbb{D}$ with reduced weight k, if $\overline{E}_{k)}(a,\mathbb{D},f)=\overline{E}_{k)}(a,\mathbb{D},g)$.

We denote the deficiency of $a\in\widehat{\mathbb{C}}$ with respect to a meromorphic function f on the unit disc $\mathbb{D}$ by

$$\delta(a,f)=\delta(0,f-a)=\liminf_{r\to1^-}\frac{m(r,\frac{1}{f-a})}{T(r,f)}=1-\limsup_{r\to1^-}\frac{N(r,\frac{1}{f-a})}{T(r,f)},$$

and denote the reduced deficiency by

$$\Theta(a,f)=\Theta(0,f-a)=1-\limsup_{r\to 1^-}\frac{\overline{N}(r,\frac{1}{f-a})}{T(r,f)}.$$

We now show our main result below which is an analog of a result on the plane $\mathbb{C}$ obtained by H.-X. Yi [17] (see also Theorem 3.34 in [16]).

Theorem 2.1 *Let $f_1(z), f_2(z)$ be non-constant non-admissible functions satisfying $0<D(f_1), D(f_2)<\infty$, $a_j (j=1,2,\ldots,q)$ be q distinct complex numbers in $\widehat{\mathbb{C}}$, and let $k_j (j=1,2,\ldots,q)$ be positive integers or ∞ satisfying*

$$k_1\ge k_2\ge\cdots\ge k_q \tag{1}$$

and

$$\overline{E}_{k_j)}(a_j,\mathbb{D},f_1)=\overline{E}_{k_j)}(a_j,\mathbb{D},f_2),\quad (j=1,2,\ldots,q). \tag{2}$$

Furthermore, let

$$\Theta(f_i)=\sum_a\Theta(0,f_i-a)-\sum_{j=1}^{q}\Theta(0,f_i-a_j),(i=1,2),$$

and

$$A_1=\frac{\sum_{j=1}^{m-1}\delta(0,f_1-a_j)}{k_m+1}+\sum_{j=m}^{q}\frac{k_j+\delta(0,f_1-a_j)}{k_j+1}+\frac{(m-2)k_m}{k_m+1}-\frac{k_n}{k_n+1}+\Theta(f_1)-2,$$

$$A_2=\frac{\sum_{j=1}^{n-1}\delta(0,f_2-a_j)}{k_n+1}+\sum_{j=n}^{q}\frac{k_j+\delta(0,f_2-a_j)}{k_j+1}+\frac{(n-2)k_n}{k_n+1}-\frac{k_m}{k_m+1}+\Theta(f_2)-2,$$

where m and n are positive integers in $\{1,2,\ldots,q\}$ and a is an arbitrary complex number. If

$$\min\{A_1,A_2\}\ge\frac{2}{D(f_1)+D(f_2)},\quad \textit{and}\quad \max\{A_1,A_2\}>\frac{2}{D(f_1)+D(f_2)}. \tag{3}$$

Then $f_1(z)\equiv f_2(z)$.

Proof: Suppose that $f_1(z)\not\equiv f_2(z)$. We assume that a_j $(j=1,2,\ldots,q)$ are finite complex numbers, otherwise, we will consider a suitable Möbious transformation. Without loss of generality, we may assume that infinite b satisfy $\Theta(0,f_1-b)>0$ and $b\ne a_j$ $(j=1,2,\ldots,q)$. We denote them by b_k $(k=1,2,\ldots,\infty)$. Obviously, $\Theta(f_1)=\sum_{k=1}^{\infty}\Theta(0,f_1-b_k)$. Thus there exists a p such that $\sum_{k=1}^{p}\Theta(0,f_1-b_k)>\Theta(f_1)-\varepsilon$ holds for ε (>0). From Theorem 1.3 we have

$$(p+q-2)T(r,f_1)\le\sum_{k=1}^{p}\overline{N}(r,\frac{1}{f_1-b_k})+\sum_{j=1}^{q}\overline{N}(r,\frac{1}{f_1-a_j})+\log\frac{1}{1-r}+S(r,f_1). \tag{4}$$

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By the definition of reduced deficiency, we have

$$\overline{N}(r, \frac{1}{f_1 - b_k}) < (1 - \Theta(0, f_1 - b_k))\, T(r, f_1) + S(r, f_1). \tag{5}$$

From Lemma 3.4 in [16] and the definition of deficiency, we get

$$\begin{aligned}\overline{N}(r, \frac{1}{f_1 - a_j}) &\leq \frac{k_j}{k_j+1}\overline{N}_{k_j)}(r, \frac{1}{f_1 - a_j}) + \frac{1}{k_j+1}N(r, \frac{1}{f_1 - a_j})\\ &< \frac{k_j}{k_j+1}\overline{N}_{k_j)}(r, \frac{1}{f_1 - a_j}) + \frac{1}{k_j+1}\,(1 - \delta(0, f_1 - a_j))\, T(r, f_1)\\ &\quad +S(r, f_1).\end{aligned}$$

From the above inequalities and (5)-(6), we get

$$\begin{aligned}(p+q-2)T(r,f_1) \leq &\left\{\sum_{k=1}^{p}(1 - \Theta(0, f_1 - b_k))\right\} T(r, f_1) + \sum_{j=1}^{q}\frac{k_j}{k_j+1}\overline{N}_{k_j)}(r, \frac{1}{f_1 - a_j})\\ &+ \left\{\sum_{j=1}^{q}\frac{1 - \delta(0, f_1 - a_j)}{k_j+1}\right\} T(r, f_1) + \log\frac{1}{1-r} + S(r, f_1).\end{aligned}$$

From (2) we have

$$1 \geq \frac{k_1}{k_1+1} \geq \frac{k_2}{k_2+1} \geq \cdots \geq \frac{k_q}{k_q+1} \geq \frac{1}{2}.$$

Hence we can deduce that

$$\begin{aligned}(p+q-2)T(r,f_1) \leq &(p - \Theta(f_1) + \varepsilon)\, T(r, f_1) + \sum_{j=1}^{q}\frac{k_m}{k_m+1}\overline{N}_{k_j)}(R, \frac{1}{f_1 - a_j})\\ &+ \left\{\sum_{j=1}^{m-1}\left(\frac{k_j}{k_j+1} - \frac{k_m}{k_m+1}\right)(1 - \delta(0, f_1 - a_j))\right\} T(r, f_1)\\ &+ \left\{\sum_{j=1}^{q}\frac{1 - \delta(0, f_1 - a_j)}{k_j+1}\right\} T(r, f_1) + \log\frac{1}{1-r} + S(r, f_1),\end{aligned}$$

and thus,

$$\left(\frac{(m-1)k_m}{k_m+1} + B_1 - \varepsilon\right) T(r, f_1) \leq \sum_{j=1}^{q}\frac{k_m}{k_m+1}\overline{N}_{k_j)}(r, \frac{1}{f_1 - a_j}) + \log\frac{1}{1-r} + S(r, f_1),$$

where

$$B_1 = \frac{\sum_{j=1}^{m-1}\delta(0, f_1 - a_j)}{k_m+1} + \sum_{j=m}^{q}\frac{k_j + \delta(0, f_1 - a_j)}{k_j+1} + \Theta(f_1) - 2.$$

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By similar discussion, we have

$$\left(\frac{(n-1)k_n}{k_n+1}+B_2-\varepsilon\right)T(r,f_2)\le\sum_{j=1}^{q}\frac{k_n}{k_n+1}\overline{N}_{k_j)}(r,\frac{1}{f_2-a_j})+\log\frac{1}{1-r}+S(r,f_2),$$

where

$$B_2=\frac{\sum_{j=1}^{n-1}\delta(0,f_2-a_1)}{k_n+1}+\sum_{j=n}^{q}\frac{k_j+\delta(0,f_2-a_j)}{k_j+1}+\Theta(f_2)-2.$$

Hence

$$\begin{aligned}&\left(\frac{(m-1)k_m}{k_m+1}+B_1-\varepsilon\right)T(r,f_1)+\left(\frac{(n-1)k_n}{k_n+1}+B_2-2\varepsilon\right)T(r,f_2)\\ \le\ &\sum_{j=1}^{q}\frac{k_m}{k_m+1}\overline{N}_{k_j)}(r,\frac{1}{f_1-a_j})+\sum_{j=1}^{q}\frac{k_n}{k_n+1}\overline{N}_{k_j)}(r,\frac{1}{f_2-a_j})\\ &+2\log\frac{1}{1-r}+S(r,f_1)+S(r,f_2).\end{aligned}$$

From the condition (2) and the first fundamental theorem, we have

$$\begin{aligned}\max\left\{\sum_{j=1}^{q}\overline{N}_{k_j)}(r,\frac{1}{f_1-a_j}),\sum_{j=1}^{q}\overline{N}_{k_j)}(r,\frac{1}{f_2-a_j})\right\}&\le N(r,\frac{1}{f_1-f_2})\\ &\le T(r,\frac{1}{f_1-f_2})+O(1)\\ &\le T(r,f_1)+T(r,f_2)+O(1).\end{aligned}$$

Therefore, from the above discussion we obtain

$$\begin{aligned}&\left(\frac{(m-1)k_m}{k_m+1}+B_1-\varepsilon\right)T(r,f_1)+\left(\frac{(n-1)k_n}{k_n+1}+B_2-\varepsilon\right)T(r,f_2)\\ \le\ &\left(\frac{k_m}{k_m+1}+\frac{k_n}{k_n+1}\right)(T(r,f_1)+T(r,f_2))+2\log\frac{1}{1-r}+S(r,f_1)+S(r,f_2).\end{aligned}$$

hence,

$$(A_1-\varepsilon)\,T(r,f_1)+(A_2-\varepsilon)\,T(r,f_2)\le 2\log\frac{1}{1-r}+S(r,f_1)+S(r,f_2). \tag{6}$$

Since $0<D(f_1),D(f_2)<\infty$, we have $S(r,f_1)=o\left(\log\frac{1}{1-r}\right),S(r,f_2)=o\left(\log\frac{1}{1-r}\right)$. And from the definition of index, for any ε satisfying

$$0<2\varepsilon<\min\left\{D(f_1),D(f_2),\max\{A_1,A_2\}-\frac{2}{D(f_1)+D(f_2)}\right\}, \tag{7}$$

there exists a sequence $\{r_t\}\to 1^-$ such that

$$T(r_t,f_1)>(D(f_1)-\varepsilon)\log\frac{1}{1-r_t},\quad T(r_t,f_2)>(D(f_2)-\varepsilon)\log\frac{1}{1-r_t}, \tag{8}$$

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for all $t \to \infty$. From (6), (7) and (8), we have

$$(9)\qquad [(D(f_1)-\varepsilon)(A_1-\varepsilon)+(D(f_2)-\varepsilon)(A_2-\varepsilon)-2]\log\frac{1}{1-r_t} < o\left(\log\frac{1}{1-r_t}\right).$$

From (9) and ε being arbitrary, the above inequality contradicts to (3). Therefore, we complete the proof of Theorem 2.1. □

From Theorem 2.1, we obtain the following corollaries.

Corollary 2.1 *Let $a_j (j = 1, 2, \ldots, q)$ be q distinct complex numbers in $\widehat{\mathbb{C}}$, and $k_j (j = 1, 2, \ldots, q)$ be positive integers or ∞ satisfying (2), and let $f_1(z), f_2(z)$ be non-constant non-admissible functions satisfying $0 < D(f_1), D(f_2) < \infty$ and (2). Set*

$$A_1 = \sum_{j=m}^{q}\frac{k_j}{k_j+1} + \frac{(m-2)k_m}{k_m+1} - \frac{k_n}{k_n+1} - 2,$$

$$A_2 = \sum_{j=n}^{q}\frac{k_j}{k_j+1} + \frac{(n-2)k_n}{k_n+1} - \frac{k_m}{k_m+1} - 2,$$

where m and n are positive integers in $\{1, 2, \ldots, q\}$. If

$$\min\{A_1, A_2\} \geq \frac{2}{D(f_1)+D(f_2)} \quad \text{and} \quad \max\{A_1, A_2\} > \frac{2}{D(f_1)+D(f_2)},$$

then $f_1(z) \equiv f_2(z)$.

Proof: Since $\Theta(f_i) \geq 0$ $(i = 1, 2)$ and $\delta(0, f_1 - a_j) \geq 0$ $(j = 1, 2, \ldots, q)$, the assertion follows from Theorem 2.1. □

Corollary 2.2 *Let $a_j (j = 1, 2, \ldots, q)$ be q distinct complex numbers in $\widehat{\mathbb{C}}$, and $k_j (j = 1, 2, \ldots, q)$ be positive integers or ∞ satisfying (2), and let $f_1(z), f_2(z)$ be non-constant non-admissible functions satisfying $0 < D(f_1), D(f_2) < \infty$ and (2). If*

$$A = \sum_{j=m}^{q}\frac{k_j}{k_j+1} + \frac{(m-3)k_m}{k_m+1} - 2 > 0,$$

where m is a positive integers in $\{1, 2, \ldots, q\}$, then $f_1(z) \equiv f_2(z)$.

Proof: Let $n = m$, Corollary 2.2 follows immediately from Corollary 2.1. □

Corollary 2.3 *Let $a_j (j = 1, 2, \ldots, q)$ be q distinct complex numbers in $\widehat{\mathbb{C}}$, and $k_j (j = 1, 2, \ldots, q)$ be positive integers or ∞ satisfying (2), and let $f_1(z), f_2(z)$ be non-constant non-admissible functions satisfying $0 < D(f_1), D(f_2) < \infty$ and (2). If*

$$\sum_{j=3}^{q}\frac{k_j}{k_j+1} > 2 + \frac{2}{D(f_1)+D(f_2)},$$

then $f_1(z) \equiv f_2(z)$.

Remark 2.1 *From Corollary 2.3, we can get that* $f_1 \equiv f_2$ *can not hold, if two non-admissible functions* f_1, f_2 *share four distinct values* IM *and* $0 < D(f_1), D(f_2) < \infty$.

Note that $1 \geq \frac{k_1}{k_1+1} \geq \frac{k_2}{k_2+1} \geq \cdots \geq \frac{k_q}{k_q+1} \geq \frac{1}{2}$, from Corollary 2.3, we can get the following corollary immediately.

Corollary 2.4 *Let* $a_j (j = 1, 2, \ldots, q)$ *be* q *distinct complex numbers in* $\widehat{\mathbb{C}}$, *and* $k_j (j = 1, 2, \ldots, q)$ *be positive integers or* ∞ *satisfying (2), and let* $f_1(z), f_2(z)$ *be non-constant non-admissible functions satisfying* $0 < D(f_1), D(f_2) < \infty$ *and (2). Set* $D := \min\{D(f_1), D(f_2)\}$,

(*i*) *if* $D > 1$, $q = 7$ *and* $k_7 \geq 2$, *then* $f_1(z) \equiv f_2(z)$;
(*ii*) *if* $D > 1$, $q = 6$ *and* $k_6 \geq 4$, *then* $f_1(z) \equiv f_2(z)$;
(*iii*) *if* $D > 2$ *and* $q = 7$, *then* $f_1(z) \equiv f_2(z)$;
(*iv*) *if* $D > 3$, $q = 6$ *and* $k_3 \geq 2$, *then* $f_1(z) \equiv f_2(z)$;
(*v*) *if* $D > 6$, $q = 5$, $k_3 \geq 3$ *and* $k_5 \geq 2$, *then* $f_1(z) \equiv f_2(z)$;
(*vi*) *if* $D > 10$, $q = 5$ *and* $k_4 \geq 4$, *then* $f_1(z) \equiv f_2(z)$;
(*vii*) *if* $D > 12$, $q = 5$, $k_3 \geq 5$ *and* $k_4 \geq 3$, *then* $f_1(z) \equiv f_2(z)$;
(*viii*) *if* $D > 42$, $q = 5$, $k_3 \geq 6$ *and* $k_4 \geq 2$, *then* $f_1(z) \equiv f_2(z)$.

3 The problem of non-admissible function and admissible function in the unit disc $\mathbb{D}$ sharing some values

We now show that an admissible function can share sufficiently many values concerning multiple values with another non-admissible function as follows.

Theorem 3.1 *If* f_1 *is admissible and* f_2 *is a non-admissible satisfying* $\lim_{r\to 1^-} T(r, f_2) = \infty$, $a_j (j = 1, 2, \ldots, q)$ *be* q *distinct complex numbers, and let* $k_j (j = 1, 2, \ldots, q)$ *be positive integers or* ∞ *satisfying (1). Set* $A_3 = B_1 + \frac{(m-2)k_m}{k_m+1}$. *Then (2) and* $A_3 > 0$ *do not hold at same time, where* B_1 *is stated as in Section 2.*

To prove the above theorem, we require the following lemma.

Lemma 3.1 (see [11, Lemma 1]). *Let* $f(z), g(z)$ *satisfy* $\lim_{r\to 1^-} T(r, f) = \infty$ *and* $\lim_{r\to 1^-} T(r, g) = \infty$. *If there is a* $K \in (0, \infty)$ *with*

$$T(r, f) \leq KT(r, g) + S(r, f) + S(r, g),$$

then each $S(r, f)$ *is also an* $S(r, g)$.

The proof of Theorem 3.1: Suppose that (2) and $A_3 > 0$ can hold at the same time, since $f_1(z)$ is an admissible function, using the same argument as in Theorem 2.1 and from Theorem 1.1 and Lemma 3.4 in [16], for any $\varepsilon (0 < 2\varepsilon < A_3)$, we have

$$(10) \qquad \left(\frac{(m-1)k_m}{k_m+1} + B_1 - \varepsilon \right) T(r, f_1) \leq \sum_{j=1}^{q} \frac{k_m}{k_m+1} \overline{N}_{k_j)}(r, \frac{1}{f_1 - a_j}) + S(r, f_1),$$

where B_1 is stated as in section 2.

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From (2), we have

$$\begin{aligned}(11)\qquad \sum_{j=1}^{q}\overline{N}_{k_j)}(r,\frac{1}{f_1-a_j}) \le & N(r,\frac{1}{f_1-f_2}) \le T(r,\frac{1}{f_1-f_2})+O(1)\\ \le & T(r,f_1)+T(r,f_2)+O(1).\end{aligned}$$

From (10) and (11), we get

$$(12)\qquad \left(\frac{(m-2)k_m}{k_m+1}+B_1-\varepsilon\right)T(r,f_1)\le\frac{k_m}{k_m+1}T(r,f_2)+S(r,f_1).$$

Since $0<\varepsilon<A_3$, we have $\frac{(m-2)k_m}{k_m+1}+B_1-\varepsilon>0$. From (12), we have

$$(13)\qquad T(r,f_1)\le\frac{1}{A_3-\varepsilon}\frac{k_m}{k_m+1}T(r,f_2)+S(r,f_1).$$

From $\frac{1}{A_3-\varepsilon}\frac{k_m}{k_m+1}>0$ and Lemma 3.1, we can get that each $S(r,f_1)$ is also an $S(r,f_2)$. Since $f_1(z)$ is admissible and $f_2(z)$ is non-admissible, we can get $T(r,f_2)=S(r,f_1)$. Thus, we have

$$(14)\qquad T(r,f_2)=S(r,f_1)=S(r,f_2)=o(T(r,f_2)).$$

This is a contradiction. Hence, we can get that (2) and $A_3>0$ do not hold at the same time. Thus, this completes the proof of Theorem 3.1.

From Theorem 3.1, we obtain the following corollaries.

Corollary 3.1 *If f_1 is admissible and f_2 is a non-admissible satisfying* $\lim_{r\to1^-}T(r,f_2)=\infty$, *$a_j(j=1,2,\dots,q)$ be q distinct complex numbers, and let $k_j(j=1,2,\dots,q)$ be positive integers or ∞ satisfying (1). Then (2) and*

$$\sum_{j=m+1}^{q}\frac{k_j}{k_j+1}+\frac{(m-1)k_m}{k_m+1}-2>0$$

do not hold at same time.

Proof: Since $\Theta(f_i)\ge0$ $(i=1,2)$ and $\delta(0,f_1-a_j)\ge0$ $(j=1,2,\dots,q)$, the assertion follows from Theorem 3.1. □

Corollary 3.2 *If f_1 is admissible and f_2 is inadmissible satisfying* $\lim_{r\to1^-}T(r,f_2)=\infty$, *$a_j(j=1,2,\dots,q)$ be q distinct complex numbers. Then*

(i) $f_1(z),f_2(z)$ can share at most three values a_1,a_2,a_3 IM in $\mathbb{D}$;

(ii) $f_1(z),f_2(z)$ can not share the following situations in $\mathbb{D}$ when a_1 with reduced weight $k(k\ge6)$, a_2 with reduced weight 6, a_3 with reduced weight 2 and the fourth value a_4 with reduced weight 1;

(iii) $f_1(z),f_2(z)$ can not share the following situations in $\mathbb{D}$ when a_1 with reduced weight $k(k\ge3)$, a_2 with reduced weight 3 and a_3,a_4 with reduced weight 2;

(iv) $f_1(z),f_2(z)$ can not share two values a_1,a_2 in $\mathbb{D}$ with reduced weight $k(k\ge2)$, and the other three values a_3,a_4,a_5 in $\mathbb{D}$ with reduced weight 1;

(v) $f_1(z),f_2(z)$ can share at most five values $a_j(j=1,2,\dots,5)$ in $\mathbb{D}$ with reduced weight 1.

Proof: (i) Suppose that $f_1(z), f_2(z)$ share four values $a_j(j = 1, 2, 3, 4)$ IM, that is, $k_j = \infty(j = 1, 2, 3, 4)$. Since $f_1(z)$ is admissible, from Theorem 1.1, we have

$$2T(r, f_1) \le \sum_{j=1}^{4} N\left(r, \frac{1}{f_1 - a_j}\right) + S(r, f_1). \tag{15}$$

From the assumptions of Corollary 3.2(i), we have

$$\sum_{j=1}^{4} N\left(r, \frac{1}{f_1 - a_j}\right) \le N\left(r, \frac{1}{f_1 - f_2}\right) \le T(r, f_1) + T(r, f_2) + O(1). \tag{16}$$

From (15) and (16), we have

$$T(r, f_1) \le T(r, f_2) + S(r, f_1). \tag{17}$$

By Lemma 3.1 and similar proof of Theorem 3.1, we have

$$T(r, f_2) = S(r, f_1) = S(r, f_2) = o(T(r, f_2)).$$

From this and $\lim_{r\to 1^-} T(r, f_2) = \infty$, we can get a contradiction.

Thus, this completes the proof of Corollary 3.2(i).

By similar proof of Corollary 3.2 (i), we can prove (ii), (iii) and (iv) of Corollary 3.2 easily. Here we omit the detail.

(v) Suppose that f_1, f_2 share six values $a_j(j = 1, 2, \dots, 6)$ with reduced weight 1, that is,

$$\overline{E}_{1)}(a_j, \mathbb{D}, f_1) = \overline{E}_{1)}(a_j, \mathbb{D}, f_2), \quad (j = 1, 2, \dots, 6),$$

and $k_1 = k_2 = \cdots = k_6 = 1$. Then, we can deduce that

$$\sum_{j=2}^{6} \frac{k_j}{k_j + 1} - 2 = 5 \times \frac{1}{2} - 2 = \frac{1}{2} > 0.$$

From this and the conclusion of Theorem 3.1, we get a contradiction.

Thus, this completes the proof of Corollary 3.2. □

4 Some Remarks

It is very interesting to consider distinct small functions instead of distinct complex numbers. Li and Qiao[8] proved that Nevanlinna's five values theorem is also true for five meromorphic functions a_j $(j = 1, 2, 3, 4, 5)$ on $\mathbb{C}$ which satisfy $T(R, a_j) = o(T(R, f_1) + T(R, f_2))$ as $R \to \infty$. For some further results related to small functions, we refer to Yao[15], Yi[18], Thai and Tan[10], Cao and Yi[1, 3]. *Thus it may be interesting to consider the substitute of distinct complex numbers by distinct small functions within the results of this paper.*

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Global Dynamics of a Certain Two-dimensional Competitive System of Rational Difference Equations with Quadratic Terms

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Abstract. We investigate global dynamics of the following system of difference equations

$$\begin{cases} x_{n+1} &= \frac{x_n^2}{a+y_n^2} \\ y_{n+1} &= \frac{y_n^2}{b+x_n^2} \end{cases}, \quad n=0,1,2,\ldots$$

where the parameters a, b are positive numbers and the initial conditions x_0, y_0 are arbitrary nonnegative numbers. We show that this system has an interesting dynamics which depends on the part of parametric space. The obtained ynamics is very different than the dynamics of the corresponding linear fractional system. In particular, we show that the system always exhibit Allee's effect.

Keywords. Allee effect, Basin of attraction, competitive map, global stable manifold, monotonicity.

AMS 2000 Mathematics Subject Classification: Primary: 39A10, 39A11 Secondary: 37E99, 37D10.

1 Introduction

In this paper we study global dynamics of the following rational system of difference equations

$$\begin{cases} x_{n+1} &= \frac{x_n^2}{a+y_n^2} \\ y_{n+1} &= \frac{y_n^2}{b+x_n^2} \end{cases}, \quad n=0,1,2,\ldots \tag{1}$$

where the parameters a, b are positive numbers and initial conditions x_0, y_0 are arbitrary nonnegative numbers.

System (1) is a competitive system, and our results are based on recent results about competitive systems in the plane, see [17, 18]. System (1) can be used as a mathematical model for competition in population dynamics.

The related competitive systems of the form

$$\begin{cases} x_{n+1} &= \frac{x_n}{a+y_n} \\ y_{n+1} &= \frac{y_n}{b+x_n} \end{cases}, \quad n=0,1,2,\ldots \tag{2}$$

and

$$\begin{cases} x_{n+1} &= \frac{x_n}{a+y_n^2} \\ y_{n+1} &= \frac{y_n}{b+x_n^2} \end{cases}, \quad n=0,1,2,\ldots \tag{3}$$

were considered in [4, 6, 7] and global dynamics was derived. Systems (2) and (3) have similar dynamics which depends on the part of parametric space. In general, there are nine dynamic scenarios for systems (2) and (3) depending on the region of parametric space. Dynamics of System (1) is quite different and there are only three dynamic scenarios which describe the global behavior of

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 19, NO.1, 2015, COPYRIGHT 2015 EUDOXUS PRESS, LLC

this system given by our main result. The common feature of all three scenarios is the appearance of the so called Allee's effect, which means that in each case the zero equilibrium has a substantial basin of attraction which contains an open set, see [22] for related competitive system. This feature was not present in any of nine dynamic scenarios for systems (2) and (3). In addition, the non-hyperbolic case for System (1) is characterized by the existence of the unique interior non-hyperbolic equilibrium, while in the case of systems (2) and (3), non-hyperbolic equilibrium points were always all points on one or both coordinate axes.

Our main result for System (1) is the following theorem.

Theorem 1 *All solutions of System* (1) *are eventually componentwise monotone.*

(a) Assume that System (1) *has no interior equilibrium point, that is assume that* $D(a,b) > 0$, *where*

$$\begin{aligned} D(a,b) = \quad & 16b + \tfrac{128}{27}b^3\left(\tfrac{1}{9}\sqrt{3}\sqrt{64b^3+27}+1\right)^{\frac{2}{3}} + \tfrac{16}{3}b^2\left(\tfrac{1}{9}\sqrt{3}\sqrt{64b^3+27}+1\right)^{\frac{4}{3}} \\ & -8\left(\tfrac{1}{9}\sqrt{3}\sqrt{64b^3+27}+1\right)^{\frac{5}{3}} + \left(\tfrac{1}{9}\sqrt{3}\sqrt{64b^3+27}+1\right)^{\frac{8}{3}} \\ & +16a\left(\tfrac{1}{9}\sqrt{3}\sqrt{64b^3+27}+1\right)^{\frac{4}{3}} + \tfrac{256}{27}b^4 + \tfrac{16}{9}\sqrt{3}b\sqrt{64b^3+27}. \end{aligned}$$

Then there exist the sets $\mathcal{W}^s(E_x)$ *and* $\mathcal{W}^s(E_y)$ *which are the graphs of a strictly non-decreasing continuous functions of the first variable on* $[0,\infty)$ *(and so are manifolds) with the following properties:*

$$\begin{aligned} \mathcal{B}(E_0) &= \{(x_0,y_0) : \mathcal{W}^s(E_y) \preceq_{se} (x_0,y_0) \preceq_{se} \mathcal{W}^s(E_x)\}, \\ \mathcal{B}((\infty,0)) &= \{(x_0,y_0) : \mathcal{W}^s(E_x) \preceq_{se} (x_0,y_0)\}, \\ \mathcal{B}((0,\infty)) &= \{(x_0,y_0) : (x_0,y_0) \preceq_{se} \mathcal{W}^s(E_y)\}, \\ \mathcal{B}(E_x) &= \mathcal{W}^s(E_x), \quad \mathcal{B}(E_y) = \mathcal{W}^s(E_y), \end{aligned}$$

where $\preceq_{se}$ *denotes south-east ordering defined by* (5) *below.*

(b) Assume that System (1) *has one interior equilibrium point* E, *that is assume that the condition* $4\bar{x}\bar{y} = 1$ *or* $D(a,b) = 0$ *holds. Then* E *is the non-hyperbolic equilibrium and there are two continuous non-decreasing curves* $\mathcal{W}^s(E_x)$ *and* $\mathcal{W}^s(E_y)$ *emanating from* E_x *and* E_y *respectively, with end points at* E. *Furthermore, there are continuous non-decreasing curves* C_u *and* C_l *(which may coincide) emanating from* E *such that the boundaries of the basins of attraction of* $(\infty,0)$ *and* $(0,\infty)$ *are given as:*

$$\partial\mathcal{B}((\infty,0)) = C_l \cup \mathcal{W}^s(E_x), \quad \partial\mathcal{B}((0,\infty)) = C_u \cup \mathcal{W}^s(E_y).$$

The basins of attraction of different equilibrium points and the boundary points are given as:

$$\begin{aligned} \mathcal{B}(E_0) &= \{(x_0,y_0) : \mathcal{W}^s(E_y) \preceq_{se} (x_0,y_0) \preceq_{se} \mathcal{W}^s(E_x)\}, \\ \mathcal{B}((\infty,0)) &= \{(x_0,y_0) : \partial\mathcal{B}((\infty,0)) \preceq_{se} (x_0,y_0)\}, \\ \mathcal{B}((0,\infty)) &= \{(x_0,y_0) : (x_0,y_0) \preceq_{se} \partial\mathcal{B}((0,\infty))\}, \\ \mathcal{B}(E) &= \{(x_0,y_0) : C_u \preceq_{se} (x_0,y_0) \preceq_{se} C_l\}. \end{aligned}$$

(c) Assume that System (1) *has two interior equilibrium points* E_+ *and* E_-, *that is assume that* $D(a,b) < 0$. *Then* E_- *is a repeller and* E_+ *is a saddle point. There exist three continuous non-decreasing curves* $\mathcal{W}^s(E_x)$, $\mathcal{W}^s(E_y)$ *and* $\mathcal{W}^s(E_+)$ *emanating from* E_x, E_y, *and* E_+ *respectively, with end points at* E_-. *The basins of attraction* $\mathcal{B}(E_0), \mathcal{B}(E_x)$ *and* $\mathcal{B}(E_y)$ *are as in (a) and* $\mathcal{B}((\infty,0))$ *and* $\mathcal{B}((0,\infty))$ *are as in (b). Furthermore,*

$$\partial\mathcal{B}((\infty,0)) = \mathcal{W}^s(E_x) \cup \mathcal{W}^s(E_+), \quad \partial\mathcal{B}((0,\infty)) = \mathcal{W}^s(E_y) \cup \mathcal{W}^s(E_+).$$

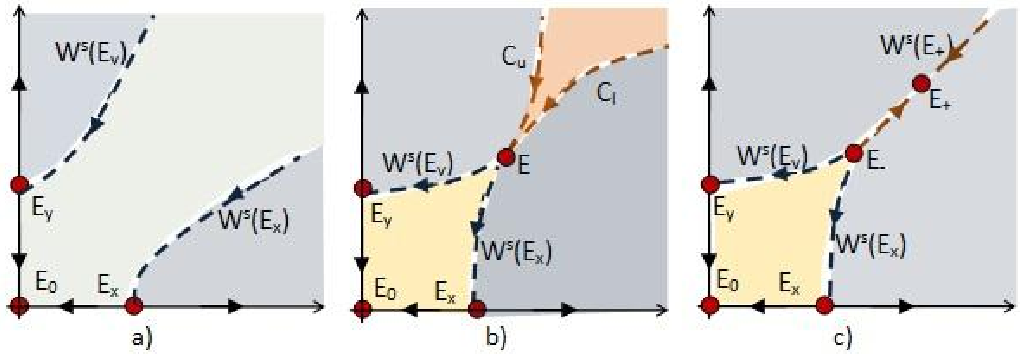

Figure 1: Visual illustration of Theorem 1.

2 Preliminaries

A first order system of difference equations

$$\left\{\begin{array}{lcl} x_{n+1} & = & f(x_n, y_n) \\ y_{n+1} & = & g(x_n, y_n) \end{array}\right. , \quad n = 0, 1, 2, \ldots \tag{4}$$

where $\mathcal{S} \subset \mathbb{R}^2$, $(f, g) : \mathcal{S} \to \mathcal{S}$, f, g are continuous functions is *competitive* if $f(x, y)$ is non-decreasing in x and non-increasing in y, and $g(x, y)$ is non-increasing in x and non-decreasing in y. If both f and g are nondecreasing in x and y, the system (4) is *cooperative*. Competitive and cooperative maps are defined similarly. *Strongly competitive* systems of difference equations or strongly competitive maps are those for which the functions f and g are coordinate-wise strictly monotone.

Competitive and cooperative systems have been investigated by many authors, see [7, 8, 13, 15, 16, 17, 18, 20, 21, 23, 24]. Special attention to discrete competitive and cooperative systems in the plane was given in [1, 2, 7, 8, 9, 16, 17, 18, 20, 24]. One of the reasons for paying special attention to two-dimensional discrete competitive and cooperative systems is their applicability and the fact that many examples of mathematical models in biology and economy which involve competition or cooperation are models which involve two species. Another reason is that the theory of two-dimensional discrete competitive and cooperative systems is very well developed, unlike such theory for three and higher dimensional systems. Part of the reason for this situation is de Mottoni-Schiaffino theorem given below, which provides relatively simple scenarios for possible behavior of many two-dimensional discrete competitive and cooperative systems. However, this does not mean that one can not encounter chaos in such systems as has been shown by H. Smith, see [24].

Here we give some basic notions about monotonic maps in plane.

We define a *partial order* $\preceq_{se}$ on $\mathbf{R}^2$ (so-called south-east ordering) so that the positive cone is the fourth quadrant, i.e. this partial order is defined by:

$$\begin{pmatrix} x^1 \\ y^1 \end{pmatrix} \preceq_{se} \begin{pmatrix} x^2 \\ y^2 \end{pmatrix} \Leftrightarrow \left\{ \begin{array}{l} x^1 \leqslant x^2 \\ y^1 \geqslant y^2. \end{array}\right. \tag{5}$$

Similarly, we define the north-east ordering as:

$$\begin{pmatrix} x^1 \\ y^1 \end{pmatrix} \preceq_{ne} \begin{pmatrix} x^2 \\ y^2 \end{pmatrix} \Leftrightarrow \left\{ \begin{array}{l} x^1 \leqslant x^2 \\ y^1 \leqslant y^2. \end{array}\right. \tag{6}$$

A map F is called *competitive* if it is non-decreasing with respect to $\preceq_{se}$, that is, if the following holds:

$$\begin{pmatrix} x^1 \\ y^1 \end{pmatrix} \preceq \begin{pmatrix} x^2 \\ y^2 \end{pmatrix} \Rightarrow F\begin{pmatrix} x^1 \\ y^1 \end{pmatrix} \preceq F\begin{pmatrix} x^2 \\ y^2 \end{pmatrix}. \tag{7}$$

For each $\mathbf{v} = (v^1, v^2) \in \mathbf{R}_+^2$, define $\mathcal{Q}_i(\mathbf{v})$ for $i = 1, .., 4$ to be the usual four quadrants based at v and numbered in a counterclockwise direction, e.g., $\mathcal{Q}_1(\mathbf{v}) = \{(x, y) \in \mathbf{R}_+^2 : v^1 \leq x, v^2 \leq y\}$.

For $S \subset R_+^2$ let S° denote the *interior* of S.

The following definition is from [24].

Definition 1 *Let R be a nonempty subset of $\mathbb{R}^2$. A competitive map $T : R \to R$ is said to satisfy condition (O+) if for every x, y in R, $T(x) \preceq_{ne} T(y)$ implies $x \preceq_{ne} y$, and T is said to satisfy condition (O−) if for every x, y in R, $T(x) \preceq_{ne} T(y)$ implies $y \preceq_{ne} x$.*

The following theorem was proved by DeMottoni-Schiaffino [9] for the Poincaré map of a periodic competitive Lotka-Volterra system of differential equations. Smith generalized the proof to competitive and cooperative maps [23].

Theorem 2 *Let R be a nonempty subset of $\mathbb{R}^2$. If T is a competitive map for which (O+) holds then for all $x \in R$, $\{T^n(x)\}$ is eventually componentwise monotone. If the orbit of x has compact closure, then it converges to a fixed point of T. If instead (O−) holds, then for all $x \in R$, $\{T^{2n}\}$ is eventually componentwise monotone. If the orbit of x has compact closure in R, then its omega limit set is either a period-two orbit or a fixed point.*

It is well known that a stable period-two orbit and a stable fixed point may coexist, see Hess [12].

The following result is from [24], with the domain of the map specialized to be the cartesian product of intervals of real numbers. It gives a sufficient condition for conditions $(O+)$ and $(O-)$.

Theorem 3 *Let $R \subset \mathbb{R}^2$ be the cartesian product of two intervals in $\mathbb{R}$. Let $T : R \to R$ be a C' competitive map. If T is injective and* $\det J_T(x) > 0$ *for all $x \in R$ then T satisfies (O+). If T is injective and* $\det J_T(x) < 0$ *for all $x \in R$ then T satisfies (O−).*

Theorems 2 and 3 are quite applicable as we have shown in [3, 4, 5, 17, 18, 20], in the case of competitive systems in the plane consisting of linear fractional equations.

The following result is from [18], which generalizes the corresponding result for hyperbolic case from [17]. Related results have been obtained by H. L. Smith in [23].

Theorem 4 *Let $\mathcal{R}$ be a rectangular subset of $\mathbb{R}^2$ and let T be a competitive map on $\mathcal{R}$. Let $\overline{x} \in \mathcal{R}$ be a fixed point of T such that $(\mathcal{Q}_1(\overline{x}) \cup \mathcal{Q}_3(\overline{x})) \cap \mathcal{R}$ has nonempty interior (i.e., $\overline{x}$ is not the NW or SE vertex of $\mathcal{R}$).*

Suppose that the following statements are true.

a. *The map T is strongly competitive on* $\text{int}((\mathcal{Q}_1(\overline{x}) \cup \mathcal{Q}_3(\overline{x})) \cap \mathcal{R})$.

b. *T is C^2 on a relative neighborhood of $\overline{x}$.*

c. *The jacobian of T at $\overline{x}$ has real eigenvalues λ, μ such that $|\lambda| < \mu$, where λ is stable and the eigenspace E^λ associated with λ is not a coordinate axis.*

d. *Either $\lambda \geq 0$ and*

$$T(x) \neq \overline{x} \quad \text{and} \quad T(x) \neq x \quad \text{for all } x \in \text{int}((\mathcal{Q}_1(\overline{x}) \cup \mathcal{Q}_3(\overline{x})) \cap \mathcal{R}),$$

or $\lambda < 0$ and

$$T^2(x) \neq x \quad \text{for all } x \in \text{int}((\mathcal{Q}_1(\overline{x}) \cup \mathcal{Q}_3(\overline{x})) \cap \mathcal{R})$$

Then there exists a curve $\mathcal{C}$ in $\mathcal{R}$ such that

(i) *$\mathcal{C}$ is invariant and a subset of $\mathcal{W}^s(\overline{x})$.*

(ii) *the endpoints of $\mathcal{C}$ lie on $\partial\mathcal{R}$.*

(iii) $\overline{x} \in \mathcal{C}$.

(iv) $\mathcal{C}$ *the graph of a strictly increasing continuous function of the first variable,*

(v) $\mathcal{C}$ *is differentiable at* $\overline{x}$ *if* $\overline{x} \in \mathrm{int}(\mathcal{R})$ *or one sided differentiable if* $\overline{x} \in \partial\mathcal{R}$*, and in all cases* $\mathcal{C}$ *is tangential to* E^{λ} *at* $\overline{x}$*,*

(vi) $\mathcal{C}$ *separates* $\mathcal{R}$ *into two connected components, namely*

$$\mathcal{W}_{-} := \{x \in \mathcal{R} : \exists y \in \mathcal{C} \textit{ with } x \preceq y\}$$

and

$$\mathcal{W}_{+} := \{x \in \mathcal{R} : \exists y \in \mathcal{C} \textit{ with } y \preceq x\}$$

(vii) $\mathcal{W}_{-}$ *is invariant, and* $\mathrm{dist}(T^n(x), \mathcal{Q}_2(\overline{x})) \to 0$ *as* $n \to \infty$ *for every* $x \in \mathcal{W}_{-}$.

(viii) $\mathcal{W}_{+}$ *is invariant, and* $\mathrm{dist}(T^n(x), \mathcal{Q}_4(\overline{x})) \to 0$ *as* $n \to \infty$ *for every* $x \in \mathcal{W}_{+}$.

The following result is a direct consequence of the Trichotomy Theorem of Dancer and Hess, see [17] and [12], and is helpful for determining the basins of attraction of the equilibrium points.

Corollary 1 *If the nonnegative cone of* $\preceq$ *is a generalized quadrant in* $\mathbb{R}^n$*, and if* T *has no fixed points in the ordered interval* $I(u_1, u_2) = \{u : u_1 \preceq_{se} u \preceq_{se} u_2\}$ *other than* u_1 *and* u_2*, then the interior of* $I(u_1, u_2)$ *is either a subset of the basin of attraction of* u_1 *or a subset of the basin of attraction of* u_2*.*

The next results gives the existence and uniqueness of invariant curves emanating from a *non-hyperbolic point of unstable type*, that is a non-hyperbolic point where second eigenvalue is outside interval $[-1, 1]$. See [19].

Theorem 5 *Let* $\mathcal{R} = (a_1, a_2) \times (b_1, b_2)$*, and let* $T : \mathcal{R} \to \mathcal{R}$ *be a strongly competitive map with a unique fixed point* $\bar{\mathbf{x}} \in \mathcal{R}$*, and such that* T *is continuously differentiable in a neighborhood of* $\bar{\mathbf{x}}$*. Assume further that at the point* $\bar{\mathbf{x}}$ *the map* T *has associated characteristic values* μ *and* ν *satisfying* $1 < \mu$ *and* $-\mu < \nu < \mu$*.*

Then there exist curves $\mathcal{C}_1$*,* $\mathcal{C}_2$ *in* $\mathcal{R}$ *and there exist* $\mathbf{p}_1, \mathbf{p}_2 \in \partial\mathcal{R}$ *with* $\mathbf{p}_1 <<_{se} \bar{\mathbf{x}} <<_{se} \mathbf{p}_2$ *such that*

(i) *For* $\ell = 1, 2$*,* $\mathcal{C}_\ell$ *is invariant, north-east strongly linearly ordered, such that* $\bar{\mathbf{x}} \in \mathcal{C}_\ell$ *and* $\mathcal{C}_\ell \subset \mathcal{Q}_3(\bar{\mathbf{x}}) \cup \mathcal{Q}_1(\bar{\mathbf{x}})$*; the endpoints* $\mathbf{q}_\ell$*,* $\mathbf{r}_\ell$ *of* $\mathcal{C}_\ell$*, where* $\mathbf{q}_\ell \preceq_{ne} \mathbf{r}_\ell$*, belong to the boundary of* $\mathcal{R}$*. For* $\ell, j \in \{1, 2\}$ *with* $\ell \neq j$*,* $\mathcal{C}_\ell$ *is a subset of the closure of one of the components of* $\mathcal{R} \setminus \mathcal{C}_j$*. Both* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *are tangential at* $\bar{\mathbf{x}}$ *to the eigenspace associated with* ν*.*

(ii) *For* $\ell = 1, 2$*, let* B_ℓ *be the component of* $\mathcal{R} \setminus \mathcal{C}_\ell$ *whose closure contains* $\mathbf{p}_\ell$*. Then* B_ℓ *is invariant. Also, for* $\mathbf{x} \in B_1$*,* $T^n(\mathbf{x})$ *accumulates on* $\mathcal{Q}_2(\mathbf{p_1}) \cap \partial\mathcal{R}$*, and for* $x \in B_2$*,* $T^n(\mathbf{x})$ *accumulates on* $\mathcal{Q}_4(\mathbf{p_2}) \cap \partial\mathcal{R}$*.*

(iii) *Let* $\mathcal{D}_1 := \mathcal{Q}_1(\bar{\mathbf{x}}) \cap \mathcal{R} \setminus (\mathcal{B}_1 \cup \mathcal{B}_2)$ *and* $\mathcal{D}_2 := \mathcal{Q}_3(\bar{\mathbf{x}}) \cap \mathcal{R} \setminus (\mathcal{B}_1 \cup \mathcal{B}_2)$*.*
Then $\mathcal{D}_1 \cup \mathcal{D}_2$ *is invariant.*

Corollary 2 *Let a map* T *with fixed point* $\bar{\mathbf{x}}$ *be as in Theorem 5. Let* $\mathcal{D}_1$*,* $\mathcal{D}_2$ *be the sets as in Theorem 5. If* T *satisfies* (O_+)*, then for* $\ell = 1, 2$*,* $\mathcal{D}_\ell$ *is invariant, and for every* $\mathbf{x} \in \mathcal{D}_\ell$*, the iterates* $T^n(\mathbf{x})$ *converge to* $\bar{\mathbf{x}}$ *or to a point of* $\partial\mathcal{R}$*. If* T *satisfies* (O_-)*, then* $T(\mathcal{D}_1) \subset \mathcal{D}_2$ *and* $T(\mathcal{D}_2) \subset \mathcal{D}_1$*. For every* $\mathbf{x} \in \mathcal{D}_1 \cup \mathcal{D}_2$*, the iterates* $T^n(\mathbf{x})$ *either converge to* $\bar{\mathbf{x}}$*, or converge to a period-two point, or to a point of* $\partial\mathcal{R}$*.*

3 Linearized Stability Analysis

The equilibrium points of System (1) satisfy the following system of equations:

$$\overline{x} = \frac{\overline{x}^2}{a+\overline{y}^2}, \quad \overline{y} = \frac{\overline{y}^2}{b+\overline{y}^2}. \tag{8}$$

All solutions of System (8) with at least one zero component are given as: $E_0(0,0), E_x(a,0)$, and $E_y(0,b)$.

The equilibrium points with strictly positive coordinates satisfy the following system of equations:

$$(E1): \ \ a+y^2-x=0, \quad (E2): \ \ b+x^2-y=0. \tag{9}$$

By eliminating variable y from system (9) we obtain that x satisfies the following quartic equation

$$x^4+2bx^2-x+a+b^2=0. \tag{10}$$

Left-hand side of (10) is a quartic polynomial which coefficients have two changes of sign. Consequently, by Descartes' rule of sign Eq.(10) have either $0,1$ or 2 positive roots and no negative roots. More precisely if we set

$$f(x)=x^4+2bx^2-x+a+b^2,$$

then $f'(x)=4x^3+4bx-1, f''(x)=12x^2+4b$ and $f(0)=a+b^2>0$ and $f(\infty)=\infty$. The polynomial $f(x)$ has a single minimum at $[0,\infty)$ attained at the point

$$x_{\min} = \sqrt[3]{\sqrt{\frac{1}{27}b^3+\frac{1}{64}}+\frac{1}{8}} - \frac{1}{3}\frac{b}{\sqrt[3]{\sqrt{\frac{1}{27}b^3+\frac{1}{64}}+\frac{1}{8}}}.$$

Now, $f(x)$ has $0,1$, or 2 positive equilibrium points if and only if $f(x_{\min})>0, f(x_{\min})=0$, or $f(x_{\min})<0$, which is equivalent to the conditions that $D(a,b)>0, D(a,b)=0$, or $D(a,b)<0$.

These equilibrium points are denoted as E, in the case of one equilibrium point, and E_+ and E_-, where $E_- \preceq_{ne} E_+$, in the case of two equilibrium points. In view of (9) the interior equilibrium points are intersection of two equilibrium curves $(E1)$ and $(E2)$, which are two orthogonal parabolas. It is clear from Figure 3 that the slope of $(E1)$ at E_- is greater than the slope of $(E2)$ at E_-, that is

$$m_{E1}(E_-)=\frac{1}{2\bar{y}} > m_{E2}(E_-)=2\bar{x},$$

which is equivalent to $4\bar{x}\bar{y}<1$. Similarly, the slope of $(E1)$ at E_+ is smaller than the slope of $(E2)$ at E_+, which is quivalent to $4\bar{x}\bar{y}>1$. If two equlibrium points E_+ and E_- coincide at E then $m_{E1}(E_-)=\frac{1}{2\bar{y}}=m_{E2}(E_-)=2\bar{x}$, which is equivalent to $4\bar{x}\bar{y}=1$.

The map associated with the System (1) has the form:

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{x^2}{a+y^2} \\ \frac{y^2}{b+y^2} \end{pmatrix}.$$

The Jacobian matrix of T is

$$J_T(x,y) = \begin{pmatrix} \frac{2x}{a+y^2} & -\frac{2x^2y}{(a+y^2)^2} \\ -\frac{2xy^2}{(b+x^2)^2} & \frac{2y}{b+x^2} \end{pmatrix}. \tag{11}$$

The Jacobian matrices of T evaluated in E_0, E_x, E_y are given as

$$J_T(E_0)=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad J_T(E_x)=\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \quad J_T(E_x)=\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}.$$

By using the equilibrium equation, we obtain that the Jacobian matrix of T evaluated in an equilibrium $E(\bar{x},\bar{y})$ with positive coordinates has the form:

$$J_T(E)=\begin{pmatrix} 2 & -2\bar{y} \\ -2\bar{x} & 2 \end{pmatrix}. \tag{12}$$

The characteristic equation of (12) is given as

$$\lambda^2-4\lambda+4-4\bar{x}\bar{y}=0$$

and the eigenvalues of (12) are

$$\lambda_{\pm}=2\pm 2\sqrt{\bar{x}\bar{y}}. \tag{13}$$

Theorem 6 *System* (1) *has always three equilibrium points* $E_0(0,0), E_x(a,0), E_y(0,b)$, *where* E_0 *is locally asymptotically stable and* $E_x(a,0)$ *and* $E_y(0,b)$ *are saddle points. In addition, System* (1) *has exactly one interior equilibrium point* $E(\bar{x},\bar{y})$, *when* $4\bar{x}\bar{y}=1$, *which is the non-hyperbolic equilibrium point of unstable type. If* $4\bar{x}\bar{y}>1$, *then System* (1) *has exactly two interior equilibrium points* $E_-(\bar{x}_-,\bar{y}_-)$, *which is repeller and* $E_+(\bar{x}_+,\bar{y}_+)$, *which is a saddle point.*

Proof. The local character of the equilibrium points $E_0(0,0), E_x(a,0), E_y(0,b)$ is obvious. In view of

$$x_{n+1}y_{n+1}=\frac{x_n^2}{a+y_n^2}\frac{y_n^2}{b+x_n^2}\le 1, \quad n=0,1,\dots$$

we conclude that $x_ny_n\le 1$ for $n=1,2,\dots$ and so $\bar{x}\bar{y}\le 1$. In view of Eq.(13) we conclude that $\lambda_+>2$ and $\lambda_-\in[0,1)$ if and only if $4\bar{x}\bar{y}>1$, which shows that $E_+(\bar{x}_+,\bar{y}_+)$ is a saddle point. Furthermore, $\lambda_->1$ if and only if $4\bar{x}\bar{y}<1$, which shows that $E_-(\bar{x}_-,\bar{y}_-)$ is a repeller. Finally, $\lambda_-=1$ if and only if $4\bar{x}\bar{y}=1$, in which case E is a non-hyperbolic point. □

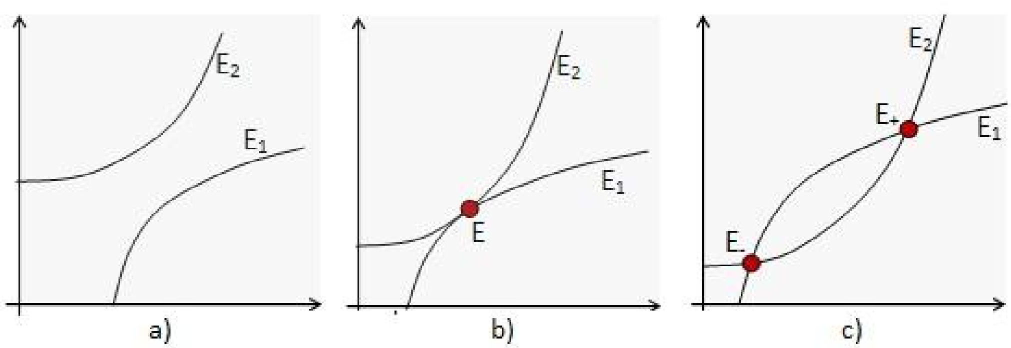

Figure 2: Visual illustration of Theorem 6.

4 Global Dynamics

In this section we prove that the map which corresponds to System (1) is injective and it satisfies (O^+) condition. We will also give the global dynamics of the map on the coordinate axes.

Theorem 7 *The map* T *which corresponds to System* (1) *is injective.*

Proof. Indeed,

$$T\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = T\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \Leftrightarrow \begin{pmatrix} \frac{x_1^2}{a+y_1^2} \\ \frac{y_1^2}{b+x_1^2} \end{pmatrix} = \begin{pmatrix} \frac{x_2^2}{a+y_2^2} \\ \frac{y_2^2}{b+x_2^2} \end{pmatrix}$$

which is equivalent to

$$a(x_1^2 - x_2^2) = -b(y_1^2 - y_2^2) = x_2^2 y_1^2 - x_1^2 y_2^2,$$

which immediately implies that $x_1 = x_2$ and $y_1 = y_2$. □

Theorem 8 *The map T which corresponds to System (1) satisfies (O^+) condition. All solutions of System (1) are eventually componentwise monotonic. All bounded solutions converge to an equilibrium point.*

Proof.

Assume that

$$T\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \leq_{ne} T\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{x_1^2}{a+y_1^2} \\ \frac{y_1^2}{b+x_1^2} \end{pmatrix} \leq_{ne} \begin{pmatrix} \frac{x_2^2}{a+y_2^2} \\ \frac{y_2^2}{b+x_2^2} \end{pmatrix}.$$

The last inequality is equivalent to

$$\begin{aligned} a(x_1^2 - x_2^2) &\leq x_2^2 y_1^2 - x_1^2 y_2^2 \\ b(y_1^2 - y_2^2) &\leq x_1^2 y_2^2 - x_2^2 y_1^2. \end{aligned} \tag{14}$$

Suppose $x_2 < x_1$. Then $y_1^2 x_2^2 > y_2^2 x_1^2$, which implies $y_1 > y_2$. On the other hand, in view of (14) we obtain $b(y_1^2 - y_2^2) < 0$ and so $y_1 < y_2$, which is a contradiction.

Suppose $x_2 = x_1$. Then $y_1 \geq y_2$, which in view of (14) implies $y_1 = y_2$.

Suppose $x_2 > x_1$. If $y_2 < y_1$ then $x_2^2 > x_1^2, y_2^2 < y_1^2$ and so $y_1^2 x_2^2 > y_2^2 x_1^2$, which in view of (14) implies $y_1 < y_2$, which is a contradiction. Consequently, $y_1 \leq y_2$ and so in all cases

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \leq_{ne} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}.$$

Thus we conclude that all solutions of (1) are eventually componentwise monotonic for all values of parameters.

□

Theorem 9 *The positive parts of x-axis and y-axis are unstable manifolds of the equilibrium points E_x and E_y respectively.*

Proof. Since $x_0 = 0$ (resp. $y_0 = 0$) implies $x_n = 0$ (resp. $y_n = 0$) for every $n \geq 1$ we conclude that the positive parts of x-axis and y-axis are invariant sets. Furthermore $x_0 = 0$ (resp. $y_0 = 0$) implies $y_{n+1} = \frac{y_n^2}{b}$ (resp. $x_{n+1} = \frac{x_n^2}{a}$) for every $n \geq 1$ and so $y_n = \frac{y_0^{2^n}}{b^{2^n - 1}}$ (resp. $x_n = \frac{x_0^{2^n}}{a^{2^n - 1}}$). Thus we have that $x_0 < a$ ($x_0 > a$ or $x_0 = a$) gives $x_n \to 0$ ($x_n \to \infty$ or $x_n = a$) as $n \to \infty$. Similarly $y_0 < b$ ($y_0 > b$ or $y_0 = b$) gives $y_n \to 0$ ($y_n \to \infty$ or $y_n = b$) as $n \to \infty$.

□

Next we present the proof of Theorem 1.

Proof. of Theorem 1 The eventual componentwise monotonicity of solutions of System (1) follows from Theorem 8.

(a) Local stability of all equilibrium points follows from Theorem 6.

The existence and the properties of the global stable manifolds $\mathcal{W}^s(E_x), \mathcal{W}^s(E_y)$ are guaranteed by Theorems 4 and 7. Thus, the regions $\mathcal{B}((\infty, 0))$ and $\mathcal{B}((0, \infty))$ are invariant and in view of Theorem 8 every solution is eventually monotonic. Consequently, in view of Theorem 4 and uniqueness of stable manifold every solution which starts in $\mathcal{B}((\infty, 0))$ (resp. $\mathcal{B}((0, \infty))$) is asymptotic to $(\infty, 0)$ (resp. $(0, \infty)$).

Let (x_0, y_0) be an arbitrary initial point between $\mathcal{W}^s(E_x)$ and $\mathcal{W}^s(E_y)$. First, assume that $x_0 > \bar{x}$.

Then $(x_0, y_{W_y}) \preceq_{se} (x_0, y_0) \preceq_{se} (x_0, y_{W_x})$, where $(x_0, y_{W_y}) \in \mathcal{W}^s(E_y), (x_0, y_{W_x}) \in \mathcal{W}^s(E_x)$, and so $T^n(x_0, y_{W_y}) \preceq_{se} T^n(x_0, y_0) \preceq_{se} T^n(x_0, y_{W_x})$. Since $T^n(x_0, y_{W_y}) \to E_y$ and $T^n(x_0, y_{W_x}) \to E_x$ as $n \to \infty$, we conclude that $T^n(x_0, y_0)$ eventually enters the ordered interval $I(E_y, E_x)$. Thus there exists $N > 0$ such that $T^N(x_0, y_0) = (x_N, y_N) \in intI(E_y, E_x)$. Then $(0, y_N) \preceq_{se} (x_N, y_N) \preceq_{se} (x_N, 0)$ and so $T^k(0, y_N) \preceq_{se} T^k(x_N, y_N) \preceq_{se} T^k(x_N, 0)$ for $k \geq N$, which implies that $E_0 = \lim_{k\to\infty} T^k(0, y_N) \preceq_{se} \lim_{k\to\infty} T^k(x_N, y_N) \preceq_{se} \lim_{k\to\infty} T^k(x_N, 0) = E_0$ and so $\lim_{k\to\infty} T^k(x_0, y_0) = E_0$.

Second, assume that $x_0 = \bar{x}$ and $y_0 > 0$. Then, by strong monotonicity of $T, x_1 < x_0$. Now, $(x_1, y_{W_y}) \preceq_{se} (x_1, y_1) \preceq_{se} (x_1, 0)$, where $(x_1, y_{W_y}) \in \mathcal{W}^s(E_y)$, which by monotonicity of T, implies that $T^n(x_1, y_{W_y}) \preceq_{se} T^n(x_1, y_1) \preceq_{se} T^n(x_1, 0)$. In a similar way as in the proof of the first case we show that $T^n(x_1, y_1)$ eventually enters the ordered interval $I(E_y, E_x)$, in which case it converges to E_0.

Third, assume that $x_0 < \bar{x}$ and $y_0 > 0$. Then, $(x_0, y_{W_y}) \preceq_{se} (x_0, y_0) \preceq_{se} (x_0, 0)$, where $(x_0, y_{W_y}) \in \mathcal{W}^s(E_y)$, which by monotonicity of T, implies that $T^n(x_0, y_{W_y}) \preceq_{se} T^n(x_0, y_0) \preceq_{se} T^n(x_0, 0)$. In a similar way as in the proof of the first case we show that $T^n(x_0, y_0)$ eventually enters the ordered interval $I(E_y, E_x)$, in which case, it converges to E_0.

(b) Local stability of all equilibrium points follows from Theorem 6.

The existence and the properties of the global stable manifolds $\mathcal{W}^s(E_x), \mathcal{W}^s(E_y)$ are guaranteed by Theorems 4 and 7. The existence and the properties of the curves C_l and C_u are guaranteed by Corollary 2. The regions $\mathcal{B}((\infty, 0))$ and $\mathcal{B}((0, \infty))$ are invariant and, by Theorem 8 all solutions are eventually componentwise monotonic. Since the basins of attraction of all equilibrium points in those regions are uniquely determined, then every solution which starts in $\mathcal{B}((\infty, 0))$ (resp. $\mathcal{B}((0, \infty))$) is asymptotic to $(\infty, 0)$ (resp. $(0, \infty)$).

Let (x_0, y_0) be an arbitrary initial point between $\mathcal{W}^s(E_x)$ and $\mathcal{W}^s(E_y)$. First, assume that $x_0 > \bar{x}$.

Then $(x_0, y_{W_y}) \preceq_{se} (x_0, y_0) \preceq_{se} (x_0, y_{W_x})$, where $(x_0, y_{W_y}) \in \mathcal{W}^s(E_y), (x_0, y_{W_x}) \in \mathcal{W}^s(E_x)$, and so $T^n(x_0, y_{W_y}) \preceq_{se} T^n(x_0, y_0) \preceq_{se} T^n(x_0, y_{W_x})$ for $n \geq 0$. Since $T^n(x_0, y_{W_y}) \to E_x$ and $T^n(x_0, y_{W_x}) \to E_x$ as $n \to \infty$, we conclude that $T^n(x_0, y_0)$ eventually enters the interior of the ordered interval $I(E_y, E_x)$, in which case, in a similar way as in the case (a), it converges to E_0.

The case $x_0 \leq \bar{x}$, is treated in exactly the same as the analogue case of (a).

Finally, let (x_0, y_0) be an arbitrary initial point between the curves C_u and C_l.

Then $(x_0, y_l) \preceq_{se} (x_0, y_0) \preceq_{se} (x_0, y_u)$, where $(x_0, y_l) \in C((0, \infty)), (x_0, y_u) \in C((\infty, 0))$, and so $T^n(x_0, y_l) \preceq_{se} T^n(x_0, y_0) \preceq_{se} T^n(x_0, y_u)$.

Since $T^n(x_0, y_l) \to E$ and $T^n(x_0, y_u) \to E$ as $n \to \infty$, we conclude that $T^n(x_0, y_0) \to E$.

(c) Local stability of all equilibrium points follows from Theorem 6.

The existence and the properties of the global stable manifolds $\mathcal{W}^s(E_x), \mathcal{W}^s(E_y)$ and $\mathcal{W}^s(E_+)$ are guaranteed by Theorems 4 and 7. The proof for the basin of attraction $\mathcal{B}(E_0)$ is same as in the case (a). Assume that an arbitrary initial point (x_0, y_0) is above $\partial\mathcal{B}((\infty, 0))$ in south-east

ordering. In view of Theorem 8 every solution is eventually componentwise monotonic and so it must be asymptotic to $(\infty, 0)$ as the other three equilibrium points have uniquely determined basins of attractions. More precisely, in view of Theorem 4 there exists the unstable manifold $\mathcal{W}^u(E_+)$, which is passing through E_+ and is continuous, non-increasing curve contained in $\mathcal{Q}_2(E_+) \cup \mathcal{Q}_4(E_+)$ with the property that all solutions which start below $\partial\mathcal{B}((\infty, 0))$ or above $\partial\mathcal{B}((0, \infty))$ are asymptotic to $\mathcal{W}^u(E_+)$ as $n \to \infty$. Similar reasoning applies if an arbitrary initial point (x_0, y_0) is below $\partial\mathcal{B}((0, \infty))$ in south-east ordering.

□

Remark 1 In the special case $a = b$ an immediate checking shows that the line $y = x$ is an invariant set and that all interior equilibrium points (if any) belongs to that line, which in view of Theorem 4 shows that $y = x$ is the stable manifold in that case. In fact, in this case the coordinates of the equilibrium points are computable and we have that for $a < 1/4$ the interior equilibrium points are

$$E_- \left(\frac{1 - \sqrt{1 - 4a}}{2}, \frac{1 - \sqrt{1 - 4a}}{2} \right), \quad E_+ \left(\frac{1 + \sqrt{1 - 4a}}{2}, \frac{1 + \sqrt{1 - 4a}}{2} \right),$$

and for $a = 1/4$ the interior equilibrium point is $E(1/2, 1/2)$, while in the case $a > 1/4$ there is no an interior equilibrium point. In any case the line $y = x$ is an invariant set which in the case $a < 1/4$ becomes the global stable manifold of E_+.

Based on our simulations we pose the following conjecture.

Conjecture 1 *We conjecture that $C_l = C_u$ in Part b) of Theorem 1.*

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 19, NO.1, 2015, COPYRIGHT 2015 EUDOXUS PRESS, LLC

HIGHER-ORDER q-DAEHEE POLYNOMIALS

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Abstract. Recently, q-Daehee polynomials and numbers are introduced (see [8]). In this paper, we consider the higher-order q-Daehee numbers and polynomials and give some new relations and identities between higher-order q-Daehee polynomials and higher-order q-Bernoulli polynomials.

1. Introduction

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. The p-adic norm $|\cdot|_p$ is normalized as $|p|_p = 1/p$. Let q be an indeterminate in $\mathbb{C}_p$ with $|1-q| < p^{-1/(p-1)}$, and let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the p-adic q-integral on $\mathbb{Z}_p$ (q-Volkenborn integral on $\mathbb{Z}_p$) is defined by Kim to be

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x, \tag{1.1}$$

where $[x]_q = \frac{1-q^x}{1-q}$ (see [3]). Let $f_1(x) = f(x+1)$. Then, by (1.1), we get

$$qI_q(f_1) - I_q(f) = (q-1)f(0) + \frac{q-1}{\log q} f'(0), \tag{1.2}$$

where $f'(0) = \frac{d}{dx} f(x) \mid_{x=0}$ (see [3]). From (1.2), we note that

$$\int_{\mathbb{Z}_p} e^{xt} d\mu_q(x) = \frac{q-1+\frac{q-1}{\log q}t}{qe^t - 1}. \tag{1.3}$$

Now, we define the q-Bernoulli numbers which are given by

$$\frac{q-1+\frac{q-1}{\log q}t}{qe^t - 1} = \sum_{n\geq 0} B_{n,q} \frac{t^n}{n!}, \tag{1.4}$$

(see [8]). Note that $\lim_{q\to 1} B_{n,q} = B_n$ the n-th Bernoulli number. From (1.3) and (1.4), we have

$$\int_{\mathbb{Z}_p} x^n d\mu_q(x) = B_{n,q}, \quad n \geq 0. \tag{1.5}$$

We can also define the q-Bernoulli polynomials which are given by the generating function to be

$$\frac{q-1+\frac{q-1}{\log q}t}{qe^t - 1} e^{xt} = \sum_{n\geq 0} B_{n,q}(x) \frac{t^n}{n!}. \tag{1.6}$$

2000 *Mathematics Subject Classification.* 05A40.

Key words and phrases. Bernoulli numbers, Bernoulli polynomials, Daehee numbers, Daehee polynomials.

Thus, by (1.4) and (1.5), we see that

$$B_{n,q}(x)=\sum_{\ell=0}^{n}\binom{n}{\ell}B_{\ell,q}x^{n-\ell}. \tag{1.7}$$

From (1.2), we can derive the following equation:

$$\int_{\mathbb{Z}_p} e^{(x+y)t}d\mu_q(y)=\frac{q-1+\frac{q-1}{\log q}t}{qe^t-1}e^{xt}=\sum_{n\ge 0}B_{n,q}(x)\frac{t^n}{n!}. \tag{1.8}$$

By (1.8), we get

$$\int_{\mathbb{Z}_p}(x+y)^n d\mu_q(y)=B_{n,q}(x),\quad n\ge 0. \tag{1.9}$$

For $\alpha\in\mathbb{N}$, the higher-order q-Bernoulli polynomials are defined by the generating function to be

$$\left(\frac{q-1+\frac{q-1}{\log q}t}{qe^t-1}\right)^{\alpha}e^{xt}=\sum_{n\ge 0}B_{n,q}^{(\alpha)}(x)\frac{t^n}{n!}. \tag{1.10}$$

Note that $\lim_{q\to 1}B_{n,q}^{(\alpha)}(x)=B_n^{(\alpha)}(x)$ the n-th higher-order Bernoulli polynomial which are defined by the generating function to be $\left(\frac{t}{e^t-1}\right)^{\alpha}e^{xt}=\sum_{n\ge 0}B_n^{(\alpha)}(x)\frac{t^n}{n!}$ (see [1-13]).

As is known, the Daehee polynomials are defined by the generating function to be $\frac{\log(1+t)}{t}(1+t)^x=\sum_{\neq 0}D_n(x)\frac{t^n}{n!}$ (see [5]). In [5], the Daehee polynomials of order α are also defined by the generating function to be

$$\left(\frac{\log(1+t)}{t}\right)^{\alpha}(1+t)^x=\sum_{\neq 0}D_n^{(\alpha)}(x)\frac{t^n}{n!},$$

(see [4, 5, 11, 12]). Recently, q-Daehee polynomials are defined by the generating function to be

$$\frac{q-1+\frac{q-1}{\log q}\log(1+t)}{qt+q-1}(1+t)^x=\sum_{n\ge 0}D_{n,q}(x)\frac{t^n}{n!} \tag{1.11}$$

(see [8]). From (1.2) and (1.11), we have

$$\int_{\mathbb{Z}_p}(1+t)^{x+y}d\mu_q(y)=\frac{q-1+\frac{q-1}{\log q}\log(1+t)}{qt+q-1}(1+t)^x=\sum_{n\ge 0}D_{n,q}(x)\frac{t^n}{n!}. \tag{1.12}$$

In viewpoint of (1.12), we consider the higher-order q-Daehee polynomials and give new relations and identities between higher-order q-Daehee polynomials and higher-order q-Bernoulli polynomials.

2. Higher-order q-Daehee polynomials

We assume that $t\in\mathbb{C}_p$ with $|t|_p<p^{-1/(p-1)}$. For $k\in\mathbb{N}$, let us define the higher-order q-Daehee polynomials as follows:

$$\left(\frac{q-1+\frac{q-1}{\log q}\log(1+t)}{qt+q-1}\right)^{k}(1+t)^x=\sum_{n\ge 0}D_{n,q}^{(k)}(x)\frac{t^n}{n!}. \tag{2.1}$$

From (1.2), we can derive the following equation:

$$\begin{aligned}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{x_1+\cdots+x_k+x}d\mu_q(x_1)\cdots d\mu_q(x_k) &= \left(\frac{q-1+\frac{q-1}{\log q}\log(1+t)}{qt+q-1}\right)^k(1+t)^x \\ &= \sum_{n\geq 0}D_{n,q}^{(k)}(x)\frac{t^n}{n!}. \qquad (2.2)\end{aligned}$$

Thus, by (2.2), we get

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_k+x)_n d\mu_q(x_1)\cdots d\mu_q(x_k) = D_{n,q}^{(k)}(x), \quad n\geq 0, \qquad (2.3)$$

which implies

$$D_{n,q}^{(k)}(x) = \sum_{\ell=0}^{n}S_1(n,\ell)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_k+x)^n d\mu_q(x_1)\cdots d\mu_q(x_k), \qquad (2.4)$$

where $S_1(n,\ell)$ is the Stirling number of the first kind. We observe that

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}e^{(x_1+\cdots+x_k+x)t}d\mu_q(x_1)\cdots d\mu_q(x_k) = \left(\frac{q-1+\frac{q-1}{\log q}\log(1+t)}{qt+q-1}\right)^k(1+t)^x. \qquad (2.5)$$

By (1.10), (2.4) and (2.5), we deduce the following result.

Theorem 1. *For $n\geq 0$ and $k\in\mathbb{N}$,*

$$D_{n,q}^{(k)}(x) = \sum_{\ell=0}^{n}S_1(n,\ell)B_{\ell,q}^{(k)}(x).$$

When $x=0$, $D_{n,q}^{(k)} = D_{n,q}^{(k)}(0)$ are called the higher-order q-Daehee numbers.

Corollary 1. *For $n\geq 0$ and $k\in\mathbb{N}$,*

$$D_{n,q}^{(k)} = \sum_{\ell_1+\cdots+\ell_k=n}\binom{n}{\ell_1,\ldots,\ell_k}D_{\ell_1,q}\cdots D_{\ell_k,q} = \sum_{j=0}^{n}S_1(n,j)B_{j,q}^{(k)},$$

where $B_{n,q}^{(k)}$ is the n-th q-Bernoulli number of order k.

In (2.1), by replacing t by e^t-1, we have

$$\left(\frac{q-1+\frac{q-1}{\log q}t}{qe^t-1}\right)^k e^{tx} = \sum_{n\geq 0}D_{n,q}^{(k)}(x)\frac{(e^t-1)^n}{n!} = \sum_{m\geq 0}\left(\sum_{n=0}^{m}D_{n,q}^{(k)}(x)S_2(n,m)\right)\frac{t^m}{m!}, \qquad (2.6)$$

where $S_2(n,m)$ is the Stirling number of the second kind. Therefore, by (1.10) and (2.6), we obtain the following result.

Theorem 2. *For $m\geq 0$ and $k\in\mathbb{N}$,*

$$B_{m,q}^{(k)}(x) = \sum_{\ell=0}^{m}S_2(m,\ell)D_{\ell,q}^{(k)}(x).$$

Now, we consider the higher-order q-Daehee polynomials of the second kind as follows: for $|t-2|_p < p^{-1/(p-1)}$ and $t \neq 1$,

$$\begin{aligned} \widehat{D}_{n,q}(x) &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (-x_1 - \cdots - x_k + x)_n d\mu_q(x_1)\cdots d\mu_q(x_k) \\ &= \sum_{\ell=0}^{n} (-1)^{n-\ell} S_1(n,\ell) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_k - x)^{\ell} d\mu_q(x_1)\cdots d\mu_q(x_k) \\ &= \sum_{\ell=0}^{n} (-1)^{n-\ell} S_1(n,\ell) B_{\ell,q}^{(k)}(-x), \end{aligned} \tag{2.7}$$

which gives the following result.

Theorem 3. *For $n \geq 0$ and $k \in \mathbb{N}$,*

$$\widehat{D}_{n,q}^{(k)}(x) = \sum_{\ell=0}^{n} (-1)^{n-\ell} S_1(n,\ell) B_{\ell,q}^{(k)}(-x).$$

From (2.7), we have

$$\begin{aligned} \sum_{n\geq 0} \widehat{D}_{n,q}(x)\frac{t^n}{n!} &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{n\geq 0} \binom{x_1 + \cdots + x_k + n - 1}{n} t^n d\mu_q(x_1)\cdots d\mu_q(x_k) \\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1-t)^{-x_1 \cdots - x_k + x} d\mu_q(x_1)\cdots d\mu_q(x_k) \\ &= \left(\frac{1-t}{q-1+t}(q - 1 + \frac{1-q}{\log q}\log(1-t))\right)^k (1-t)^x. \end{aligned} \tag{2.8}$$

By (2.8), we get

$$\begin{aligned} \sum_{n\geq 0} \widehat{D}_{n,q}(x)\frac{(1-e^{-t})^n}{n!} &= \left(\frac{e^{-t}}{q-e^{-t}}(q-1+\frac{q-1}{\log q}t)\right)^k e^{-xt} = \left(\frac{q-1+\frac{q-1}{\log q}t}{qe^t - 1}\right)^k e^{-xt} \\ &= \sum_{m\geq 0} B_{m,q}^{(k)}(-x)\frac{t^m}{m!}. \end{aligned} \tag{2.9}$$

Also, we have

$$\sum_{n\geq 0} \widehat{D}_{n,q}(x)\frac{(1-e^{-t})^n}{n!} = \sum_{m\geq 0} \left(\sum_{n=0}^{m} \widehat{D}_{n,q}^{(k)}(x)(-1)^{m-n} S_2(m,n)\right) \frac{t^m}{m!}. \tag{2.10}$$

Therefore, by (2.9) and (2.10), we obtain the following result.

Theorem 4. *For $m \geq 0$ and $k \in \mathbb{N}$,*

$$B_{m,q}^{(k)}(-x) = \sum_{\ell=0}^{m} (-1)^{m-\ell} S_2(m,\ell) \widehat{D}_{\ell,q}^{(k)}(x).$$

By (2.7), we get

$$\widehat{D}_{n,q}^{(k)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_k + x)^{(n)} d\mu_q(x_1)\cdots d\mu_q(x_k), \tag{2.11}$$

where $x^{(n)} = x(x+1)\cdots(x+n-1)$. Thus, from (2.11), we have

$$\begin{aligned}
\widehat{D}^{(k)}_{n,q}(x) &= (-1)^n \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(-x_1-\cdots-x_k+x)_n d\mu_q(x_1)\cdots d\mu_q(x_k) \\
&= (-1)^n \sum_{\ell=0}^{n} S_1(n,\ell)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(-x_1-\cdots-x_k+x)^{\ell} d\mu_q(x_1)\cdots d\mu_q(x_k) \\
&= (-1)^n \sum_{\ell=0}^{n}\sum_{m=0}^{\ell} S_1(n,\ell)\binom{\ell}{m}x^{\ell-m}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(-x_1-\cdots-x_k)^{m} d\mu_q(x_1)\cdots d\mu_q(x_k) \\
&= (-1)^n \sum_{\ell=0}^{n}\sum_{m=0}^{\ell} S_1(n,\ell)\binom{\ell}{m}x^{\ell-m}B^{(k)}_{m,q} \\
&= \sum_{\ell=0}^{n}(-1)^{n-\ell}S_1(n,\ell)B^{(k)}_{\ell,q}(-x).
\end{aligned} \tag{2.12}$$

Therefore, by (2.12), we obtain the following theorem.

Theorem 5. *For $n \geq 0$ and $k \in \mathbb{N}$,*

$$\widehat{D}^{(k)}_{n,q}(x) = \sum_{\ell=0}^{n}(-1)^{n-\ell}S_1(n,\ell)B^{(k)}_{\ell,q}(-x).$$

Now, we observe that

$$\begin{aligned}
(-1)^n\frac{\widehat{D}_{n,q}(x)}{n!} &= (-1)^n\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\sum_{n\geq 0}\binom{x_1+\cdots+x_k+x}{n}d\mu_q(x_1)\cdots d\mu_q(x_k) \\
&= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\binom{-x_1-\cdots-x_k-x+n-1}{n}d\mu_q(x_1)\cdots d\mu_q(x_k) \\
&= \sum_{m=0}^{n}\binom{n-1}{m-1}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\binom{-x_1-\cdots-x_k-x}{m}d\mu_q(x_1)\cdots d\mu_q(x_k) \\
&= \sum_{m=1}^{n}\frac{1}{m!}\binom{n-1}{n-m}m!\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\binom{-x_1-\cdots-x_k-x}{m}d\mu_q(x_1)\cdots d\mu_q(x_k) \\
&= \sum_{m=1}^{n}\frac{1}{m!}\binom{n-1}{n-m}\widehat{D}^{(k)}_{m,q}(-x).
\end{aligned} \tag{2.13}$$

Therefore, by (2.13), we obtain the following result.

Theorem 6. *For $n \geq 0$ and $k \in \mathbb{N}$,*

$$(1-)^n\frac{\widehat{D}^{(k)}_{n,q}(x)}{n!} = \sum_{\ell=1}^{n}\frac{(-1)^{\ell}}{\ell!}\binom{n-1}{n-\ell}\widehat{D}^{(k)}_{\ell,q}(-x).$$

Note that, by (2.7), we have

$$\begin{aligned}
\frac{\widehat{D}_{n,q}(x)}{n!} &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\sum_{n\geq 0}\binom{x_1+\cdots+x_k+n-1-x}{n}d\mu_q(x_1)\cdots d\mu_q(x_k) \\
&= \sum_{m=0}^{n}\binom{n-1}{m-1}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\binom{x_1+\cdots+x_k-x}{m}d\mu_q(x_1)\cdots d\mu_q(x_k) \\
&= \sum_{m=1}^{n}\frac{1}{m!}\binom{n-1}{n-m}m!\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\binom{x_1+\cdots+x_k-x}{m}d\mu_q(x_1)\cdots d\mu_q(x_k) \\
&= \sum_{m=1}^{n}\frac{1}{m!}\binom{n-1}{n-m}\widehat{D}^{(k)}_{m,q}(-x).
\end{aligned} \tag{2.14}$$

Therefore, by (2.13), we obtain the following theorem.

Theorem 7. *For $n \geq 0$ and $k \in \mathbb{N}$,*

$$\frac{\widehat{D}^{(k)}_{n,q}(x)}{n!} = \sum_{\ell=1}^{n}\frac{1}{\ell!}\binom{n-1}{n-\ell}\widehat{D}^{(k)}_{\ell,q}(-x).$$

Acknowledgment This research was supported by Kyungpook National University BK 21 plus fund 2014.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 19, NO.1, 2015, COPYRIGHT 2015 EUDOXUS PRESS, LLC

Approximate fuzzy ternary homomorphisms and fuzzy ternary derivations on fuzzy ternary Banach algebras

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Abstract. Using the fixed point method, we prove the Hyers-Ulam stability of ternary homomorphisms and ternary derivations in fuzzy ternary Banach algebras associated to the following additive functional equation

$$\sum_{k=2}^{n}(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k}+1}^{n})f(\sum_{i\neq 1, i\neq i_1, i_2 \cdots, i_{n-k+1}}^{n} x_i - \sum_{r=1}^{n-k+1} x_{i_r}) + f(\sum_{i=1}^{n} x_i) = 2^{n-1} f(x_1)$$

for a fixed integer n with $n \geq 2$.

1. Introduction and preliminaries

The theory of fuzzy sets was introduced by Zadeh in 1965 [53]. Fuzzy set theory is a powerful hand set for modeling uncertainty and vagueness in various problems arising in the field of science and engineering. The fuzzy topology proves to be a very useful tool to deal with such situations where the use of classical theories breaks down. In 1984, Katsaras [18] introduced an idea of a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. In the same year, Wu and Fang [50] introduced a notion fuzzy normed apace to give a generalization of the Kolmogoroff normalized theorem for fuzzy topological vector spaces. In 1992, Felbin [10] introduced an alternative definition of a fuzzy norm on a vector space with an associated metric of Kaleva and Seikkala type [16]. Some mathematics have define fuzzy normed on a vector form various point of view [25, 41, 51]. In particular, Bag and Samanta [2] following Cheng and Mordeson [6], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric of Kramosil and Michalek type [24]. They established a decomposition theorem of fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [3].

A classical equation in the theory of functional equations is the following: "when is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation?". If the problem accepts a solution, we

[0]**2010 Mathematics Subject Classification**: 39B52, 39B82, 45L05, 46S40, 47S40, 47H10.

[0]**Keywords**: Hyers-Ulam stability; fuzzy ternary Banach algebra; additive functional equation; fuzzy ternary homomorphism; fuzzy ternary derivation; fixed point method.

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say that the equation is stable. The first problem concerning group homomorphisms was raised by Ulam [49] in 1940. In the next year, Hyers [12] gave a first affirmative answer to the question of Ulam in context of Banach spaces. In 1978, Rassias [45] proved a generalization of the Hyers' theorem for additive mappings. Furthermore, in 1994, Găvruţa [11] provided a further generalization of Rassias' theorem in which he replaced the bound $\varepsilon(\|x\|^p + \|y\|^p)$ in by a general control function $\varphi(x, y)$. Recently, several stability results have been obtained for various equations and mappings with more general domains and ranges have been investigated by a number of authors and there are many interesting results concerning this problem [1, 13, 14, 15, 22, 23, 26, 27, 28, 29, 31, 32, 33, 34, 35, 42, 43, 46, 47, 48, 52].

In the following, we will give some notations that are needed in this paper.

Definition 1.1. *Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0, 1]$ is said to be a fuzzy norm on X if for all $x, y \in X$ and all $t, s \in \mathbb{R}$,*
$(N1)$ $N(x, t) = 0$ for $t \leq 0$;
$(N2)$ $N(x, t) = 1$ for all $t > 0$ if and only if $x = 0$;
$(N3)$ $N(cx, t) = P(x, \frac{t}{|c|})$ for each $c \neq 0$;
$(N4)$ $N(x + y, s + t) \geq \min\{N(x, t), N(y, s)\}$;
$(N5)$ $N(x, \cdot)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1$;
(N6) $N(x, \cdot)$ is continuous on $\mathbb{R}$ for $x \neq 0$.

The pair (X, N) is called a fuzzy normed linear space.

One may regard $N(x, t)$ as the truth value of the statement "the norm of x is less than or equal to the real number t".

Example 1.2. *Let $(X, \|\cdot\|)$ be a normed linear space and $\alpha, \beta > 0$. Then*

$$N(x,t) = \begin{cases} \frac{\alpha t}{\beta t + \|x\|}, & t > 0, \quad x \in X, \\ 0, & t \leq 0, \quad x \in X \end{cases}$$

is a fuzzy norm on X.

Example 1.3. *Let $(X, \|\cdot\|)$ be a normed linear space. Then*

$$N(x,t) = \begin{cases} 0, & t < 0, \\ \frac{t}{\|x\|}, & 0 < t \leq \|x\|, \quad x \in X, \\ 1, & t > \|x\|, \quad x \in X \end{cases}$$

is a fuzzy norm on X.

Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to be convergent if there exists $x \in X$ such that $lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In that case, x is called the limit of the sequence $\{x_n\}$ and we denote it by $N - lim_{n\to\infty} x_n = x$.

A sequence $\{x_n\}$ in X is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy normed is said to be complete and the fuzzy normed vector space is called a fuzzy Banach space.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at point $x_0 \in X$ if, for each sequence $\{x_n\}$ converging to x_0 in X, the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then f is said to be continuous on X [3].

Ternary algebraic operations have propounded originally in nineteenth century by several mathematicians such as Cayley [5] who introduced the notion of cubic matrix which in turn was generalized by Kapranov, Gelfand and Zelevinskii in 1990 [17]. The application of ternary algebra in supersymmetry is presented in [20] and in Yang-Baxter equation in [37].

Let X be a complex linear space equipped a mapping $[\cdot,\cdot,\cdot] : X \times X \times X \to X$ with $(x, y, z) \to [x, y, z]$ that is linear in variables x, y, z and satisfies the associative identity, i.e., $[x, y, [z, u, v]] = [x, [y, z, u], v] = [[x, y, z], u, v]$ for all $x, y, z, u, v \in X$. The pair $(X, [\cdot,\cdot,\cdot])$ is called a *ternary algebra*. The ternary algebra $(X, [\cdot,\cdot,\cdot])$ is called unital if it has an identity element, i.e., an element $e \in X$ such that $[x, e, e] = [e, e, x] = x$ for all $x \in X$.

The linear space X is called a *ternary normed algebra* if X is a ternary algebra and there exists a norm $\|\cdot\|$ on X which satisfies $\|[x, y, z]\| \le \|x\|\|y\|\|z\|$ for all $x, y, z \in X$. Whenever the ternary algebra X is unital with unit element e, we repute $\|e\| = 1$. A normed ternary algebra X is called a *ternary Banach algebra* if $(X, \|\cdot\|)$ is a Banach space.

Definition 1.4. *Let X be a ternary algebra and (X, N) be a fuzzy normed space.*
(1) *The fuzzy normed space (X, N) is called a fuzzy ternary normed algebra if*

$$N([x, y, z], tsr) \ge N(x, t)N(y, s)N(z, r)$$

for all $x, y \in X$ and all positive real numbers t, s, r.
(2) *A complete fuzzy ternary normed algebra is called a fuzzy ternary Banach algebra.*

Example 1.5. *Let $(X, \|\cdot\|)$ be a ternary normed (Banach) algebra. Let*

$$N(x,t) = \begin{cases} \frac{t}{t+\|x\|}, & t > 0, \quad x \in X, \\ 0, & t \le 0, \quad x \in X. \end{cases}$$

Then $N(x, t)$ is a fuzzy norm on X and (X, N) is a fuzzy ternary normed (Banach) algebra.

Definition 1.6. *Let (X, N) and (Y, N') be two fuzzy ternary Banach algebras.*
(1) *The $\mathbb{C}$-linear mapping $f : (X, N) \to (Y, N')$ is called a fuzzy ternary homomorphism if*

$$f([x, y, z]) = [f(x), f(y), f(z)]$$

for all $x, y, z \in X$.
(2) *The $\mathbb{C}$-linear mapping $f : (X, N) \to (X, N)$ is called a fuzzy ternary derivation if*

$$f([x, y, z]) = [f(x), y, z] + [x, f(y), z] + [x, y, f(z)]$$

for all $x, y, z \in X$.

Let X be a nonempty set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if d satisfies
(1) $d(x, y) = 0$ if and only if $x = y$ for $x, y \in X$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Let (X, d) be a generalized metric space. An operator $T : X \to X$ satisfies a Lipschitz condition with Lipschitz constant L, if there exists a constant $L \ge 0$ such that $d(Tx, Ty) \le Ld(x, y)$ for all $x, y \in X$. If the Lipschitz constant L is less than 1, then the

operator T is called a strictly contractive operator. We recall the following theorem by Diaz and Margolis.

Theorem 1.7. ([30, 44]) *Let (X, d) be a complete generalized metric space and $J : X \to X$ be a strictly contractive mapping with Lipshitz constant $L < 1$. Then, for each given $x \in X$, either*

$$d(J^n x, J^{n+1}x) = \infty \quad \text{for all} \quad n \geq 0,$$

or there exists a natural number n_0 such that
(1) $d(J^n x, J^{n+1}x) < \infty$ for all $n \geq n_0$;
(2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;
(3) y^* is the unique fixed point of J in the set $Y = \{y \in X : d(J^{n_0}, y) < \infty\}$;
(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

In 1996, Isac and Rassias [14] were the first to provide applications of stability theory of functional equations for the proof of new fixed-point theorems with applications. By using the fixed point method, the stability problems of several functional equations have been extensively investigated by a number of authors (see [4, 8, 9, 19, 39, 40]).

Recently, Ebadian and Ghobadipour [7] considered the Hyers-Ulam stability of double derivations on Banach algebras and Lie *-double derivations on Lie C*-algebras associated with the following additive functional equation

$$\sum_{k=2}^{n}\Big(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k}+1}^{n}\Big)f\Big(\sum_{i\neq 1, i\neq i_1, i_2, \cdots, i_{n-k+1}}^{n} x_i - \sum_{r=1}^{n-k+1} x_{i_r}\Big) + f\Big(\sum_{i=1}^{n} x_i\Big) = 2^{n-1} f(x_1) \tag{1.1}$$

for a fixed integer n with $n \geq 2$.

In this paper, we investigate the Hyers-Ulam stability of fuzzy ternary homomorphisms and fuzzy ternary derivations on fuzzy ternary Banach algebras associated with the additive functional equation (1.1).

Throughout this article, assume that (X, N) and (Y, N') be fuzzy ternary Banach algebras.

2. Approximate fuzzy ternary homomorphisms in fuzzy ternary Banach algebras

In this section, we prove the Hyers-Ulam stability of fuzzy ternary homomorphisms in fuzzy ternary Banach algebras related to the additive functional equation (1.1).

Theorem 2.1. *Let $\varphi : X^n \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(2x_1, 2x_2, \cdots, 2x_n) \leq 2L\varphi(x_1, x_2, \cdots, x_n) \tag{2.1}$$

for all $x_1, x_2, \cdots, x_n \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$N\Big(\sum_{k=2}^{n}\Big(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k}+1}^{n}\Big)f\Big(\sum_{i\neq 1, i\neq i_1, i_2, \cdots, i_{n-k+1}}^{n} \mu x_i - \sum_{r=1}^{n-k+1} \mu x_{i_r}\Big) + f\Big(\sum_{i=1}^{n} \mu x_i\Big) - 2^{n-1}\mu f(x_1), t\Big) \geq \frac{t}{t + \varphi(x_1, x_2, \cdots, x_n)} \tag{2.2}$$

$$N(f([x,y,z]) - [f(x), f(y), f(z)], t) \geq \frac{t}{t + \varphi(x, y, z, 0, \cdots, 0)} \tag{2.3}$$

for all $x, y, z, x_1, x_2, \cdots, x_n \in X$, all $\mu \in \mathbb{T}^1 := \{u \in \mathbb{C} : |u| = 1\}$ and all $t > 0$. Then $H(x) = N - \lim_{m\to\infty} \frac{1}{2^m} f(2^m x)$ exists for each $x \in X$, and defines a unique fuzzy ternary homomorphism $H : X \to Y$ such that

$$N(f(x) - H(x), t) \geq \frac{2^{n-1}(1-L)t}{2^{n-1}(1-L)t + \varphi(x, x, 0, \cdots, 0)} \tag{2.4}$$

for all $x \in X$ and all $t > 0$.

Proof. Consider the set $\Omega := \{g : X \to Y, g(0) = 0\}$ and introduce the generalized metric

$$d(g, h) = \inf\{\eta \in \mathbb{R}^+ : N(g(x) - h(x), \eta t) \geq \frac{t}{t + \varphi(x, x, 0, \cdots, 0)}\}$$

where $\inf \emptyset = +\infty$. The proof of the fact that (Ω, d) is a complete generalized metric space can be found in [4]. Now we consider the mapping $J : \Omega \to \Omega$ defined by

$$Jg(x) := \frac{1}{2} g(2x)$$

for all $g \in \Omega$ and $x \in X$. Let $\varepsilon > 0$ and $f, g \in \Omega$ be given such that $d(g, h) < \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t + \varphi(x, x, 0, \cdots, 0)},$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N(\frac{1}{2} g(2x) - \frac{1}{2} h(2x), L\varepsilon t) \\ &= N(g(2x) - h(2x), 2L\varepsilon t) \\ &\geq \frac{2Lt}{2Lt + \varphi(2x, 2x, 0, 0, \cdots, 0)} \\ &\geq \frac{2Lt}{2Lt + 2L\varphi(x, x, 0, \cdots, 0)} \\ &= \frac{t}{t + \varphi(x, 0, \cdots, 0)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g, h) < \varepsilon$ implies that $d(Jg, Jh) < L\varepsilon$ for all $g, h \in \Omega$, that is, J is a self-mapping of Ω with the Lipschitz constant L. We use the following relation

$$1 + \sum_{k=1}^{n-k} \binom{n-k}{k} = \sum_{k=0}^{n-k} \binom{n-k}{k} = 2^{n-k}$$

for all $n > k$. Letting $\mu = 1$ and putting $x_1 = x_2 = x$ and $x_3 = x_4 = \cdots = x_n = 0$ in (2.2), we have

$$N(2^{n-2} f(2x) - 2^{n-1} f(x), t) \geq \frac{t}{t + \varphi(x, x, 0, \cdots, 0)} \tag{2.5}$$

for all $x \in X$ and all $t > 0$. Then

$$N\left(\frac{1}{2} f(2x) - f(x), \frac{t}{2^{n-1}}\right) \geq \frac{t}{t + \varphi(x, x, 0, \cdots, 0)} \tag{2.6}$$

for all $x \in X$ and all $t > 0$. It follows from (2.6) that $d(f, Jf) = \frac{1}{2^{n-1}}$. By Theorem 1.7, there exists a mapping $H : X \to Y$ such that the following holds:
(1) H is a fixed point of J, that is,

$$H(2x) = 2H(x) \tag{2.7}$$

for all $x \in X$. The mapping H is a unique fixed point of J in the set $\Delta = \{h \in \Omega : d(g, h) < \infty\}$. This implies that H is a unique mapping satisfying (2.7) such that there exists $\eta \in (0, \infty)$ satisfying

$$N(f(x) - H(x), \eta t) \geq \frac{t}{t + \varphi(x, x, 0, \cdots, 0)}$$

for all $x \in X$ and all $t > 0$.
(2) $d(J^m f, H) \to 0$ as $m \to \infty$. This implies the equality

$$N - \lim_{m\to\infty} \frac{1}{2^m} f(2^m x) = H(x)$$

exists for each $x \in X$,
(3) $d(f, H) \leq \frac{1}{1-L} d(f, Jf)$, which implies inequality

$$d(f, H) \leq \frac{1}{2^{n-1}(1 - L)}$$

and so

$$N(f(x) - H(x), t) \geq \frac{2^{n-1}(1 - L)t}{2^{n-1}(1 - L)t + \varphi(x, x, 0, \cdots, 0)}.$$

Thus (2.4) holds.

It follows from (2.1) and (2.2) that

$$\begin{aligned}
&N(\sum_{k=2}^{n}(\sum_{i_1=2}^{k} \sum_{i_2=i_1+1}^{k+1} \cdots \sum_{i_{n-k+1}=i_{n-k}+1}^{n})H(\sum_{i\neq 1, i\neq i_1, i_2, \cdots, i_{n-k+1}}^{n} \mu x_i - \sum_{r=1}^{n-k+1} \mu x_{i_r}) \\
&\qquad + H(\sum_{i=1}^{n} \mu x_i) - 2^{n-1} H(\mu x_1), t) \\
&= \lim_{m\to\infty} N(\frac{1}{2^m} \sum_{k=2}^{n}(\sum_{i_1=2}^{k} \sum_{i_2=i_1+1}^{k+1} \cdots \sum_{i_{n-k+1}=i_{n-k}+1}^{n})f(\sum_{i\neq 1, i\neq i_1, i_2, \cdots, i_{n-k+1}}^{n} \mu 2^m x_i - \sum_{r=1}^{n-k+1} \mu 2^m x_{i_r}) \\
&+ f(\sum_{i=1}^{n} \mu 2^m x_i) - 2^{n-1} f(\mu 2^m x_1), t) \\
&\geq \lim_{m\to\infty} \frac{2^m t}{2^m t + \varphi(2^m x_1, 2^m x_2, \cdots, 2^m x_n)} \\
&\geq \lim_{m\to\infty} \frac{t}{t + \frac{1}{2^m} \varphi(2^m x_1, 2^m x_2, \cdots, 2^m x_n)} \\
&\geq \lim_{m\to\infty} \frac{t}{t + L^m \varphi(x_1, x_2, \cdots, x_n)} = 1
\end{aligned}$$

for all $x_1, x_2, \cdots, x_n \in X$, all $t > 0$ and all $\mu \in \mathbb{T}^1$. Thus

$$\sum_{k=2}^{n}(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k}+1}^{n})H(\sum_{i\neq 1, i\neq i_1, i_2, \cdots, i_{n-k+1}}^{n}\mu x_i - \sum_{r=1}^{n-k+1}\mu x_{i_r}) + H(\sum_{i=1}^{n}\mu x_i)$$
$$= 2^{n-1}\mu H(x_1)$$

for all $x_1, x_2, \cdots, x_n \in X$. By [21], $H : X \to Y$ is Cauchy additive, that is, $H(x + y) = H(x) + H(y)$ for all $x, y \in X$. By a similar method to the proof of [30], one can show that the mapping $H : X \to Y$ is $\mathbb{C}$-linear.

By (2.3), we have

$$\begin{aligned} N(\frac{1}{2^{3m}}f([2^m x, 2^p y, 2^m z]) - \frac{1}{2^{3m}}[f(2^m x), f(2^m y), f(2^m y)], t) \\ \geq \frac{2^{3m}t}{2^{3m}t + \varphi(2^m x, 2^m y, 2^m z, \cdots, 0)} \\ \geq \frac{2^{3m}t}{2^{3m}t + 2^m L^m \varphi(x, y, z, \cdots, 0)} \end{aligned}$$

for all $x, y, z \in X$ and all $t > 0$. Since

$$\lim_{m\to\infty}\frac{2^{3m}t}{2^{3m}t + 2^m L^m \varphi(x, y, z, \cdots, 0)} = 1$$

for all $x, y, z \in X$ and all $t > 0$,

$$H([x, y, z]) = [H(x), H(y), H(z)]$$

for all $x, y, z \in X$. This means that H is a fuzzy ternary homomorphism. This complete the proof. □

Corollary 2.2. *Let X be a ternary Banach algebra with norm $\|\cdot\|$, $\delta \geq 0$ and p be a real number with $0 < p < 1$. Let $f : X \to Y$ be a mapping satisfying*

$$N(\sum_{k=2}^{n}(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k}+1}^{n})f(\sum_{i\neq 1, i\neq i_1, i_2, \cdots, i_{n-k+1}}^{n}\mu x_i - \sum_{r=1}^{n-k+1}\mu x_{i_r}) + f(\sum_{i=1}^{n}\mu x_i) - 2^{n-1}f(\mu x_1), t) \geq \frac{t}{t + \delta\sum_{i=1}^{n}\|x_i\|^p} \tag{2.8}$$

$$N(f([x, y, z]) - [f(x), f(y), f(z)], t) \geq \frac{t}{t + \delta(\|x\|^p + \|y\|^p + \|z\|^p)} \tag{2.9}$$

for all $x, y, z, x_1, x_2, \cdots, x_n \in X$, all $\mu \in \mathbb{T}^1$ and all $t > 0$. Then there exists a unique fuzzy ternary homomorphism $H : X \to Y$ such that

$$N(f(x) - H(x), t) \geq \frac{2^{n-1}(1 - 2^{p-1})t}{2^{n-1}(1 - 2^{p-1})t + 2\delta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.1 by taking $\varphi(x_1, x_2, \cdots, x_n) := \delta\sum_{i=1}^{n}\|x_i\|^p$ for all $x_1, x_2, \cdots, x_n \in X$. It follows from (2.8) that $f(0) = 0$. Choosing $L = 2^{p-1}$, we get the desired result. □

Theorem 2.3. *Let $\varphi : X^n \to [0,\infty)$ be a function such that there exists an $L<1$ with*

$$\varphi\left(\frac{x_1}{2},\frac{x_2}{2},\cdots,\frac{x_n}{2}\right) \le \frac{L}{2}\varphi(x_1,x_2,\cdots,x_n)$$

for all $x_1,x_2,\cdots,x_n \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0)=0$, (2.2) and (2.3). Then $H(x) = N-\lim_{m\to\infty} 2^m f\left(\frac{1}{2^m}x\right)$ exists for each $x \in X$, and defines a unique fuzzy ternary homomorphism $H : X \to Y$ such that

$$N(f(x)-H(x),t) \ge \frac{2^{n-1}(1-L)t}{2^{n-1}(1-L)t + L\varphi(x,x,0,\cdots,0)}$$

for all $x \in X$ and all $t>0$.

Proof. Let (S,d) be the generalized metric space in the proof of Theorem 2.1.

Consider the linear mapping $J : \Omega \to \Omega$ defined by

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $g \in \Omega$ and $x \in X$. We can conclude that J is a strictly contractive self-mapping of S with the Lipschitz constant L.

It follows from (2.5) that

$$N\left(f(x)-2f(\frac{x}{2}),\frac{Lt}{2^{n-1}}\right) \ge \frac{t}{t+\varphi(x,x,0,\cdots,0)}$$

It follows that $d(f,Jf) = \frac{L}{2^{n-1}}$.

By Theorem 1.7, there exists a mapping $H : X \to Y$ such that the following holds:
(1) H is a fixed point of J, that is,

$$H\left(\frac{x}{2}\right) = \frac{1}{2}H(x) \tag{2.10}$$

for all $x \in X$. The mapping H is a unique fixed point of J in the set $\Delta = \{h \in \Omega : d(g,h) < \infty\}$. This implies that H is a unique mapping satisfying (2.10) such that there exists $\eta \in (0,\infty)$ satisfying

$$N(f(x)-H(x),\eta t) \ge \frac{t}{t+\varphi(x,x,0,\cdots,0)}$$

for all $x \in X$ and all $t>0$.
(2) $d(J^m f,H) \to 0$ as $m \to \infty$. This implies the equality

$$N-\lim_{m\to\infty} 2^m f\left(\frac{1}{2^m}x\right) = H(x)$$

exists for each $x \in X$,
(3) $d(f,H) \le \frac{1}{1-L}d(f,Jf)$, which implies inequality

$$d(f,H) \le \frac{L}{2^{n-1}(1-L)}$$

and so

$$N(f(x)-H(x),t) \ge \frac{2^{n-1}(1-L)t}{2^{n-1}(1-L)t + L\varphi(x,x,0,\cdots,0)}.$$

The rest the proof is similar to the proof of Theorem 2.1. □

Corollary 2.4. *Let X be a ternary Banach algebra with norm $\|\cdot\|$, $\delta \geq 0$ and p be a real number with $p > 1$. Let $f : X \to Y$ be a mapping satisfying (2.8) and (2.9). Then there exists a unique fuzzy ternary homomorphism $H : X \to Y$ such that*

$$N(f(x) - H(x), t) \geq \frac{2^{n-1}(1-2^{1-p})t}{2^{n-1}(1-2^{1-p})t + 2^{2-p}\delta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.3 by taking $\varphi(x_1, x_2, \cdots, x_n) := \delta \sum_{i=1}^{n} \|x_i\|^p$ for all $x_1, x_2, \cdots, x_n \in X$. It follows from (2.8) that $f(0) = 0$. Choose $L = 2^{1-p}$, we get the desired result. □

3. Approximate fuzzy ternary derivations on fuzzy ternary Banach algebras

In this section, we prove the Hyers-Ulam stability of fuzzy ternary derivations on fuzzy ternary Banach algebras related to the additive bfunctional equation (1.1).

Theorem 3.1. *Let $\varphi : X^n \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(2x_1, 2x_2, \cdots, 2x_n) \leq 2L\varphi(x_1, x_2, \cdots, x_n)$$

for all $x_1, x_2, \cdots, x_n \in X$. Let $f : X \to X$ be a mapping satisfying $f(0) = 0$, (2.2) and

$$N(f([x, y, z]) - [f(x), y, z] - [x, f(y), z] - [x, y, f(z)]), t) \geq \frac{t}{t + \varphi(x, y, z, \cdots, 0)} \quad (3.1)$$

for all $x, y, z, x_1, x_2, \cdots, x_n \in X$ and all $t > 0$. Then $D(x) = N - \lim_{m\to\infty} \frac{1}{2^m} f(2^m x)$ exists for each $x \in X$, and defines a unique fuzzy ternary derivation $D : X \to X$ such that

$$N(f(x) - D(x), t) \geq \frac{2^{n-1}(1-L)t}{2^{n-1}(1-L)t + \varphi(x, x, 0, \cdots, 0)} \quad (3.2)$$

for all $x \in X$ and all $t > 0$.

Proof. By the same reasoning as in the proof of Theorem 2.1, the mapping $D : X \to X$ is a unique $\mathbb{C}$-linear mapping satisfying (3.2).

Now, we show that $D : X \to X$ is a fuzzy ternary derivation. By (3.1), we have

$$\begin{aligned} &N(\frac{1}{2^{3m}} f([2^m x, 2^m y, 2^m z]) - \frac{1}{2^{3m}}([f(2^m x), y, z] - [x, f(2^m y), z] - [x, y, f(2^m z)]), t) \\ &\quad \geq \frac{2^{3m}t}{2^{3m}t + \varphi(2^m x, 2^m y, 2^m z, \cdots, 0)} \\ &\quad \geq \frac{2^{3m}t}{2^{3m}t + 2^m L^m \varphi(x, y, z, \cdots, 0)} \end{aligned}$$

for all $x, y, z \in X$ and all $t > 0$. Since

$$\lim_{m\to\infty} \frac{2^{3m}t}{2^{3m}t + 2^m L^m \varphi(x, y, z, \cdots, 0)} = 1$$

for all $x, y, z \in X$ and all $t > 0$,

$$D([x, y, z]) = [D(x), y, z] + [x, D(y), z] + [x, y, D(z)]$$

for all $x, y, z \in X$. This means that D is a fuzzy ternary derivation. □

Theorem 3.2. *Let $\varphi : X^n \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(\frac{x_1}{2}, \frac{x_2}{2}, \cdots, \frac{x_n}{2}) \leq \frac{1}{2} L\varphi(x_1, x_2, \cdots, x_n)$$

for all $x_1, x_2, \cdots, x_n \in X$. Let $f : X \to X$ be a mapping satisfying $f(0) = 0$, (2.2) and (3.1). Then $D(x) = N - \lim_{m\to\infty} 2^m f\left(\frac{1}{2^m}x\right)$ exists for each $x \in X$, and defines a unique fuzzy ternary derivation $D : X \to X$ such that

$$N(f(x) - D(x), t) \geq \frac{2^{n-1}(1 - L)t}{2^{n-1}(1 - L)t + L\varphi(x, x, 0, \cdots, 0)}$$

for all $x \in X$ and all $t > 0$.

Acknowledgments

C. Park and D. Y. Shin were supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2012R1A1A2004299) and (NRF-2010-0021792), respectively.

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Fuzzy stability of functional equations in n-variable fuzzy Banach spaces

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Abstract. In this paper, we prove a generalized fuzzy version of the Hyers-Ulam stability for the functional equations $f(\varphi(X)) = \phi(X)f(X) + \psi(X)e$ and $f(\varphi(X)) = \phi(X)f(X)$ by using the direct method and the fixed point method, where X denotes an n-variable. Furthermore, we can apply the obtained results to some well-known functional equations such as γ-function, β-function, and G-function type's equations.

Keywords: Fuzzy Banach space; Hyers-Ulam stability; Fixed point; Functional equation.

1. Introduction and preliminaries

In 1940, Ulam [60] gave a talk and proposed a number of unsolved problems. In particular, he asked the following question: "when and under what condition does a solution of a functional equation near an approximately solution of that exist?"

Today, this question is considered as the source of the stability of functional equations. In 1941, Hyers [18] formulated and proved the Ulam's problem for the Cauchy's functional equation on Banach spaces. The result of Hyers was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [53] for linear mappings by considering an unbounded Cauchy difference. In 1994, Găvruţa [16] provided a further generalization of Rassias' theorem in which he replaced the unbounded Cauchy difference by a general control function for the existence of a unique linear mapping.

Since then, many interesting results concerning the stability of different functional equations have been obtained by numerous authors (cf. [4, 17, 19, 20, 22, 23, 24, 29, 30, 34, 36, 37, 47, 49, 54, 55, 56, 57]).

In this paper, we investigate the following functional equations

$$f(\varphi(X)) = \phi(X)f(X) + \psi(X)e, \tag{1.1}$$
$$f(\varphi(X)) = \phi(X)f(X) \tag{1.2}$$

where φ, ϕ, ψ are given functions, while f is a unknown function and $X = (x_1, \cdots, x_n)$, in the fuzzy normed spaces setting. We first state how to construct these equations. The functional equation $f(x + 1) = xf(x)$, is called the gamma type functional equation, which is considered by Jung (cf. [23, 24]). This equation could be generalized the gamma type functional equation $f(x+p) = \varphi(x)f(x)$ and the beta type functional equation $f(x+p, y+p) = \varphi(x, y)f(x, y)$ [27], and also this last case could be extended again to the functional equation (1.1). For more details see [59].

[0]2010 Mathematics Subject Classification: Primary 39B52, 47H10, 46S40, 47S40, 34K36, 26E50.
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In 2004, Kim [28] studied the stability of the functional equation (1.1) in n-variables in Banach spaces. Recently, the first author and others have worked on some different functional equations in fuzzy normed spaces by using the direct method and the fixed point method (cf. [21, 35, 46, 50, 58]).

In this paper, using the ideas from the papers of Kim (cf. [27, 28]), we adopt the direct and fixed point methods to prove a generalized fuzzy version of the Hyers-Ulam stability for the functional equations (1.1) and (1.2) in n-variables in fuzzy Banach spaces.

To more clarify the reader, now we give briefly some useful information, definitions and fundamental results of fixed point theory and then fuzzy normed spaces, respectively.

Fixed point theory has a basic role in applications of some branches of mathematics. In 1996, for the first time, Isac and Rassias [20] provided applications of stability theory of functional equations for the proof of new fixed point theorems with applications. For more study about the stability problems of several functional equations, by using fixed point methods, one can refer to (cf. [6, 7, 8, 10, 11, 12, 13, 14, 25, 33, 41, 43, 44, 45, 48, 51, 52]).

Definition 1.1. Let $\mathcal{X}$ be a set. A function $d : \mathcal{X} \times \mathcal{X} \to [0, \infty]$ is called a *generalized metric* on $\mathcal{X}$ if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in \mathcal{X}$;
(3) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in \mathcal{X}$.

Theorem 1.2. ([6, 9]) *Let $(\mathcal{X}, d)$ be a complete generalized metric space and let $\mathcal{J} : \mathcal{X} \to \mathcal{X}$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in \mathcal{X}$, either*

$$d(\mathcal{J}^n x, \mathcal{J}^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(\mathcal{J}^n x, \mathcal{J}^{n+1} x) < \infty, \qquad \forall n \ge n_0$;
(2) *the sequence $\{\mathcal{J}^n x\}$ converges to a fixed point y^* of $\mathcal{J}$;*
(3) *y^* is the unique fixed point of $\mathcal{J}$ in the set $\mathcal{Y} = \{y \in \mathcal{X} \mid d(\mathcal{J}^{n_0} x, y) < \infty\}$;*
(4) *$d(y, y^*) \le \frac{1}{1-L} d(y, \mathcal{J} y)$ for all $y \in \mathcal{Y}$.*

In 1984, Katsaras [26] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy normed on a vector space from various points of view (cf. [2, 15, 26, 32, 38, 61]). In particular, Bag and Samanta [2] following Cheng and Mordeson [5], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [31]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [3].

Definition 1.3. ([2, 38, 39, 40]) Let $\mathcal{X}$ be a complex vector space. A function $\mathcal{N} : \mathcal{X} \times \mathbb{R} \to [0, 1]$ is called a *fuzzy norm* on $\mathcal{X}$ if for all $x, y \in \mathcal{X}$ and all $s, t \in \mathbb{R}$,

(N_1) $\mathcal{N}(x, t) = 0$ for $t \le 0$;
(N_2) $x = 0$ if and only if $\mathcal{N}(x, t) = 1$ for all $t > 0$;
(N_3) $\mathcal{N}(cx, t) = \mathcal{N}(x, \frac{t}{|c|})$ if $c \ne 0$;
(N_4) $\mathcal{N}(x + y, s + t) \ge \min\{\mathcal{N}(x, s), \mathcal{N}(y, t)\}$;
(N_5) $\mathcal{N}(x, \cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} \mathcal{N}(x, t) = 1$;
(N_6) for $x \ne 0$, $\mathcal{N}(x, \cdot)$ is continuous on $\mathbb{R}$.

The pair $(\mathcal{X}, \mathcal{N})$ is called a *fuzzy normed vector space.*

Example 1.4. *Let $(\mathcal{X}, \|\cdot\|)$ be a normed linear space and $\alpha, \beta > 0$. Then*

$$\mathcal{N}_{\mathcal{X}}(x, t) = \begin{cases} \frac{\alpha t}{\alpha t + \beta \|x\|} & t > 0, x \in \mathcal{X} \\ 0 & t \le 0, x \in \mathcal{X} \end{cases}$$

is a fuzzy norm on $\mathcal{X}$, which is called the induced fuzzy norm of $(\mathcal{X}, \|\cdot\|)$. In this case $(\mathcal{X}, \mathcal{N}_{\mathcal{X}})$ is called an induced fuzzy normed space.

Definition 1.5. Let $\mathcal{A}$ be a complex algebra and $(\mathcal{A}, \mathcal{N})$ be a fuzzy normed space. The fuzzy norm $\mathcal{N}$ is called an *algebraic fuzzy norm* on $\mathcal{A}$ if,
$(N7)$ $\mathcal{N}(xy, st) \geq \mathcal{N}(x, s) \cdot \mathcal{N}(y, t)$
for all $x, y \in \mathcal{A}$ and all $s, t \in \mathbb{R}$.

The pair $(\mathcal{A}, \mathcal{N})$ is called a *fuzzy normed algebra*.

Example 1.6. *Let $(\mathcal{A}, \|\cdot\|)$ be a normed algebra. Let*

$$\mathcal{N}(a, t) = \begin{cases} \frac{t}{t+\|a\|} & t > 0, \quad a \in \mathcal{A} \\ 0 & t \leq 0, \quad a \in \mathcal{A}. \end{cases}$$

Then $\mathcal{N}(a, t)$ is a fuzzy norm on $\mathcal{A}$ and $(\mathcal{A}, \mathcal{N}(a, t))$ is a fuzzy normed algebra.

Definition 1.7. ([2, 38, 39, 40]) Let $(\mathcal{X}, \mathcal{N})$ be a fuzzy normed vector space. A sequence $\{x_n\}$ in $\mathcal{X}$ is said to *be convergent* or *converge* if there exists an $x \in \mathcal{X}$ such that $\lim_{n\to\infty} \mathcal{N}(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by $\mathcal{N}$-$\lim_{n\to\infty} x_n = x$.

Definition 1.8. ([2, 38, 39, 40]) Let $(\mathcal{X}, \mathcal{N})$ be a fuzzy normed vector space. A sequence $\{x_n\}$ in $\mathcal{X}$ is called *Cauchy* if for each $\epsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $\mathcal{N}(x_{n+p} - x_n, t) > 1 - \epsilon$.

It is well known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*. A complete fuzzy normed algebra is called a *fuzzy Banach algebra*.

We say that a mapping $f : \mathcal{X} \to \mathcal{Y}$ between fuzzy normed vector spaces $\mathcal{X}$ and $\mathcal{Y}$ is continuous at a point $x_0 \in \mathcal{X}$ if for each sequence $\{x_n\}$ converging to x_0 in $\mathcal{X}$, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : \mathcal{X} \to \mathcal{Y}$ is continuous at each $x \in \mathcal{X}$, then $f : \mathcal{X} \to \mathcal{Y}$ is said to be *continuous* on $\mathcal{X}$ (see [3]).

Throughout this paper, suppose that $(\mathcal{B}, \mathcal{N})$ is a unital fuzzy Banach algebra (with unit e) over a field $\mathcal{K}$, where $\mathcal{K}$ is either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. $\beta > 0$ and $\epsilon > 0$ will be fixed positive real numbers and $\mathbb{R}_+$ denotes the set of all nonnegative real numbers. Assume that $(\mathcal{C}, \mathcal{N})$ is a fuzzy Banach space over $\mathcal{K}$. Given a nonempty set S and a mapping $\varphi : S^n \to S^n$, we define $\varphi_0(X) := X$ and $\varphi_n(X) := \varphi(\varphi_{n-1}(X))$ for all positive integers n and all $X \in S^n$. In addition, functions $\phi : S^n \to \mathcal{K} \setminus \{0\}$, and $\varepsilon : S^n \to \mathbb{R}_+$ and a mapping $\psi : S^n \to \mathcal{B}$ are defined.

2. Generalization of Hyers-Ulam stability to a fuzzy version for (1.1) and (1.2): fixed point method

In this section, by using the fixed point method, we prove a generalized fuzzy version of Hyers-Ulam stability for the functional equation (1.1) in n-variables and fuzzy Banach spaces.

Theorem 2.1. *Let ε, ψ, φ and ϕ be given as were explained in the introduction such that there exists an $L < 1$ with*

$$\varepsilon(\varphi(X)) \leq L\ |\phi(X)|\ \varepsilon(X)$$

for all $X \in S^n$. Let $f : S^n \to \mathcal{B}$ be a mapping satisfying

$$\mathcal{N}\left(f\left(\varphi(X)\right) - \phi(X)f(X) - \psi(X)e,\ t\right) \geq \frac{t}{t + \varepsilon\left(X\right)} \tag{2.1}$$

for all $X \in S^n$ and all $t > 0$. Then there exists a unique solution $H : S^n \to \mathcal{B}$ of (1.1) *such that*

$$H(X) = \mathcal{N}\text{-}\lim_{l\to\infty} \frac{f(\varphi_l(X))}{\prod_{j=0}^{l-1}\phi(\varphi_j(X))} - \sum_{k=0}^{l-1} \frac{\psi(\varphi_k(X))e}{\prod_{j=0}^{k}\phi(\varphi_j(X))}, \tag{2.2}$$

$$\mathcal{N}(f(X) - H(X),\ t) \geq \frac{(1-L)\ |\phi(X)|\ t}{(1-L)\ |\phi(X)|\ t + \varepsilon(X)} \tag{2.3}$$

for all $X \in S^n$ and all $t > 0$.

Proof. We consider the set $\mathcal{Q} := \{g : S^n \to \mathcal{B}\}$ and define the following generalized metric d:

$$d(g,h) = \inf\left\{\mu \in \mathbb{R}_+ :\ \mathcal{N}(g(X) - h(X),\ \mu t) \geq \frac{t}{t+\varepsilon(X)},\ \forall X \in S^n,\ \forall t > 0\right\}$$

where, as usual, $\inf \emptyset = +\infty$. It is easy to show that $(\mathcal{Q}, d)$ is complete (see the proof of [42, Lemma 2.1]). Consider the linear mapping $\mathcal{J}_{\varphi,\psi,\phi} : \mathcal{Q} \to \mathcal{Q}$, as follows:

$$\mathcal{J}_{\varphi,\psi,\phi}(g) := \frac{g(\varphi) - \psi e}{\phi}$$

for all $g \in \mathcal{Q}$, where the functions φ, ψ, ϕ are defined. For convenience, we write $\mathcal{J}(g)$, instead of $\mathcal{J}_{\varphi,\psi,\phi}(g)$. By the definition of $\mathcal{J}(g)$, we can get

$$\mathcal{J}^l\left(g(X)\right) = \frac{g(\varphi_l(X))}{\prod_{j=0}^{l-1}\phi(\varphi_j(X))} - \sum_{k=0}^{l-1} \frac{\psi(\varphi_k(X))e}{\prod_{j=0}^{k}\phi(\varphi_j(X))} \tag{2.4}$$

for all $g \in \mathcal{Q}$, all $X \in S^n$ and all positive integers l.

From (2.1) and (N_3), it follows that

$$\mathcal{N}\left(\mathcal{J}(f(X)) - f(X),\ \frac{t}{|\phi(X)|}\right) = \mathcal{N}\left(\frac{f\left(\varphi(X)\right) - \psi(X)e}{\phi(X)} - f(X),\ \frac{t}{|\phi(X)|}\right) \geq \frac{t}{t+\varepsilon(X)}$$

for all $X \in S^n$, which means

$$d\left(\mathcal{J}(f), f\right) \leq \frac{1}{|\phi(X)|} \tag{2.5}$$

for all $X \in S^n$. Assume that $g, h \in \mathcal{Q}$ are given with $d(g,h) = \epsilon$. Then we have

$$\mathcal{N}\left(g(X) - h(X),\ \epsilon t\right) \geq \frac{t}{t+\varepsilon(X)}$$

for all $X \in S^n$ and all $t > 0$. By the definition of $\mathcal{J}(g)$ and then substituting X and t by $\varphi(X)$ and $L\ |\phi(X)|\ t$ respectively, we obtain

$$\begin{aligned}
&\mathcal{N}\left(\mathcal{J}(g(X)) - \mathcal{J}(h(X)),\ L\epsilon t\right) \\
&\quad = \mathcal{N}\left(\frac{g(\varphi(X)) - h(\varphi(X))}{\phi(X)},\ L\epsilon t\right) = \mathcal{N}\left(g(\varphi(X)) - h(\varphi(X)),\ L\ |\phi(X)|\ \epsilon t\right) \\
&\quad \geq \frac{L\ |\phi(X)|\ t}{L\ |\phi(X)|\ t + \varepsilon(\varphi(X))} \geq \frac{L\ |\phi(X)|\ t}{L\ |\phi(X)|\ t + L\ |\phi(X)|\ \varepsilon(X)} \\
&\quad = \frac{t}{t+\varepsilon(X)}
\end{aligned}$$

for all $X \in S^n$ and all $t > 0$, which leads us to the inequality

$$d(\mathcal{J}(g), \mathcal{J}(h)) \leq L\epsilon = Ld(g,h)$$

for all $g, h \in \mathcal{Q}$.

Therefore, $\mathcal{J}$ is a strictly contractive mapping with Lipschitz constant $L < 1$.

According to Theorem 1.2:

(1) $\mathcal{J}$ has a fixed point, i.e., there exists a mapping $H : S^n \to \mathcal{B}$ such that $\mathcal{J}(H) = H$, and so

$$H(X) = \frac{H(\varphi(X)) - \psi(X)e}{\phi(X)} \tag{2.6}$$

for all $X \in S^n$. The mapping H is also the unique fixed point of $\mathcal{J}$ in the set

$$\mathcal{M} = \{g \in \mathcal{Q} : d(f, g) < \infty\}.$$

This implies that H is a unique mapping satisfying (2.6), moreover there exists a $\mu \in (0, \infty)$ such that

$$\mathcal{N}(f(X) - H(X),\ \mu t) \geq \frac{t}{t + \varepsilon(X)}$$

for all $X \in S^n$;

(2) The sequence $\{\mathcal{J}^n(g)\}$ converges to H, for each given $g \in \mathcal{Q}$. Thus $d\left(\mathcal{J}^n(f), H\right) \to 0$ as $n \to \infty$. This signifies the equality

$$H(X) = \mathcal{N}\text{-}\lim_{n\to\infty} \mathcal{J}^n\left(f(X)\right)$$

for all $X \in S^n$. From this and (2.4), we deduce that (2.2) holds;

(3) $d(g, H) \leq \frac{1}{1-L} d\left(g, \mathcal{J}(g)\right)$ for all $g \in \mathcal{M}$. So it follows from (2.5) that

$$\begin{aligned} d(f, H) \leq \frac{1}{1-L} d\left(f, \mathcal{J}(f)\right) \quad &\leq \quad \frac{1}{(1-L)|\phi(X)|}, \\ \mathcal{N}\left(f(X) - H(X),\ \frac{t}{(1-L)|\phi(X)|}\right) \quad &\geq \quad \frac{t}{t + \varepsilon(X)} \end{aligned}$$

for all $X \in S^n$ and all $t > 0$. Putting $(1 - L)\ |\phi(X)|\ t$ instead of t in the above inequality, we get the inequality (2.3).

Now, we show that $H : S^n \to \mathcal{B}$ satisfies (1.1) as follows:

By (2.4), one can show easily that $\mathcal{J}^n(\varphi(X)) = \phi(X)\mathcal{J}^{n+1}(X) + \psi(X)e$. By (N_4), for any fixed $t > 0$, we have

$$\begin{aligned} &\mathcal{N}\left(H(\varphi(X)) - \phi(X)H(X) - \psi(X)e,\ t\right) \\ &\qquad \geq \min\left\{\mathcal{N}\left(H(\varphi(X)) - \mathcal{J}^n(\varphi(X)),\ \frac{t}{3}\right),\ \mathcal{N}\left(\mathcal{J}^{n+1}(X)\phi(X) - H(X)\phi(X),\ \frac{t}{3}\right),\right. \\ &\qquad\qquad \left.\mathcal{N}\left(\mathcal{J}^n(\varphi(X)) - \phi(X)\mathcal{J}^{n+1}(X) - \psi(X)e,\ \frac{t}{3}\right)\right\} \end{aligned}$$

in which, every three terms on the right-hand side tend to one as $n \to \infty$. Hence

$$\mathcal{N}\left(H(\varphi(X)) - \phi(X)H(X) - \psi(X)e,\ t\right) = 1$$

for all $t > 0$ and all $X \in S^n$, which means by (N_2) that $H(\varphi(X)) - \phi(X)H(X) - \psi(X)e = 0$ for all $X \in S^n$. So the mapping $H : S^n \to \mathcal{B}$ satisfies (1.1), and the proof is complete. □

Corollary 2.2. *Let ε, φ and ϕ be given in the introduction as were explained such that there exists an $L < 1$ with*

$$\varepsilon(\varphi(X)) \leq L\ |\phi(X)|\ \varepsilon(X)$$

for all $X \in S^n$. Let $f : S^n \to \mathcal{C}$ be a mapping satisfying

$$\mathcal{N}\left(f\left(\varphi(X)\right) - \phi(X)f(X),\ t\right) \geq \frac{t}{t + \varepsilon\left(X\right)}$$

for all $X \in S^n$ and all $t > 0$. Then there exists a unique solution $H : S^n \to \mathcal{C}$ of (1.2) such that

$$H(X) = \mathcal{N}\text{-}\lim_{l\to\infty} \frac{f(\varphi_l(X))}{\prod_{j=0}^{l-1} \phi(\varphi_j(X))},$$

$$\mathcal{N}(f(X) - H(X),\ t) \geq \frac{(1-L)\ |\phi(X)|\ t}{(1-L)\ |\phi(X)|\ t + \varepsilon(X)}$$

for all $X \in S^n$ and all $t > 0$.

Proof. Putting $\psi(X) = 0$, and then applying Theorem 2.1, we get easily the desired result. □

The above results include more cases, for example, to obtain simpler results, we can put $\varphi(X) = X + P$, $S = (0, \infty)$ and $\mathcal{B} = \mathbb{R}$. In this case, $X = (x_1, x_2, \cdots, x_n)$ and $P = (p_1, p_2, \cdots, p_n)$ are supposed. For more details, see [28].

3. Generalization of Hyers-Ulam stability to a fuzzy version for (1.1) and (1.2): direct method

In this section, by using the direct method, we prove a generalized fuzzy version of Hyers-Ulam stability for the functional equations (1.1) and (1.2) in n-variables and fuzzy Banach spaces.

Consider the following condition:

If φ, ϕ and ε be given as were explained, then

$$\omega(X) := \sum_{k=0}^{\infty} \frac{\varepsilon\left(\varphi_k(X)\right)}{\prod_{j=0}^{k} |\phi(\varphi_j(X))|} < \infty \tag{3.1}$$

for all $X \in S^n$.

The condition (3.1) will be used in this section to express our theorems and proofs.

Theorem 3.1. *Let φ, ϕ and ε satisfy (3.1). Let $f : S^n \to \mathcal{B}$ be a mapping satisfying*

$$\lim_{t\to\infty} \mathcal{N}\left(f(\varphi(X)) - \phi(X)f(X) - \psi(X)e,\ t\,\varepsilon(X)\right) = 1 \tag{3.2}$$

for all $X \in S^n$. Then there exists a unique solution $g : S^n \to \mathcal{B}$ of (1.1) such that

$$\lim_{t\to\infty} \mathcal{N}(f(X) - g(X),\ t\,\omega(X)) = 1 \tag{3.3}$$

for all $X \in S^n$.

Proof. We start with defining the functions $\omega_l : S^n \to \mathbb{R}_+$ and the mappings $g_l : S^n \to \mathcal{B}$ as follows:

$$\begin{aligned} \omega_l(X) &:= \sum_{k=0}^{l-1} \frac{\varepsilon(\varphi_k(X))}{\prod_{j=0}^{k} |\phi(\varphi_j(X))|}, \\ g_l(X) &:= \frac{f(\varphi_l(X))}{\prod_{j=0}^{l-1} \phi(\varphi_j(X))} - \sum_{k=0}^{l-1} \frac{\psi(\varphi_k(X))e}{\prod_{j=0}^{k} \phi(\varphi_j(X))} \end{aligned}$$

for all $X \in S^n$ and all positive integers l.

By (3.2), for a given $\epsilon > 0$, one can assert that there is some $t_0 > 0$ such that

$$\begin{aligned} \mathcal{N}\left(f(\varphi(X)) - \phi(X)f(X) - \psi(X)e,\ t\,\varepsilon(X)\right) &\geq 1 - \epsilon, \\ \mathcal{N}\left(\frac{f(\varphi(X))}{\phi(X)} - f(X) - \frac{\psi(X)e}{\phi(X)},\ t\,\frac{\varepsilon(X)}{|\phi(X)|}\right) &\geq 1 - \epsilon \end{aligned} \tag{3.4}$$

for all $t \geq t_0$ and all $X \in S^n$. From the definition of g_l and then replacing X by $\varphi_l(X)$ in (3.4), we obtain

$$\begin{aligned} &\mathcal{N}\left(g_{l+1}(X) - g_l(X),\ t\,\frac{\varepsilon\left(\varphi_l(X)\right)}{\prod_{j=0}^{l} |\phi\left(\varphi_j(X)\right)|}\right) \\ &= \mathcal{N}\left(\prod_{j=0}^{l-1} \phi\left(\varphi_j(X)\right)[g_{l+1}(X) - g_l(X)],\ t\,\frac{\varepsilon\left(\varphi_l(X)\right)}{|\phi\left(\varphi_l(X)\right)|}\right) \\ &= \mathcal{N}\left(\frac{f(\varphi_{l+1}(X))}{\phi(\varphi_l(X))} - f(\varphi_l(X)) - \frac{\psi(\varphi_l(X))e}{\phi(\varphi_l(X))},\ t\,\frac{\varepsilon(\varphi_l(X))}{|\phi(\varphi_l(X))|}\right) \geq 1 - \epsilon \end{aligned} \tag{3.5}$$

for all $t \geq t_0$ and all $X \in S^n$. Using the induction method, we prove that

$$\mathcal{N}\left(g_l(X) - f(X),\ t\,\omega_l(X)\right) \ \geq\ 1 - \epsilon \tag{3.6}$$

for all $t \geq t_0$, all $X \in S^n$ and all positive integers l. The case $l = 1$ is the inequality (3.4), and so we assume that the inequality (3.6) holds true for some l. The case $l + 1$ is an immediate consequence of (3.5) and (3.6). Indeed, we have

$$\begin{aligned}
&\mathcal{N}\left(g_{l+1}(X) - f(X),\ t\,\omega_{l+1}(X)\right)\\
&\quad \geq \min\left\{\mathcal{N}\left(g_{l+1}(X) - g_l(X),\ t\ \frac{\varepsilon\left(\varphi_l(X)\right)}{\prod_{j=0}^{l} |\phi\left(\varphi_j(X)\right)|}\right),\right.\\
&\qquad\qquad \left.\mathcal{N}\left(g_l(X) - f(X),\ t\sum_{k=0}^{l-1}\frac{\varepsilon(\varphi_k(X))}{\prod_{j=0}^{k}|\phi(\varphi_j(X))|}\right)\right\}\\
&\quad \geq \min\{1-\epsilon, 1-\epsilon\} = 1-\epsilon,
\end{aligned}$$

which ends the induction method.

Now we claim that $\{g_l(X)\}$ is a Cauchy sequence. In order to verify that, by (3.5), for $l+p > l > 0$ and $p > 0$, we have

$$\begin{aligned}
&\mathcal{N}\left(g_{l+p}(X) - g_l(X),\ t\sum_{k=l}^{l+p-1}\frac{\varepsilon\left(\varphi_k(X)\right)}{\prod_{j=0}^{k}|\phi(\varphi_j(X))|}\right)\\
&\quad \geq \min_{l \leq i \leq l+p-1}\left\{\mathcal{N}\left(g_{i+1}(X) - g_i(X),\ t\ \frac{\varepsilon\left(\varphi_i(X)\right)}{\prod_{j=0}^{i}|\phi(\varphi_j(X))|}\right)\right\}\\
&\quad \geq \min\left\{1-\epsilon, \cdots, 1-\epsilon\right\} = 1-\epsilon
\end{aligned} \tag{3.7}$$

for all $t \geq t_0$ and all $X \in S^n$ (especially for $t = t_0$). The condition (3.1) implies that for a given $\delta > 0$, there is an $l_0 \in \mathbb{N}$ such that

$$t_0 \sum_{k=l}^{l+p-1}\frac{\varepsilon(\varphi_k(X))}{\prod_{j=0}^{k}|\phi(\varphi_j(X))|} \ <\ \delta$$

for all $l \geq l_0$ and all $p > 0$. From this, (3.7), (N_5) and (N_6) for the function $\mathcal{N}\left(g_{l+p}(X) - g_l(X),\ \cdot\right)$, we deduce that

$$\mathcal{N}\left(g_{l+p}(X) - g_l(X),\ \delta\right) \geq \mathcal{N}\left(g_{l+p}(X) - g_l(X),\ t_0\sum_{k=l}^{l+p-1}\frac{\varepsilon\left(\varphi_k(X)\right)}{\prod_{j=0}^{k}|\phi(\varphi_j(X))|}\right) \geq 1-\epsilon$$

for all $l \geq l_0$ and all $p > 0$. Therefore the sequence $\{g_l(X)\}$ is Cauchy in $\mathcal{B}$. Since $\mathcal{B}$ is a fuzzy Banach space, the sequence $\{g_l(X)\}$ converges to some $g(X) \in \mathcal{B}$, and so we can define for all $X \in S^n$, a function $g : S^n \to \mathcal{B}$ by

$$g(X) := \mathcal{N} - \lim_{l\to\infty} g_l(X).$$

In other words, $\lim_{l\to\infty}\mathcal{N}(g_l(X) - g(X)\ ,\ t) = 1$ for all $t > 0$.

By the same argument which was done for $\mathcal{J} : S^n \to \mathcal{B}$ in the proof of Theorem 2.1, we can easily show that the mapping $g : S^n \to \mathcal{B}$ satisfies (1.1).

In continue, we will also show that $g : S^n \to \mathcal{B}$ satisfies the equality (3.3).

Let $t \geq t_0$, $t' > 0$ and $0 < \epsilon < 1$. Comparing ω with ω_l and using (N_5) and (N_6) for the function $\mathcal{N}(f(X) - g(X),\ \cdot)$, we obtain that

$$\begin{aligned}
&\mathcal{N}(f(X) - g(X),\ t\,\omega(X) + t')\\
&\quad \geq \mathcal{N}(f(X) - g(X),\ t\,\omega_l(X) + t')\\
&\quad \geq \min\left\{\mathcal{N}\left(f(X) - g_l(X),\ t\,\omega_l(X)\right),\ \mathcal{N}\left(g_l(X) - g(X),\ t'\right)\right\}
\end{aligned}$$

for all $X \in S^n$, and so by (3.6) and the fact that $\lim_{l\to\infty} \mathcal{N}(g_l(X) - g(X),\ t') = 1$, we obtain

$$\mathcal{N}(f(X) - g(X),\ t\ \omega(X) + t') \geq 1 - \epsilon$$

if $l \in \mathbb{N}$ is large enough. Let $t' \to 0$ and $t \to \infty$. Then the above inequality clearly signifies that (3.3) holds.

To finish the proof, it is just necessary to show that the mapping $g : S^n \to \mathcal{B}$ is unique. So let $h : S^n \to \mathcal{B}$ be another mapping satisfying (1.1) and (3.3).

It follows from (1.1) for g and h that

$$g(X) - h(X) = \frac{g(\varphi(X)) - h(\varphi(X))}{\phi(X)}$$

for all $X \in S^n$. Substituting X by $\varphi(X)$ continually, we lead to

$$g(X) - h(X) = \frac{g(\varphi_l(X)) - h(\varphi_l(X))}{\prod_{j=0}^{l-1} \phi(\varphi_j(X))} \tag{3.8}$$

for all $X \in S^n$. Fix $c > 0$. Given $\epsilon > 0$, the inequality (3.3) decides some $t_0 > 0$ for h and g such that

$$\begin{aligned} \mathcal{N}(f(X) - g(X),\ t\ \omega(X)) &\geq 1 - \epsilon, \\ \mathcal{N}(f(X) - h(X),\ t\ \omega(X)) &\geq 1 - \epsilon \end{aligned}$$

for all $t \geq t_0$ and all $X \in S^n$ (specially for $t = t_0$). From (3.1), we know that there is a large enough integer $l_0 > 0$ such that

$$t_0 \sum_{k=l}^{\infty} \frac{\varepsilon(\varphi_k(X))}{\prod_{j=0}^{k} |\phi(\varphi_j(X))|} < \frac{c}{2} \tag{3.9}$$

for all $l \geq l_0$. By (3.8), (3.9) and the fact that

$$\begin{aligned} \sum_{k=l}^{\infty} \frac{\varepsilon(\varphi_k(X))}{\prod_{j=0}^{k} |\phi(\varphi_j(X))|} &= \left(\sum_{k=0}^{\infty} \frac{\varepsilon(\varphi_{l+k}(X))}{\prod_{j=0}^{k} |\phi(\varphi_{l+j}(X))} \right) \cdot \frac{1}{\prod_{j=0}^{l-1} |\phi(\varphi_j(X))|} \\ &= \omega(\varphi_l(X)) \cdot \frac{1}{\prod_{j=0}^{l-1} |\phi(\varphi_j(X))|}, \end{aligned}$$

we obtain

$$\begin{aligned} \mathcal{N}(g(X) - h(X),\ c) &= \mathcal{N}\left(\frac{g(\varphi_l(X)) - h(\varphi_l(X))}{\prod_{j=0}^{l-1} \phi(\varphi_j(X))},\ c \right) \\ &\geq \min\left\{ \mathcal{N}\left(\frac{g(\varphi_l(X)) - f(\varphi_l(X))}{\prod_{j=0}^{l-1} \phi(\varphi_j(X))},\ \frac{c}{2} \right), \right. \\ &\qquad \left. \mathcal{N}\left(\frac{f(\varphi_l(X)) - h(\varphi_l(X))}{\prod_{j=0}^{l-1} \phi(\varphi_j(X))},\ \frac{c}{2} \right) \right\} \\ &\geq \min\{ \mathcal{N}(g(\varphi_l(X)) - f(\varphi_l(X)),\ t_0\ \omega(\varphi_l(X))), \\ &\qquad \mathcal{N}(f(\varphi_l(X)) - h(\varphi_l(X)),\ t_0\ \omega(\varphi_l(X))) \} \\ &\geq 1 - \epsilon \end{aligned}$$

for all $X \in S^n$ and all positive integers l, which shows the uniqueness of g. □

Now let φ and ϕ satisfy

$$\mu(X) := \sum_{k=0}^{\infty} \prod_{j=0}^{k} \frac{1}{|\phi(\varphi_j(X))|} < \infty \tag{3.10}$$

for all $X \in S^n$. Putting $\varepsilon(X) = \beta > 0$ in Theorem 3.1, we have the following corollary:

Corollary 3.2. *Let φ, ϕ satisfy* (3.10) *and $f : S^n \to \mathcal{B}$ a mapping satisfying*

$$\lim_{t\to\infty} \mathcal{N}\left(f(\varphi(X)) - \phi(X)f(X) - \psi(X)e,\ t\ \beta\right) = 1$$

for all $X \in S^n$. Then there exists a unique solution $g : S^n \to \mathcal{B}$ of (1.1) *such that*

$$\lim_{t\to\infty} \mathcal{N}(f(X) - g(X),\ t\ \mu(X)) = 1$$

for all $X \in S^n$.

If we put $\psi(X) = 0$ in Theorem 3.1 and Corollary 3.2, then we get the results for (1.2).

Remark 3.3. All the obtained results could be applied to the gamma type, the G-function, Schröder, and the beta type functional equations. In other words, we can set many different type and suitable functions instead of $\varphi(X)$, $\varepsilon(X)$ and $\phi(X)$ to obtain more interesting and useful results. For more applications, examples and details, we refer the reader to (cf. [22, 23, 24, 27, 28]).

Acknowledgments

D. Y. Shin and C. Park were supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2010-0021792) and (NRF-2012R1A1A2004299), respectively.

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Volume 19, Number 2 August 2015
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

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(twelve times annually)

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<u>Book</u>

2. G.G.Lorentz, (title of book in italics) *Bernstein Polynomials* (2nd ed.), Chelsea,New York,1986.

<u>Contribution to a Book</u>

3. M.K.Khan, Approximation properties of beta operators,in(title of book in italics) *Progress in Approximation Theory* (P.Nevai and A.Pinkus,eds.), Academic Press, New York,1991,pp.483-495.

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The similarity and application for generalized interval-valued fuzzy soft sets

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Abstract

In this paper, by combining the interval-valued fuzzy set and fuzzy soft set, we define generalized interval-valued fuzzy soft sets which is an extension to the fuzzy soft set. The complement, union, intersection and sum operations are also investigated. We have further studied the similarity between two generalized interval-valued fuzzy soft sets. Finally, application of generalized interval-valued fuzzy soft sets in decision making problem has been shown.

Key words: Interval-valued fuzzy set; Fuzzy soft set; Generalized interval-valued fuzzy soft sets; Similarity measure; Decision making

1 Introduction

Molodtsov [1] initiated a novel concept called soft sets as a new mathematical tool for dealing with uncertainties. The soft set theory is free from many difficulties that have troubled the usual theoretical approaches. It has been found that fuzzy sets, rough sets, and soft sets are closely related concepts [2]. Soft set theory has potential applications in many different fields including the smoothness of functions, game theory, operational research, Perron integration, probability theory, and measurement theory [1,3]. Research works on soft sets are very active and progressing rapidly in these years. Maji et al. [4] defined several operations on soft sets and made a theoretical study on the theory of soft sets. Feng et al. [5] applied soft set theory to the study of semirings and initiated the notion of soft semirings. Furthermore, based on [4], Ali et al. [6] introduced some new

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operations on soft sets and improved the notion of complement of soft set. They proved that certain De Morgans laws hold in soft set theory. Qin and Hong [7] introduced the notion of soft equality and established lattice structures and soft quotient algebras of soft sets. Maji et al [8] presented the notion of generalized fuzzy soft sets theory which is based on a combination of the fuzzy set and soft set models. Yang et al. [9] presented the concept of the interval-valued fuzzy soft sets by combining interval-valued fuzzy set [10–12] and soft set models. Feng et al. [13]provided a framework to combine fuzzy sets, rough sets and soft sets all together, which gives rise to several interesting new concepts such as rough soft sets, soft rough sets and soft rough fuzzy sets. Park et al [14] discussed some properties of equivalence soft set relations. By combining the multi-fuzzy set and soft set models, Yang et al. [15] presented the concept of the multi-fuzzy soft set, and provided its application in decision making under an imprecise environment. Shabir [16] presented a new approach to soft rough sets by combining the rough set and soft set.

The purpose of this paper is to combine the interval-valued fuzzy sets and generalized fuzzy soft soft, from which we can obtain a new soft set model: generalized interval-valued fuzzy soft set theory. Intuitively, generalized interval-valued fuzzy soft set theory presented in this paper is an extension of interval-valued fuzzy soft set and generalized fuzzy soft set. We have further studied the similarity between two generalized interval-valued fuzzy soft sets. We finally present examples which show that the decision making method of generalized interval-valued fuzzy soft set can be successfully applied to many problems that contain uncertainties.

The rest of this paper is organized as follows. The following section briefly reviews some background on soft set, fuzzy soft set and interval-valued fuzzy set. In section 3, the concept of generalized interval-valued fuzzy soft set is presented. The complement, union, intersection and sum operations on the generalized interval-valued fuzzy soft set are then defined. Also their some interesting properties have been investigated. In section 4, similarity between two generalized interval-valued fuzzy soft sets has been discussed. An application of generalized interval-valued fuzzy soft set in decision making problem has been shown in section 5. Section 6 concludes the paper.

2 Preliminaries

In this section we give few definitions regarding soft sets.

Definition 2.1 ([1]) *Let U be an initial universe set and E be a universe set of parameters. A pair (F, A) is called a soft set over U if $A \subset E$ and $F : A \to P(U)$, where $P(U)$ is the set of all subsets of U.*

Definition 2.2 ([17]) *Let U be an initial universe set and E be a universe set of parameters. A pair (F, A) is called a fuzzy soft set over U if $A \subset E$ and $F : A \to F(U)$, where $F(U)$ is the set of all fuzzy subsets of U.*

Definition 2.3 ([10]) *An interval-valued fuzzy set $\hat{X}$ on an universe U is a mapping such that*

$$\hat{X} : U \to Int([0,1]),$$

where Int([0,1]) stands for the set of all closed subintervals of [0,1].

For the sake of convenience, the set of all interval-valued fuzzy sets on U is denoted by $IVF(U)$. Suppose that $\hat{X} \in IVF(U), \forall x \in U, \mu_{\hat{X}}(x) = [\mu^-_{\hat{X}}(x), \mu^+_{\hat{X}}(x)]$ is called the degree of membership an element x to $\hat{X}$. $\mu^-_{\hat{X}}(x)$ and $\mu^+_{\hat{X}}(x)$ are referred to as the lower and upper degrees of membership an element x to $\hat{X}$ where $0 \leq \mu^-_{\hat{X}}(x) \leq \mu^+_{\hat{X}}(x) \leq 1$.

The basic operations on $IVF(U)$ are defined as follows : for all $\hat{X}, \hat{Y} \in IVF(U)$, then
(1) the complement of $\hat{X}$ is denoted by $\hat{X}^C$ where
$\mu^C_{\hat{X}}(x) = 1 - \mu_{\hat{X}}(x) = [1 - \mu^+_{\hat{X}}(x), 1 - \mu^-_{\hat{X}}(x)]$;
(2) the intersection of $\hat{X}$ and $\hat{Y}$ is denoted by $\hat{X} \cap \hat{Y}$ where
$\mu_{\hat{X}\cap\hat{Y}}(x) = inf[\mu_{\hat{X}}(x), \mu_{\hat{Y}}(x)] = [inf(\mu^-_{\hat{X}}(x), \mu^-_{\hat{Y}}(x)), inf(\mu^+_{\hat{X}}(x), \mu^+_{\hat{Y}}(x))]$;
(3) the union of $\hat{X}$ and $\hat{Y}$ is denoted by $\hat{X} \cup \hat{Y}$ where
$\mu_{\hat{X}\cup\hat{Y}}(x) = sup[\mu_{\hat{X}}(x), \mu_{\hat{Y}}(x)] = [sup(\mu^-_{\hat{X}}(x), \mu^-_{\hat{Y}}(x)), sup(\mu^+_{\hat{X}}(x), \mu^+_{\hat{Y}}(x))]$;
(4) the sum of $\hat{X}$ and $\hat{Y}$ is denoted by $\hat{X} \oplus \hat{Y}$ where
$\mu_{\hat{X}\oplus\hat{Y}}(x) = [inf\{1, (\mu^-_{\hat{X}}(x) + \mu^-_{\hat{Y}}(x))\}, inf\{1, (\mu^+_{\hat{X}}(x) + \mu^+_{\hat{Y}}(x))\}]$;

3 Generalized interval-valued fuzzy soft set

3.1 Concept of generalized interval-valued fuzzy soft set

In this subsection, we give a modified definition of interval-valued fuzzy soft set.

Definition 3.1 *Let U be an initial universe and E be a set of parameters. The pair (U, E) is called a soft universe. Suppose that $F : E \to IVF(U)$, and f is an interval-valued fuzzy subset of E, i.e.$f : E \to Int([0,1])$, we say that F_f is a generalized interval-valued fuzzy soft set(GIVFSS, in short) over the soft universe (U, E) if and only if F_f is a mapping given by*

$$F_f : E \to IVF(U) \times Int([0,1]),$$

where $F_f(e) = (F(e), f(e)), F(e) \in IVF(U)$, and $f(e) \in Int([0,1])$

Here for each parameter $e_i, F_f(e_i) = (F(e_i), f(e_i))$ indicates not only the range of belongingness of the elements of U in $F(e_i)$ but also the range of possibility of such be belongingness which is represented by $f(e_i))$.

For all $e \in E, F_f(e)$ is actually a generalized interval-valued fuzzy set of (U, E), where $x \in U$ and $e \in E$, it can be written as:

$$F_f(e) = (F(e), f(e)),$$

where $F(e) = \{< x, [\mu^-_{F(e)}(x), \mu^+_{F(e)}(x)] >: x \in U\}, f(e) = [\mu^-_{f(e)}(x), \mu^+_{f(e)}(x)]$.

Remark 3.2 *A generalized interval-valued fuzzy soft set is also a special case of a soft set because it is still a mapping from parameters to $IVF(U) \times Int([0, 1])$. If $\forall e \in E, \forall x \in U, \mu^-_{F(e)}(x) = \mu^+_{F(e)}(x)$, and $\mu^-_{f(e)}(x) = \mu^+_{f(e)}(x)$, then F_f will be degenerated to be a generalized fuzzy soft sets [8].*

Example 3.3 *Let U be a set of three houses under consideration of a decision maker to purchase, which is denoted by $U = \{h_1, h_2, h_3\}$. Let E be a parameter set, where $E = \{e_1, e_2, e_3\} = \{expensive; beautiful; in\ the\ green\ surroundings\}$. Suppose that $f(e_1) = [0.6, 0.9], f(e_2) = [0.3, 0.5], f(e_3) = [0.1, 0.2]$. We define a function $F_f : E \to IVF(U) \times Int([0, 1])$, given by as follows:*

$$F_f(e_1) = (\{\frac{[0.6, 0.7]}{h_1}, \frac{[0.3, 0.5]}{h_2}, \frac{[0.6, 0.8]}{h_3}, \}, [0.6, 0.9])$$

$$F_f(e_2) = (\{\frac{[0.1, 0.3]}{h_1}, \frac{[0.2, 0.4]}{h_2}, \frac{[0.8, 1]}{h_3}, \}, [0.3, 0.5])$$

$$F_f(e_3) = (\{\frac{[0.4, 0.7]}{h_1}, \frac{[0.5, 0.8]}{h_2}, \frac{[0.2, 0.4]}{h_3}, \}, [0.1, 0.2])$$

Then F_f is a GIVFSS over (U, E).

By the above example, we can see that the precise evaluation for each parameter is unknown while the lower and upper limits of such an evaluation is given. Meanwhile, the precise evaluation for each object on each parameter is unknown while the lower and upper limits of such an evaluation is given. For example, we can describe the degree of three"expensive" houses by the interval [0.6,0.9]. In other words, the concept of "expensive" is vague, but it can be characterized by the interval [0.6,0.9].

In matrix from this can be expressed as

$$F_f = \begin{pmatrix} [0.6,0.7] & [0.3,0.5] & [0.6,0.8] & [0.6,0.9] \\ [0.1,0.3] & [0.2,0.4] & [0.8,1] & [0.3,0.5] \\ [0.4,0.7] & [0.5,0.8] & [0.2,0.4] & [0.1,0.2] \end{pmatrix}$$

where the ith row vector represent $F_f(e_i)$, the ith column vector represent x_i, the last column represent the range of f and it will be called membership interval-valued matrix of F_f.

Definition 3.4 *Let F_f and G_g be two GIVFSSs over (U, E). Now F_f is said to be a generalized interval-valued fuzzy soft subset of G_g if and only if*

(1) f is an interval-valued fuzzy subset of g;

(2) $F(e)$ is also an interval-valued fuzzy subset of $G(e), \forall e \in E$.

In this case, we write $F_f \subseteq G_g$.

Example 3.5 *Consider the GIVFSS F_f over (U, E) given in Example 3.3. Let G_g be another GIVFSS over (U, E) defined as follows:*

$$G_g(e_1) = (\{\frac{[0.4, 0.6]}{h_1}, \frac{[0, 0.4]}{h_2}, \frac{[0.5, 0.7]}{h_3}, \}, [0.4, 0.8])$$

$$G_g(e_2) = (\{\frac{[0, 0.2]}{h_1}, \frac{[0.1, 0.3]}{h_2}, \frac{[0.6, 0.9]}{h_3}, \}, [0.2, 0.4])$$

$$G_g(e_3) = (\{\frac{[0.2, 0.6]}{h_1}, \frac{[0.3, 0.6]}{h_2}, \frac{[0, 0.3]}{h_3}, \}, [0, 0.2])$$

Clearly, we have $G_g \subseteq F_f$.

Definition 3.6 *Let F_f and G_g be two GIVFSSs over (U, E) . Now F_f and G_g are said to be a generalized interval-valued fuzzy soft equal if and only if*

(1) F_f is a generalized interval-valued fuzzy soft subset of G_g;

(2) G_g is a generalized interval-valued fuzzy soft subset of F_f,

which can be denoted by $F_f = G_g$.

3.2 Operations on generalized interval-valued fuzzy soft set

Definition 3.7 *The complement of F_f denoted by F_f^C and is defined by $F_f^C = G_g$, where $G(e) = F^C(e), g(e) = f^C(e)$.*

From the above definition, we can see that $(F_f^C)^C = F_f$.

Example 3.8 *Consider the GIVFSS G_g over (U, E) defined in Example 3.5. Thus, by Definition 3.7, we have*

$$G_g^C = \begin{pmatrix} [0.4,0.6] & [0.6,1] & [0.3,0.5] & [0.2,0.6] \\ [0.8,1] & [0.7,0.9] & [0.1,0.4] & [0.6,0.8] \\ [0.4,0.8] & [0.4,0.7] & [0.7,1] & [0.8,1] \end{pmatrix}$$

Definition 3.9 *The union operation on the two GIVFSSs F_f and G_g, denoted by $F_f \cup G_g$, is defined by a mapping given by $H_h : E \to IVF(U) \times Int([0, 1])$, such that $H_h(e) = (H(e), h(e))$, where $H(e) = F(e) \cup G(e), h(e) = f(e) \cup g(e)$.*

Definition 3.10 *The intersection operation on the two GIVFSSs F_f and G_g, denoted by $F_f \cap G_g$, is defined by a mapping given by $H_h : E \to IVF(U) \times Int([0, 1])$, such that $H_h(e) = (H(e), h(e))$, where $H(e) = F(e) \cap G(e), h(e) = f(e) \cap g(e)$.*

Definition 3.11 *The sum operation on the two GIVFSSs F_f and G_g, denoted by $F_f \oplus G_g$, is defined by a mapping given by $H_h : E \to IVF(U) \times Int([0,1])$, such that $H_h(e) = (H(e), h(e))$, where $H(e) = F(e) \oplus G(e), h(e) = f(e) \oplus g(e)$.*

Example 3.12 *Let us consider the GIVFSS F_f in Example 3.3. Let G_g be another GIVFSS over (U, E) defined as follows:*

$$G_g(e_1) = (\{\frac{[0.5, 0.8]}{h_1}, \frac{[0, 0.2]}{h_2}, \frac{[0.2, 0.9]}{h_3}, \}, [0.5, 1])$$

$$G_g(e_2) = (\{\frac{[0, 0.4]}{h_1}, \frac{[0.3, 0.5]}{h_2}, \frac{[0.9, 1.0]}{h_3}, \}, [0.4, 0.6])$$

$$G_g(e_3) = (\{\frac{[0.5, 0.6]}{h_1}, \frac{[0.6, 0.7]}{h_2}, \frac{[0.1, 0.3]}{h_3}, \}, [0.2, 0.4])$$

Then

$$F_f \cup G_g = \begin{pmatrix} [0.6,0.8] & [0.3,0.5] & [0.6,0.9] & [0.6,1.0] \\ [0.1,0.4] & [0.3,0.5] & [0.9,1.0] & [0.4,0.6] \\ [0.5,0.7] & [0.6,0.8] & [0.2,0.4] & [0.2,0.4] \end{pmatrix}$$

$$F_f \cap G_g = \begin{pmatrix} [0.5,0.7] & [0,0.2] & [0.2,0.8] & [0.5,0.9] \\ [0,0.3] & [0.2,0.4] & [0.8,1.0] & [0.3,0.5] \\ [0.4,0.6] & [0.5,0.7] & [0.1,0.3] & [0.1,0.2] \end{pmatrix}$$

$$F_f \oplus G_g = \begin{pmatrix} [1.0,1.0] & [0.3,0.7] & [0.8,1] & [1.0,1.0] \\ [0.1,0.7] & [0.5,0.9] & [1.0,1.0] & [0.7,1.0] \\ [0.9,1.0] & [1.0,1.0] & [0.3,0.7] & [0.3,0.6] \end{pmatrix}$$

Definition 3.13 *A GIVFSS is said to a generalized $F-$empty interval-valued fuzzy soft set, denoted by $F_{\hat{0}}$, if $F_{\hat{0}} : E \to IVF(U) \times Int([0,1])$, such that $F_{\hat{0}}(e) = (F(e), \hat{0}(e))$, where $\hat{0}(e) = [0,0], \forall e \in E$.*

If $F(e) = \varnothing$, then the generalized $F-$empty interval-valued fuzzy soft set is called a generalized empty interval-valued fuzzy soft set, denoted by $\varnothing_{\hat{0}}$.

Definition 3.14 *A GIVFSS is said to a generalized $F-$universal interval-valued fuzzy soft set, denoted by $F_{\hat{1}}$, if $F_{\hat{1}} : E \to IVF(U) \times Int([0,1])$, such that $F_{\hat{1}}(e) = (F(e), \hat{1}(e))$, where $\hat{1}(e) = [1,1], \forall e \in E$.*

If $F(e) = U$, then the generalized $F-$universal interval-valued fuzzy soft set is called a generalized universal interval-valued fuzzy soft set, denoted by $U_{\hat{1}}$.

From the Definition 3.13 and 3.14, obviously we have

(1) $\varnothing_{\hat{0}} \subseteq F_{\hat{0}} \subseteq F_f \subseteq F_{\hat{1}} \subseteq U_{\hat{1}}$

(2) $\varnothing_{\hat{0}}^C = U_{\hat{1}}$.

Proposition 3.15 *Let F_f be a GIVFSS over (U, E), then the following holds:*
(1) $F_f \cup \varnothing_{\hat{0}} = F_f, F_f \cap \varnothing_{\hat{0}} = \varnothing_{\hat{0}}$,
(2) $F_f \cup U_{\hat{1}} = U_{\hat{1}}, F_f \cap U_{\hat{1}} = F_f$,
(3) $F_f \cup F_{\hat{0}} = F_f, F_f \cap F_{\hat{0}} = F_{\hat{0}}$,
(4) $F_f \cup F_{\hat{1}} = F_{\hat{1}}, F_f \cap F_{\hat{1}} = F_f$,
(5) $F_f \oplus \varnothing_{\hat{0}} = F_f, F_f \oplus U_{\hat{1}} = U_{\hat{1}}$.

Proof. Straightforward. □

Remark 3.16 *Let F_f be a GIVFSS over (U, E), if $F_f \neq U_{\hat{1}}$ or $F_f \neq \varnothing_{\hat{0}}$, then $F_f \cup F_f^C \neq U_{\hat{1}}$, and $F_f \cap F_f^C \neq \varnothing_{\hat{0}}$.*

Proposition 3.17 *Let $F_f G_g$ and H_h be any three GIVFSSs over (U, E) , then the following holds:*
(1) $F_f \cup G_g = G_g \cup F_f$,
(2) $F_f \cap G_g = G_g \cap F_f$,
(3) $F_f \cup (G_g \cup H_h) = (F_f \cup G_g) \cup H_h$,
(4) $F_f \cap (G_g \cap H_h) = (F_f \cap G_g) \cap H_h$,
(5) $F_f \oplus G_g = G_g \oplus F_f$.

Proof. The properties follow from definition. □

Proposition 3.18 *Let F_f and G_g be two GIVFSSs over (U, E) . Then De-Morgan's laws are valid:*
(1) $(F_f \cup G_g)^C = F_f^C \cap G_g^C$,
(2) $(F_f \cap G_g)^C = F_f^C \cup G_g^C$.

Proof. For all $e \in E$,

$$\begin{aligned}(F_f \cup G_g)^C &= ((F(e) \cup G(e), f(e) \cup g(e)))^C \\ &= (F^C(e) \cap G^C(e)), (f^C(e) \cap g^C(e)) \\ &= (F^C(e), f^C(e)) \cap (G^C(e) \cap g^C(e)) = F_f^C \cap G_g^C\end{aligned}$$

Likewise, the proof of (2) can be made similarly. □

Proposition 3.19 *Let F_f , G_g and H_h be any three GIVFSSs over (U, E) . Then,*
(1) $F_f \cup (G_g \cap H_h) = (F_f \cup G_g) \cap (F_f \cup H_h)$,
(2) $F_f \cap (G_g \cup H_h) = (F_f \cap G_g) \cup (F_f \cap H_h)$,

Proof. The proof follows from definition and distributive property of interval-valued fuzzy set. □

4 Similarity between two generalized interval-valued fuzzy soft sets

In this section, a measure of similarity between two GIVFSSs has been given.

Let $U = \{x_1, x_2, \cdots, x_n\}$ be the universal set of elements and $E = \{e_1, e_2, \ldots, e_m\}$ be the universal set of parameters. Let F_f and $G_g \in GIVFSS(U, E)$, where $F_f = \{(F(e_i), f(e_i)), i = 1, 2 \cdots, m\}$ and $G_g = \{(G(e_i), g(e_i)), i = 1, 2 \cdots, m\}$.

Thus $\hat{F} = \{F(e_i), i = 1, 2 \cdots, m\}$, and $\hat{G} = \{G(e_i), i = 1, 2 \cdots, m\}$ are two families of interval-valued fuzzy sets. Now the similarity between $\hat{F}$ and $\hat{G}$ is found first and it is denoted by $M(\hat{F}, \hat{G})$. Next the similarity between interval-valued fuzzy sets f and g is found and is denoted by $m(f, g)$. Then the similarity between the two GIVFSSs F_f and G_g is denoted by $S(F_f, G_g) = M(\hat{F}, \hat{G}) \cdot m(f, g)$.

Here, $M(\hat{F}, \hat{G}) = \max_i M_i(\hat{F}, \hat{G})$, where

$$M_i(\hat{F}, \hat{G}) = 1 - \frac{1}{n} \sum_{j=1}^{n} max\{|\mu^-_{F(e_i)}(x_j) - \mu^-_{G(e_i)}(x_j)|, |\mu^+_{F(e_i)}(x_j) - \mu^+_{G(e_i)}(x_j)|\}.$$

Also

$$m(f, g) = 1 - \frac{1}{m} \sum_{i=1}^{m} max\{|\mu^-_f(e_i) - \mu^-_g(e_i)|, |\mu^+_f(e_i) - \mu^+_g(e_i)|\}.$$

Example 4.1 *Consider the following two GIVFSSs where $U = \{x_1, x_2, x_3, x_4\}$, and $E = \{e_1, e_2, e_3\}$.*

$$F_f = \begin{pmatrix} [0.6,0.7] & [0.3,0.5] & [0.6,0.8] & [0.6,0.9] & [0.6,0.7] \\ [0.1,0.3] & [0.2,0.4] & [0.8,1.0] & [0.3,0.5] & [0.8,0.9] \\ [0.4,0.7] & [0.5,0.8] & [0.2,0.4] & [0.1,0.2] & [0.4,0.8] \end{pmatrix}$$

$$G_g = \begin{pmatrix} [0.5,0.6] & [0.4,0.8] & [0.2,0.3] & [0.7,1.0] & [0.5,0.8] \\ [0.0,0.2] & [0.1,0.5] & [0.7,0.8] & [0.2,0.6] & [0.6,0.8] \\ [0.5,0.6] & [0.6,0.7] & [0.3,0.6] & [0.2,0.6] & [0.3,0.5] \end{pmatrix}$$

Here $m(f, g) = 0.8$.

And $M_1(\hat{F}, \hat{G}) = 0.75, M_2(\hat{F}, \hat{G}) = 0.875, M_3(\hat{F}, \hat{G}) = 0.8$,

$\therefore M(\hat{F}, \hat{G}) = 0.875$.

Hence the similarity between the two GIVFSSs F_f and G_g will be $S(F_f, G_g) = 0.7$.

From the above definition, we can easily obtain the following conclusions.

Proposition 4.2 *Let F_f , G_g and H_h be any three GIVFSSs over (U, E). Then the following holds:*

(1) $S(F_f, G_g) = S(G_g, F_f)$,

(2) $0 \leq S(F_f, G_g) \leq 1$,

(3) $F_f = G_g \Rightarrow S(F_f, G_g) = 1$,

(4) $F_f \subseteq G_g \subseteq H_h \Rightarrow S(F_f, H_h) \leq S(G_g, H_h)$.

5 Application of generalized interval-valued fuzzy soft set

In this section, we define an aggregate interval-valued fuzzy set of a GIVFSS. We also define GIVFSS aggregation operator that produce an aggregate interval-valued fuzzy set from a GIVFSS over (U, E).

Definition 5.1 *Let F_f be a GIVFSS over (U, E), i.e.$F_f \in GIVFSS(U, E)$. Then GIVFSS-aggregation operator, denoted by $GIVFSS_{agg}$, is defined by*

$GIVFSS_{agg} : GIVFSS(U, E) \to IVF(U)$,

$GIVFSS_{agg} F_f = H$,

where

$$H = \{\frac{\mu_H(u)}{u}, u \in U\}$$

which is an interval-valued fuzzy set over U. The value H is called aggregate interval-valued fuzzy set of F_f.

Here, the membership degree $\mu_H(u_i)$ of u_i is defined as follows

$$\mu_H(u_i) = [a_i^-, a_i^+] = [\bigoplus_{x\in E} \mu_f^-(x)\mu_{F(x)}^-(u_i), \bigoplus_{x\in E} \mu_f^+(x)\mu_{F(x)}^+(u_i)]$$

From the above definition, it is noted that the $GIVFSS_{agg}$ on the interval-valued fuzzy set is an operation by which several approximate functions of a GIVFSS are combined to produce a single interval-valued fuzzy set that is the aggregate interval-valued fuzzy set of the GIVFSS. Once an aggregate interval-valued fuzzy set has been arrived at, it may be necessary to choose the best single alternative from this set. Therefore, we can construct a GIVFSS-decision making method by the following algorithm.

Step 1 Construct a GIVFSS over (U, E),

Step 2 Find the aggregate interval-valued fuzzy set H,

Step 3 Compute the score r_i of u_i such that

$$r_i = \sum_{u_i \in U} ((a_i^- - a_j^-) + (a_i^+ - a_j^+)),$$

Step 4 Find the largest value in S where $S = \max_{u_i\in U}\{r_i\}$.

Example 5.2 *Assume that a company want to fill a position. There are six candidates who form the set of alternatives, $U = \{u_1, u_2, \cdots, u_6\}$. The hiring committee consider a set of parameters, $E = \{x_1, x_2, x_3\}$. The parameters $x_i (i = 1, 2, 3)$ stand for "experience", "computer knowledge" and "young age", respectively.*

After a serious discussion each candidate is evaluated from point of view of the goals and the constraint according to a chose subset $f = \{\frac{[0.5,0.7]}{x_1}, \frac{[0.8,1]}{x_2}, \frac{[0.6,0.8]}{x_3}\}$ of E. Finally, the committee constructs the following GIVFSS over (U, E).

Step 1 Let the constructed GIVFSS, F_f , be as follows,

$$F_f = \begin{pmatrix} [0.0,0.0] & [0.3,0.4] & [0.4,0.6] & [0.9,1.0] & [0.1,0.2] & [0.0,0.0] & [0.5,0,7] \\ [0.4,0.6] & [0.2,0.4] & [0.7,0.8] & [0.3,0.6] & [0.0,0.0] & [0.0,0.0] & [0.8,1.0] \\ [0.0,0.2] & [0.4,0.6] & [0.0,0.0] & [0.0,0.0] & [0.6,0.8] & [0.2,0.5] & [0.6,0.8] \end{pmatrix}$$

Step 2 The aggregate interval-valued fuzzy set can be found as

$$H = \{\frac{[0.32, 0.76]}{u_1}, \frac{[0.55, 1]}{u_2}, \frac{[0.76, 1]}{u_3}, \frac{[0.69, 1]}{u_4}, \frac{[0.41, 0.78]}{u_5}, \frac{[0.12, 0.4]}{u_6}\}$$

Step 3 For all $u_i \in U$, compute the score r_i of u_i such that $r_i = \sum_{u_i \in U}((a_i^- - a_j^-) + (a_i^+ - a_j^+))$.

Thus, we have $r_1 = -1.31, r_2 = 1.51, r_3 = 2.77, r_4 = 2.35, r_5 = -0.65, r_6 = -4.67$.

Step 4 The decision is anyone of the elements in S where $S = \max_{u_i \in U}\{r_i\}$.

In our example, the candidate u_3 is the best choice because $\max_{u_i \in U}\{r_i\} = \{u_3\}$. This result is reasonable because we can see that $\mu_H(u_3) \geq \mu_H(u_i)$, where $i = 1, 2, 4, 5, 6$.

6 Conclusion

Soft set theory, proposed by Molodtsov, has been regarded as an effective mathematical tool to deal with uncertainty. However, it is difficult to be used to represent the fuzziness of problem. In order to handle these types of problem parameters, some fuzzy extensions of soft set theory are presented, yielding fuzzy soft set theory. In this paper, the notion of generalized interval-valued fuzzy soft set theory is proposed. Our generalized interval-valued fuzzy soft set theory is a combination of a generalized fuzzy soft set theory and an interval-valued fuzzy set theory. In other words, our generalized interval-valued fuzzy soft set theory is an extension of generalized interval-valued fuzzy soft set theory. The complement, union, intersection and sum operations are defined on generalized interval-valued fuzzy soft sets. The basic properties of the generalized interval-valued fuzzy soft sets are also presented and discussed. Similarity measure of two generalized interval-valued fuzzy soft sets is discussed. Finally, an application of this theory has been applied to solve a decision making problem.

In further research, the parameterization reduction of generalized interval-valued fuzzy soft sets is an important and interesting issue to be addressed.

Acknowledgements

The authors would like to thank the anonymous referees for their valuable comments and suggestions. This work is supported by the Fundamental Research Funds for the

Central Universities of Northwest University for Nationalities (No. 31920130012) and the Young Research Foundation of Northwest University for Nationalities (No.12XB29).

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Distribution reductions in α-maximal tolerance level SF decision information systems *

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February 23, 2014

Abstract: In this paper, we introduce α-maximal tolerance levels in SF decision information systems, establish the rough set models and obtain distribution reductions of α-maximal tolerance level SF decision information systems.

Keywords: SF decision information system; α-maximal tolerance level; ε-inconsistent; Distribution reduction.

1 Introduction

Rough set theory, proposed by Pawlak [1], is a new mathematical tool for data reasoning. It may be seen as an extension of classical set theory and has been successfully applied to machine learning, intelligent systems, inductive reasoning, pattern recognition, mereology, image processing, signal analysis, knowledge discovery, decision analysis, expert systems and many other fields [2, 3, 4, 5].

Rough sets are based on an assumption that every object in the universe is associated with some information. Objects characterized by the same information are indiscernible with the available information about them. The indiscernibility relation generated in this way is the mathematical basis for rough set theory. Rough set theory has used successfully in the analysis of data in information systems.

Set-valued information systems as important information systems have been gained further research in recent years. Wang et al. [7] and Song et al. [8] studied set-valued information system based on tolerance relations. But compatible class has some shortcomings. Guan et al. [9, 10] introduced the concept of maximum compatible classes and made up for the deficiency of the compatible class.

Fuzzy decision information systems are decision information systems under the fuzzy environment. There are a lot of research achievements on fuzzy decision information systems (see [11, 12, 13]).

*This work is supported by the National Social Science Foundation of China (No. 12BJL087).

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Set-valued fuzzy decision information systems are set-valued decision information systems under the fuzzy environment. α-maximal tolerance level set-valued fuzzy decision information systems are their generalization.

In this paper, we investigate deeply attribute reductions in α-maximal tolerance level set-valued fuzzy decision information systems.

2 Preliminaries

Throughout this paper, "set-valued fuzzy" denote briefly by "SF", U denotes a finite and nonempty set called the universe, 2^U denotes the power set of U, $F(U)$ denotes the set of all fuzzy sets in U, I denotes $[0,1]$ and B^A denotes $\{f|f : A \to B\}$. For convenience, stipulate

$$U = \{x_1, x_2, \cdots, x_n\},\ A = \{a_1, a_2, \cdots, a_m\},\ D = \{d_1, d_2, \cdots, d_p\},\ \varepsilon, \alpha \in [0, 1].$$

Definition 2.1 ([14]). *(U, A, F, D, G) is called an SF decision information system, where (U, A, F) is a set-valued information system, i.e., U is the universe, A is the condition attribute set, $F = \{f_a \in (2^{V_a} - \{\emptyset\})^U :\ a \in A\}$ is the relationship set between U and A, f_a is the information function of a, V_a is the range of a; D is the decision attribute set; $G = \{g_d \in I^U :\ d \in D\}$ is the relationship set between U and D, $g_d \in F(U)$ is the information function of d.*

Example 2.2. *Table 1 gives an IF decision information system where $U = \{x_1, x_2, \cdots, x_8\}$, $A = \{a_1, a_2, a_3\}$, $D = \{d_1, d_2, d_3\}$.*

Table 1: The SF decision information systems (U, A, F, D, G)

U	a_1	a_2	a_3	d_1	d_2	d_3
x_1	$\{0,1\}$	$\{2\}$	$\{0,1\}$	0	0.3	0.4
x_2	$\{1,2\}$	$\{1\}$	$\{0\}$	0.4	0.8	0.7
x_3	$\{0\}$	$\{0,1\}$	$\{1\}$	0.2	1	0.1
x_4	$\{0\}$	$\{2\}$	$\{0,1\}$	0.6	0.1	0.2
x_5	$\{1\}$	$\{1,2\}$	$\{1,2\}$	0.6	0.8	0.9
x_6	$\{1\}$	$\{1\}$	$\{0,2\}$	1	0.5	0.5
x_7	$\{0,1\}$	$\{0,2\}$	$\{1,2\}$	0.7	0.2	0.6
x_8	$\{0,1\}$	$\{0,1\}$	$\{2\}$	0.5	0.3	0.3

Definition 2.3 ([10]). *Let R be a tolerance relation on U.*

(1) *$X \in 2^U$ is called a compatible class of R, if for any $x, y \in X$, xRy.*

(2) *$X \in 2^U$ is called a maximum compatible class of R, if X is a compatible class of R and for any $x \in U - X$, there exists $y \in X$ such that $(x, y) \notin R$*

The set of all maximum compatible classes of R is denoted by $CCR(U)$

For $x \in U$, denote

$$CCR(x) = \{K \in CCR(U) :\ x \in K\}.$$

Definition 2.4 ([7, 8]). *Let (U, A, F, D, G) be a SF decision information system. For $B \subseteq A$, define a tolerance relation T_B of B on U as follows:*

$$T_B = \{(x, y) \in U \times U :\ f_b(x) \cap f_b(y) \neq \emptyset\ (\forall\ b \in B)\}.$$

If xT_By for any $x, y \in U$, then x, y are called compatible on B.

Denote

$$T_B(x) = \{y \in U : \ xT_By\},$$
$$CCT_B(x) = \{K \in CCT_B(U) : \ x \in K\}.$$

For $b \in B$, $x \in U$, denote

$$T_b = T_{\{b\}}, \quad T_b(x) = T_{\{b\}}(x);$$
$$CCT_b(U) = CCT_{\{b\}}(U), \quad CCT_b(x) = CCT_{\{b\}}(x).$$

It is easy to verify that

$$T_B = \bigcap_{b \in B} T_b, \ T_B(x) = \bigcap_{b \in B} T_b(x).$$

Example 2.5. *Consider the SF decision information system (U, A, F, D, G) in Example 2.3.*

Put $C = \{a_1, a_2, a_3\}$. Then
$T_C(x_1) = \{x_1\}$, $T_C(x_2) = \{x_2, x_3\}$, $T_C(x_3) = \{x_2, x_3, x_4, x_6\}$,
$T_C(x_4) = \{x_3, x_4, x_6\}$, $T_C(x_5) = \{x_5, x_6\}$, $T_C(x_6) = \{x_3, x_4, x_5, x_6\}$;
$CCT_C(x_1) = \{\{x_1\}\}$, $CCT_C(x_2) = \{\{x_2, x_3\}\}$,
$CCT_C(x_3) = \{\{x_2, x_3\}, \{x_3, x_4, x_6\}\}$, $CCT_C(x_4) = \{\{x_3, x_4, x_6\}\}$,
$CCT_C(x_5) = \{\{x_5, x_6\}\}$, $CCT_C(x_6) = \{\{x_3, x_4, x_6\}, \{x_5, x_6\}\}$.

Definition 2.6 ([15])**.** *Let (U, A, F, D, G) be a SF decision information system. $\forall \ x_i, x_j \in U$, define*

$$S_D(x_i, x_j) = \bigwedge\{1 - \mid f_d(x_i) - f_d(x_j) \mid : d \in D\},$$
$$S_D^\varepsilon = \{(x_i, x_j) \in U \times U : \ S_D(x_i, x_j) \geqslant \varepsilon\},$$

where ε is called the decision tolerance.

Remark 2.7. *S_D^ε is a tolerance relation on U. Specially, $S_D^1 = \{(x_i, x_j) \in U \times U : \ f_d(x_i) = f_d(x_j) \ (\forall \ d \in D)\}$ is an equivalence relation on U.*

Denote

$$[x_i]_D^\varepsilon = \{x_j \in U : \ (x_i, x_j) \in S_D^\varepsilon\},$$
$$\mathcal{D}^\varepsilon = U/S_D^\varepsilon = \{[x_i]_D^\varepsilon \ : \ x_i \in U\}.$$

They mean the compatible class of x_i on S_D^ε and the fuzzy decision classification on U, respectively.

Example 2.8. *Consider the SF decision information system (U, A, F, D, G) in Example 2.3.*

Put $D = \{d_1, d_2, d_3\}$ and $\varepsilon = 0.7$. Then

$$\mathcal{D}^{0.7} = \{[x_i]_D^{0.7} \ : \ i = 1, 2, \cdots, 8\},$$

where

$$[x_1]_D^{0.7} = \{x_1\}, \quad [x_2]_D^{0.7} = \{x_2, x_5\}, \quad [x_3]_D^{0.7} = \{x_3\},$$
$$[x_4]_D^{0.7} = \{x_4, x_8\}, \quad [x_5]_D^{0.7} = \{x_2, x_5\}, \quad [x_6]_D^{0.7} = \{x_6, x_7\},$$
$$[x_7]_D^{0.7} = \{x_6, x_7, x_8\}, \quad [x_8]_D^{0.7} = \{x_4, x_7, x_8\}.$$

3 α-maximal tolerance level SF decision information systems

Definition 3.1. *Let (U, A, F, D, G) be a SF decision information system. For $x_i, x_j \in U$, $b \in B \subseteq A$, $\alpha \in [0, 1]$, denote*

$$\Gamma_b(x_i, x_j) = \frac{|CCT_b(x_i) \cap CCT_b(x_j)|}{|CCT_b(x_i) \cup CCT_b(x_j)|}.$$

where $|\cdot|$ means the cardinality. Denote

$$\widetilde{T_B^\alpha} = \{(x_i, x_j) \in U \times U : \Gamma_b(x_i, x_j) \geqslant \alpha \ (\forall\ b \in B)\}.$$

Then α is called the maximal tolerance level.

It is easy to verify that $\widetilde{T_B^\alpha}$ is a tolerance relation on U.

For $b \in B$, denote

$$\widetilde{T_b^\alpha} = \widetilde{T_{\{b\}}^\alpha}.$$

Denote

$$\widetilde{T_B^\alpha}(x) = \{y \in U : \ (x, y) \in \widetilde{T_B^\alpha}\},$$

$$\widetilde{T_B^\alpha}(U) = \{\widetilde{T_B^\alpha}(x) : \ x \in U\}.$$

Obviously $\widetilde{T_B^1} = \{(x_i, x_j) \in U \times U : \ CCT_b(x_i) = CCT_b(x_j) \ (\forall\ b \in B)\}$ is an equivalence relation on U.

Example 3.2. *Consider the SF decision information system (U, A, F, D, G) in Example 2.3.*

Put $C = \{a_1, a_2, a_3\}$. Then

$$\widetilde{T_C^{0.4}}(x_1) = \{x_1, x_4\}, \quad \widetilde{T_C^{0.4}}(x_2) = \{x_2, x_6\}, \quad \widetilde{T_C^{0.4}}(x_3) = \{x_3\},$$

$$\widetilde{T_C^{0.4}}(x_4) = \{x_1, x_4\}, \quad \widetilde{T_C^{0.4}}(x_5) = \{x_5\}, \quad \widetilde{T_C^{0.4}}(x_6) = \{x_2, x_5, x_6, x_8\},$$

$$\widetilde{T_C^{0.4}}(x_7) = \{x_7\}, \quad \widetilde{T_C^{0.4}}(x_8) = \{x_6, x_8\}.$$

Proposition 3.3. *Let $x \in U$, $C \subseteq B \subseteq A$. Then*

(1) $\widetilde{T_A^\alpha} \subseteq \widetilde{T_B^\alpha} \subseteq \widetilde{T_C^\alpha}$, $\quad \widetilde{T_A^\alpha}(x) \subseteq \widetilde{T_B^\alpha}(x) \subseteq \widetilde{T_C^\alpha}(x)$.

(2) $\widetilde{T_B^\alpha} = \bigcap\limits_{b \in B} \widetilde{T_b^\alpha}$.

Proof. The proof is not difficult, so we omit it. □

Definition 3.4. *Let (U, A, F, D, G) be a α-maximal tolerance level SF decision information system where ε is the decision tolerance. If $\widetilde{T_A^\alpha} \subseteq S_D^\varepsilon$, then it is called a α-maximal tolerance level ε-consistent SF decision information system; Otherwise, it is called a α-maximal tolerance level ε-inconsistent SF decision information system.*

Obviously, inconsistent (resp. consistent) SF decision information systems are 1-maximal tolerance level 1-inconsistent (resp. 1-consistent) SF decision information systems.

Example 3.5. *Consider the SF decision information system (U, A, F, D, G) in Example 2.3.*

Note that

$$\widetilde{T_A^{0.4}} \subseteq S_D^{0.4}, \qquad \widetilde{T_A^{0.4}} \not\subseteq S_D^{0.7}.$$

Then (U, A, F, D, G) are both 0.4-maximal tolerance level 0.4-consistent SF decision information systems and 0.4-maximal tolerance level 0.7-inconsistent SF decision information systems.

4 Distribution reductions

Definition 4.1. *Let (U, A, F, D, G) be a α-maximal tolerance level SF decision information system. For $B \subseteq A$, $X \in 2^U$, define*

$$\underline{CCT_B^\alpha}(X) = \{x \in U : \widetilde{T_B^\alpha}(x) \subseteq X\},$$

$$\overline{CCT_B^\alpha}(X) = \{x \in U : \widetilde{T_B^\alpha}(x) \cap X \neq \emptyset\}.$$

Then $\underline{CCT_B^\alpha}(X)$ (resp. $\overline{CCT_B^\alpha}(X)$) is called the α-level lower (resp. upper) approximation of X.

Proposition 4.2. *Let (U, A, F, D, G) be a α-maximal tolerance level SF decision information system. If $C \subseteq B \subseteq A$, then $\forall\ X \in 2^U$,*

$$\underline{CCT_C^\alpha}(X) \subseteq \underline{CCT_B^\alpha}(X) \subseteq X \subseteq \overline{CCT_B^\alpha}(X) \subseteq \overline{CCT_C^\alpha}(X).$$

Proof. This holds by Proposition 3.3. □

Definition 4.3. *Let (U, A, F, D, G) be a α-maximal tolerance level ε-inconsistent SF decision information system and let $\mathcal{D}^\varepsilon = \{D_i^\varepsilon : i \leqslant r\}$ be the classification of fuzzy decision on U. For $B \subseteq A$, define*

$$L_B(\alpha, \varepsilon) = (\underline{CCT_B^\alpha}(D_1^\varepsilon), \underline{CCT_B^\alpha}(D_2^\varepsilon), \cdots, \underline{CCT_B^\alpha}(D_r^\varepsilon)),$$

$$H_B(\alpha, \varepsilon) = (\overline{CCT_B^\alpha}(D_1^\varepsilon), \overline{CCT_B^\alpha}(D_2^\varepsilon), \cdots, \overline{CCT_B^\alpha}(D_r^\varepsilon)),$$

they are called α-level lower distribution function and α-level upper distribution function, respectively.

(1) If $L_B(\alpha, \varepsilon) = L_A(\alpha, \varepsilon)$, then B is called a α-level lower ε-distribution coordinate set.

(2) If B is a α-level lower ε-distribution coordinate set and $\forall\ B' \subsetneq B$, $L_{B'}(\alpha, \varepsilon) \neq L_A(\alpha, \varepsilon)$, then B is called a α-level lower distribution ε-reduction.

(3) If $H_B(\alpha, \varepsilon) = H_A(\alpha, \varepsilon)$, then B is called a α-level upper ε-distribution coordinate set.

(4) If B is a α-level upper ε-distribution coordinate set and $\forall\ B' \subsetneq B$, $H_{B'}(\alpha, \varepsilon) \neq H_A(\alpha, \varepsilon)$, then B is called a α-level upper distribution ε-reduction.

5

Denote

$$(G_B^\alpha)^\varepsilon(x) = \{\ D^\varepsilon \in \mathcal{D}^\varepsilon :\ \widetilde{T_B^\alpha}(x) \subseteq D^\varepsilon\},$$

$$(M_B^\alpha)^\varepsilon(x) = \{\ D^\varepsilon \in \mathcal{D}^\varepsilon :\ \widetilde{T_B^\alpha}(x) \cap D^\varepsilon \neq \emptyset\}.$$

Proposition 4.4. *Let (U, A, F, D, G) be a α-maximal tolerance level ε-inconsistent SF decision information system and let $\mathcal{D}^\varepsilon$ be the classification of fuzzy decision on U.*

(1) $\forall\ D^\varepsilon \in \mathcal{D}^\varepsilon$,

$$\underline{CCT_B^\alpha}(D^\varepsilon) = \{x \in U :\ D^\varepsilon \in (G_B^\alpha)^\varepsilon(x)\}.$$

(2) $\forall\ D^\varepsilon \in \mathcal{D}^\varepsilon$,

$$\overline{CCT_B^\alpha}(D^\varepsilon) = \{x \in U :\ D^\varepsilon \in (M_B^\alpha)^\varepsilon(x)\}.$$

(3) *If $C \subseteq B \subseteq A$, then $\forall\ x \in U$,*

$$(G_C^\alpha)^\varepsilon(x) \subseteq (G_B^\alpha)^\varepsilon(x), \quad (M_C^\alpha)^\varepsilon(x) \supseteq (M_B^\alpha)^\varepsilon(x).$$

Proof. (1) $\forall\ x \in \underline{CCT_B^\alpha}(D^\varepsilon)$, we have $\widetilde{T_B^\alpha}(x) \subseteq D^\varepsilon$. Then

$$D^\varepsilon \in (G_B^\alpha)^\varepsilon(x).$$

Thus $x \in \{x \in U :\ D^\varepsilon \in (G_B^\alpha)^\varepsilon(x)\}$.

On the other hand, $\forall\ x \in \{x \in U :\ D^\varepsilon \in (G_B^\alpha)^\varepsilon(x)\})$, we have

$$D^\varepsilon \in (G_B^\alpha)^\varepsilon(x).$$

Then $\widetilde{T_B^\alpha}(x) \subseteq D^\varepsilon$.
Thus

$$x \in \underline{CCT_B^\alpha}(D^\varepsilon).$$

Hence

$$\underline{CCT_B^\alpha}(D^\varepsilon) = \{x \in U :\ D^\varepsilon \in (G_B^\alpha)^\varepsilon(x)\}.$$

(2) The proof is similar to (1).

(3) The proof is not difficult, so we omit it. □

Theorem 4.5. *Let (U, A, F, D, G) be a α-maximal tolerance level ε-inconsistent SF decision information system. Then*

(1) *$B \subseteq A$ is a α-level lower distribution ε-coordinate set $\Longleftrightarrow$*

$$\forall\ x \in U,\ (G_B^\alpha)^\varepsilon(x) = (G_A^\alpha)^\varepsilon(x).$$

(2) *$B \subseteq A$ is a α-level upper distribution ε-coordinate set $\Longleftrightarrow$*

$$\forall\ x \in U,\ (M_B^\alpha)^\varepsilon(x) = (M_A^\alpha)^\varepsilon(x).$$

Proof. (1) Since $\mathcal{D}^\varepsilon$ is the classification of fuzzy decision on U, we have

$$(G_B^\alpha)^\varepsilon(x) \subseteq D^\varepsilon \Longleftrightarrow x \in \underline{CCT_B^\alpha}(D^\varepsilon) \Longleftrightarrow D^\varepsilon \in (G_B^\alpha)^\varepsilon(x).$$

Thus B is a α-level lower distribution ε-coordinate set $\Longleftrightarrow$

$$(G_B^\alpha)^\varepsilon(x) \subseteq D^\varepsilon \Longleftrightarrow (G_A^\alpha)^\varepsilon(x) \subseteq D^\varepsilon \quad (\forall\ x \in U,\ D^\varepsilon \in \mathcal{D}^\varepsilon)$$

$$\Longleftrightarrow\ (G_B^\alpha)^\varepsilon(x) = (G_A^\alpha)^\varepsilon(x) \quad (\forall\ x \in U).$$

(2) The proof is similar to (1). □

Definition 4.6. *Let (U, A, F, D, G) be a α-maximal tolerance level ε-inconsistent SF decision information system and let $\mathcal{D}^\varepsilon$ be the classification of fuzzy decision on U. Define*

$$\underline{P}_\varepsilon^\alpha = \{(x_i, x_j) \in U \times U :\ \exists\ D^\varepsilon \in \mathcal{D}^\varepsilon,\ x_i \in \underline{CCT_A^\alpha}(D^\varepsilon),\ x_j \notin D^\varepsilon\},$$

$$\overline{P}_\varepsilon^\alpha = \{(x_i, x_j) \in U \times U :\ \exists\ D^\varepsilon \in \mathcal{D}^\varepsilon,\ x_i \notin \overline{CCT_A^\alpha}(D^\varepsilon),\ x_j \in D^\varepsilon\};$$

$$\underline{W}_\varepsilon^\alpha(x_i, x_j) = \begin{cases} \{a \in A :\ (x_i, x_j) \notin \widetilde{T_a^\alpha}\}, & (x_i, x_j) \in \underline{P}_\varepsilon^\alpha, \\ \emptyset, & (x_i, x_j) \notin \underline{P}_\varepsilon^\alpha. \end{cases}$$

$$\overline{W}_\varepsilon^\alpha(x_i, x_j) = \begin{cases} \{a \in A :\ (x_i, x_j) \notin \widetilde{T_a^\alpha}\}, & (x_i, x_j) \in \overline{P}_\varepsilon^\alpha, \\ \emptyset, & (x_i, x_j) \notin \overline{P}_\varepsilon^\alpha; \end{cases}$$

$$\underline{\mathcal{W}}_\varepsilon^\alpha = (\underline{W}_\varepsilon^\alpha(x_i, x_j))_{n\times n}, \qquad \overline{\mathcal{W}}_\varepsilon^\alpha = (\overline{W}_\varepsilon^\alpha(x_i, x_j))_{n\times n}.$$

Then $\underline{W}_\varepsilon^\alpha(x_i, x_j)$ (resp. $\overline{W}_\varepsilon^\alpha(x_i, x_j)$) is called the α-level lower (resp. upper) distribution ε-discernibility attribute set of x_i and x_j on $\mathcal{D}^\varepsilon$, $\underline{\mathcal{W}}_\varepsilon^\alpha$ (resp. $\overline{\mathcal{W}}_\varepsilon^\alpha$) is called the α-level lower (resp. upper) distribution ε-discernibility matrix on $\mathcal{D}^\varepsilon$.

Theorem 4.7. *Let (U, A, F, D, G) be a α-maximal tolerance level ε-inconsistent SF decision information system. Then*

(1) *$B \subseteq A$ is a α-level lower ε-distribution coordinate set $\Longleftrightarrow$*

$$B \cap \underline{W}_\varepsilon^\alpha(x_i, x_j) \neq \emptyset \textit{ whenever } \underline{W}_\varepsilon^\alpha(x_i, x_j) \neq \emptyset.$$

(2) *$B \subseteq A$ is a α-level upper distribution ε-coordinate set $\Longleftrightarrow$*

$$B \cap \overline{W}_\varepsilon^\alpha(x_i, x_j) \neq \emptyset \textit{ whenever } \overline{W}_\varepsilon^\alpha(x_i, x_j) \neq \emptyset.$$

Proof. (1) Necessity. Suppose that there exists $(x_i, x_j) \in \underline{P}_\varepsilon^\alpha$ such that

$$B \cap \underline{W}_\varepsilon^\alpha(x_i, x_j) = \emptyset \ \textit{ and } \ \underline{W}_\varepsilon^\alpha(x_i, x_j) \neq \emptyset.$$

Since B is a α-level lower distribution ε-coordinate set and $D^\varepsilon \in \mathcal{D}^\varepsilon$, we have

$$B \subseteq A - \underline{W}_\varepsilon^\alpha(x_i, x_j).$$

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This implies that $(x_i, x_j) \in \widetilde{T_B^\alpha}$. Then

$$x_j \in \widetilde{T_B^\alpha}(x_i).$$

Note that $(x_i, x_j) \in \underline{P}_\varepsilon^\alpha$. Then $x_i \in \underline{CCT_A^\alpha}(D^\varepsilon)$, $x_j \notin D^\varepsilon$.

Since B is a α-level lower distribution ε-coordinate set, we have

$$\underline{CCT_B^\alpha}(D^\varepsilon) = \underline{CCT_A^\alpha}(D^\varepsilon) \ \ (\forall\ D^\varepsilon \in \mathcal{D}^\varepsilon).$$

Then $x_i \in \underline{CCT_B^\alpha}(D^\varepsilon)$. So $\widetilde{T_B^\alpha}(x_i) \subseteq D^\varepsilon$.

Note that $x_j \in \widetilde{T_B^\alpha}(x_i)$. Then $x_j \in D^\varepsilon$. This is a contradiction.

Sufficiency. Suppose that B is not a α-level lower ε-distribution coordinate set. Then

$$L_B(\alpha, \varepsilon) \neq L_A(\alpha, \varepsilon).$$

Thus, there exists $D_0^\varepsilon \in \mathcal{D}^\varepsilon$ such that

$$\underline{CCT_B^\alpha}(D_0^\varepsilon) \neq \underline{CCT_A^\alpha}(D_0^\varepsilon).$$

$B \subseteq A$ implies that $\underline{CCT_B^\alpha}(D_0^\varepsilon) \subsetneq \underline{CCT_A^\alpha}(D_0^\varepsilon)$.
Pick

$$x \in \underline{CCT_A^\alpha}(D_0^\varepsilon) - \underline{CCT_B^\alpha}(D_0^\varepsilon).$$

Since $\widetilde{T_B^\alpha}(x_i) \nsubseteq D_0^\varepsilon$, we have

$$\widetilde{T_B^\alpha}(x_i) - D_0^\varepsilon \neq \emptyset.$$

This illustrates that there exists $x_j \in U - D_0^\varepsilon$ such that $x_j \in \widetilde{T_B^\alpha}(x_i)$.
Then

$$(x_i, x_j) \in \widetilde{T_B^\alpha} = \bigcap_{b \in B} \widetilde{T_B^\alpha}.$$

This implies that

$$(x_i, x_j) \in \widetilde{T_B^\alpha} \ \ (\forall\ b \in B).$$

But $(x_i, x_j) \in \underline{P}_\varepsilon^\alpha$. So $B \cap \underline{W}_\varepsilon^\alpha(x_i, x_j) = \emptyset$. This is a contradiction.

Hence B is a α-level lower distribution ε-coordinate set.

(2) The proof is similar to (1). □

Definition 4.8. *Let (U, A, F, D, G) be a α-maximal tolerance level ε-inconsistent SF decision information system, $B \subseteq A$, $\mathcal{D}^\varepsilon$ be the fuzzy classification of decision on U. Suppose that $\underline{W}_\varepsilon^\alpha(x_i, x_j)$ and $\overline{W}_\varepsilon^\alpha(x_i, x_j)$ are the α-level lower distribution ε-discernibility attribute set and α-level upper distribution ε-discernibility attribute set of x_i and x_j on $\mathcal{D}^\varepsilon$, respectively. Denote*

$$\underline{\triangle}_\varepsilon^\alpha = \bigwedge\{\bigvee \underline{W}_\varepsilon^\alpha(x_i, x_j) : (x_i, x_j) \in \underline{P}_\varepsilon^\alpha\},$$

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$$\overline{\triangle}_{\varepsilon}^{\alpha} = \bigwedge\{\bigvee \overline{W}_{\varepsilon}^{\alpha}(x_i, x_j) : (x_i, x_j) \in \overline{P}_{\varepsilon}^{\alpha}\}.$$

Then $\underline{\triangle}_{\varepsilon}^{\alpha}$ (resp. $\overline{\triangle}_{\varepsilon}^{\alpha}$) is called the α-level lower (resp. upper) distribution ε-discernibility function.

Theorem 4.9. *Let (U, A, F, D, G) be a α-maximal tolerance level ε-inconsistent SF decision information system. The minimal disjunctive normal form of its α-level lower, upper distributionmε-discernibility function are respectively*

$$\underline{\triangle}_{\varepsilon}^{\alpha} = \bigvee_{m=1}^{t} (\bigwedge_{n=1}^{q_m^{(1)}} a_{l_n}^{(1)}), \quad \overline{\triangle}_{\varepsilon}^{\alpha} = \bigvee_{m=1}^{t} (\bigwedge_{n=1}^{q_m^{(2)}} a_{l_n}^{(2)}).$$

Denote

$$B_{k_m}^{(i)} = \{a_{l_n}^{(i)} : n = 1, 2, \cdots, q_m^{(i)}\} \quad (i = 1, 2).$$

Then $\forall\ m \leq t$, $B_{k_m}^{(1)}$ (resp. $B_{k_m}^{(2)}$) is a α-level lower (resp. upper) distribution ε-reduction.

Proof. This hods by using Boolean reasoning theorem. $\square$

Example 4.10. *Consider 0.4-maximal tolerance level 0.7-inconsistent SF decision information systems (U, A, F, D, G) in Example 3.5.*

The 0.4-level lower distribution 0.7-discernibility attribute matrix is

$$(\underline{\mathcal{W}}_{0.7}^{0.4}) = \begin{pmatrix} \emptyset & \emptyset & \{a_2\} & \emptyset & \{a_3\} & \emptyset & \{a_3\} & \{a_2, a_3\} \\ \emptyset & \emptyset & \{a_1, a_3\} & \emptyset & \emptyset & \emptyset & \{a_2, a_3\} & \emptyset \\ \{a_2\} & \{a_1, a_3\} & \emptyset & \{a_2\} & \{a_1, a_2\} & \{a_1, a_3\} & \{a_2\} & \{a_3\} \\ \emptyset & \emptyset & \{a_2\} & \emptyset & C & \emptyset & \{a_3\} & \emptyset \\ \{a_3\} & \emptyset & \{a_1, a_2\} & C & \emptyset & \{a_2, a_3\} & \{a_2\} & \{a_2\} \\ \emptyset & \emptyset & \{a_1, a_3\} & \emptyset & \{a_2, a_3\} & \emptyset & \{a_2, a_3\} & \emptyset \\ \{a_3\} & \{a_2, a_3\} & \{a_2\} & \{a_3\} & \{a_2\} & \{a_2, a_3\} & \emptyset & \emptyset \\ \{a_2, a_3\} & \emptyset & \{a_3\} & \emptyset & \{a_2\} & \emptyset & \emptyset & \emptyset \end{pmatrix}.$$

The 0.4-level lower distribution 0.7-discernibility function is

$$\underline{\triangle}_{0.7}^{0.4} = a_2 \wedge a_3 \wedge (a_2 \vee a_3) \wedge (a_1 \vee a_3) \wedge (a_1 \vee a_2) \wedge (a_1 \vee a_2 \vee a_3) = a_2 \wedge a_3.$$

Hence $B = \{a_2, a_3\}$ is the 0.4-level lower distribution 0.7-reduction.

5 Conclusions

In this paper, we have researched distribution reductions in α-maximal tolerance level ε-inconsistent SF decision information systems systems. In future work, we will investigate knowledge acquisition in α-maximal tolerance level SF decision information systems.

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Separation Problem for Second Order Elliptic Differential Operators on Riemannian Manifolds

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Abstract

In this paper we aim to study the separation problem for the second order elliptic differential expression of the form $E = \triangle_M + b_i \frac{\partial^2}{\partial x^{i^2}} + q$, with the real-valued positive continuous functions b_i, on a complete Riemannian manifold (M, g) with metric g, where $\triangle_M$ is the scalar Laplacian on M and $q \geq 0$ is a locally square integrable function on M.

In the terminology of Everitt and Giertz, E is said to be separated in $L^2(M)$ if for $u \in L^2(M)$ such that $Eu \in L^2(M)$, we have $b_i \frac{\partial^2 u}{\partial x^{i^2}}$ and $qu \in L^2(M)$. We give sufficient conditions for E to be separated in $L^2(M)$.

Keywords: Separation, Second order elliptic differential operator, Riemannian manifold

AMS Subject Classification: 47F05, 58J99.

1 Introduction

The separation property of differential operators was first introduced in 1971 by Everitt and Giertz [9]. Many recent results of separation of differential operators may be found in Brown [6], Mohamed and Atia [15,16], Atia and Mahmoud [3] and Atia [1].

Zayed et al [18] have studied the separation of the Laplace-Beltrami differential operator of the form

$Au = -\frac{1}{\sqrt{\det g(x)}} \frac{\partial}{\partial x_i} \left[\sqrt{\det g(x)} g^{-1}(x) \frac{\partial u}{\partial x_j} \right] + V(x)u(x)$, for every $x \in \Omega \subset R^n$, in the Hilbert space $H = L_2(\Omega, H_1)$ with the operator potential $V(x) \in C^1(\Omega, L(H_1))$, where $L(H_1)$ is the space of all bounded linear operators on the Hilbert space $H_1$and $g(x) = g_{ij}(x)$ is the Riemannian matrix and $g^{-1}(x)$ is the inverse of the matrix $g(x)$. Atia [2] has proved the separation of a bi-harmonic differential expression of the form $A = \triangle\triangle + q$ on a complete Riemannian manifold (M, g) with metric g, where $\triangle$ is the laplacian on M and

$q \geq 0$ is a locally square integrable function on M. For more studies about the separation property see for example [4,12,13,14]. Let (M, g) be a Riemannian manifold without boundary (i.e. M is a $C^{\infty} - manifold$ without boundary and (g_{jk}) is a Riemannian metric on M) and $dimM = n$. We will assume that M is connected. We will also assume that we are given a positive smooth measure $d\mu$, i.e. in any local coordinates $x^1, x^2, \ldots, x^n$ there exists a strictly positive $C^{\infty}-$density $\rho(x)$ such that $d\mu = \rho(x)\, dx^1 dx^2 \ldots dx^n$.

In the sequel, $L^2(M)$ is the space of complex-valued square integrable functions on M with the inner product:

$$(u, v) = \int\limits_M \left(uv^-\right) d\mu, \tag{1}$$

and $\|.\|$ is the norm in $L^2(M)$ corresponding to the inner product (1). We use the notation $L^2\left(\Lambda^1 T^* M\right)$ for the space of complex-valued square integrable 1-forms on M with the inner product:

$$(W, \Psi)_{L^2(\Lambda^1 T^* M)} = \int\limits_M \left\langle W, \overline{\Psi} \right\rangle d\mu, \tag{2}$$

where for 1-forms $W = W_j dx^j$ and $\Psi = \Psi_k dx^k$, we define $\langle W, \Psi \rangle = g^{jk} W_j \Psi_k$, where (g^{jk}) is the inverse matrix to (g_{jk}), and $\overline{\Psi} = \overline{\Psi_k} dx^k$. (Above we use the standard Einstein summation convention). The notation $\|.\|_{L^2(\Lambda^1 T^* M)}$ stands for the norm in $L^2\left(\Lambda^1 T^* M\right)$ corresponding to the inner product (2). In what follows, by $C^{\infty}(M)$ we denote the space of smooth functions on M, by $C_c^{\infty}(M)$ −the space of smooth compactly supported functions on M, by $\Omega^1(M)$ −the space of smooth 1-forms on M and by $\Omega_c^1(M)$ −the space of smooth compactly supported 1-forms on M.

In the sequel, the operator $d : C^{\infty}(M) \to \Omega^1(M)$ is the standard differential and $d^* : \Omega^1(M) \to C^{\infty}(M)$ is the formal adjoint of d defined by the identity: $(du, w)_{L^2(\Lambda^1 T^* M)} = (u, d^* w)\,,\ u \in C_c^{\infty}(M)\ and\ w \in \Omega^1(M)$. By $\Delta_M = d^* d$ we will denote the scalar laplacian on M.

We consider the second order elliptic differential expression:

$$E = \triangle_M + b_i \frac{\partial^2}{\partial x^{i^2}} + q, \tag{3}$$

with the real-valued positive continuous functions b_i, where $\triangle_M$ is the scalar Laplacian on M, $0 \leq q \in L^2_{loc}(M)$ is a real-valued function and the expression $b_i \frac{\partial^2}{\partial x^{i^2}}$ makes sense in a coordinate neighborhood U with coordinates $x^1, x^2, \ldots, x^n$.

Definition 1 *The set* D_1 :

Let E be as in (3), we will use the notation

$$D_1 = \{u \in L^2(M) : Eu \in L^2(M)\}. \tag{4}$$

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Remark 2 *In general, it is not true that for all* $u \in D_1$ *we have* $\Delta_M u \in L^2(M)$, $b_i \frac{\partial^2 u}{\partial x^{i^2}}$ *and* $qu \in L^2(M)$ *separately. Using the terminology of Everitt and Giertz [9], we will say that the differential expression* $E = \triangle_M + b_i \frac{\partial^2}{\partial x^{i^2}} + q$ *is separated in* $L^2(M)$ *when the following statement holds true: for all* $u \in D_1$ *we have* $b_i \frac{\partial^2 u}{\partial x^{i^2}}$ *and* $qu \in L^2(M)$.

2 The Main Result

We now state the main result.

Theorem 3 *Assume that there exists a function* $0 \leq v \in C^1(M)$ *such that*

$$v(x) \leq q(x) \leq cv(x) \tag{5}$$

and

$$|dv(x)| \leq \sigma v^{\frac{3}{2}}(x), \quad \forall x \in M \ , \tag{6}$$

where $c>0$ *and* $0 \leq \sigma < 2$ *are constants and the notation* $|dv(x)|$ *denotes the norm of* $dv(x) \in T_x^* M$ *with respect to the inner product in* $T_x^* M$ *induced by the metric g. Assume that (M,g) is connected* $C^\infty-$*Riemannian manifold without boundary with metric g and a positive smooth measure* $d\mu$. *Additionally, assume that (M,g) is complete. Then*

$$\|\Delta_M u\| + \left\| b_i \frac{\partial^2 u}{\partial x^{i^2}} \right\| + \|qu\| \leq C \ (\|Eu\| + \|u\|), \quad for\ all \quad u \in D_1, \tag{7}$$

where $C \geq 0$ *is a constant independent of* u. *And so the differential expression* E *is separated in* $L^2(M)$.

Remark 4 *Theorem (3) extends a result of Biomatov* [4, *theorem* 4] *concerning the separation property for the schrodinger operator* $-\Delta + q$ *in* $L^2(R^n)$, *where* Δ *is the standard Laplacian on* R^n *with standard metric and measure and* $0 \leq q \in C^1(R^n)$.

Definition 5 *Differential expression* E_v

Let $0 \leq v \in C^1(M)$, *by* E_v *we will denote the differential expression* $E_v = \triangle_M + b_i \frac{\partial^2}{\partial x^{i^2}} + v$.

In the two preliminary lemmas, we will adope the scheme of Biomatov [4] and Everitt and Giertz [9] to our context. In the proof of Theorem 3 we use the positivity preserving property of resolvents of self-adjoint closures of $E_v \mid_{C_c^\infty(M)}$ and $E \mid_{C_c^\infty(M)}$.

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Lemma 6 *Assume that (M,g) is connected $C^{\infty}-$Riemannian manifold without boundary, with metric g and positive smooth measure $d\mu$. Assume that $0 \leq v \in C^1(M)$ satisfies(6) with $\sigma \in [0,2)$. Then the following inqualities hold:*

$$\|\Delta_M u\| + \left\|b_i \frac{\partial^2 u}{\partial x^{i^2}}\right\| + \|vu\| \leq c^* \|E_v u\|, \quad \forall u \in C_c^{\infty}(M) \tag{8}$$

and

$$\left\|v^{\frac{1}{2}} du\right\|_{L^2(\Lambda^1 T^* M)} \leq c^* \|E_v u\|, \quad \forall u \in C_c^{\infty}(M) \tag{9}$$

where E_v as in Def. 5 and $c^*$is a constant depending on n and σ.

Proof. *We will first prove that the following equality hold for any $\beta > 0$:*

$$\begin{aligned}\|E_v u\|^2 &= \|vu\|^2 + \left\|b_i \frac{\partial^2 u}{\partial x^{i^2}}\right\|^2 + \beta \|\Delta_M u\|^2 + (1-\beta)\operatorname{Re}(\Delta_M u, E_v u) \\ &\quad + (1+\beta)\operatorname{Re}(\Delta_M u, vu) + (1+\beta)\operatorname{Re}\left(\Delta_M u, b_i \frac{\partial^2 u}{\partial x^{i^2}}\right) \\ &\quad + 2\operatorname{Re}\left(b_i \frac{\partial^2 u}{\partial x^{i^2}}, vu\right),\end{aligned} \tag{10}$$

for every $u \in C_c^{\infty}(M)$. Let $\beta > 0$ be arbitrary, By the definition of E_v, for every $u \in C_c^{\infty}(M)$, we have

$$\begin{aligned}\|E_v u\|^2 &= \left(\triangle_M u + b_i \frac{\partial^2 u}{\partial x^{i^2}} + vu, \triangle_M u + b_i \frac{\partial^2 u}{\partial x^{i^2}} + vu\right) \\ &= (\Delta_M u, \Delta_M u) + \left(\Delta_M u, b_i \frac{\partial^2 u}{\partial x^{i^2}}\right) + (\Delta_M u, vu) + \left(b_i \frac{\partial^2 u}{\partial x^{i^2}}, \Delta_M u\right) \\ &\quad + \left(b_i \frac{\partial^2 u}{\partial x^{i^2}}, b_i \frac{\partial^2 u}{\partial x^{i^2}}\right) + \left(b_i \frac{\partial^2 u}{\partial x^{i^2}}, vu\right) + (vu, \Delta_M u) + \left(vu, b_i \frac{\partial^2 u}{\partial x^{i^2}}\right) + (vu, vu),\end{aligned}$$

since $(\Delta_M u, vu) + (vu, \Delta_M u) = 2\operatorname{Re}(\Delta_M u, vu)$, $\left(b_i \frac{\partial^2 u}{\partial x^{i^2}}, vu\right) + \left(vu, b_i \frac{\partial^2 u}{\partial x^{i^2}}\right) = 2\operatorname{Re}\left(b_i \frac{\partial^2 u}{\partial x^{i^2}}, vu\right)$ and $\left(\Delta_M u, b_i \frac{\partial^2 u}{\partial x^{i^2}}\right) + \left(b_i \frac{\partial^2 u}{\partial x^{i^2}}, \Delta_M u\right) = 2\operatorname{Re}\left(\Delta_M u, b_i \frac{\partial^2 u}{\partial x^{i^2}}\right)$. Then

$$\begin{aligned}\|E_v u\|^2 &= \|vu\|^2 + \|\Delta_M u\|^2 + \left\|b_i \frac{\partial^2 u}{\partial x^{i^2}}\right\|^2 \\ &\quad + 2\operatorname{Re}(\Delta_M u, vu) + 2\operatorname{Re}\left(b_i \frac{\partial^2 u}{\partial x^{i^2}}, vu\right) + 2\operatorname{Re}\left(\Delta_M u, b_i \frac{\partial^2 u}{\partial x^{i^2}}\right) \\ &= \|vu\|^2 + \left\|b_i \frac{\partial^2 u}{\partial x^{i^2}}\right\|^2 + \beta \|\Delta_M u\|^2 + (1-\beta)\|\Delta_M u\|^2 \\ &\quad + 2\operatorname{Re}(\Delta_M u, vu) + 2\operatorname{Re}\left(b_i \frac{\partial^2 u}{\partial x^{i^2}}, vu\right) + 2\operatorname{Re}\left(\Delta_M u, b_i \frac{\partial^2 u}{\partial x^{i^2}}\right) \\ &= \|vu\|^2 + \left\|b_i \frac{\partial^2 u}{\partial x^{i^2}}\right\|^2 + \beta \|\Delta_M u\|^2 + (1-\beta)\operatorname{Re}(\Delta_M u, \Delta_M u) \\ &\quad + 2\operatorname{Re}(\Delta_M u, vu) + 2\operatorname{Re}\left(b_i \frac{\partial^2 u}{\partial x^{i^2}}, vu\right) + 2\operatorname{Re}\left(\Delta_M u, b_i \frac{\partial^2 u}{\partial x^{i^2}}\right)\end{aligned}$$

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$$\begin{aligned} &= \|vu\|^2 + \left\|b_i \frac{\partial^2 u}{\partial x^{i^2}}\right\|^2 + \beta \|\Delta_M u\|^2 + (1-\beta)\operatorname{Re}\left(\Delta_M u, E_v u - b_i \frac{\partial^2 u}{\partial x^{i^2}} - vu\right) \\ &\quad + 2\operatorname{Re}(\Delta_M u, vu) + 2\operatorname{Re}\left(b_i \frac{\partial^2 u}{\partial x^{i^2}}, vu\right) + 2\operatorname{Re}\left(\Delta_M u, b_i \frac{\partial^2 u}{\partial x^{i^2}}\right) \\ &= \|vu\|^2 + \left\|b_i \frac{\partial^2 u}{\partial x^{i^2}}\right\|^2 + \beta \|\Delta_M u\|^2 + (1-\beta)\operatorname{Re}(\Delta_M u, E_v u) \\ &\quad + (1+\beta)\operatorname{Re}\left(\Delta_M u, b_i \frac{\partial^2 u}{\partial x^{i^2}}\right) + (1+\beta)\operatorname{Re}(\Delta_M u, vu) + 2\operatorname{Re}\left(b_i \frac{\partial^2 u}{\partial x^{i^2}}, vu\right), \end{aligned}$$

where$(.,.)$ *is as in (1) and* $\|.\|$ *is the corresponding norm in* $L^2(M)$. *Since* $u \in C_c^\infty(M)$, *using integration by parts and the product rule, we have*

$$\begin{aligned} \operatorname{Re}(\Delta_M u, vu) &= \operatorname{Re}(d^* du, vu) = \operatorname{Re}(du, d(vu))_{L^2(\Lambda^1 T^* M)} \\ &= \operatorname{Re}(du, vdu)_{L^2(\Lambda^1 T^* M)} + \operatorname{Re}(du, udv)_{L^2(\Lambda^1 T^* M)} \\ &= \operatorname{Re} z + W \end{aligned} \tag{11}$$

where

$$z = \int_M \langle du, \overline{u}(dv)\rangle \, d\mu, \tag{12}$$

and

$$W = \left(v^{\frac{1}{2}} du, v^{\frac{1}{2}} du\right)_{L^2(\Lambda^1 T^* M)}, \tag{13}$$

from (11) we get

$$(1+\beta)\operatorname{Re}(\Delta_M u, vu) = (1+\beta)\operatorname{Re} z + (1+\beta)W \geq -(1+\beta)|z| + (1+\beta)W. \tag{14}$$

We will now estimate $|z|$, *where* z *is as in (12). Using Cauchy Schwarz inequality and the inequality*

$$2ab \leq ka^2 + k^{-1} b^2, \tag{15}$$

where a, b *and* k *are positive real numbers, we get for any* $\delta > 0$: $|z| \leq \int_M |du|\,|dv|\,|u|\, d\mu$, *from (6), we get* $|z| \leq \sigma \int_M v^{\frac{3}{2}} |du|\,|u|\, d\mu = \sigma \int_M \left|v^{\frac{1}{2}} du\right| |vu|\, d\mu \leq \frac{\beta\delta}{2} \left\|v^{\frac{1}{2}} du\right\|^2_{L^2(\Lambda^1 T^* M)} + \frac{\sigma^2}{2\beta\delta}\|vu\|^2$, *then* $-(1+\beta)|z| \geq -\frac{(1+\beta)\beta\delta}{2}\left\|v^{\frac{1}{2}} du\right\|^2_{L^2(\Lambda^1 T^* M)} - \frac{(1+\beta)\sigma^2}{2\beta\delta}\|vu\|^2$, *so from (14), we obtain*

$$\begin{aligned} (1+\beta)\operatorname{Re}(\Delta_M u, vu) \geq\ & -\frac{(1+\beta)\beta\delta}{2}\left\|v^{\frac{1}{2}} du\right\|^2_{L^2(\Lambda^1 T^* M)} - \frac{(1+\beta)\sigma^2}{2\beta\delta}\|vu\|^2 \\ & + (1+\beta)\left\|v^{\frac{1}{2}} du\right\|^2_{L^2(\Lambda^1 T^* M)} + (1+\beta)\operatorname{Re}\left(\Delta_M u, b_i \frac{\partial^2 u}{\partial x^{i^2}}\right) \\ & + (1+\beta)\operatorname{Re}(\Delta_M u, vu) + 2\operatorname{Re}\left(b_i \frac{\partial^2 u}{\partial x^{i^2}}, vu\right), \end{aligned} \tag{16}$$

Using (15), for any $\alpha > 0$ *we get* $|\mathrm{Re}\,(\Delta_M u, E_v u)| \leq |(\Delta_M u, E_v u)| \leq \frac{\alpha}{2} \|\Delta_M u\|^2 + \frac{1}{2\alpha} \|E_v u\|^2$*, so*

$$(1-\beta)\,|\mathrm{Re}\,(\Delta_M u, E_v u)| \leq \frac{(1-\beta)\alpha}{2} \|\Delta_M u\|^2 + \frac{(1-\beta)}{2\alpha} \|E_v u\|^2 . \tag{17}$$

Similarly, for any $\gamma > 0$*, we get* $\left|\mathrm{Re}\left(\Delta_M u, b_i \frac{\partial^2 u}{\partial x^{i^2}}\right)\right| \leq \left|\left(\Delta_M u, b_i \frac{\partial^2 u}{\partial x^{i^2}}\right)\right| \leq \frac{\gamma}{2} \|\Delta_M u\|^2 + \frac{1}{2\gamma} \left\|b_i \frac{\partial^2 u}{\partial x^{i^2}}\right\|^2$*, so*

$$(1+\beta)\left|\mathrm{Re}\left(\Delta_M u, b_i \frac{\partial^2 u}{\partial x^{i^2}}\right)\right| \leq \frac{(1+\beta)\gamma}{2} \|\Delta_M u\|^2 + \frac{(1+\beta)}{2\gamma} \left\|b_i \frac{\partial^2 u}{\partial x^{i^2}}\right\|^2 , \tag{18}$$

and for any $\theta > 0$ *we get*

$$\left|\mathrm{Re}\left(b_i \frac{\partial^2 u}{\partial x^{i^2}}, vu\right)\right| \geq -\frac{\theta}{2} \left\|b_i \frac{\partial^2 u}{\partial x^{i^2}}\right\|^2 - \frac{1}{2\theta} \|vu\|^2 . \tag{19}$$

Combining (16), (17), (18) and (19), we obtain

$$\begin{aligned} \|E_v u\|^2 \quad \geq \quad & \|vu\|^2 + \left\|b_i \frac{\partial^2 u}{\partial x^{i^2}}\right\|^2 + \beta \|\Delta_M u\|^2 - \frac{|1-\beta|\,\alpha}{2} \|\Delta_M u\|^2 \\ & -\frac{(1+\beta)\gamma}{2} \|\Delta_M u\|^2 - \frac{|1-\beta|}{2\alpha} \|E_v u\|^2 \\ & +(1+\beta)(1-\frac{\beta\delta}{2}) \left\|v^{\frac{1}{2}} du\right\|^2_{L^2(\Lambda^1 T^* M)} - \frac{(1+\beta)}{2\gamma} \left\|b_i \frac{\partial^2 u}{\partial x^{i^2}}\right\|^2 \\ & -\theta \left\|b_i \frac{\partial^2 u}{\partial x^{i^2}}\right\|^2 - \frac{(1+\beta)\sigma^2}{2\beta\delta} \|vu\|^2 - \frac{1}{\theta} \|vu\|^2 . \end{aligned} \tag{20}$$

From (20), we get

$$\begin{aligned} \left(1 + \frac{|1-\beta|}{2\alpha}\right) \|E_v u\|^2 \quad \geq \quad & \left(1 - \frac{1}{\theta} - \frac{(1+\beta)\sigma^2}{2\beta\delta}\right) \|vu\|^2 \\ & + \left(\beta - \frac{|1-\beta|\,\alpha}{2} - \frac{(1+\beta)\gamma}{2}\right) \|\Delta_M u\|^2 \\ & + \left((1+\beta)(1-\frac{\beta\delta}{2})\right) \left\|v^{\frac{1}{2}} du\right\|^2_{L^2(\Lambda^1 T^* M)} \\ & + \left(|1-\theta| - \frac{(1+\beta)}{2\gamma}\right) \left\|b_i \frac{\partial^2 u}{\partial x^{i^2}}\right\|^2 . \end{aligned} \tag{21}$$

The inequalities (8) and (9) will immediately follow from (21) if

$$(1+\beta)\sigma^2 < 2\beta\delta \left(1 - \frac{1}{\theta}\right),\ \beta\delta < 2,\ (1+\beta) < 2\gamma\,|1-\theta| \ \text{ and } \ |1-\beta|\,\alpha + (1+\beta)\gamma < 2\beta. \tag{22}$$

Since, by hypothesis $0 \leq \sigma < 2$*, there exist numbers* $\beta > 0,\ \alpha > 0,\ \delta > 0,\ \gamma > 0$ *and* $\theta > 1$ *such that the inequalities (22) hold. This concludes the proof of the lemma.* ■

In the sequel, we will also use the following terms and notations.

Definition 7 *Minimal operators S and T:*

Let E be as in (3) and E_v be as in def 5. We define the minimal operators S and T in $L^2(M)$ associated to E and E_v by the formulas $Su = Eu$ and $Tu = E_v u$ with domains Dom(S)=Dom(T) $=C_c^\infty(M)$. Since S and T are symmetric operators, it follows that S and T is closable, see for example, section V.3.3 in [9]*. In what follows, we will denote by $\widetilde{S}$ and $\widetilde{T}$ the closures in $L^2(M)$ of the operators S and T respectively.*

Definition 8 *Maximal operators H and K:*

Let E be as in (3). We define the maximal operator H in $L^2(M)$ associated to E by the formula $Hu = S^*u$, where S^* is the adjoint of the operator S in $L^2(M)$. In the case when $q \in L^2_{loc}(M)$ is real-valued, it is well known that $Dom(H) = D_1$, where $D_1$is as in (4)*. Let E_v be as in def. 5, we define the maximal operator K in $L^2(M)$ associated to E_v by the formula $Ku = T^*u$, and we have $Dom(K) = \{u \in L^2(M) : E_v u \in L^2(M)\}$.*

Remark 9 *By lemma 5.1 from* [17] *it follows that $Dom(K) \subset W^{2,2}_{loc}(M)$.*

Definition 10 *Essential self-adjointness of S and T:*

If (M,g) is a complete Riemannian manifold with metric g and positive smooth measure $d\mu$ and if $0 \le q \in L^2_{loc}(M)$, then the operator S is essentially self-adjoint in $L^2(M)$ (see [5], [10] and [11]). In this case we have $\widetilde{S} = S^*$. In particular, since $0 \le v \in C^1(M) \subset L^2_{loc}(M)$, it follows that the operator T is essentially self-adjoint in $L^2(M)$.

Lemma 11 *Assume that (M,g) is connected $C^\infty-$Riemannian manifold without boundary, with metric g and positive smooth measure $d\mu$. Additionally, assume that (M,g) is complete. Assume that $0 \le v \in C^1(M)$ satisfies (5). Then the inequalities (8) and (9) hold for all $u \in Dom(K)$, where K is as in definition 8.*

Proof. Under the hypotheses of this lemma, by definition 10 it follows that T is essentially self-adjoint and $K = T^* = \widetilde{T}$. In particular, $Dom(K) = Dom\left(\widetilde{T}\right)$. Let $u \in Dom(K)$. *Then* $\exists$ seq $\{u_k\} \in C_c^\infty(M)$ such that $u_k \to u$ and $E_v u_k \to \widetilde{T}u$. Since by lemma (6) the seq $\{u_k\}$ satisfies (8) and (9), it follows that the sequences $\{\Delta_M u_k\}$, $\{b_i \frac{\partial^2 u}{\partial x^{i2}}\}$*and* $\{vu_k\}$ are Cauchy sequences in $L^2(M)$ and $\{v^{\frac{1}{2}} du_k\}$ is a Cauchy seq. in $L^2\left(\Lambda^1 T^* M\right)$. We will show that

$$vu_k \to vu \text{ in } L^2(M) \text{ as } k \to \infty. \tag{23}$$

Since $\{vu_k\}$ is a Cauchy seq. in $L^2(M)$, it follows that $\{vu_k\}$ converges to $s \in L^2(M)$. Let Φ be an arbitrary element of $C_c^\infty(M)$, then

$$0 = (vu_k, \Phi) - (u_k, v\Phi) \to (s, \Phi) - (u, v\Phi) = (s - vu, \Phi), \tag{24}$$

where $(.,.)$ is as in (1), since $C_c^\infty(M)$ is dense in $L^2(M)$, we get $s = vu$ and (23) is proven. If (M,g) is complete, it is well known that Δ_M is essentially self adjoint on $C_c^\infty(M)$ and we have the following equality $\widetilde{(\Delta_M |_{C_c^\infty(M)})} = \Delta_{M,\max}$ where $\Delta_{M,\max}u = \Delta_M u$ with the domain $\mathrm{Dom}(\Delta_{M,\max}) = \{u \in L^2(M) : \Delta_M u \in L^2(M)\}$, see for example, theorem 3.5 in [8]. Since $u_k \to u$ in $L^2(M)$ and since $\{\Delta_M u_k\}$ is a Cauchy seq. in $L^2(M)$, by the definition of $\widetilde{(\Delta_M |_{C_c^\infty(M)})}$ it follows that $u \in Dom(\widetilde{(\Delta_M |_{C_c^\infty(M)})})$. Since $\widetilde{(\Delta_M |_{C_c^\infty(M)})} = \Delta_{M,\max}u$, we have

$$\Delta_M u_k \to \Delta_M u \ \ in \ \ L^2(M) \ \ as \ k \to \infty. \tag{25}$$

Since $\{\Delta_M u_k\}$ and $\{u_k\}$ are Cauchy seqs in $L^2(M)$, and since $\|du_k\|^2_{L^2(\Lambda^1 T^*M)} = (du_k, du_k)_{L^2(\Lambda^1 T^*M)} = (\Delta_M u_k, u_k) \le \|\Delta_M u_k\| \|u_k\|$,it follows that $\{du_k\}$ is a Cauchy seq. in $L^2(\Lambda^1 T^*M)$, and hence, du_k converges to some element $w \in L^2(\Lambda^1 T^*M)$. Let $\Psi \in \Omega_c^1(M)$ be arbitrary, then by integration by parts (see, for example, lemma 8.8 in [5]) and remark 9, we get

$$\begin{aligned} 0 &= (du_k, \Psi)_{L^2(\Lambda^1 T^*M)} - (u_k, d^*\Psi) \\ &\to (w, \Psi)_{L^2(\Lambda^1 T^*M)} - (u, d^*\Psi) \\ &= (w, \Psi)_{L^2(\Lambda^1 T^*M)} - (du, \Psi)_{L^2(\Lambda^1 T^*M)}, \end{aligned} \tag{26}$$

where $(.,.)$ is the inner product in $L^2(M)$. From (26) we get $du = w \in L^2(\Lambda^1 T^*M)$, and hence

$$du_k \to du = w \ in \ L^2(\Lambda^1 T^*M), \ as \ k \to \infty. \tag{27}$$

Since $\{v^{\frac{1}{2}} du_k\}$ is a Cauchy seq. in $L^2(\Lambda^1 T^*M)$, using (27) and (24), we obtain

$$v^{\frac{1}{2}} du_k \to v^{\frac{1}{2}} du \ in \ \ L^2(\Lambda^1 T^*M) \ \ as \ k \to \infty. \tag{28}$$

Since $u_k \to u$ in $L^2(M)$ and since $\{b_i \frac{\partial^2 u_k}{\partial x^{i2}}\}$ is a Cauchy seq. in $L^2(M)$, then we get

$$b_i \frac{\partial^2 u_k}{\partial x^{i2}} \to b_i \frac{\partial^2 u}{\partial x^{i2}}. \tag{29}$$

Using (23) ,(25), (28) and (29) and taking limits as $k \to \infty$ in all terms in (8) and (9) (with u replaced by u_k), shows that (8) and (9) hold for all $u \in Dom(K) = Dom(\widetilde{T})$. This concludes the proof of the lemma. ■

Definition 12 *Operators R_1 and R_2*

Let S and T be as in def. 7 and let H and K as in def. 8. By def. 10 the operators $H = \widetilde{S}$ and $K = \widetilde{T}$ are non-negative self adjoint in $L^2(M)$. Thus,

$$R_1 = (H+1)^{-1} : L^2(M) \to L^2(M)$$

and

$$R_2 = (K+1)^{-1} : L^2(M) \to L^2(M), \tag{30}$$

are bounded linear operators.

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Definition 13 *Positivity preserving property*

Let (X, μ) be a measure space. A bounded linear operator $A : L^2(X, \mu) \to L^2(X, \mu)$ is said to be positivity preserving if every $u \in L^2(X, \mu)$ such that $u \geq 0$ on X, we have $Au \geq 0$ on X.

Remark 14 *Let $A : L^2(X, \mu) \to L^2(X, \mu)$ be positivity preserving bounded linear operator. Then the following inequality holds for all $u \in L^2(X, \mu)$*

$$|(Au)(x)| \leq A|u(x)| \quad \text{on } X, \tag{31}$$

where $|.|$ denotes the absolute value of a complex number. For the proof of (31), see the proof of the inequality $(X.103)$ in [13].

In the sequel, we will use the following proposition.

Proposition 15 *Assume that (M, g) is (not necessarily complete) $C^\infty-$ Riemannian manifold without boundary. Assume that M is connected and oriented. Assume that $Q_0 \in L^2_{loc}(M)$ is real valued. Additionally assume that $\left(\left(\Delta_M + b_i \frac{\partial^2}{\partial x^{i2}} + Q_0\right) u, u\right) \geq 0 \ \forall u \in C_c^\infty(M)$. Let S_0 be the friedrichs extension of $\left(\Delta_M + b_i \frac{\partial^2}{\partial x^{i2}} + Q_0\right)|_{C_c^\infty(M)}$. Assume that λ is a positive real number. Then the operator $(S_0 + \lambda)^{-1}$ is positivity preserving. (See [12]).*

From proposition 15 we obtain the following corollary.

Corollary 16 *Assume that (M, g) is a complete $C^\infty-$Riemannian manifold without boundary. Assume that M is connected and oriented. Assume that $0 \leq q \in L^2_{loc}(M)$ and $0 \leq v \in C^1(M)$. Let $R_1$and R_2 be as in (30). Then the operators $R_1$and $R_2$are positivity preserving.*

Proof. *Since (M, g) is complete, by def. 10, it follows that S and T are non-negative essentially self-adjoint operators in $L^2(M)$. Thus the friedrichs extensions of $\left(\Delta_M + b_i \frac{\partial^2}{\partial x^{i2}} + q\right)|_{C_c^\infty(M)}$ and $\left(\Delta_M + b_i \frac{\partial^2}{\partial x^{i2}} + v\right)|_{C_c^\infty(M)}$ are H and K, respectively. By proposition (15) it follows that $R_1$and R_2 are positivity preserving operators in $L^2(M)$.* ∎

Now we introduce the proof of our main theorem.

Proof. of Theorem (3): By lemma 11 it follows that

$$\|vu\| \leq c^* \|Ku\| \quad \forall u \in Dom(K). \tag{32}$$

Let R_2 be as in (30), and let $v : L^2(M) \to L^2(M)$,denote the maximal multiplication operator corresponding to the function v. From (32) it follows that $vR_2 : L^2(M) \to L^2(M)$ is a bounded linear operator. Let $q : L^2(M) \to L^2(M)$,denote the maximal multiplication operator corresponding to the function q. Since $vR_2 : L^2(M) \to L^2(M)$,is a bounded linear operator, by (5) it follows that $qR_2 : L^2(M) \to L^2(M)$, is a bounded linear operator. We will

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now show that qR_1 is a bounded linear operator: $L^2(M) \to L^2(M)$. First we will show that

$$qR_1 f \leq qR_2 f \qquad for\ every\ 0 \leq f \in L^2(M). \tag{33}$$

Since $R_1 f \in D_1 \subset L^2(M)$ and since $q \in L^2_{loc}(M)$, it follows that $qR_1 f \in L^1_{loc}(M)$ is a function. Let $0 \leq f \in L^2(M)$ be arbitrary. Since $R_2 f \in Dom(K) = \{z \in L^2(M) : E_v z \in L^2(M)\}$, by lemma (11) we have $(\Delta_M (R_2 f))$, $b_i \frac{\partial^2 R_2 f}{\partial x^{i^2}}$ and $(v(R_2 f)) \in L^2(M)$. Since $qR_2 : L^2(M) \to L^2(M)$ is a bounded linear operator, we get $(qR_2 f) \in L^2(M)$. Thus $R_2 f \in D_1$, and hence

$$qR_2 f = qR_1 (H+1) R_2 f\ . \tag{34}$$

By the definition of R_2 we have

$$qR_1 f = qR_1 (K+1) R_2 f\ . \tag{35}$$

From (34)and (35) we have

$$\begin{aligned} qR_2 f - qR_1 f &= qR_1 ((H+1) R_2 f - (K+1) R_2 f) \\ &= qR_1 ((H-K) R_2 f) = qR_1 (q-v) R_2 f\ . \end{aligned} \tag{36}$$

Since R_2 is positivity preserving in $L^2(M)$, we have $R_2 f \geq 0$. Since vR_2 and qR_2 are bounded linear operators $L^2(M) \to L^2(M)$ and since $q - v \geq 0$, then

$$0 \leq (q-v) R_2 f \in L^2(M). \tag{37}$$

Since R_1 is positivity preserving in $L^2(M)$, from (37) we get

$$R_1 ((q-v) R_2 f) \geq 0. \tag{38}$$

Since $q \geq 0$, from (36) and (38) we get $qR_2 f - qR_1 f = qR_1 (q-v) R_2 f \geq 0$, and (33) is proven. Let $w \in L^2(M)$ be arbitrary. Since $R_1 : L^2(M) \to L^2(M)$, is positivity preserving bounded linear operator, by (31) we have the following inequality:

$$|R_1 w| \leq R_1 |w|. \tag{39}$$

Since $q \geq 0$, from (39) and (33) we obtain the inequality,

$$|qR_1 w| = q |R_1 w| \leq qR_1 |w| \leq qR_2 |w|. \tag{40}$$

Since $qR_2 : L^2(M) \to L^2(M)$,is a bounded linear operator, from (40) it follows that, $\|qR_1 w\| \leq \|qR_2 |w|\| \leq C_3 \|w\|$, $\quad \forall w \in L^2(M)$,where C_3 is a constant. Hence qR_1 is a bounded linear operator $L^2(M) \to L^2(M)$. Consequently $b_i \frac{\partial^2 R_1}{\partial x^{i^2}}$ is a bounded linear operator $L^2(M) \to L^2(M)$. Let $u \in D_1$ be arbitrary. Then $qu = qR_1 (u + Eu)$, and $b_i \frac{\partial^2 u}{\partial x^{i^2}} = b_i \frac{\partial^2}{\partial x^{i^2}} R_1 (u + Eu)$ where $Eu = \Delta_M u + b_i \frac{\partial^2 u}{\partial x^{i^2}} + qu$. Since $qR_1 : L^2(M) \to L^2(M)$ and $b_i \frac{\partial^2 R_1}{\partial x^{i^2}} : L^2(M) \to L^2(M)$,are bounded linear operators, we have

$$\begin{aligned} \|qu\| &= \|qR_1 (u + Eu)\| \\ &\leq C_1 \|u + Eu\| \leq C_1 (\|u\| + \|Eu\|), \quad \forall u \in D_1 \end{aligned} \tag{41}$$

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and

$$\begin{aligned}\left\|b_i\frac{\partial^2 u}{\partial x^{i^2}}\right\| &= \left\|b_i\frac{\partial^2}{\partial x^{i^2}}R_1\left(u+Eu\right)\right\| \\ &\leq C_2\left\|u+Eu\right\| \leq C_2\left(\|u\|+\|Eu\|\right), \quad \forall u\in D_1 \qquad (42)\end{aligned}$$

where $C_1, C_2 \geq 0$ are constants. Using (41) and (42) we get

$$\begin{aligned}\|\Delta_M u\| &= \left\|Eu - b_i\frac{\partial^2 u}{\partial x^{i^2}} - qu\right\| \\ &\leq \|Eu\| + \left\|b_i\frac{\partial^2 u}{\partial x^{i^2}}\right\| + \|qu\| \\ &\leq C_3\left(\|u\|+\|Eu\|\right), \qquad \forall u\in D_1 \qquad (43)\end{aligned}$$

where $C_3 \geq 0$ is a constant. From (41), (42) and (43), we obtain (7). This concludes the proof of the Theorem. ■

Acknowledgement 17 *This paper was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, under grant No. (16-662-D1432). The authors, therefore, acknowledge with thanks DSR technical and financial support.*

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 19, NO.2, 2015, COPYRIGHT 2015 EUDOXUS PRESS, LLC

AN INTEGRAL-TYPE OPERATOR FROM WEIGHTED BERGMAN-PRIVALOV SPACES TO BLOCH-TYPE SPACES

YONG REN

Abstract

Let $H(B)$ denote the space of all holomorphic functions on the unit ball B of $\mathbb{C}^n$. Let φ be a holomorphic self-map of B and $g \in H(B)$ such that $g(0) = 0$. In this paper, we study the following integral-type operator, recently introduced by Stević in [16]

$$P_\varphi^g f(z) = \int_0^1 f(\varphi(tz))g(tz)\frac{dt}{t}, \quad f \in H(B).$$

The boundedness and compactness of the operator P_φ^g from weighted Bergman-Privalov spaces to Bloch-type spaces in the unit ball are studied.

1 Introduction

Let B be the unit ball of $\mathbb{C}^n$, ∂B its boundary, $d\sigma$ the normalized surface measure on ∂B and dv be the normalized Lebesgue measure on B. We denote by $H(B)$ the class of all holomorphic functions on B. $H(B)$ is a Fréchet space (locally convex, metrizable and complete) with respect to the compact-open topology. By Montel's theorem, bounded sets in $H(B)$ are relatively compact and hence bounded sequences in $H(B)$ admit convergent subsequences. Convergence in this space will be referred to as local uniform ($l.u.$) convergence.

A positive continuous function μ on the interval $[0, 1)$ is called normal if there is a $\delta \in [0, 1)$ and s and t, $0 < s < t$ such that (see, e.g. [13])

$$\frac{\mu(r)}{(1-r)^s} \text{ is decreasing on } [\delta, 1) \text{ and } \lim_{r\to 1}\frac{\mu(r)}{(1-r)^s} = 0;$$
$$\frac{\mu(r)}{(1-r)^t} \text{ is increasing on } [\delta, 1) \text{ and } \lim_{r\to 1}\frac{\mu(r)}{(1-r)^t} = \infty.$$

Let $p \in [1, \infty)$, $\alpha > -1$. An $f \in H(B)$ is said to belong to the weighted Bergman-Privalov space, denoted by $\mathcal{AN}_\alpha^p = \mathcal{AN}_\alpha^p(B)$, if

$$\|f\|_{\mathcal{AN}_\alpha^p}^p = \int_B \big[\log(1 + |f(z)|)\big]^p dv_\alpha(z) < \infty,$$

2000 *Mathematics Subject Classification.* Primary 47B38, Secondary 30D45.
Key words and phrases. Integral-type operator, weighted Bergman-Privalov space, Bloch-type space, boundedness, compactness.

where $dv_\alpha(z) = c_\alpha(1-|z|^2)^\alpha dv(z)$ and c_α is chosen such that $v_\alpha(B) = 1$. For each $f \in \mathcal{AN}^p_\alpha$ the following inequality holds ([12])

$$\log(1+|f(z)|) \le \frac{(1+|z|)^{(n+1+\alpha)/p}}{(1-|z|)^{(n+1+\alpha)/p}} \|f\|_{\mathcal{AN}^p_\alpha}. \tag{1}$$

Hence the topology of $\mathcal{AN}^p_\alpha$ is stronger than that of locally uniform convergence.

Let μ be a normal function on $[0,1)$. An $f \in H(B)$ is said to belong to the Bloch-type space, denoted by $\mathcal{B}_\mu = \mathcal{B}_\mu(B)$, if

$$b_\mu(f) := \sup_{z\in B} \mu(|z|)\,|\Re f(z)| < \infty.$$

where $\Re f(z) = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z)$ is the radial derivative of f. The Bloch-type space is a Banach space with the norm $\|f\|_{\mathcal{B}_\mu} = |f(0)| + b_\mu(f)$. Let $\mathcal{B}_{\mu,0}$ denote the subspace of $\mathcal{B}_\mu$ consisting of those $f \in \mathcal{B}_\mu$ for which

$$\lim_{|z|\to 1} \mu(|z|)|\Re f(z)| = 0.$$

This space is called the little Bloch-type space. When $\mu(r) = 1-r^2$, we get the classical Bloch space and little Bloch space respectively. For more information of the Bloch space, see for example [33].

Let φ be a holomorphic self-map of B. The composition operator induced by φ is

$$(C_\varphi f)(z) = (f\circ\varphi)(z), \quad f \in H(B).$$

It is interesting to provide a function theoretic characterization when φ induce a bounded or compact composition operator on various spaces. The book [2] contains much information on this topic.

Let φ be a holomorphic self-map of B and $g \in H(B)$ such that $g(0) = 0$. For $f \in H(B)$, $z \in B$, the following integral-type operator

$$P^g_\varphi f(z) = \int_0^1 f(\varphi(tz))g(tz)\frac{dt}{t}, \tag{2}$$

was recently introduced by Stević in [16] and studied also in [17, 19, 21, 25, 27, 32]. The operator is a natural generalization of an operator in the unit disk (see [6, 8, 9, 22]). It is clear that so called Riemann-Stieltjes (or Extended Cesàro) operator denoted by T_g is a particular case of operator P^g_φ. It was studied, e.g. in [1, 3, 4, 5, 7, 10, 11, 14, 31, 36](see also the references therein). For some other related integral-type operators see also [15, 18, 20, 23, 24, 26, 34, 35, 38, 39].

In this paper we study the boundedness and compactness of the integral-type operator P^g_φ from weighted Bergman-Privalov spaces to Bloch-type spaces and little Bloch-type spaces in the unit ball. As a corollary, we obtain the characterization of the Riemann-Stieltjes operator T_g from weighted Bergman-Privalov spaces to Bloch-type spaces and little Bloch-type spaces. These results are new even in the unit disk.

Constants are denoted by C in this paper, they are positive and may differ from one occurrence to the other.

2 Main results and proofs

In this section we will give our main results and proofs. First we state several auxiliary results which will be used in the proofs of main results.

The first lemma was proved in [16] similar to a result in [3].

Lemma 1. *Suppose $f, g \in H(B)$, $g(0) = 0$ and φ is a holomorphic self-map of B. Then*

$$\Re[P_\varphi^g(f)](z) = f(\varphi(z))g(z).$$

The following result can be found, for example, in [15].

Lemma 2. *Let μ be a normal function on $[0, 1)$. A closed set K in $\mathcal{B}_\mu$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1} \sup_{f\in K} \mu(|z|)|\Re f(z)| = 0.$$

A subset T of $\mathcal{AN}_\alpha^p$ is called bounded if it is bounded for the defining $F-$norm $\|\cdot\|_{\mathcal{AN}_\alpha^p}$. Given a Banach space X, we say that a linear map $L : \mathcal{AN}_\alpha^p \to X$ is bounded if $L(T) \subset X$ is bounded for every bounded subsets T of $\mathcal{AN}_\alpha^p$. We say that L is compact if $L(T) \subset X$ is relatively compact for every bounded subsets $T \subset \mathcal{AN}_\alpha^p$.

The following criterion for compactness follows from arguments similar, e.g. to those in [2] and [7, 30].

Lemma 3. *Let $p \in [1, \infty)$, $\alpha > -1$, $g \in H(B)$, $g(0) = 0$, μ be a normal function on $[0, 1)$ and φ be a holomorphic self-map of B. Then $P_\varphi^g : \mathcal{AN}_\alpha^p \to \mathcal{B}_\mu$ is compact if and only if for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in $\mathcal{AN}_\alpha^p$ which converges to zero l.u., we have $\|P_\varphi^g f_k\|_{\mathcal{B}_\mu} \to 0$ as $k \to \infty$.*

Now, we are in a position to formulate and prove the main results of this paper.

Theorem 1. *Let $p \in [1, \infty)$, $\alpha > -1$, $g \in H(B)$, $g(0) = 0$, μ be a normal function on $[0, 1)$ and φ be a holomorphic self-map of B. Then $P_\varphi^g : \mathcal{AN}_\alpha^p \to \mathcal{B}_\mu$ is bounded if and only if for all $c > 0$,*

$$\sup_{z\in B} \mu(|z|)|g(z)| \exp\Big[\frac{c}{(1 - |\varphi(z)|^2)^{\frac{n+1+\alpha}{p}}}\Big] < \infty. \tag{3}$$

Proof. Assume that (3) holds. Then for any $f \in \mathcal{AN}_\alpha^p$, by (2) there is a $c_0 > 0$ such that

$$|f(z)| \le \exp\Big[\frac{c_0\|f\|_{\mathcal{AN}_\alpha^p}}{(1 - |z|^2)^{\frac{n+1+\alpha}{p}}}\Big]. \tag{4}$$

By Lemma 1 and (4), for a bounded set $T \subset \mathcal{AN}_\alpha^p$, say by L, we have

$$\begin{aligned}\|P_\varphi^g f\|_{\mathcal{B}_\mu} &= |(P_\varphi^g f)(0)| + \sup_{z\in B}\mu(|z|)|\Re(P_\varphi^g f)(z)| \\ &= \sup_{z\in B}\mu(|z|)|f(\varphi(z))||g(z)| \\ &\le \sup_{z\in B}\mu(|z|)|g(z)|\exp\Big[\frac{c_0 L}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}}}\Big] < \infty. \end{aligned} \tag{5}$$

Here we used the fact that $(P_\varphi^g f)(0) = 0$. From (5) we see that $P_\varphi^g : \mathcal{AN}_\alpha^p \to \mathcal{B}_\mu$ is bounded.

Conversely, suppose that $P_\varphi^g : \mathcal{AN}_\alpha^p \to \mathcal{B}_\mu$ is bounded. For $w \in B$ and any $c > 0$, set

$$f_w(z) = \exp\Big[c\{\frac{1-|\varphi(w)|^2}{(1-\langle z,\varphi(w)\rangle)^2}\}^{\frac{n+1+\alpha}{p}}\Big]. \tag{6}$$

It is easy to see that (see, e.g. [29]) $f_w \in \mathcal{AN}_\alpha^p$ and $\sup_{w\in B}\|f_w\|_{\mathcal{AN}_\alpha^p}^p \le C$. Therefore we have

$$\begin{aligned}\infty &> \sup_{w\in B}\|P_\varphi^g f_w\|_{\mathcal{B}_\mu} \ge \|P_\varphi^g f_w\|_{\mathcal{B}_\mu} \\ &= |(P_\varphi^g f_w)(0)| + \sup_{z\in B}\mu(|z|)|\Re(P_\varphi^g f_w)(z)| \\ &\ge \mu(|w|)|g(w)|\exp\Big[\frac{c}{(1-|\varphi(w)|^2)^{\frac{n+1+\alpha}{p}}}\Big],\end{aligned}$$

for each $w \in B$, which implies (3), finishing the proof of Theorem 1. $\square$

Theorem 2. *Let $p \in [1,\infty)$, $\alpha > -1$, $g \in H(B)$, $g(0) = 0$, μ be a normal function on $[0,1)$ and φ be a holomorphic self-map of B. Then $P_\varphi^g : \mathcal{AN}_\alpha^p \to \mathcal{B}_\mu$ is compact if and only if*

$$\|g\|_\mu := \sup_{z\in B}\mu(|z|)|g(z)| < \infty \tag{7}$$

and for every $c > 0$,

$$\lim_{|\varphi(z)|\to 1}\mu(|z|)|g(z)|\exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}}}\Big] = 0. \tag{8}$$

Proof. Assume that (7) and (8) hold. From these conditions, we easily get (3). Hence $P_\varphi^g : \mathcal{AN}_\alpha^p \to \mathcal{B}_\mu$ is bounded from Theorem 1.

Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{AN}_\alpha^p$ with $\sup_{k\in\mathbb{N}}\|f_k\|_{\mathcal{AN}_\alpha^p} \le Q$ and that f_k converges to 0 *l.u.*. By (8) we have that for every $\varepsilon > 0$ and $c > 0$, there is a $\delta \in (0,1)$ such that

$$\mu(|z|)|g(z)|\exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}}}\Big] < \varepsilon \tag{9}$$

when $\delta < |\varphi(z)| < 1$. From (7) and (9) with $c = c_0Q$, we have

$$\begin{aligned}
&\|P^g_\varphi f_k\|_{\mathcal{B}_\mu} = \sup_{z\in B}\mu(|z|)|\Re(P^g_\varphi f_k)(z)| \\
\leq\ & \Big(\sup_{\{z\in B:\,|\varphi(z)|\leq\delta\}} + \sup_{\{z\in B\,:\,\delta<|\varphi(z)|<1\}}\Big)\mu(|z|)|g(z)||f_k(\varphi(z))| \\
\leq\ & \|g\|_\mu \sup_{|w|\leq\delta}|f_k(w)| + \sup_{\{z\in B\,:\,\delta<|\varphi(z)|<1\}}\mu(|z|)|g(z)|\exp\Big[\frac{c_0Q}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}}}\Big] \\
\leq\ & \|g\|_\mu \sup_{|w|\leq\delta}|f_k(w)| + \varepsilon. \qquad (10)
\end{aligned}$$

By the assumption we have $\lim_{k\to\infty}\sup_{|w|\leq\delta}|f_k(w)| = 0$. Using this fact and letting $k\to\infty$ in (10), we obtain that $\limsup_{k\to\infty}\|P^g_\varphi f_k\|_{\mathcal{B}_\mu} \leq \varepsilon$. Since ε is an arbitrary positive number it follows that

$$\limsup_{k\to\infty}\|P^g_\varphi f_k\|_{\mathcal{B}_\mu} = 0.$$

It follows from Lemma 3 that the implication follows.

Conversely, suppose that $P^g_\varphi : \mathcal{AN}^p_\alpha \to \mathcal{B}_\mu$ is compact. Then it is clear that $P^g_\varphi : \mathcal{AN}^p_\alpha \to \mathcal{B}_\mu$ is bounded and hence (3) holds. Taking $f(z)\equiv 1$, we see that (7) holds.

Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in B such that $\lim_{k\to\infty}|\varphi(z_k)| = 1$. Since (3) holds, we have

$$\lim_{k\to\infty}\mu(|z_k|)|g(z_k)| = \lim_{|\varphi(z_k)|\to 1}\mu(|z_k|)|g(z_k)| = 0. \qquad (11)$$

For any $c > 0$, set

$$f_k(z) = \exp\Big[c\{\frac{1-|\varphi(z_k)|^2}{(1-\langle z,\varphi(z_k)\rangle)^2}\}^{\frac{n+1+\alpha}{p}}\Big] - 1. \qquad (12)$$

From the proof of Theorem 1, we see that $f_k \in \mathcal{AN}^p_\alpha$ and $\sup_{k\in\mathbb{N}}\|f_k\|^p_{\mathcal{AN}^p_\alpha} \leq C$. Moreover, f_k converges to 0 $l.u.$. Therefore, by Lemma 3 we have

$$\lim_{k\to\infty}\|P^g_\varphi f_k\|_{\mathcal{B}_\mu} = 0. \qquad (13)$$

In addition,

$$\begin{aligned}
\|P^g_\varphi f_k\|_{\mathcal{B}_\mu} &= \sup_{z\in B}\mu(|z|)|\Re(P^g_\varphi f_k)(z)| \\
&\geq \mu(|z_k|)|g(z_k)|\Big(\exp\Big[\frac{c}{(1-|\varphi(z_k)|^2)^{\frac{n+1+\alpha}{p}}}\Big] - 1\Big),
\end{aligned}$$

which together with (11) and (13) imply

$$\lim_{k\to\infty}\mu(|z_k|)|g(z_k)|\exp\Big[\frac{c}{(1-|\varphi(z_k)|^2)^{\frac{n+1+\alpha}{p}}}\Big] = 0. \qquad (14)$$

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From which we see that (8) holds. This completes the proof of Theorem 2. □

From Theorems 1 and 2, we easily obtain the following result.

Corollary 1. *Let $p \in [1,\infty)$, $\alpha > -1$, $g \in H(B)$, $g(0) = 0$, μ be a normal function on $[0,1)$ and φ be a holomorphic self-map of B. Then the following statements are equivalent.*

(i) $P_\varphi^g : \mathcal{AN}_\alpha^p \to \mathcal{B}_\mu$ is bounded;

(ii) $P_\varphi^g : \mathcal{AN}_\alpha^p \to \mathcal{B}_\mu$ is compact;

(iii)

$$\|g\|_\mu := \sup_{z\in B} \mu(|z|)|g(z)| < \infty \tag{15}$$

and for every $c > 0$,

$$\lim_{|\varphi(z)|\to 1} \mu(|z|)|g(z)| \exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}}}\Big] = 0. \tag{16}$$

Theorem 3. *Let $p \in [1,\infty)$, $\alpha > -1$, $g \in H(B)$, $g(0) = 0$, μ be a normal function on $[0,1)$ and φ be a holomorphic self-map of B. Then the following statements are equivalent.*

(i) $P_\varphi^g : \mathcal{AN}_\alpha^p \to \mathcal{B}_{\mu,0}$ is bounded;

(ii) $P_\varphi^g : \mathcal{AN}_\alpha^p \to \mathcal{B}_{\mu,0}$ is compact;

(iii) For all $c > 0$,

$$\lim_{|z|\to 1} \mu(|z|)|g(z)| \exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}}}\Big] = 0. \tag{17}$$

Proof. $(iii) \Rightarrow (ii)$. Suppose that (17) holds. From Theorem 1 we see that $P_\varphi^g : \mathcal{AN}_\alpha^p \to \mathcal{B}_\mu$ is bounded. By (5) we have

$$\mu(|z|)|\Re(P_\varphi^g f)(z)| \le \mu(|z|)|g(z)| \exp\Big[\frac{c_0\|f\|_{\mathcal{AN}_\alpha^p}}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}}}\Big]. \tag{18}$$

which together with the boundedness of $P_\varphi^g : \mathcal{AN}_\alpha^p \to \mathcal{B}_\mu$ imply that $P_\varphi^g : \mathcal{AN}_\alpha^p \to \mathcal{B}_{\mu,0}$ is bounded. In addition, taking the supremum in (18) over the unit ball of the space $\mathcal{AN}_\alpha^p$, then letting $|z| \to 1$, we obtain

$$\lim_{|z|\to 1} \sup_{\|f\|_{\mathcal{AN}_\alpha^p}\le 1} \mu(|z|)|\Re(P_\varphi^g f)(z)| = 0. \tag{19}$$

From Lemma 2 and (19), we see that $P_\varphi^g : \mathcal{AN}_\alpha^p \to \mathcal{B}_{\mu,0}$ is compact.

$(ii) \Rightarrow (i)$. This implication is clear.

$(i) \Rightarrow (iii)$. Suppose that $P_\varphi^g : \mathcal{AN}_\alpha^p \to \mathcal{B}_{\mu,0}$ is bounded. Then $P_\varphi^g : \mathcal{AN}_\alpha^p \to \mathcal{B}_\mu$ is bounded. Taking $f(z) \equiv 1$, then by the boundedness of the operator $P_\varphi^g : \mathcal{AN}_\alpha^p \to \mathcal{B}_{\mu,0}$ we get

$$\lim_{|z|\to 1} \mu(|z|)|g(z)| = 0. \tag{20}$$

From Corollary 1 we see that $P_\varphi^g : \mathcal{AN}_\alpha^p \to \mathcal{B}_\mu$ is compact, by Theorem 2 we have

$$\lim_{|\varphi(z)|\to 1} \mu(|z|)|g(z)| \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}}}\right] = 0. \tag{21}$$

(20) together with (21) imply that (17) holds. The proof is complete. □

From Theorems 1-3, we immediately obtained the following corollaries.

Corollary 2. *Let $p \in [1,\infty)$, $\alpha > -1$, $g \in H(B)$ and μ be a normal function on $[0,1)$. Then the following statements are equivalent.*
(i) $T_g : \mathcal{AN}_\alpha^p \to \mathcal{B}_\mu$ is bounded;
(ii) $T_g : \mathcal{AN}_\alpha^p \to \mathcal{B}_{\mu,0}$ is bounded;
(iii) $T_g : \mathcal{AN}_\alpha^p \to \mathcal{B}_\mu$ is compact;
(iv) $T_g : \mathcal{AN}_\alpha^p \to \mathcal{B}_{\mu,0}$ is compact;
(v) For all $c > 0$,

$$\lim_{|z|\to 1} \mu(|z|)|\Re g(z)| \exp\left[\frac{c}{(1-|z|^2)^{\frac{n+1+\alpha}{p}}}\right] = 0.$$

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 19, NO.2, 2015, COPYRIGHT 2015 EUDOXUS PRESS, LLC

ON THE (r,q)-BERNOULLI AND (r,q)-EULER NUMBERS AND POLYNOMIALS

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Abstract. In this paper, we construct new extension of q-Bernoulli and q-Euler numbers and polynomials. These results are generalizations for several well known identities on q-Bernoulli and q-Euler numbers and polynomials, and Carlitz's q-Bernoulli numbers.

1. Introduction

In this paper, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will respectively denote the ring of integers, the field of rational numbers, the ring of p-adic integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. For any $s \in \mathbb{C}_p$, we normally assume $|s-1|_p < p^{-\frac{1}{p-1}}$ so that $s^x = e^{x\log s}$ for $|x|_p < 1$. Also, we use the notation $[x]_r = \frac{1-r^x}{1-r}$ for any x with $|x|_p < 1$ and $r \in \mathbb{Z}$. Fix $d \in \mathbb{Z}$ and p be a prime number. Set

$$\begin{aligned} X_d &= \varprojlim_{N} \mathbb{Z}/dp^N\mathbb{Z}, \\ X^* &= \bigcup_{\substack{0<a<dp \\ (a,p)=1}} (a + dp\mathbb{Z}_p), \\ a + dp^N\mathbb{Z}_p &= \left\{ x \in X | x \equiv a \pmod{dp^N} \right\}, \end{aligned}$$

where $a \in \mathbb{Z}$ lies in $0 \le a < dp^N$. The distribution on X is given by $\mu_q\left(a + dp^N\mathbb{Z}_p\right) = \frac{q^a}{[dp^N]}$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the p-adic q-integral on $\mathbb{Z}_p$ (q-Volkenborn integral on $\mathbb{Z}_p$) is defined by Kim [8] to be

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x. \tag{1.1}$$

Then, by (1.1), we get

$$qI_q(f_1) - I_q(f) = (q-1)f(0) + \frac{q-1}{\log q} f'(0), \ (p \text{ is any prime number}), \tag{1.2}$$

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0), \ (p \text{ is any odd prime number}), \tag{1.3}$$

where $f_1(x) = f(x+1)$ and $f'(0) = \frac{d}{dx} f(x)\mid_{x=0}$ (see [8]).

As is well known, q-Bernoulli polynomials are defined by the generating function to be

$$\frac{t}{qe^t - 1} e^{xt} = \sum_{n\ge 0} B_{n,q}(x) \frac{t^n}{n!},$$

2000 *Mathematics Subject Classification.* 05A40.

Key words and phrases. Bernoulli numbers, Bernoulli polynomials, Carliz's q-Bernoulli numbers, Euler numbers, Euler polynomials.

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(see [5]). When $x = 0$, $B_{n,q} = B_{n,q}(0)$ are called the n-th q-Bernoulli numbers. In [5] derived the Witts formula for q-Bernoulli numbers and polynomials:

$$\int_{\mathbb{Z}_p} q^y y^n d\mu_1(y) = B_{n,q} \text{ and } \int_{\mathbb{Z}_p} q^y (x+y)x^n d\mu_1(y) = B_{n,q}(x).$$

In [9], the q-Euler polynomials are defined by the generating function to be

$$\frac{2}{qe^t+1}e^{xt} = \sum_{n\geq 0} E_{n,q}(x)\frac{t^n}{n!}.$$

When $x = 0$, $E_{n,q} = E_{n,q}(0)$ are called the n-th q-Euler numbers. In [2] derived the Witts formula for q-Euler numbers and polynomials:

$$\int_{\mathbb{Z}_p} q^y y^n d\mu_{-1}(y) = E_{n,q} \text{ and } \int_{\mathbb{Z}_p} q^y (x+y)x^n d\mu_{-1}(y) = E_{n,q}(x).$$

In [3, 4], Carlitz defined the q-extension of Bernoulli numbers as $\beta_{0,q} = 1$, $q(q\beta+1)^n - \beta_{n,q} = \delta_{n,1}$ with the usual convention of replacing β_q^ℓ by $\beta_{\ell,q}$, where $\delta_{n,k}$ is the Kronecker's symbol. This shows

$$\beta_{n,q} = \frac{1}{(1-q)^n}\sum_{\ell=0}^{n}(-1)^\ell\binom{n}{\ell}\frac{\ell+1}{[\ell+1]_q}. \tag{1.4}$$

Also, Carlitz [3, 4] have defined the q-extension of Bernoulli polynomials as

$$\beta_{n,q}(x) = \sum_{\ell=0}^{n}\binom{n}{\ell}[x]_q^{n-\ell}q^{\ell x}\beta_{\ell,q}.$$

In [6] it was shown that the distribution μ_q yields

$$\beta_{n,q} = \int_{\mathbb{Z}_p}[y]_q^n d\mu_q(y) \text{ and } \beta_{n,q}(x) = \int_{\mathbb{Z}_p}[x+y]_q^n d\mu_q(y). \tag{1.5}$$

Moreover, he showed that $\beta_{n,q}$ can be represented as an integral by q-analogue μ_q of ordinary p-adic invariant measure. In [11], modified the above numbers, $\beta_{n,q}$, as $\beta'_{0,q} = \frac{q-1}{\log q}$ and $(q\beta'_q+1)^k - \beta'_{n,q} = \delta_{n,1}$ with the usual convention of replacing ${\beta'}_q^\ell$ by $\beta'_{\ell,q}$. Recently, Seo at al. [10] defined the modified q-Bernoulli polynomials by the generating function to be

$$e^{((q-1)x-1)t}\sum_{j\geq 0}\frac{j+1}{[j+1]_q}\frac{t^j}{j!} = \sum_{\neq 0}\tilde{\beta}_{n,q}(x)(q-1)^n\frac{t^n}{n!},$$

and showed that (see end of Section 4).

$$\int_{\mathbb{Z}_p}(x+[y]_q)^n d\mu_q(y) = \tilde{\beta}_{n,q}(x). \tag{1.6}$$

The aim of this paper is to study new extensions of q-Bernoulli and q-Euler numbers and polynomials, see the next sections. More precisely, we define an (r,q)-extension of Bernoulli polynomials by the generating function to be

$$(1-q)P_{r,q}(t) - t\frac{1-q}{\log q}\frac{\log r}{1-r}P_{r,qr}(t) = \sum_{n\geq 0}B_n(x|r,q)\frac{t^n}{n!} \tag{1.7}$$

and we define an (r,q)-extension of Euler polynomials by the generating function to be

$$(1+q)P_{r,-q}(t) = \sum_{n\geq 0}E_n(x|r,q)\frac{t^n}{n!}, \tag{1.8}$$

where the function $P_{r,q}(t)$ is defined to be $\sum_{j\geq 0} q^j e^{[j]_r t}$. Note that

$$P_{r,q}(t) = \sum_{n\geq 0}\sum_{j\geq 0} q^j [j]_r^n \frac{t^n}{n!} = \sum_{n\geq 0}\sum_{i=0}^{n} \frac{(-1)^i}{(1-r)^n}\binom{n}{i}\sum_{j\geq 0} q^j r^{ij}\frac{t^n}{n!}$$
$$= \sum_{n\geq 0}\left(\sum_{i=0}^{n}\frac{(-1)^i}{(1-r)^n(1-qr^i)}\binom{n}{i}\right)\frac{t^n}{n!}. \tag{1.9}$$

Hence, by (1.7), we have

$$B_n(x|q,q) = (1-q)\left(\sum_{i=0}^{n}\frac{(-1)^i}{(1-q)^n(1-q^{i+1})}\binom{n}{i}\right) - \frac{1-q}{\log q}\frac{\log q}{1-q}\left(\sum_{i=0}^{n}\frac{(-1)^i}{(1-q)^n(1-q^{i+2})}n\binom{n}{i}\right)$$
$$= \sum_{i=0}^{n}\frac{(-1)^i}{(1-q)^n[i+1]_q}\binom{n}{i} + \sum_{i=1}^{n}\frac{(-1)^i i}{(1-q)^n[i+1]_q)}\binom{n}{i}$$
$$= \frac{1}{(1-q)^n}\sum_{i=0}^{n}\frac{(-1)^i(i+1)}{[i+1]_q}\binom{n}{i},$$

which implies that $\lim_{r\to q} B_n(r,q) = \beta_{n,q}$, see (1.4). In other words, the (r,q)-Bernoulli numbers are extension of the q-Bernoulli numbers as defined by Carlitz [3, 4] in (1.4). In section 2, we consider several properties of (r,q)-Bernoulli numbers. Also, by (1.8), we have

$$E_n(x|q,q) = (1+q)P_{r,-q}(t)$$
$$= \frac{1+q}{(1-q)^{n+1}}\sum_{i=0}^{n}\frac{(-1)^i}{[i+1]_q}\binom{n}{i},$$

which are called the q-Euler numbers and are denoted by $e_{n,q}$. Thus, $\lim_{r\to q} E_n(r,q) = e_{n,q}$. In Section 3, we consider several properties of (r,q)-Euler numbers. Moreover, these polynomials, $B_n(x|r,q)$ and $E(x|r,q)$, are useful to study various identities of Carlitz's (r,q)-Bernoulli, and identities of (r,q)-Bernoulli and (r,q)-Euler polynomials, which are (r,q)-extension of the above identities.

2. The (r, q)-Bernoulli polynomials

Define $\beta_n(r,q) = \int_X [x]_r^n d\mu_q(x)$. Since μ_q is a distribution on X, we see that

$$\int_{\mathbb{Z}_p} [x]_r^n d\mu_q(x) = \lim_{N\to\infty}\frac{1}{[p^N]_q}\sum_{a=0}^{p^N-1}[a]_r^n q^a = \beta_n(r,q) \tag{2.1}$$

which has a sense as we see readily that the limit is convergent. We define (r,q)-Bernoulli number $B_n(r,q) \in \mathbb{C}_p$ by making use of the above integral

$$B_n(r,q) = \int_{\mathbb{Z}_p}[x]_r^n d\mu_q(x).$$

Clearly, $\lim_{r\to q} B_n(r,q) = B_n(q)$ are the q-Bernoulli numbers as defined in Kim [6], and $\lim_{\substack{r\to 1\\ q\to 1}} B_n(r,q) = B_n$ are the Bernoulli numbers.

Our first goal is to show that this definition is equivalent to the definition that given in (1.7). Let $F_{r,q}(t) = \sum_{n\geq 0} B_n(r,q)\frac{t^n}{n!}$ be the generating function of $B_n(r,q)$. Then

$$F_{r,q}(t) = \lim_{N\to\infty}\frac{1}{[p^N]_q}\sum_{a=0}^{p^N-1}\left(\sum_{n\geq 0}[a]_r^n\frac{t^n}{n!}\right)q^a = \lim_{N\to\infty}\frac{1}{[p^N]_q}\sum_{a=0}^{p^N-1}q^a e^{[a]_r t}.$$

Note that

$$\begin{aligned}F_{r,q}(rt) =&\frac{1}{qe^t}\lim_{N\to\infty}\frac{1}{[p^N]_q}\sum_{a=0}^{p^N-1}q^{a+1}e^{[a+1]_rt}\\ =&\frac{1}{qe^t}\lim_{N\to\infty}\frac{1}{[p^N]_q}\sum_{a=1}^{p^N}q^{a}e^{[a]_rt}\\ =&\frac{1}{qe^t}\left(F_{r,q}(t)+\lim_{N\to\infty}\frac{q^{p^N}e^{[p^N]_rt}-1}{[p^N]_q}\right)\\ =&\frac{1}{qe^t}\left(F_{r,q}(t)+\lim_{h\to 0}\frac{q^{h}e^{[h]_rt}-1}{[h]_q}\right)\\ =&\frac{1}{qe^t}\left(F_{r,q}(t)+(1-q)\frac{\log q+t\frac{\log r}{1-r}}{-\log q}\right).\end{aligned}$$

Thus, we obtain that the generating function $F_{r,q}(t)$ satisfies

$$F_{r,q}(t) = qe^tF_{r,q}(rt) + (1-q) - \frac{t(1-q)\log r}{(1-r)\log q}.$$

By applying this q-difference equation, we obtain

$$\begin{aligned}F_{r,q}(t) =&\sum_{j\geq 0}(1-q-r^jt\alpha)q^je^{(1+r+\cdots+r^{j-1})t}\\ =&\sum_{j\geq 0}(1-q)q^je^{\frac{1-r^j}{1-r}t}-t\alpha\sum_{j\geq 0}(qr)^je^{\frac{1-r^j}{1-r}t}\\ =&(1-q)P_{r,q}(t)-t\alpha P_{r,qr}(t),\end{aligned}\tag{2.2}$$

where $\alpha = \frac{(1-q)\log r}{(1-r)\log q}$. By (2.2), we obtain the following theorem.

Theorem 2.1. *We have*

$$F_{r,q}(t) = \sum_{n\geq 0}B_n(r,q)\frac{t^n}{n!} = (1-q)P_{r,q}(t) - t\frac{1-q}{\log q}\frac{\log r}{1-r}P_{r,qr}(t).$$

If $r = q = 1$ then $F_{1,1}(t) = -t\sum_{j\geq 0}e^{jt} = \frac{-t}{1-e^t} = \frac{t}{e^t-1}$, which it is the generating function of Bernoulli numbers. Note that the above theorem shows that (1.7) is equivalent to (2.1) and the (q,q)-Bernoulli numbers are the same as q-Bernoulli numbers, see (1.7).

Now, we use of (1.2) to obtain a recurrence relation for $B_n(r,q)$. Let $f(x) = [x]_r^n$. Obviously, $B_0(r,q) = 1$ and $B_1(r,q) = \frac{q-\alpha}{1-rq}$. Let $n \geq 2$, by (1.2) we have

$$-q\int_{\mathbb{Z}_p}[x+1]_r^n d\mu_q(x) + B_n(r,q) = 0.$$

Thus, by using the fact that $[x+1]_r = 1 + r[x]_r$, we obtain

$$-q\sum_{j=0}^{n}\binom{n}{j}r^j\int_{\mathbb{Z}_p}[x]_r^j d\mu_q(x) + B_n(r,q) = 0,$$

which leads to the following result.

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Corollary 2.2. *For all $n \geq 2$,*

$$B_n(r,q) = q\sum_{j=0}^{n}\binom{n}{j}r^jB_j(r,q)$$

with $B_0(r,q)=1$ and $B_1(r,q)=\frac{q-\alpha}{1-rq}$.

The above corollary gives

$$\frac{B_n(r,q)}{n!} = q\sum_{j=0}^{n}\frac{1}{(n-j)!}\frac{r^jB_j(r,q)}{j!}$$

with $n\geq 2$. If multiplying by t^n and summing over $n \geq 2$, we obtain

$$F_{r,q}(t)-\frac{q-\alpha}{1-rq}t-1 = q\left(e^tF_{r,q}(rt)-1-t-r\frac{q-\alpha}{1-rq}t\right).$$

Hence $F_{r,q}(t) = qe^tF_{r,q}(rt)+1-q-\alpha t$, which agrees with Theorem 2.1.

Now, we define the (r,q)-Bernoulli polynomials as

$$B_n(x|r,q) = \int_{\mathbb{Z}_p}[x+y]_r^n d\mu_q(y).$$

Note that $B_n(0|r,q) = B_n(r,q)$. Since $[x+y]_r = [x]_r + r^x[y]_r$, we obtain

$$\begin{aligned} B_n(x|r,q) &= \int_{\mathbb{Z}_p}[x+y]_r^n d\mu_q(y) \\ &= \sum_{j=0}^{n}\binom{n}{j}[x]_r^{n-j}r^{xj}\int_{\mathbb{Z}_p}[y]_r^j d\mu_q(y) \\ &= \sum_{j=0}^{n}\binom{n}{j}[x]_r^{n-j}r^{xj}B_j(r,q), \end{aligned}$$

which implies the following result.

Theorem 2.3. *For all $n \geq 0$,*

$$B_n(x|r,q) = \sum_{j=0}^{n}\binom{n}{j}[x]_r^{n-j}r^{xj}B_j(r,q).$$

Note that the above theorem with $r=q$ reduces to

$$B_n(x|q,q) = \sum_{j=0}^{n}\binom{n}{j}[x]_q^{n-j}q^{xj}B_j(q,q),$$

as has been shown by Carlitz [3, 4] and by Kim [6].

By Theorem 2.1 and (1.9), we obtain the following formula.

Theorem 2.4. *The n-th (r,q)-Bernoulli number is given by*

$$B_n(r,q) = \frac{1}{(1-r)^n}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{1-q}{1-qr^j}\left(1+j\frac{\log r}{\log q}\right).$$

Note that the above theorem with $r=q$ reduces to the q-Bernoulli numbers as defined in [3, 4] and investigated in [6], see (1.7).

3. The (r,q)-Euler polynomials

Define $e_n(r,q)=\int_X [x]_r^n d\mu_{-q}(x)$. Since μ_{-q} is a distribution on X, we see that

$$\int_{\mathbb{Z}_p}[x]_r^n d\mu_{-q}(x)=\lim_{N\to\infty}\frac{1}{[p^N]_{-q}}\sum_{a=0}^{p^N-1}[a]_r^n(-q)^a=\beta_n(r,q) \tag{3.1}$$

which has a sense as we see readily that the limit is convergent. Note that in this case p is an odd prime number. We define (r,q)-Euler number $E_n(r,q)\in\mathbb{C}_p$ by making use of the above integral

$$E_n(r,q)=\int_{\mathbb{Z}_p}[x]_r^n d\mu_{-q}(x).$$

Clearly, $\lim_{r\to q}E_n(r,-q)=E_n(q)$ are the q-Euler numbers as defined in Kim [6], and $\lim_{\substack{r\to 1\\ q\to 1}}E_n(r,q)=E_n$ are the Euler numbers.

Our first goal is to show that (1.8) and (3.1) are equivalents. By the definitions, $E_0(r,q)=1$. Now, let us consider the case $n\geq 1$. By (1.3), we have $q\int_{\mathbb{Z}_p}[x+1]_r^n d\mu_{-q}(x)+E_n(r,q)=0$, which, by $[x+1]_r=1+r[x]_r$, implies

$$E_n(r,q)=-q\sum_{j=0}^{n}\left(\binom{n}{j}r^j\int_{\mathbb{Z}_p}[x]_r^j d\mu_{-q}(x)\right).$$

Thus, we can state the following recurrence relation.

Lemma 3.1. *For all $n\geq 1$,*

$$E_n(r,q)=-q\sum_{j=0}^{n}\binom{n}{j}r^jE_j(r,q)$$

with $E_0(r,q)=1$.

Define $G_{r,q}(t)=\sum_{n\geq 0}\frac{E_n(r,q)}{n!}t^n$ to be the generating function for the (r,q)-Euler numbers. The above lemma gives

$$\sum_{n\geq 1}\frac{E_n(r,q)}{n!}t^n=-q\sum_{n\geq 1}\sum_{j=0}^{n}\frac{t^n}{(n-j)!}\frac{r^jt^jE_j(r,q)}{j!},$$

and thus

$$G_{r,q}(t)-1=-q\left(e^tG_{r,q}(rt)-1\right).$$

which is equivalent to $G_{r,q}(t)=1+q-qe^tG_{r,q}(rt)$. Iterating this equation infinite number of times shows

$$\begin{aligned}G_{r,q}(t)=&1+q-qe^t\left(1+q-qe^{rt}G_{r,q}(r^2t)\right)\\=&1+q-q(1+q)e^t+q^2e^{(r+1)t}\left(1+q-qe^{r^2t}\right)G_{r,q}(r^3t)\\=&1+q-q(1+q)e^t+q^2e^{(r+1)t}-q^3(1+q)e^{(r^2+r+1)}+\cdots\\=&(1+q)\sum_{j\geq 0}(-q)^je^{[j]_rt}=(1+q)P_{r,-q}(t).\end{aligned}$$

Hence, we have the following result.

Theorem 3.2. *We have*

$$G_{r,q}(t)=\sum_{n\geq 0}E_n(r,q)\frac{t^n}{n!}=(1+q)P_{r,-q}(t).$$

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If $r = q = 1$, then

$$G_{1,1}(t) = (1+1)\sum_{j\ge 0}(-1)^j e^{jt} = \frac{2}{1+e^t},$$

which it is the generating function for the Euler numbers.

Now, we define the (r,q)-Euler polynomials as $E_n(x|r,q) = \int_{\mathbb{Z}_p}[x+y]_r^n d\mu_{-q}(y)$. Using similar argument as in the proof of Theorem 2.3, we obtain the following theorem.

Theorem 3.3. *For all $n \ge 0$,*

$$E_n(x|r,q) = \sum_{j=0}^{n}\binom{n}{j}[x]_r^{n-j} r^{jx} E_j(r,q).$$

Now, we consider an explicit formula for the n-th (r,q)-Euler number. By (3.1), we have

$$\begin{aligned}
E_n(r,q) &= \lim_{N\to\infty}\frac{1}{[p^N]_{-q}}\sum_{j=0}^{p^N-1}(-q)^j[j]_r^n \\
&= \frac{1}{(1-r)^n}\sum_{i=0}^{n}(-1)^i\binom{n}{i}\left(\lim_{N\to\infty}\frac{1}{[p^N]_{-q}}\sum_{j=0}^{p^N-1}(-q)^j r^{ji}\right) \\
&= \frac{1}{(1-r)^n}\sum_{i=0}^{n}(-1)^i\binom{n}{i}\left(\lim_{N\to\infty}\frac{1+q}{1+q^{p^N}}\frac{1+(qr^i)^{p^N}}{1+qr^i}\right) \\
&= \frac{1}{(1-r)^n}\sum_{i=0}^{n}(-1)^i\binom{n}{i}\frac{1+q}{1+qr^i},
\end{aligned}$$

which implies the following formula.

Theorem 3.4. *The n-th (r,q)-Euler number is given by*

$$E_n(r,q) = \frac{1}{(1-r)^n}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{1+q}{1+qr^j}.$$

4. Modifying Bernoulli and Euler numbers and polynomials

Following Kim [7], we define the modified (r,q)-Bernoulli number as

$$\tilde{B}_n(r,q) = \int_{\mathbb{Z}_p} q^{-x}[x]_r^n d\mu_q(x).$$

By similar arguments as in the proof of Theorem 2.4, we obtain the following formula

$$\tilde{B}_n(r,q) = \frac{1-q}{(1-r)^n\log q}\left(\log r\sum_{i=1}^{n}\frac{i\binom{n}{i}(-1)^i}{1-r^i} - 1\right).$$

By (1.2) with $n \ge 2$, we have

$$-q\int_{\mathbb{Z}_p} q^{-x}[x+1]_r^n d\mu_q(x) + \int_{\mathbb{Z}_p} q^{-x}[x]_r^n d\mu_q(x) = 0,$$

which, by $[x+1]_r = 1 + r[x]_r$, implies

$$-\sum_{j=0}^{n}\binom{n}{j}\int_{\mathbb{Z}_p} q^{-x}r^j[x]_r^j d\mu_q(x) + \tilde{B}_n(r,q) = 0.$$

Thus, we obtain the following theorem.

Theorem 4.1. *For $n \geq 2$,*

$$\tilde{B}_n(r,q) = \sum_{j=0}^{n} \binom{n}{j} r^j \tilde{B}_j(r,q)$$

with $\tilde{B}_0(r,q) = \frac{q-1}{\log q}$ and $\tilde{B}_1(r,q) = \frac{q-1}{(1-r)\log q}\left(1 + \frac{\log r}{1-r}\right)$.

Define $\tilde{F}_{r,q}(t) = \sum_{j\geq 0} \tilde{B}_n(r,q)\frac{t^n}{n!}$. Then the above theorem gives

$$\sum_{n\geq 2} \frac{\tilde{B}_n(r,q)}{n!} t^n = \sum_{n\geq 2}\sum_{j=0}^{n} \frac{t^{n-j}}{(n-j)!}\frac{r^j \tilde{B}_j(r,q)}{j!} t^j,$$

and so

$$\tilde{F}_{r,q}(t) - \tilde{B}_0(r,q) - \tilde{B}_1(r,q)t = e^t \tilde{F}_{r,q}(rt) - \tilde{B}_0(r,q) - t(\tilde{B}_0(r,q) + r\tilde{B}_1(r,q)).$$

By using the values of $\tilde{B}_0(r,q)$, $\tilde{B}_1(r,q)$ and $\alpha = \frac{(1-q)\log r}{(1-r)\log q}$, we obtain

$$\tilde{F}_{r,q}(t) = -\alpha t \sum_{j\geq 0} r^j e^{(1+r+\cdots+r^{j-1})} = -\alpha t \sum_{j\geq 0} r^j e^{[r]_j t} = -\alpha t P_{r,r}(t),$$

which leads to the following result.

Theorem 4.2. *We have*

$$\tilde{F}_{r,q}(t) = -t\frac{(1-q)\log r}{(1-r)\log q} P_{r,r}(t).$$

If we put $r = q = 1$, then $\alpha = 1$ and the above theorem gives

$$\tilde{F}_{1,1}(t) = -t \sum_{j\geq 0} e^{jt} = \frac{-t}{1-e^t} = \frac{t}{e^t - 1},$$

which is the same generating function for the Bernoulli numbers.

Now, we consider the case of modified (r,q)-Euler numbers. We define the modified (r,q)-Euler number as

$$\tilde{E}_n(r,q) = \int_{\mathbb{Z}_p} q^{-x}[x]_r^n d\mu_{-q}(x),$$

where p is an odd prime. By the definition of p-adic q-integral, we obtain (Similar to the proof of Theorem 4.1) the following result.

Theorem 4.3. *For all $n \geq 0$,*

$$\tilde{E}_n(r,q) = \frac{1+q}{(1-r)^n} \sum_{i=0}^{n} \frac{(-1)^i \binom{n}{i}}{1+r^i}.$$

By (1.3) with $f(x) = q^{-x}[x]_r^n$, we obtain that

$$q\int_{\mathbb{Z}_p} q^{-x-1}[x+1]_r^n d\mu_{-q}(x) + \int_{\mathbb{Z}_p} q^{-x}[x]_r^n d\mu_{-q}(x) = 0,$$

which, by $[x+1]_r = 1 + r[x]_r$, implies the following result.

Theorem 4.4. *For $n \geq 1$,*

$$\tilde{E}_n(r,q) = -\sum_{j=0}^{n} \binom{n}{j} r^j \tilde{E}_j(r,q)$$

with $\tilde{E}_0(r,q) = \frac{1+q}{2}$.

Define $\tilde{G}_{r,q}(t)=\sum_{n\geq 0}\tilde{E}_n(r,q)\frac{t^n}{n!}$. Then Theorem 4.4 gives

$$\sum_{n\geq 1}\frac{\tilde{E}_n(r,q)}{n!}t^n=-\sum_{n\geq 1}\sum_{j=0}^{n}\frac{t^{n-j}}{(n-j)!}\frac{r^j\tilde{E}_j(r,q)}{j!}t^j$$

and so

$$\tilde{E}_{r,q}(t)=-e^t\tilde{E}_{r,q}(rt)+1+q.$$

By iterating it, we obtain the following result.

Theorem 4.5. *We have*

$$\tilde{G}_{r,q}(t)=(1+q)P_{r,-1}(t)=(1+q)\sum_{n\geq 0}\sum_{j\geq 0}(-1)^j[r]_j^n\frac{t^n}{n!}.$$

By applying the above theorem for $r=q=1$, we get that the generating function for the Euler numbers is given by

$$\tilde{G}_{1,1}(t)=2\sum_{j\geq 0}(-1)^je^{jt}=\frac{2}{1+e^t},$$

which agrees with the definitions.

We end this paper, by the following comment. Several authors defined various types of modification of q-Bernoulli and q-Euler numbers. For instance, in [10] has been defined the modified q-Bernoulli polynomials as

$$\sum_{n\geq 0}\widehat{B}_n(x|q)(q-1)^n\frac{t^n}{n!}=e^{((q-1)x-1)t}\sum_{i\geq 0}\frac{i+1}{[i+1]_q}\frac{t^i}{i!},$$

where has been shown

$$\widehat{B}_n(x|q)=\int_{\mathbb{Z}_p}(x+[y]_q)^n d\mu_q(x).$$

This can be extended by defining the modified (r,q)-Bernoulli polynomials as

$$\widehat{B}_n(x|r,q)=\int_{\mathbb{Z}_p}(x+[y]_r)^n d\mu_q(x).$$

In this case, we have

$$\begin{aligned}\sum_{n\geq 0}\widehat{B}_n(x|r,q)(q-1)^n\frac{t^n}{n!}&=\int_{\mathbb{Z}_p}e^{(q-1)(x+[y]_r)t}d\mu_q(x)\\&=e^{((q-1)x-1)t}\int_{\mathbb{Z}_p}e^{(q-1)(1+[y]_r)t}d\mu_q(x)\end{aligned}$$

Acknowledgment This research was supported by Kyungpook National University BK 21 plus fund 2014.

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Solving fuzzy fractional partial differential equations by fuzzy Laplace-Fourier transforms

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ABSTRACT

In this paper, the solutions of linear fuzzy fractional partial differential equations (FFPDEs) under Caputo-type H-differentiability have been investigated. To this end, the fuzzy Laplace and fuzzy Fourier transform were used to obtain the solutions of FFPDEs. Then, some new results regarding the relation between some types of differentiability have been obtained. Finally, some applicable examples are solved in order to show the ability of the proposed method.

Keywords: fuzzy-valued function, Caputo-type H-differentiability, fuzzy Laplace and fuzzy Fourier transforms.

1. Introduction

The Fourier transform was firstly introduced by N. Wiener in 1929 [1] in the form of fractional order of the Fourier operator. Recently, fractional Fourier transform has gained popularity due to its applications in dynamical systems, quantum mechanics, stochastic processes, chemistry, signal processing, optics and other subjects.

In this paper, we utilize Caputo-type differentiability by using Hukuhara difference so-called Caputo-type H-differentiability. This procedure is employed to derive a concept which is constructed based on the combination of greatly generalized differentiability [2] and Caputo-type derivative [3, 4].

In here, we try to solve FFPDEs under Caputo-type H-differentiability using fuzzy Laplace and fuzzy Fourier transforms. So, at first, we define Caputo-type H-differentiability which is a direct extension of Caputo-type derivatives with respect to Hukuhara difference, and then, we will investigate the Laplace and Fourier transform of fractional derivatives which is important tool to solve FFPDEs with fuzzy Laplace and fuzzy Fourier transforms.

2. Preliminaries and notations

The basic definition of fuzzy numbers is given in [11]

Definition 2.1. A fuzzy number or (or interval) u in parametric form is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r)$, $\overline{u}(r)$, $0 \leq r \leq 1$, which satisfy the following requirements [5]

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1. $\underline{u}(r)$ is a bounded non-decreasing left continuous function in $(0,1]$ and right continuous at 0.
2. $\overline{u}(r)$ is a bounded non-decreasing left continuous function in $(0,1]$ and right continuous at 0.
3. $\underline{u}(r) \le \overline{u}(r), 0 \le r \le 1$

In particular, if $\underline{u}, \overline{u}$ are linear functions we have a triangular fuzzy number. A crisp number r is simply represented by $\underline{u}(r) = \overline{u}(r) = u, \;\; 0 \le r \le 1$.

Definition 2.2. For arbitrary fuzzy numbers $u = \left(\underline{u}(r), \overline{u}(r)\right)$, $v = \left(\underline{v}(r), \overline{v}(r)\right)$ we have algebraic operations below [5]:

1. $ku = \begin{cases} \left(k\underline{u}, k\overline{u}\right) & k \ge 0 \\ \left(k\overline{u}, k\underline{u}\right) & k < 0 \end{cases}$
2. $u + v = \left(\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r)\right)$
3. $u - v = \left(\underline{u}(r) - \overline{v}(r), \overline{u}(r) - \underline{v}(r)\right)$
4. $u.v = (\min s, \max s)$, which

$$s = \{\underline{u}, \underline{v}, \underline{u}\overline{v}, \overline{u}\underline{v}, \overline{uv}\}.$$

3. Caputo-type H-differentiability

In this section, the concept of fuzzy Caputo derivatives has been reviewed [7, 9, 10]. Also, we denote $\mathbb{C}^{\mathbb{F}}[a,b]$ as a space of all fuzzy-valued functions which are continuous on $[a,b]$. Also, we denote the space of all Lebesgue-integrable fuzzy-value functions on the bounded interval $[a,b] \in \mathbb{R}$ by $L^{\mathbb{F}}[a,b]$, we denote the space of fuzzy-value functions $f(x)$ which have continuous H-derivative up to order $n-1$ on $[a,b]$ such that $f^{(n-1)}(x) \in A\mathbb{C}^{\mathbb{F}}([a,b])$ by $A\mathbb{C}^{(n)\mathbb{F}}([a,b])$.

Definition 3.1. Let $f \in \mathbb{C}^{\mathbb{F}}[a,b] \cap L^{\mathbb{F}}[a,b]$, the fuzzy Riemann-Liouville integral of fuzzy-valued function f is defined as following:

$$\left(I_{a^+}^{\beta} f\right)(x) = \frac{1}{\Gamma(\beta)} \int_a^x \frac{f(t)dt}{(x-t)^{1-\beta}}, x > a, 0 < \beta < 1. \tag{3.1}$$

Since, $f(x;r) = \left[\underline{f}(x;r), \overline{f}(x;r)\right]$, for all $0 \le r \le 1$, then we can indicate the fuzzy Riemann-Liouville integral of fuzzy-valued function f based on the lower and upper functions as following:

Theorem 3.1 Let $f \in \mathbb{C}^{\mathbb{F}}[a,b] \cap L^{\mathbb{F}}[a,b]$, the fuzzy Riemann-Liouville integral of fuzzy-valued function f is defined as following:

$$\left(I_{a^+}^{\beta} f\right)(x;r) = \left[\left(I_{a^+}^{\beta} \underline{f}\right)(x;r), \left(I_{a^+}^{\beta} \overline{f}\right)(x;r)\right], 0 \le r \le 1, \tag{3.2}$$

where

$$\left(I_{a^+}^{\beta} \underline{f}\right)(x;r) = \frac{1}{\Gamma(\beta)} \int_a^x \frac{\underline{f}(t;r)dt}{(x-t)^{1-\beta}}, 0 \le r \le 1, \tag{3.3}$$

and

$$\left(I_{a^+}^{\beta}\overline{f}\right)(x;r) = \frac{1}{\Gamma(\beta)}\int_a^x \frac{f(t;r)dt}{(x-t)^{1-\beta}}, 0 \le r \le 1, \tag{3.4}$$

Now, we define fuzzy Riemann-Liouville fractional derivatives of order $n-1<\beta<n$ for fuzzy-valued function f [12, 13].

Definition 3.2 Let $f^n \in \mathbb{C}^{(n)\mathbb{F}}[a,b] \cap L_1^{\mathbb{F}}[a,b]$, and $x_0 \in (a,b)$ and $\Phi(x) = \frac{1}{\Gamma(n-\beta)}\int_a^x \frac{f(t)dt}{(x-t)^{\beta-n+1}}$ ($n = [\beta]+1, x>a$). We say that f is fuzzy Riemann-Liouville fractional differentiable of order β, at x_0, if there exists an element $\left(D_{a^+}^{\beta}f\right)(x_0) \in \mathbb{E}$, such that for all $h>0$ sufficiently small

i. $\left(D_{a^+}^{\beta}f\right)(x_0) = \lim_{h\to 0}\frac{\Phi^{(n-1)}(x_0+h)\ominus\Phi^{(n-1)}(x_0)}{h} = \lim_{h\to 0}\frac{\Phi^{(n-1)}(x_0)\ominus\Phi^{(n-1)}(x_0-h)}{h}$

or

ii. $\left(D_{a^+}^{\beta}f\right)(x_0) = \lim_{h\to 0}\frac{\Phi^{(n-1)}(x_0)\ominus\Phi^{(n-1)}(x_0+h)}{-h} = \lim_{h\to 0}\frac{\Phi^{(n-1)}(x_0-h)\ominus\Phi^{(n-1)}(x_0)}{-h}$

For sake of simplicity, we say that a fuzzy-valued function f is ${}^{RL}[(i)-\beta]$-differentiable if it is differentiable as in the Definition 3.2 case (i), and is ${}^{RL}[(ii)-\beta]$-differentiable if it is differentiable as in the Definition 3.2 case (ii).

Definition 3.3. Let $f^n \in \mathbb{C}^{(n)\mathbb{F}}[a,b] \cap L_1^{\mathbb{F}}[a,b]$, f^n is integrable, then the right fuzzy Caputo derivative of f for $n-1<\beta<n$ and $x \in [a,b]$, denoted by $\left({}^{C}D_{a^+}^{\beta}f\right)(x) \in \mathbb{E}$ and defined by

$$\left({}^{C}D_{a^+}^{\beta}f\right)(x) = \frac{1}{\Gamma(n-\beta)} \odot \int_a^x (x-t)^{-\beta+n-1} \odot f^{(n)}(t)dt. \tag{3.5}$$

Theorem 3.2. Let $f^n \in \mathbb{C}^{(n)\mathbb{F}}[a,b] \cap L_1^{\mathbb{F}}[a,b]$, and $x \in (a,b)$, $n-1<\beta<n$, such that for all $0 \le r \le 1$, then

i. If f be a ${}^{C}[(i)-\beta]$--differentiable fuzzy-valued function, then:
$$\left({}^{C}D_{a^+}^{\beta}f\right)(x;r) = \left[\left({}^{C}D_{a^+}^{\beta}\underline{f}\right)(x;r), \left({}^{C}D_{a^+}^{\beta}\overline{f}\right)(x;r)\right],$$

ii. If f be a ${}^{C}[(ii)-\beta]$--differentiable fuzzy-valued function, then:
$$\left({}^{C}D_{a^+}^{\beta}f\right)(x;r) = \left[\left({}^{C}D_{a^+}^{\beta}\overline{f}\right)(x;r), \left({}^{C}D_{a^+}^{\beta}\underline{f}\right)(x;r)\right],$$

where

$$\left({}^{C}D_{a^+}^{\beta}\underline{f}\right)(x;r) = {}^{RL}D_{a^+}^{\beta}\left[\underline{f}(x;r) - \sum_{k=0}^{n-1}\frac{x^k}{k!}\underline{f}^{(k)}(a;r)\right](x;r) = \frac{1}{\Gamma(n-\beta)}\frac{d^n}{dx^n}\int_a^x \frac{\underline{f}(t;r)-\sum_{k=0}^{n-1}\frac{x^k}{k!}\underline{f}^{(k)}(a;r)}{(x-t)^{\beta-n+1}}dt,$$

and

$$\left({}^{C}D_{a^+}^{\beta}\overline{f}\right)(x;r) = {}^{RL}D_{a^+}^{\beta}\left[\overline{f}(x;r) - \sum_{k=0}^{n-1}\frac{x^k}{k!}\overline{f}^{(k)}(a;r)\right](x;r) = \frac{1}{\Gamma(n-\beta)}\frac{d^n}{dx^n}\int_a^x \frac{\overline{f}(t;r)-\sum_{k=0}^{n-1}\frac{x^k}{k!}\overline{f}^{(k)}(a;r)}{(x-t)^{\beta-n+1}}dt.$$

Theorem 3.3 (see [13]) Let $u(x;r)$ be closed- and bounded-fuzzy-valued function on $(-\infty,+\infty)$. Suppose that $\underline{u}(x;r)$ and $\overline{u}(x;r)$ are improper Riemann-integrable on $(-\infty,+\infty)$, $\forall r \in [0,1]$. Then $u(x;r) \in \mathcal{JFR}_{\mathrm{I}}$ on $(-\infty,+\infty)$ and the improper fuzzy Riemann integral $\int_{-\infty}^{+\infty} u(x;r)dx$ is a closed fuzzy number. Furthermore, we have

$$\int_{-\infty}^{+\infty} u(x;r)dx = \left[\int_{-\infty}^{+\infty}\underline{u}(x;r)dx, \int_{-\infty}^{+\infty}\overline{u}(x;r)dx\right]. \tag{3.6}$$

4. The fuzzy Laplace transforms

In this section, we consider the fuzzy Laplace transform for fuzzy-valued function, then Derivative theorem is given which is essential for determining solutions of FFPDEs.

In this way, Allahviranloo et al. [7], suggested concept of Laplace transforms for fuzzy-valued function as following:

Definition 4.1. Let u is continuous fuzzy-valued function. Suppose that $u(x,t)\odot e^{-st}$ is improper fuzzy Riemann-integrable on $[0,\infty)$, then $\int_0^\infty u(x,t)\odot e^{-st}\,dt$ is called fuzzy Laplace transforms and denoted by:

$$(\mathcal{L}_t u)(x,s) = \int_0^\infty u(x,t)\odot e^{-st}\,dt, \quad (s > 0 \text{ and integer}) \tag{4.1}$$

we have for $0 \le r \le 1$:

$$\left[\int_0^\infty u(x,t)\odot e^{-st}\,dt\right]^r = \left[\int_0^\infty \underline{u}(x,t)\odot e^{-st}\,dt, \int_0^\infty \overline{u}(x,t)\odot e^{-st}\,dt\right].$$

Also by using the definition of classical Laplace transform:

$$\mathcal{L}_t\left[\underline{u}(x,t;r)\right] = \int_0^\infty \underline{u}(x,t;r)\odot e^{-st}\,dt,$$

and

$$\mathcal{L}_t[\overline{u}(x,t;r)] = \int_0^\infty \overline{u}(x,t;r)\odot e^{-st}\,dt,$$

then, we get:

$$\left[\mathcal{L}_t[f(x)]\right]^r = \left[\mathcal{L}_t\left[\underline{f}(x;r)\right], \mathcal{L}_t\left[\overline{f}(x;r)\right]\right].$$

Definition 4.2. Inverse Fuzzy Laplace transform with respect to s is defined as following:

$$(\mathcal{L}_s^{-1}u)(x,t) = \frac{1}{2\pi}\int_{\gamma-i\infty}^{\gamma+i\infty} u(x,s)\odot e^{st}ds\,, x\in\mathbb{R}, \gamma = \Re(s) > \sigma_\varphi \tag{4.2}$$

Theorem 4.1 (Derivative theorem). Suppose that $f \in C^{(n)\mathbb{F}}[0,\infty)\cap L^{\mathbb{F}}[0,\infty)$, then

$$\mathcal{L}_t\left\{{}^{C}D_{0^+,t}^{\alpha}u(x,s)\right\} = s^\alpha\mathcal{L}_t u(x,s) \ominus \sum_{k=0}^{m-1} s^{\alpha-k-1}\frac{\partial^k u(x,0^+)}{\partial t^k},\ m-1<\alpha\le m,$$

if $D^{(k-1)}f$ is ${}^{C}[(i)-\alpha]$-differentiable, and also

$$\mathcal{L}_t\left\{{}^{C}D_{0^+,t}^{\alpha}u(x,s)\right\} = -\sum_{k=0}^{m-1} s^{\alpha-k-1}\frac{\partial^k u(x,0^+)}{\partial t^k} \ominus \left(-s^\alpha\mathcal{L}_t u(x,s)\right),\ m-1<\alpha\le m,$$

if $D^{(k-1)}f$ is ${}^{C}[(ii)-\alpha]$-differentiable.

5. The Fuzzy Fourier Transforms

In this section, we are going to define fuzzy Fourier transform for fuzzy-valued function. Moreover, we will consider the properties of the fuzzy Fourier transforms, then a derivative theorem is given in order to connect between Laplace transform of fractional derivative and corresponding fuzzy-valued function.

Definition 5.1. Let u is continuous fuzzy-valued function. Suppose that $u(x,t) \circledast e^{-i\omega x}$ is improper fuzzy Riemann-integrable on $(-\infty,+\infty)$, [13], then $\int_{-\infty}^{+\infty} u(x,t) \circledast e^{-i\omega x} dx$ is called fuzzy Fourier transforms and denoted by:

$$(\mathcal{F}_x u)(\omega,t) = \int_{-\infty}^{+\infty} u(x,t) \circledast e^{-i\omega x} dx \tag{5.1}$$

Using Theorem 3.3, we have for $0 \le r \le 1$:

$$\left[\int_{-\infty}^{+\infty} u(x,t) \circledast e^{-i\omega x} dx\right]^r = \left[\int_{-\infty}^{+\infty} \underline{u}(x,t;r) \circledast e^{-i\omega x} dx, \int_{-\infty}^{+\infty} \overline{u}(x,t) \circledast e^{-i\omega x} dx\right],$$

Also by using the definition of classical Fourier transform:

$$\left(\mathcal{F}_x \underline{u}\right)(\omega,t;r) = \int_{-\infty}^{+\infty} \underline{u}(x,t;r) \circledast e^{-i\omega x} dx,$$

and

$$(\mathcal{F}_x \overline{u})(\omega,t;r) = \int_{-\infty}^{+\infty} \overline{u}(x,t;r) \circledast e^{-i\omega x} dx,$$

then, we get:

$$[(\mathcal{F}_x u)(\omega,t)]^r = \left[\left(\mathcal{F}_x \underline{u}\right)(\omega,t;r), (\mathcal{F}_x \overline{u})(\omega,t;r)\right].$$

Definition 5.2. Inverse Fuzzy Fourier transform with respect to ω is defined as below:

$$\mathcal{F}_\omega^{-1} u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\omega x} \circledast u(\omega,t) d\omega \quad (\omega \in \mathbb{R}, t > 0) \tag{5.2}$$

6. FFPDEs under Caputo-type H-differentiability

In this section we show the notations and the basic fuzzy fractional partial differential equation associated with the wave reaction-diffusion systems. Let $u(x,t)$ be the concentration of a substance distributed in space one dimensional space and $\varphi(x,t,u)$ be a non-linear function, where t is the time-variable and x is the space-variable. It is important to note that the function $\varphi(x,t,u)$ can represent a term of the so-called Fisher–Kolmogorov equation or a term of the Ginzburg–Landau equation, which appear in field theory and superconductivity. In the first case, the non-linear term is $\varphi(x,t,u) = Au(x,t)[1-u(x,t)]$ with A a constant, whereas in the second case we have $\varphi(x,t,u) = Au(x,t)\left[1-\left(u(x,t)\right)^2\right]$ with B another constant. Here we discuss only the case $\varphi(x,t,u) \equiv \varphi(x,t)$, i.e. linear case, as in Eq. (6.1), [8].
The following fuzzy fractional partial differential equation, the so-called generalized wave reaction-diffusion equation, is considered:

$$a\left({}^{C}D_{0^+,t}^{\alpha_1} u\right)(x,t) \oplus b\left({}^{C}D_{0^+,t}^{\alpha_2} u\right)(x,t) = c\left({}^{C}D_{x,-\infty}^{\beta} u\right)(x,t) \oplus \varphi\big(x,t,u(x,t)\big), \tag{6.1}$$

$$\frac{\partial^k u(x,0^+)}{\partial x^k} = f_k(x),\ \ k = 0,1,\dots,m-1, \frac{\partial^l u(x,0^+)}{\partial x^l} = g_l(x),\ \ l = 0,1,\dots,n-1,\ \lim_{x\to\pm\infty} u(x,t) = 0, \tag{6.2}$$

where $t > 0$, $x \in \mathbb{R}$, $m-1 < \alpha_1 \le m$, $n-1 < \alpha_2 \le n$, $0 \le \beta \le 1$ and a, b, c are real constants.

6.1. Determining algebraic solutions

In this subsection, we provide the fuzzy Laplace transform and its inverse to derive solutions of FFPDE (6.1). By taking Laplace transform on the both sides of Eq. (6.1), we get the following:

$$\mathcal{L}_t\left\{a\left({}^{C}D_{0^+,t}^{\alpha_1}u\right)(x,t)\oplus b\left({}^{C}D_{0^+,t}^{\alpha_2}u\right)(x,t)\right\}=\mathcal{L}_t\left\{c\left({}^{C}D_{x,-\infty}^{\beta}u\right)(x,t)\oplus\varphi\big(x,t,u(x,t)\big)\right\},\quad(6.3)$$

Then, based on the types of Caputo-type H-differentiability we have the following cases:

Case I. Let us consider $Du(x,t)$ is a ${}^{C}[(i)-\alpha]$-differentiable function, then Eq. (6.3) is extended based on its lower and upper functions as following:

$$a\left(s^{\alpha_1}\mathcal{L}_t\{\underline{u}\}(x,s;r)-\sum_{k=0}^{m-1}s^{\alpha_1-k-1}\underline{f}_k(x;r)\right)+b\left(s^{\alpha_2}\mathcal{L}_t\{\underline{u}\}(x,s;r)-\sum_{l=0}^{n-1}s^{\alpha_2-l-1}\underline{g}_l(x;r)\right)=c\left({}^{C}D_{x,-\infty}^{\beta}\mathcal{L}_t\{\underline{u}\}\right)(x,s;r)+\mathcal{L}_t\left\{\underline{\varphi}\right\}(x,s,u(x,t);r),$$

$$(6.4)$$

$$a\left(s^{\alpha_1}\mathcal{L}_t\{\overline{u}\}(x,s;r)-\sum_{k=0}^{m-1}s^{\alpha_1-k-1}\overline{f}_k(x;r)\right)+b\left(s^{\alpha_2}\mathcal{L}_t\{\overline{u}\}(x,s;r)-\sum_{l=0}^{n-1}s^{\alpha_2-l-1}\overline{g}_l(x;r)\right)=c\left({}^{C}D_{x,-\infty}^{\beta}\mathcal{L}_t\{\overline{u}\}\right)(x,s;r)+\mathcal{L}_t\{\overline{\varphi}\}(x,s,u(x,t);r),$$

where

$$\underline{\varphi}(x,t,u(x,t);r)=\min\{\varphi(x,t,u)\,|\,u\in[\underline{u}(x,t;r),\overline{u}(x,t;r)]\},0\le r\le 1,$$

$$\overline{\varphi}(x,t,u(x,t);r)=max\{\varphi(x,t,u)\,|\,u\in[\underline{u}(x,t;r),\overline{u}(x,t;r)]\},0\le r\le 1.$$

Taking the Fourier transform in Eq. (6.4), and in order to solve the linear system (6.3), for simplify we assume that:

$$\begin{cases}\mathcal{F}_x\left\{\mathcal{L}_t\{\underline{u}(x,t;r)\}\right\}=H_1(\omega,s;r),0\le r\le 1\\ \mathcal{F}_x\{\mathcal{L}_t\{\overline{u}(x,t;r)\}\}=K_1(\omega,s;r),0\le r\le 1\end{cases},$$

where $H_1(\omega,s;r)$ and $K_1(\omega,s;r)$ are solutions of system (6.4). Then, by using the inverse Laplace transform and the inverse Fourier transform, $\underline{u}(x,t;r)$ and $\overline{u}(x,t;r)$ are computed as:

$$\underline{u}(x,t;r)=\mathcal{F}_x\left\{\mathcal{L}_t^{-1}\{H_1(\omega,s;r)\}\right\}=\frac{1}{2\pi}\underline{u}(x,0;r)\int_{-\infty}^{\infty}G_1(\omega,t)d\omega+\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^t G_1(\omega,t;\tau)\underline{\varphi}(\omega,t-\tau;r)d\tau\,d\omega,$$

$$(6.5)$$

$$\overline{u}(x,t;r)=\mathcal{F}_x\left\{\mathcal{L}_t^{-1}\{K_1(\omega,s;r)\}\right\}=\frac{1}{2\pi}\overline{u}(x,0;r)\int_{-\infty}^{\infty}G_1(\omega,t)d\omega+\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^t G_1(\omega,t;\tau)\overline{\varphi}(\omega,t-\tau;r)d\tau\,d\omega.$$

These two cases can be considered as particular cases of Eqs. (6.5)

A. Classical Green's function

In this case the homogeneous initial condition is $f(x)=\tilde{0}$. Substituting this into Eqs. (6.5)

$$[u(x,t)]^r=\big(G_1(x,t|0,0)\big)^r=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^t G_1(\omega,t;\tau)\left[\underline{\varphi}(\omega,t-\tau),\overline{\varphi}(\omega,t-\tau)\right]^r d\tau\,d\omega,$$

B. The propagator

Substituting $\varphi(x,t;r)=\tilde{0}$ and $f(x;r)=\delta(x;r)$ into Eqs. (6.5), we can write

$$\underline{u}(x,t;r)=\frac{1}{2\pi}\underline{u}(x,0;r)\int_{-\infty}^{\infty}G_1(\omega,t)d\omega,$$

$$\overline{u}(x,t;r)=\frac{1}{2\pi}\overline{u}(x,0;r)\int_{-\infty}^{\infty}G_1(\omega,t)d\omega.$$

Case II. Let us consider $Du(x,t)$ is a ${}^{C}[(i)-\alpha]$ -differentiable function, then Eq. (6.1) is extended based on its lower and upper functions as following:

$$a\left(s^{\alpha_1}\mathcal{L}_t\{\underline{u}\}(x,s;r)-\sum_{k=0}^{m-1}s^{\alpha_1-k-1}\underline{f_k}(x;r)\right)+b\left(s^{\alpha_2}\mathcal{L}_t\{\underline{u}\}(x,s;r)-\sum_{l=0}^{n-1}s^{\alpha_2-l-1}\underline{g_l}(x;r)\right)=$$
$$c\left({}^{C}D_{x,-\infty}^{\beta}\mathcal{L}_t\{\underline{u}\}\right)(x,s;r)+\mathcal{L}_t\left\{\underline{\varphi}\right\}(x,s,u(x,t);r),$$

(6.6)

$$a\left(s^{\alpha_1}\mathcal{L}_t\{\overline{u}\}(x,s;r)-\sum_{k=0}^{m-1}s^{\alpha_1-k-1}\overline{f}_k(x;r)\right)+b\left(s^{\alpha_2}\mathcal{L}_t\{\overline{u}\}(x,s;r)-\sum_{l=0}^{n-1}s^{\alpha_2-l-1}\overline{g}_l(x;r)\right)=$$
$$c\left({}^{C}D_{x,-\infty}^{\beta}\mathcal{L}_t\{\overline{u}\}\right)(x,s;r)+\mathcal{L}_t\{\overline{\varphi}\}(x,s,u(x,t);r),$$

where

$$\underline{\varphi}(x,t,u(x,t);r)=\min\left\{\varphi(x,t,u)\middle|u\in\left[\underline{u}(x,t;r),\overline{u}(x,t;r)\right]\right\},0\le r\le 1,$$

$$\overline{\varphi}(x,t,u(x,t);r)=max\left\{\varphi(x,t,u)\middle|u\in\left[\underline{u}(x,t;r),\overline{u}(x,t;r)\right]\right\},0\le r\le 1.$$

Taking the Fourier transform in Eq. (6.6), and in order to solve the linear system (6.6), for simplify we assume that:

$$\begin{cases}\mathcal{F}_x\left\{\mathcal{L}_t\{\underline{u}(x,t;r)\}\right\}=H_2(\omega,s;r),0\le r\le 1\\ \mathcal{F}_x\{\mathcal{L}_t\{\overline{u}(x,t;r)\}\}=K_2(\omega,s;r),0\le r\le 1\end{cases},$$

where $H_2(\omega,s;r)$ and $K_2(\omega,s;r)$ are solutions of system (6.6). Then, by using the inverse Laplace transform and the inverse Fourier transform, $\underline{u}(x,t;r)$ and $\overline{u}(x,t;r)$ are computed as:

$$\underline{u}(x,t;r)=\mathcal{F}_x\left\{\mathcal{L}_t^{-1}\{H_2(\omega,s;r)\}\right\}=\frac{1}{2\pi}\underline{u}(x,0;r)\int_{-\infty}^{\infty}G_1(\omega,t)d\omega+\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^t G_1(\omega,t;\tau)\underline{\varphi}(\omega,t-\tau;r)d\tau\, d\omega,$$

(6.7)

$$\overline{u}(x,t;r)=\mathcal{F}_x\left\{\mathcal{L}_t^{-1}\{K_2(\omega,s;r)\}\right\}=\frac{1}{2\pi}\overline{u}(x,0;r)\int_{-\infty}^{\infty}G_1(\omega,t)d\omega+\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^t G_1(\omega,t;\tau)\overline{\varphi}(\omega,t-\tau;r)d\tau\, d\omega.$$

These two cases can be considered as particular cases of Eqs. (6.7)

A. Classical Green's function

In this case the homogeneous initial condition is $f(x)=\tilde{0}$. Substituting this into Eqs. (6.7)

$$[u(x,t)]^r=\left(G_2(x,t|0,0)\right)^r=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^t G_2(\omega,t;\tau)\left[\underline{\varphi}(\omega,t-\tau),\overline{\varphi}(\omega,t-\tau)\right]^r d\tau\, d\omega,$$

B. The propagator

Substituting $\varphi(x,t;r)=\tilde{0}$ and $f(x;r)=\delta(x;r)$ into Eqs. (6.7), we can write

$$\underline{u}(x,t;r)=\frac{1}{2\pi}\underline{u}(x,0;r)\int_{-\infty}^{\infty}G_2(\omega,t)d\omega,$$

$$\overline{u}(x,t;r)=\frac{1}{2\pi}\overline{u}(x,0;r)\int_{-\infty}^{\infty}G_2(\omega,t)d\omega.$$

7. Examples

In this section, we handle two examples with details to solve FFPDEs under Caputo-type H-differentiability.

Example 7.1 (Fuzzy fractional reaction-diffusion equation). Let us consider the following FFPDE:

$$\begin{cases} {}^{C}D_{0^+,t}^{2\alpha}u(x,t)+2\lambda {}^{C}D_{0^+,t}^{\alpha}u(x,t)=\nu^2 {}^{C}D_{x,-\infty}^{2\beta}u(x,t)+\xi^2u(x,t)+\varphi(x,t) \\ u(x,0)=0 \; and \; u_t(x,0)=0\in\mathbb{E} \end{cases}, \tag{7.1}$$

where $t>0$ and $-\infty<x<+\infty,\ 1/2<\alpha\le 1,\ 0<\beta\le 1$.

Applying Laplace transform on the both sides of above equation, we obtain:

$$\mathcal{L}_t\{{}^{C}D_{0^+,t}^{2\alpha}u(x,t)+2\lambda {}^{C}D_{0^+,t}^{\alpha}u(x,t)\}=\mathcal{L}_t\left\{\nu^2 {}^{C}D_{x,-\infty}^{2\beta}u(x,t)+\xi^2u(x,t)+\varphi(x,t)\right\},$$

Using ${}^{C}[(i)-\alpha]$−differentiability, we get

$$\nu^2 {}^{C}D_{x,-\infty}^{2\beta}\mathcal{L}_t\{u\}(x,s;r)+\xi^2\mathcal{L}_t\{u\}(x,s;r)+\mathcal{L}_t\{\varphi\}(x,s;r)=s^{2\alpha}\mathcal{L}_t\{\underline{u}\}(x,s;r)-s^{2\alpha-1}\underline{f_0}(x;r)+ 2\lambda s^{\alpha}\mathcal{L}_t\{\underline{u}\}(x,s;r)-2\lambda s^{\alpha-1}\underline{f_0}(x;r),$$

$$\tag{7.2}$$

$$\nu^2 {}^{C}D_{x,-\infty}^{2\beta}\mathcal{L}_t\{u\}(x,s;r)+\xi^2\mathcal{L}_t\{u\}(xs;r)+\mathcal{L}_t\{\varphi\}(x,s;r)=s^{2\alpha}\mathcal{L}_t\{\underline{u}\}(x,s;r)-s^{2\alpha-1}\underline{f_0}(x;r)+ 2\lambda s^{\alpha}\mathcal{L}_t\{\underline{u}\}(x,s;r)-2\lambda s^{\alpha-1}\underline{f_0}(x;r),$$

Taking the Fourier transform in Eq. (7.2), and the Fourier transform of the fractional derivative

$$\mathcal{F}_x\{{}^{C}D_{x,-\infty}^{\mu}u(x,t)\}=-|\omega|^{\mu}\mathcal{F}_x\{u(\omega,t)\}$$

we get

$$\left(s^{2\alpha}+2\lambda s^{\alpha}-\nu^2|\omega|^{2\beta}-\xi^2\right)\mathcal{F}_x\left\{\mathcal{L}_t\{\underline{u}\}\right\}(\omega,s;r)=s^{2\alpha-1}\mathcal{F}_x\left\{\underline{f_0}\right\}(\omega;r)+ 2\lambda s^{\alpha-1}\mathcal{F}_x\left\{\underline{f_0}\right\}(\omega;r)+\mathcal{F}_x\left\{\mathcal{L}_t\left\{\underline{\varphi}\right\}\right\}(\omega,s;r),$$

$$\tag{7.3}$$

$$\left(s^{2\alpha}+2\lambda s^{\alpha}-\nu^2|\omega|^{2\beta}-\xi^2\right)\mathcal{F}_x\{\mathcal{L}_t\{\overline{u}\}\}(\omega,s;r)=s^{2\alpha-1}\mathcal{F}_x\{\overline{f}_0\}(\omega;r)+ 2\lambda s^{\alpha-1}\mathcal{F}_x\{\overline{f}_0\}(\omega;r)+\mathcal{F}_x\{\mathcal{L}_t\{\overline{\varphi}\}\}(\omega,s;r),$$

Applying inverse of Laplace transform on the both sides of Eq. (7.3), we get the following:

$$\begin{cases}\mathcal{F}_x\left\{\mathcal{L}_t\{\underline{u}\}\right\}(\omega,s;r)=\frac{s^{2\alpha-1}\mathcal{F}_x\left\{\underline{f_0}\right\}(\omega;r)}{s^{2\alpha}+2\lambda s^{\alpha}+\Lambda(\omega)}+\frac{2\lambda s^{\alpha-1}\mathcal{F}_x\left\{\underline{f_0}\right\}(\omega;r)}{s^{2\alpha}+2\lambda s^{\alpha}+\Lambda(\omega)}+\frac{\mathcal{F}_x\left\{\mathcal{L}_t\left\{\underline{\varphi}\right\}\right\}(\omega,s;r)}{s^{2\alpha}+2\lambda s^{\alpha}+\Lambda(\omega)},\\ \mathcal{F}_x\{\mathcal{L}_t\{\overline{u}\}\}(\omega,s;r)=\frac{s^{2\alpha-1}\mathcal{F}_x\{\overline{f}_0\}(\omega;r)}{s^{2\alpha}+2\lambda s^{\alpha}+\Lambda(\omega)}+\frac{2\lambda s^{\alpha-1}\mathcal{F}_x\{\overline{f}_0\}(\omega;r)}{s^{2\alpha}+2\lambda s^{\alpha}+\Lambda(\omega)}+\frac{\mathcal{F}_x\{\mathcal{L}_t\{\overline{\varphi}\}\}(\omega,s;r)}{s^{2\alpha}+2\lambda s^{\alpha}+\Lambda(\omega)}\end{cases}$$

where we have put $\Lambda(\omega)=-\nu^2|\omega|^{2\beta}-\xi^2$.

Taking the inverse fuzzy Laplace transform and inverse fuzzy Fourier transform, we get

$$\underline{u}(x,t;r)=\frac{1}{2\pi}\underline{f_0}(x;r)\sum_{i=0}^{\infty}(-2\lambda)^i t^{\alpha i}\int_{-\infty}^{\infty}e^{-i\omega x}\left(E_{2\alpha,\alpha i+1}^{(i+1)}(-\Lambda(\omega)t^{2\alpha})+\right.$$
$$\left.2\lambda t^{\alpha}E_{2\alpha,\alpha+\alpha i+1}^{(i+1)}(-\Lambda(\omega)t^{2\alpha})\right)d\omega+$$
$$\frac{1}{2\pi}\sum_{i=0}^{\infty}(-2\lambda)^i\int_0^t\xi^{\alpha i+(2\alpha-1)}\int_{-\infty}^{\infty}e^{-i\omega x}E_{2\alpha,2\alpha+\alpha i}^{(i+1)}(-\Lambda(\omega)\xi^{2\alpha})d\omega\,\underline{\varphi}(x,t-\xi;r)d\xi,$$

(7.4)

$$\overline{u}(x,t;r)=\frac{1}{2\pi}\overline{f}_0(x;r)\sum_{i=0}^{\infty}(-2\lambda)^i t^{\alpha i}\int_{-\infty}^{\infty}e^{-i\omega x}\left(E_{2\alpha,\alpha i+1}^{(i+1)}(-\Lambda(\omega)t^{2\alpha})+\right.$$
$$\left.\frac{2\lambda}{\alpha}t^{\alpha}E_{2\alpha,\alpha+\alpha i+1}^{(i+1)}(-\Lambda(\omega)t^{2\alpha})\right)d\omega+$$
$$\frac{1}{2\pi}\sum_{i=0}^{\infty}(-2\lambda)^i\int_0^t\xi^{\alpha i+(2\alpha-1)}\int_{-\infty}^{\infty}e^{-i\omega x}E_{2\alpha,2\alpha+\alpha i}^{(i+1)}(-\Lambda(\omega)\xi^{2\alpha})d\omega\,\overline{\varphi}(x,t-\xi;r)d\xi,$$

These two cases can be considered as particular cases of Eqs. (7.4)

A. Classical Green's function

In this case the homogeneous initial condition is $f(x)=\tilde{0}$. Substituting this into Eqs. (7.4)

$$[u(x,t)]^r=\left(G(x,t|0,0)\right)^r=$$
$$\frac{1}{2\pi}\sum_{i=0}^{\infty}(-2\lambda)^i\int_0^t\xi^{\alpha i+(2\alpha-1)}\int_{-\infty}^{\infty}e^{-i\omega x}E_{2\alpha,2\alpha+\alpha i}^{(i+1)}(-\Lambda(\omega)\xi^{2\alpha})d\omega\,[\varphi(x,t-\xi)]^r d\xi,$$

B. The propagator

Substituting $\varphi(x,t;r)=\tilde{0}$ and $f(x;r)=\delta(x;r)$ into Eqs. (7.4), we can write

$$\underline{u}(x,t;r)=\frac{1}{2\pi}\underline{f_0}(x;r)\sum_{i=0}^{\infty}(-2\lambda)^i t^{\alpha i}\int_{-\infty}^{\infty}e^{-i\omega x}\left(E_{2\alpha,\alpha i+1}^{(i+1)}(-\Lambda(\omega)t^{2\alpha})+\right.$$
$$\left.2\lambda t^{\alpha}E_{2\alpha,\alpha+\alpha i+1}^{(i+1)}(-\Lambda(\omega)t^{2\alpha})\right)d\omega,$$

$$\overline{u}(x,t;r)=\frac{1}{2\pi}\overline{f}_0(x;r)\sum_{i=0}^{\infty}(-2\lambda)^i t^{\alpha i}\int_{-\infty}^{\infty}e^{-i\omega x}\left(E_{2\alpha,\alpha i+1}^{(i+1)}(-\Lambda(\omega)t^{2\alpha})+\right.$$
$$\left.2\lambda t^{\alpha}E_{2\alpha,\alpha+\alpha i+1}^{(i+1)}(-\Lambda(\omega)t^{2\alpha})\right)d\omega.$$

For special case ${}^{C}[(i)-\alpha]$, let us consider, $\alpha=0.75$, $\beta=0.5$, $\lambda=1/2$, $\xi=1$,$\nu=1/2$, $[u(x,0)]^r=[f_0(x)]^r=[-1+r,1-r]$, $t=0.001$, then we get the solution for case ${}^{C}[(i)-\alpha]$ as following:

$$[u(x,0.001)]^r = \frac{1}{2\pi}[-1+r,1-r]\sum_{i=0}^{\infty}(-1)^i(0.001)^{0.75i}\int_{-\infty}^{\infty}e^{-i\omega x}\left(\sum_{k=0}^{\infty}\frac{\Gamma(i+1+k)\left(-\left(-\frac{1}{4}\omega-1\right)(0.001)^{1.5}\right)^k}{k!\Gamma(i+1)\Gamma(1.5k+0.75i+1)}+\right.$$
$$\left.(0.001)^{0.75}\sum_{k=0}^{\infty}\frac{\Gamma(i+1+k)\left(-\left(-\frac{1}{4}\omega-1\right)(0.001)^{1.5}\right)^k}{k!\Gamma(i+1)\Gamma(1.5k+1.75+0.75i)}\right)d\omega,\ 0\le r\le 1,$$

Example 7.2 (Fuzzy fractional reaction-diffusion equation). Let us consider the following FFPDE:

$$\begin{cases} b\,{}^{C}D^{\alpha}_{0^+,t}u(x,t) = c\,{}^{C}D^{2\beta}_{x,-\infty}u(x,t) - \nu^2 u(x,t) + \varphi(x,t) \\ u(x,0) = 0 \text{ and } u_t(x,0) = 0 \in \mathbb{E} \end{cases}, \tag{7.5}$$

where $t > 0$ ve $-\infty < x < +\infty$, 0< $\alpha \le 1$, $0 < \beta \le 1$.

Applying Laplace transform on the both sides of above equation, we obtain:

$$\mathcal{L}_t\left\{b\,{}^{C}D^{\alpha}_{0^+,t}u(x,t)\right\} = \mathcal{L}_t\left\{c\,{}^{C}D^{2\beta}_{x,-\infty}u(x,t) - \nu^2 u(x,t) + \varphi(x,t)\right\},$$

Using ${}^{C}[(ii)-\alpha]$−differentiability, we get

$$\begin{cases} c\,{}^{C}D^{2\beta}_{x,-\infty}\mathcal{L}_t\{\underline{u}\}(x,s;r) - \nu^2\mathcal{L}_t\{\underline{u}\}(x,s;r) + \mathcal{L}_t\left\{\underline{\varphi}\right\}(x,s;r) = bs^{\alpha}\mathcal{L}_t\{\underline{u}\}(x,s;r) - bs^{\alpha-1}\underline{f_0}(x;r) \\ c\,{}^{C}D^{2\beta}_{x,-\infty}\mathcal{L}_t\{\overline{u}\}(x,s;r) - \nu^2\mathcal{L}_t\{\overline{u}\}(x,s;r) + \mathcal{L}_t\{\overline{\varphi}\}(x,s;r) = bs^{\alpha}\mathcal{L}_t\{\overline{u}\}(x,s;r) - bs^{\alpha-1}\overline{f}_0(x;r) \end{cases}, \tag{7.6}$$

Taking the Fourier transform in Eqs. (7.6), and the Fourier transform of the fractional derivative

$$\mathcal{F}_x\left\{{}^{C}D^{\mu}_{x,-\infty}\,u(x,t)\right\} = -|\omega|^{\mu}\mathcal{F}_x\{u(\omega,t)\}$$

we get,

$$\begin{cases} \left(bs^{\alpha} - c|\omega|^{2\beta} + \nu^2\right)\mathcal{F}_x\left\{\mathcal{L}_t\{\underline{u}\}\right\}(\omega,s;r) = bs^{\alpha-1}\mathcal{F}_x\left\{\underline{f_0}\right\}(\omega;r) + \mathcal{F}_x\left\{\mathcal{L}_t\left\{\underline{\varphi}\right\}\right\}(\omega,s;r) \\ \left(bs^{\alpha} - c|\omega|^{2\beta} + \nu^2\right)\mathcal{F}_x\left\{\mathcal{L}_t\{\overline{u}\}\right\}(\omega,s;r) = bs^{\alpha-1}\mathcal{F}_x\left\{\overline{f}_0\right\}(\omega;r) + \mathcal{F}_x\left\{\mathcal{L}_t\{\overline{\varphi}\}\right\}(\omega,s;r) \end{cases}, \tag{7.7}$$

Applying inverse of Laplace transform on the both sides of Eq. (7.7), we get the following:

$$\begin{cases} \mathcal{F}_x\left\{\mathcal{L}_t\{\underline{u}\}\right\}(\omega,s;r) = \frac{bs^{\alpha-1}\mathcal{F}_x\left\{\underline{f_0}\right\}(\omega;r)}{bs^{\alpha}+\Lambda(\omega)} + \frac{\mathcal{F}_x\left\{\mathcal{L}_t\left\{\underline{\varphi}\right\}\right\}(\omega,s;r)}{bs^{\alpha}+\Lambda(\omega)} \\ \mathcal{F}_x\left\{\mathcal{L}_t\{\overline{u}\}\right\}(\omega,s;r) = \frac{bs^{\alpha-1}\mathcal{F}_x\left\{\overline{f}_0\right\}(\omega;r)}{bs^{\alpha}+\Lambda(\omega)} + \frac{\mathcal{F}_x\left\{\mathcal{L}_t\{\overline{\varphi}\}\right\}(\omega,s;r)}{bs^{\alpha}+\Lambda(\omega)} \end{cases},$$

where we have put $\Lambda(\omega) = -c|\omega|^{2\beta} + \nu^2$.

Taking the inverse fuzzy Laplace transform and inverse fuzzy Fourier transform, we get

$$\underline{u}(x,t;r) = \frac{1}{2\pi}\underline{f_0}(x;r)\int_{-\infty}^{\infty}e^{-i\omega x}E_{\alpha,1}\left(-\frac{\Lambda(\omega)}{b}t^{\alpha}\right)d\omega +$$
$$\frac{1}{2\pi}\frac{1}{b}\int_0^t\xi^{\alpha-1}\int_{-\infty}^{\infty}e^{-i\omega x}E_{\alpha,\alpha}\left(-\frac{\Lambda(\omega)}{b}\xi^{\alpha}\right)d\omega\,\underline{\varphi}(x,t-\xi;r)d\xi,$$

$$\tag{7.8}$$

$$\overline{u}(x,t;r) = \frac{1}{2\pi}\overline{f}_0(x;r)\int_{-\infty}^{\infty}e^{-i\omega x}E_{\alpha,1}\left(-\frac{\Lambda(\omega)}{b}t^{\alpha}\right)d\omega +$$
$$\frac{1}{2\pi}\frac{1}{b}\int_0^t\xi^{\alpha-1}\int_{-\infty}^{\infty}e^{-i\omega x}E_{\alpha,\alpha}\left(-\frac{\Lambda(\omega)}{b}\xi^{\alpha}\right)d\omega\,\overline{\varphi}(x,t-\xi;r)d\xi.$$

These two cases can be considered as particular cases of Eqs. (7.8).

A. Classical Green's function

In this case the homogeneous initial condition is $f(x) = \tilde{0}$. Substituting this into Eqs. (7.8)

$$[u(x,t)]^r = \left(G_2(x,t|0,0)\right)^r = \frac{1}{2\pi}\frac{1}{b}\int_0^t \xi^{\alpha-1}\int_{-\infty}^{\infty} e^{-i\omega x} E_{\alpha,\alpha}\left(-\frac{\Lambda(\omega)}{b}\xi^\alpha\right) d\omega\, [\varphi(x,t-\xi)]^r d\xi,$$

B. The propagator

Substituting $\varphi(x,t;r) = \tilde{0}$ and $f(x;r) = \delta(x;r)$ into Eqs. (7.8), we can write

$$\underline{u}(x,t;r) = \frac{1}{2\pi}\underline{f_0}(x;r)\int_{-\infty}^{\infty} e^{-i\omega x} E_{\alpha,1}\left(-\frac{\Lambda(\omega)}{b}t^\alpha\right) d\omega,$$

$$\overline{u}(x,t;r) = \frac{1}{2\pi}\overline{f}_0(x;r)\int_{-\infty}^{\infty} e^{-i\omega x} E_{\alpha,1}\left(-\frac{\Lambda(\omega)}{b}t^\alpha\right) d\omega.$$

For special case ${}^C[(ii)-\alpha]$, let us consider, $\alpha = 0.5$, $\beta = 0.5$, $b = c = 1$, $\nu = 1$, $[u(x,0)]^r = [f_0(x)]^r = [-1+r, 1-r]$, $t = 0.001$, then we get the solution for case ${}^C[(ii)-\alpha]$ as following:

$$[u(x,0.001)]^r = \frac{1}{2\pi}[-1+r,1-r]\int_{-\infty}^{\infty} e^{-i\omega x}\sum_{k=0}^{\infty}\frac{\left((\omega-1)(0.001)^{0.5}\right)^k}{\Gamma(0.5k+1)} d\omega\, , 0 \le r \le 1,$$

8. Conclusions

In this paper, the fuzzy Laplace transforms and fuzzy Fourier transforms have been studied in order to solve fuzzy fractional partial differential equations (FFPDEs) of order $0 < \alpha \le 1$ under Caputo-type H-differentiability. To this end, Caputo-type differentiability based on the Hukuhara difference was introduced and then Laplace transform of fractional derivative was discussed by Derivative theorem (Theorem 4.1). Consequently, we solved some well-known examples in fractional manner to obtain the solutions of FFPDEs under Caputo-type H-differentiability.

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Aspects of univalent holomorphic functions involving Ruscheweyh derivative and generalized Sălăgean operator

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Abstract

Making use Ruscheweyh derivative and generalized Sălăgean operator, we introduce a new class of analytic functions $\mathcal{RD}(\gamma,\lambda,\alpha,\beta)$ defined on the open unit disc, and investigate its various characteristics. Further we obtain distortion bounds, extreme points and radii of close-to-convexity, starlikeness and convexity and neighborhood property for functions belonging to the class $\mathcal{RD}(\gamma,\lambda,\alpha,\beta)$.

Keywords: Analytic functions, univalent functions, radii of starlikeness and convexity, neighborhood property, Salagean operator, Ruscheweyh operator.
2000 Mathematical Subject Classification: 30C45, 30A20, 34A40.

1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form $f(z)=z+\sum_{j=2}^{\infty}a_jz^j$, which are analytic and univalent in the open unit disc $U=\{z:z\in\mathbb{C}:|z|<1\}$. $\mathcal{T}$ is a subclass of $\mathcal{A}$ consisting the functions of the form $f(z)=z-\sum_{j=2}^{\infty}|a_j|\,z^j$. For functions $f,g\in\mathcal{A}$ given by $f(z)=z+\sum_{j=2}^{\infty}a_jz^j$, $g(z)=z+\sum_{j=2}^{\infty}b_jz^j$, we define the Hadamard product (or convolution) of f and g by $(f*g)(z)=z+\sum_{j=2}^{\infty}a_jb_jz^j,\ \ z\in U$.

Definition 1.1 *(Al Oboudi [5]) For $f\in\mathcal{A}$, $\lambda\geq 0$ and $n\in\mathbb{N}$, the operator D_λ^n is defined by $D_\lambda^n:\mathcal{A}\to\mathcal{A}$,*

$$\begin{aligned}D_\lambda^0 f(z) &= f(z)\\ D_\lambda^1 f(z) &= (1-\lambda)f(z)+\lambda z f'(z)=D_\lambda f(z),\ \ldots\\ D_\lambda^{n+1}f(z) &= (1-\lambda)D_\lambda^n f(z)+\lambda z\left(D_\lambda^n f(z)\right)'=D_\lambda\left(D_\lambda^n f(z)\right),\quad z\in U.\end{aligned}$$

Remark 1.1 *If $f\in\mathcal{A}$ and $f(z)=z+\sum_{j=2}^{\infty}a_jz^j$, then $D_\lambda^n f(z)=z+\sum_{j=2}^{\infty}\left[1+(j-1)\lambda\right]^n a_jz^j$, $z\in U$.*
If $f\in\mathcal{T}$ and $f(z)=z-\sum_{j=2}^{\infty}a_jz^j$, then $D_\lambda^n f(z)=z-\sum_{j=t+1}^{\infty}\left[1+(j-1)\lambda\right]^n a_jz^j$, $z\in U$.

Remark 1.2 *For $\lambda=1$ in the above definition we obtain the Sălăgean differential operator [8].*

Definition 1.2 *(Ruscheweyh [7]) For $f\in\mathcal{A}$, $n\in\mathbb{N}$, the operator R^n is defined by $R^n:\mathcal{A}\to\mathcal{A}$,*

$$\begin{aligned}R^0 f(z) &= f(z)\\ R^1 f(z) &= zf'(z),\ \ldots\\ (n+1)R^{n+1}f(z) &= z\left(R^n f(z)\right)'+nR^n f(z),\quad z\in U.\end{aligned}$$

Remark 1.3 *If $f\in\mathcal{A}$, $f(z)=z+\sum_{j=2}^{\infty}a_jz^j$, then $R^n f(z)=z+\sum_{j=2}^{\infty}\frac{(n+j-1)!}{n!(j-1)!}a_jz^j$, $z\in U$.*
If $f\in\mathcal{T}$, $f(z)=z-\sum_{j=2}^{\infty}a_jz^j$, then $R^n f(z)=z-\sum_{j=t+1}^{\infty}\frac{(n+j-1)!}{n!(j-1)!}a_jz^j$, $z\in U$.

Definition 1.3 *[1], [2] Let $\gamma,\lambda\geq 0$, $n\in\mathbb{N}$. Denote by $RD_{\lambda,\gamma}^n$ the operator given by $RD_{\lambda,\gamma}^n:\mathcal{A}\to\mathcal{A}$,*

$$RD_{\lambda,\gamma}^n f(z)=(1-\gamma)R^n f(z)+\gamma D_\lambda^n f(z),\qquad z\in U.$$

Remark 1.4 *If* $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, *then* $RD_{\lambda,\gamma}^{n} f(z) = z + \sum_{j=2}^{\infty} \left\{ \gamma \left[1 + (j-1)\lambda\right]^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\} a_j z^j$, $z \in U$.

If $f \in \mathcal{T}$, $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, *then* $RD_{\lambda,\gamma}^{n} f(z) = z - \sum_{j=t+1}^{\infty} \left\{ \gamma \left[1 + (j-1)\lambda\right]^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\} a_j z^j$, $z \in U$.

The operator $RD_{1,\gamma}^{n} f(z) = L_{\gamma}^{n} f(z)$ *which was introduced in [3].*

Following the work of Sh.Najafzadeh and E.Pezeshki [6] we can define the class $RD(\gamma, \alpha, \beta)$ as follows.

Definition 1.4 *For* $\gamma, \lambda \geq 0$, $0 \leq \alpha < 1$ *and* $0 < \beta \leq 1$, *let* $\mathcal{RD}(\gamma, \lambda, \alpha, \beta)$ *be the subclass of* $\mathcal{T}$ *consisting of functions that satisfying the inequality*

$$\left| \frac{RD_{\lambda,\gamma}^{\mu,n} f(z) - 1}{2\nu(RD_{\lambda,\gamma}^{\mu,n} f(z) - \alpha) - (RD_{\lambda,\gamma}^{\mu,n} f(z) - 1)} \right| < \beta \tag{1.1}$$

where

$$RD_{\lambda,\gamma}^{\mu,n} f(z) = (1-\mu) \frac{RD_{\lambda,\gamma}^{n} f(z)}{z} + \mu (RD_{\lambda,\gamma}^{n} f(z))', \quad 0 < \nu \leq 1. \tag{1.2}$$

Remark 1.5 *If* $f \in \mathcal{T}$, $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, *then*
$RD_{\lambda,\gamma}^{\mu,n} f(z) = 1 - \sum_{j=t+1}^{\infty} \left[1 + \mu (j-1)\right] \left\{ \gamma \left[1 + (j-1)\lambda\right]^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\} a_j z^{j-1}$, $z \in U$.

The class $\mathcal{RD}(\gamma, 1, \alpha, \beta) = \mathcal{L}(\gamma, \alpha, \beta)$ *defined and studied in [4]*

2 Coefficient bounds

In this section we obtain coefficient bounds and extreme points for functions is $\mathcal{RD}(\gamma, \lambda, \alpha, \beta)$.

Theorem 2.1 *Let the function* $f \in \mathcal{T}$. *Then* $f \in \mathcal{RD}(\gamma, \lambda, \alpha, \beta)$ *if and only if*

$$\sum_{j=t+1}^{\infty} (1 + \mu(j-1))[1 + \beta(2\nu - 1)] \left\{ \gamma \left[1 + (j-1)\lambda\right]^n + (1-\gamma) \frac{(n+j-1)!}{n!\,(j-1)!} \right\} a_j < 2\beta\nu(1-\alpha). \tag{2.1}$$

The result is sharp for the function $F(z)$ *defined by*

$$F(z) = z - \frac{2\beta\nu(1-\alpha)}{(1 + \mu(j-1))[1 + \beta(2\nu - 1)] \left\{ \gamma \left[1 + (j-1)\lambda\right]^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\}} z^j, \qquad j \geq t+1.$$

Proof. Suppose f satisfies (2.1). Then for $|z| < 1$, we have
$\left| RD_{\lambda,\gamma}^{\mu,n} f(z) - 1 \right| - \beta \left| 2\nu(RD_{\lambda,\gamma}^{\mu,n} f(z) - \alpha) - (RD_{\lambda,\gamma}^{\mu,n} f(z) - 1) \right| =$
$\left| - \sum_{j=t+1}^{\infty} (1 + \mu(j-1)) \left\{ \gamma \left[1 + (j-1)\lambda\right]^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\} a_j z^{j-1} \right| -$
$\beta \left| 2\nu(1-\alpha) - (2\nu - 1) \sum_{j=t+1}^{\infty} \left\{ \gamma \left[1 + (j-1)\lambda\right]^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\} \left[1 + \mu (j-1)\right] a_j z^{j-1} \right| \leq$
$\sum_{j=t+1}^{\infty} \left[1 + \mu(j-1)\right] \left\{ \gamma \left[1 + (j-1)\lambda\right]^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\} a_k - 2\beta\nu(1-\alpha) +$
$\sum_{j=t+1}^{\infty} \beta(2\nu - 1)(1 + \mu(j-1)) \left\{ \gamma \left[1 + (j-1)\lambda\right]^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\} a_j =$
$\sum_{j=t+1}^{\infty} \left[1 + \mu(j-1)\right] \left[1 + \beta(2\nu - 1)\right] \left\{ \gamma \left[1 + (j-1)\lambda\right]^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\} a_j - 2\beta\nu(1-\alpha) < 0.$
Hence, by using the maximum modulus Theorem and (1.1), $f \in \mathcal{RD}(\gamma, \lambda, \alpha, \beta)$. Conversely, assume that
$\left| \frac{RD_{\lambda,\gamma}^{\mu,n} f(z) - 1}{2\nu(RD_{\lambda,\gamma}^{\mu,n} f(z) - \alpha) - (RD_{\lambda,\gamma}^{\mu,n} f(z) - 1)} \right| = \left| \frac{- \sum_{j=t+1}^{\infty} [1 + \mu(j-1)] \left\{ \gamma [1 + (j-1)\lambda]^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\} a_j z^{j-1}}{2\nu(1-\alpha) - \sum_{j=t+1}^{\infty} [1 + \mu(j-1)](2\nu - 1) \left\{ \gamma [1 + (j-1)\lambda]^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\} a_j z^{j-1}} \right| < \beta$, $z \in U$.
Since $Re(z) \leq |z|$ for all $z \in U$, we have

$$Re \left\{ \frac{\sum_{j=t+1}^{\infty} \left[1 + \mu(j-1)\right] \left\{ \gamma \left[1 + (j-1)\lambda\right]^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\} a_j z^{j-1}}{2\nu(1-\alpha) - \sum_{j=t+1}^{\infty} \left[1 + \mu(j-1)\right] (2\nu - 1) \left\{ \gamma \left[1 + (j-1)\lambda\right]^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\} a_j z^{j-1}} \right\} < \beta. \tag{2.2}$$

By choosing choose values of z on the real axis so that $RD_{\lambda,\gamma}^{\mu,n} f(z)$ is real and letting $z \to 1$ through real values, we obtain the desired inequality (2.1). ■

Corollary 2.2 *If $f \in \mathcal{T}$ be in $\mathcal{DR}(\gamma, \lambda, \alpha, \beta)$, then*

$$a_j \leq \frac{2\beta\nu(1-\alpha)}{[1+\mu(j-1)]\,[1+\beta(2\nu-1)]\left\{\gamma\,[1+(j-1)\,\lambda]^n + (1-\gamma)\,\frac{(n+j-1)!}{n!(j-1)!}\right\}}, \quad j \geq t+1, \tag{2.3}$$

with equality only for functions of the form $F(z)$.

Theorem 2.3 *Let $f_1(z) = z$ and*

$$f_j(z) = z - \frac{2\beta\nu(1-\alpha)}{[1+\mu(j-1)]\,[1+\beta(2\nu-1)]\left\{\gamma\,[1+(j-1)\,\lambda]^n + (1-\gamma)\,\frac{(n+j-1)!}{n!(j-1)!}\right\}} z^j, \quad j \geq t+1, \tag{2.4}$$

for $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $\gamma, \lambda \geq 0$ and $0 < \nu \leq 1$. Then $f(z)$ is in the class $\mathcal{RD}(\gamma, \lambda, \alpha, \beta)$ if and only if it can be expressed in the form

$$f(z) = \sum_{j=t}^{\infty} \omega_j f_j(z), \tag{2.5}$$

where $\omega_j \geq 0$ and $\sum_{j=1}^{\infty} \omega_j = 1$.

Proof. Suppose $f(z)$ can be written as in (2.5). Then $f(z) = z - \sum_{j=t+1}^{\infty} \omega_j \frac{2\beta\nu(1-\alpha)}{[1+\mu(j-1)][1+\beta(2\nu-1)]\left\{\gamma[1+(j-1)\lambda]^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}} z^j$. Now, $\sum_{j=t+1}^{\infty} \frac{[1+\mu(j-1)][1+\beta(2\nu-1)]\left\{\gamma[1+(j-1)\lambda]^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{2\beta\nu(1-\alpha)} \omega_j \frac{2\beta\nu(1-\alpha)}{[1+\mu(j-1)][1+\beta(2\nu-1)]\left\{\gamma[1+(j-1)\lambda]^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}} = \sum_{j=t+1}^{\infty} \omega_j = 1 - \omega_1 \leq 1$.Thus $f \in \mathcal{RD}(\gamma, \lambda, \alpha, \beta)$.
Conversely, let $f \in \mathcal{RD}(\gamma, \lambda, \alpha, \beta)$. Then by using (2.3), setting $\omega_j = \frac{[1+\mu(j-1)][1+\beta(2\nu-1)]\left\{\gamma[1+(j-1)\lambda]^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{2\beta\nu(1-\alpha)} a_j$, $j \geq t+1$ and $\omega_1 = 1 - \sum_{j=2}^{\infty} \omega_j$, we have $f(z) = \sum_{j=t}^{\infty} \omega_j f_j(z)$. And this completes the proof of Theorem 2.3. ■

3 Distortion bounds

In this section we obtain distortion bounds for the class $\mathcal{RD}(\gamma, \lambda, \alpha, \beta)$.

Theorem 3.1 *If $f \in \mathcal{RD}(\gamma, \lambda, \alpha, \beta)$, then*

$$r - \frac{2\beta\nu(1-\alpha)}{(1+\mu t)[1+\beta(2\nu-1)]\left[\gamma(1+t\lambda)^n + (1-\gamma)\,\frac{(t+n)!}{n!t!}\right]} r^{t+1} \leq |f(z)| \tag{3.1}$$

$$\leq r + \frac{2\beta\nu(1-\alpha)}{(1+\mu t)[1+\beta(2\nu-1)]\left[\gamma(1+t\lambda)^n + (1-\gamma)\,\frac{(t+n)!}{n!t!}\right]} r^{t+1}$$

holds if the sequence $\{\sigma_j(\gamma, \lambda, \beta, \nu)\}_{j=t+1}^{\infty}$ is non-decreasing, and

$$1 - \frac{2\beta\nu(1-\alpha)(t+1)}{(1+\mu t)[1+\beta(2\nu-1)]\left[\gamma(1+t\lambda)^n + (1-\gamma)\,\frac{(t+n)!}{n!t!}\right]} r^{t} \leq |f'(z)| \tag{3.2}$$

$$\leq 1 + \frac{2\beta\nu(1-\alpha)(t+1)}{(1+\mu t)[1+\beta(2\nu-1)]\left[\gamma(1+t\lambda)^n + (1-\gamma)\,\frac{(t+n)!}{n!t!}\right]} r^{t}$$

holds if the sequence $\{\frac{\sigma_j(\gamma,\lambda,\beta,\nu)}{j}\}_{j=t+1}^{\infty}$ is non- decreasing, where
$\sigma_j(\gamma, \beta, \nu) = [1+\mu(j-1)]\,[1+\beta(2\nu-1)]\left\{\gamma\,[1+(j-1)\,\lambda]^n + (1-\gamma)\,\frac{(n+j-1)!}{n!(j-1)!}\right\}$.
The bounds in (3.1) and (3.2) are sharp, for $f(z)$ given by

$$f(z) = z - \frac{2\beta\nu(1-\alpha)}{(1+\mu t)[1+\beta(2\nu-1)]\left[\gamma(1+t\lambda)^n + (1-\gamma)\,\frac{(t+n)!}{n!t!}\right]} z^{t+1}, \quad z = \pm r. \tag{3.3}$$

Proof. In view of Theorem 2.1, we have

$$\sum_{j=t+1}^{\infty} a_j \leq \frac{2\beta\nu(1-\alpha)}{(1+\mu t)[1+\beta(2\nu-1)]\left[\gamma(1+t\lambda)^n+(1-\gamma)\frac{(t+n)!}{n!t!}\right]}. \tag{3.4}$$

We obtain $|z| - |z|^{t+1}\sum_{j=t+1}^{\infty} a_j \leq |f(z)| \leq |z| + |z|^{t+1}\sum_{j=t+1}^{\infty} a_j$. Thus

$$\begin{aligned} r - \frac{2\beta\nu(1-\alpha)}{(1+\mu t)[1+\beta(2\nu-1)]\left[\gamma(1+t\lambda)^n+(1-\gamma)\frac{(t+n)!}{n!t!}\right]} r^{t+1} &\leq |f(z)| \\ \leq r + \frac{2\beta\nu(1-\alpha)}{(1+\mu t)[1+\beta(2\nu-1)]\left[\gamma(1+t\lambda)^n+(1-\gamma)\frac{(t+n)!}{n!t!}\right]} r^{t+1}. \end{aligned} \tag{3.5}$$

Hence (3.1) follows from (3.5). Further, $\sum_{j=t+1}^{\infty} ja_j \leq \frac{2\beta\nu(1-\alpha)}{(1+\mu t)[1+\beta(2\nu-1)]\left[\gamma(1+t\lambda)^n+(1-\gamma)\frac{(t+n)!}{n!t!}\right]}$. Hence (3.2) follows from $1 - r^t\sum_{j=t+1}^{\infty} ja_j \leq |f'(z)| \leq 1 + r^t\sum_{j=t+1}^{\infty} ja_j$. ■

4 Radius of starlikeness and convexity

The radii of close-to-convexity, starlikeness and convexity for the class $\mathcal{RD}(\gamma,\lambda,\alpha,\beta)$ are given in this section.

Theorem 4.1 *Let the function $f \in \mathcal{T}$ belong to the class $\mathcal{RD}(\gamma,\lambda,\alpha,\beta)$, Then $f(z)$ is close -to-convex of order δ, $0 \leq \delta < 1$ in the disc $|z| < r$, where*

$$r := \inf_{j\geq t+1}\left[\frac{(1-\delta)\left[1+\mu(j-1)\right]\left[1+\beta(2\nu-1)\right]\left\{\gamma\left[1+(j-1)\lambda\right]^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{2\beta\nu j(1-\alpha)}\right]^{\frac{1}{t}}. \tag{4.1}$$

The result is sharp, with extremal function $f(z)$ given by (2.3).

Proof. For given $f \in \mathcal{T}$ we must show that

$$|f'(z) - 1| < 1 - \delta. \tag{4.2}$$

By a simple calculation we have $|f'(z)-1| \leq \sum_{j=t+1}^{\infty} ja_j |z|^t$. The last expression is less than $1-\delta$ if $\sum_{j=t+1}^{\infty} \frac{j}{1-\delta} a_j |z|^t < 1$. Using the fact that $f \in \mathcal{RD}(\gamma,\lambda,\alpha,\beta)$ if and only if $\sum_{j=t+1}^{\infty} \frac{[1+\mu(j-1)][1+\beta(2\nu-1)]\left\{\gamma[1+(j-1)\lambda]^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{2\beta\nu(1-\alpha)} a_j \leq 1$.

(4.2) holds true if $\frac{j}{1-\delta}|z|^t \leq \sum_{j=t+1}^{\infty} \frac{[1+\mu(j-1)][1+\beta(2\nu-1)]\left\{\gamma[1+(j-1)\lambda]^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{2\beta\nu(1-\alpha)}$. Or, equivalently, $|z|^t \leq \sum_{j=t+1}^{\infty} \frac{(1-\delta)[1+\mu(j-1)][1+\beta(2\nu-1)]\left\{\gamma[1+(j-1)\lambda]^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{2\beta\nu j(1-\alpha)}$, which completes the proof. ■

Theorem 4.2 *Let $f \in \mathcal{RD}(\gamma,\lambda,\alpha,\beta)$. Then*
1. f is starlike of order δ, $0 \leq \delta < 1$, in the disc $|z| < r_1$ where,
$r_1 = \inf_{j\geq t+1}\left\{\frac{(1-\delta)[1+\mu(j-1)][1+\beta(2\nu-1)]\left\{\gamma[1+(j-1)\lambda]^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{2\beta\nu(1-\alpha)(j-\delta)}\right\}^{\frac{1}{t}}$.
2. f is convex of order δ, $0 \leq \delta < 1$, in the disc $|z| < r_2$ where,
$r_2 = \inf_{j\geq t+1}\left\{\frac{(1-\delta)[1+\mu(j-1)][1+\beta(2\nu-1)]\left\{\gamma[1+(j-1)\lambda]^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{2\beta\nu j(j-1)(1-\alpha)}\right\}^{\frac{1}{t}}$.
Each of these results is sharp for the extremal function $f(z)$ given by (2.5).

Proof. 1. For $0 \leq \delta < 1$ we need to show that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 - \delta. \tag{4.3}$$

We have $\left|\frac{zf'(z)}{f(z)}-1\right| \leq \left|\frac{\sum_{j=t+1}^{\infty}(j-1)a_j|z|^t}{1-\sum_{j=t+1}^{\infty}a_j|z|^t}\right|$. The last expresion is less than $1-\delta$ if $\sum_{j=t+1}^{\infty}\frac{(j-\delta)}{1-\delta}a_j\left|z\right|^t<1$. Using the fact that $f\in\mathcal{RD}(\gamma,\lambda,\alpha,\beta)$ if and only if $\sum_{j=t+1}^{\infty}\frac{[1+\mu(j-1)][1+\beta(2\nu-1)]\left\{\gamma[1+(j-1)\lambda]^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{2\beta\nu(1-\alpha)}a_j<1$.

(4.3) holds true if $\frac{j-\delta}{1-\delta}\left|z\right|^t<\frac{[1+\mu(j-1)][1+\beta(2\nu-1)]\left\{\gamma[1+(j-1)\lambda]^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{2\beta\nu(1-\alpha)}$. Or, equivalently, $\left|z\right|^t<\frac{(1-\delta)[1+\mu(j-1)][1+\beta(2\nu-1)]\left\{\gamma[1+(j-1)\lambda]^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{2\beta\nu(1-\alpha)(j-\delta)}$, which yields the starlikeness of the family.

2. Using the fact that f is convex if and only zf' is starlike, we can prove (2) with a similar way of the proof of (1). The function f is convex if and only if

$$\left|zf''(z)\right|<1-\delta. \tag{4.4}$$

We have $\left|zf''(z)\right|\leq\left|\sum_{j=t+1}^{\infty}j(j-1)a_j\left|z\right|^{t-1}\right|<1-\delta$, equivalently with $\sum_{j=t+1}^{\infty}\frac{j(j-1)}{1-\delta}a_j\left|z\right|^{t-1}<1$. Using the fact that $f\in\mathcal{RD}(\gamma,\lambda,\alpha,\beta)$ if and only if $\sum_{j=t+1}^{\infty}\frac{[1+\mu(j-1)][1+\beta(2\nu-1)]\left\{\gamma[1+(j-1)\lambda]^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{2\beta\nu(1-\alpha)}a_j<1$.

(4.4) holds true if $\frac{j(j-1)}{1-\delta}\left|z\right|^{t-1}<\frac{[1+\mu(j-1)][1+\beta(2\nu-1)]\left\{\gamma[1+(j-1)\lambda]^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{2\beta\nu(1-\alpha)}$, or, equivalently, $\left|z\right|^{t-1}<\frac{(1-\delta)[1+\mu(j-1)][1+\beta(2\nu-1)]\left\{\gamma[1+(j-1)\lambda]^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{2\beta\nu j(j-1)(1-\alpha)}$, which yields the convexity of the family. ■

5 Neighborhood Property

In this section we study neighborhood property for functions in the class $\mathcal{RD}(\gamma,\lambda,\alpha,\beta)$.

Definition 5.1 *For functions f belong to $\mathcal{A}$ of the form and $\varepsilon\geq 0$, we define $\eta-\varepsilon$ - neighborhood of f by $N_{\varepsilon}^{\eta}(f)=\{g(z)\in\mathcal{A}: g(z)=z+\sum_{j=2}^{\infty}b_jz^j,\ \sum_{j=2}^{\infty}j^{\eta+1}\left|a_j-b_j\right|\leq\varepsilon\}$, where η is a fixed positive integer.*

By using the following lemmas we will investigate the $\eta-\varepsilon$ - neighborhood of function in $\mathcal{RD}(\gamma,\lambda,\alpha,\beta)$.

Lemma 5.1 *Let $-1\leq\beta<1$, if $g(z)=z+\sum_{j=2}^{\infty}b_jz^j$ satisfies $\sum_{j=2}^{\infty}j^{\rho+1}\left|b_j\right|\leq\frac{2\beta\nu(1-\alpha)}{1+\beta(2\nu-1)}$, then $g(z)\in\mathcal{RD}(\gamma,\lambda,\alpha,\beta)$.*

Proof. By using of Theorem 2.1, it is sufficient to show that $\frac{[1+\mu(j-1)][1+\beta(2\nu-1)]\left\{\gamma[1+(j-1)\lambda]^{\rho}+(1-\gamma)\frac{(\rho+j-1)!}{\rho!(j-1)!}\right\}}{2\beta\nu(1-\alpha)}=\frac{j^{\rho+1}}{2\beta\nu(1-\alpha)}[1+\beta(2\nu-1)]$. But $\frac{[1+\beta(2\nu-1)]\left\{\gamma[1+(j-1)\lambda]^{\rho}+(1-\gamma)\frac{(\rho+j-1)!}{\rho!(j-1)!}\right\}}{2\beta\nu(1-\alpha)}\leq\frac{j^{\rho+1}}{2\beta\nu(1-\alpha)}[1+\beta(2\nu-1)]$. Therefore it is enough to prove that $Q(j,\rho)=\frac{\gamma[1+(j-1)\lambda]^{\rho}+(1-\gamma)\frac{(\rho+j-1)!}{\rho!(j-1)!}}{j^{\rho+1}}\leq 1$, the result follows because the last inequality holds for all $j\geq t+1$. ■

Lemma 5.2 *Let $f(z)=z-\sum_{k=2}^{\infty}a_kz^k\in\mathcal{T}$, $\gamma,\lambda\geq 0$, $0\leq\alpha<1$, $0<\beta\leq 1$ and $\varepsilon\geq 0$. If $\frac{f(z)+\epsilon z}{1+\epsilon}\in\mathcal{RD}(\gamma,\lambda,\alpha,\beta)$, then $\sum_{j=t+1}^{\infty}j^{\rho+1}a_j\leq\frac{2\beta\nu(1-\alpha)(1+\epsilon)(t+1)^{\rho+1}}{[1+\mu(j-1)][1+\beta(2\nu-1)]\left[\gamma(1+t\lambda)^n+(1-\gamma)\frac{(t+n)!}{n!t!}\right]}$, where either $\rho=0$ or $\rho=1$. The result is sharp with the extremal function $f(z)=z-\frac{2\beta\nu(1-\alpha)(1+\epsilon)}{[1+\mu(j-1)][1+\beta(2\nu-1)]\left[\gamma(1+t\lambda)^n+(1-\gamma)\frac{(t+n)!}{n!t!}\right]}z^{t+1},\ \ z\in U$.*

Proof. Letting $g(z)=\frac{f(z)+\epsilon z}{1+\epsilon}$ we have $g(z)=z-\sum_{j=t+1}^{\infty}\frac{a_j}{1+\epsilon}z^j$, $z\in U$. In view of Theorem 2.3, $g(z)=\sum_{j=1}^{\infty}\eta_jg_j(z)$ where $\eta_j\geq 0$, $\sum_{j=1}^{\infty}\eta_j=1$, $g_1(z)=z$ and $g_j(z)=z-\frac{2\beta\nu(1-\alpha)(1+\epsilon)}{[1+\mu(j-1)][1+\beta(2\nu-1)]\left\{\gamma[1+(j-1)\lambda]^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}z^j$, $j\geq t+1$. So we obtain, $g(z)=\eta_1z+\sum_{j=t+1}^{\infty}\eta_j[z-\frac{2\beta\nu(1-\alpha)(1+\epsilon)}{[1+\mu(j-1)][1+\beta(2\nu-1)]\left\{\gamma[1+(j-1)\lambda]^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}z^j]=z-\sum_{j=t+1}^{\infty}\eta_k\frac{2\beta\nu(1-\alpha)(1+\epsilon)}{[1+\mu(j-1)][1+\beta(2\nu-1)]\left\{\gamma[1+(j-1)\lambda]^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}z^j$. Since $\eta_j\geq 0$ and $\sum_{j=2}^{\infty}\eta_j\leq 1$, it follows that $\sum_{j=t+1}^{\infty}a_k\leq\sup_{j\geq t+1}j^{\rho+1}\frac{2\beta\nu(1-\alpha)(1+\epsilon)}{[1+\mu(j-1)][1+\beta(2\nu-1)]\left\{\gamma[1+(j-1)\lambda]^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}$. Since whenever $\rho=0$ or $\rho=1$ we conclude $W(j,\rho,\gamma,\alpha,\beta,\epsilon)=j^{\rho+1}\frac{2\beta\nu(1-\alpha)(1+\epsilon)}{[1+\mu(j-1)][1+\beta(2\nu-1)]\left\{\gamma[1+(j-1)\lambda]^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}$, is a decreasing function of j, the result will follow. The proof is complete. ■

Theorem 5.1 *Let $\rho=0$ or $\rho=1$ and suppose $-1\leq\theta<\frac{[1+\mu(t-1)][1+\beta(2\nu-1)]\left[\gamma(1+t\lambda)^n+(1-\gamma)\frac{(t+n)!}{n!t!}\right]-2\beta\nu(1-\alpha)(1+\epsilon)(t+1)^{\eta+1}}{[1+\mu(t-1)][1+\beta(2\nu-1)]\left[\gamma(1+t\lambda)^n+(1-\gamma)\frac{(t+n)!}{n!t!}\right]}$, $0\leq\beta<1$, $f(z)\in\mathcal{T}$ and $\frac{f(z)+\epsilon z}{1+\epsilon}\in\mathcal{RD}(\gamma,\lambda,\alpha,\beta)$, then the $\eta-\varepsilon$-neighborhood of f is the subset of $\mathcal{RD}(\gamma,\lambda,\alpha,\beta)$, where $\varepsilon\leq\frac{2(1-\alpha)\left\{\theta\gamma[1+\mu(t-1)][1+\beta(2\nu-1)]\left[\gamma(1+t\lambda)^n+(1-\gamma)\frac{(t+n)!}{n!t!}\right]-\beta\gamma[1+\theta(2\nu-1)](1+\epsilon)(t+1)^{\eta+1}\right\}}{[1+\theta(2\nu-1)][1+\mu(t-1)][1+\beta(2\nu-1)]\left[\gamma(1+t\lambda)^n+(1-\gamma)\frac{(t+n)!}{n!t!}\right]}$. The result is sharp.*

Proof. For $f(z) = z - \sum_{j=2}^{\infty} |a_j| z^j$, let $g(z) = z + \sum_{j=2}^{\infty} b_j z^j$ be in $N_{\varepsilon}^{\eta}(f)$. So by Lemma 5.2, we have $\sum_{j=2}^{\infty} j^{\eta+1} |b_j| = \sum_{j=2}^{\infty} j^{\eta+1} |a_j - b_j - a_j| \leq \varepsilon + \frac{2\beta\nu(1-\alpha)(1+\epsilon)(t+1)^{\eta+1}}{[1+\mu(t-1)][1+\beta(2\nu-1)]\left[\gamma(1+t\lambda)^n+(1-\gamma)\frac{(t+n)!}{n!t!}\right]}$. By using Lemma 5.1, $g(z) \in \mathcal{RD}(\gamma, \lambda, \alpha, \beta)$ if $\varepsilon + \frac{2\beta\nu(1-\alpha)(1+\epsilon)(t+1)^{\eta+1}}{[1+\mu(t-1)][1+\beta(2\nu-1)]\left[\gamma(1+t\lambda)^n+(1-\gamma)\frac{(t+n)!}{n!t!}\right]} \leq \frac{2\theta\nu(1-\alpha)}{1+\theta(2\nu-1)}$, that is,
$\varepsilon \leq \frac{2(1-\alpha)\left\{\theta\gamma[1+\mu(t-1)][1+\beta(2\nu-1)]\left[\gamma(1+t\lambda)^n+(1-\gamma)\frac{(t+n)!}{n!t!}\right]-\beta\gamma[1+\theta(2\nu-1)](1+\epsilon)(t+1)^{\eta+1}\right\}}{[1+\theta(2\nu-1)][1+\mu(t-1)][1+\beta(2\nu-1)]\left[\gamma(1+t\lambda)^n+(1-\gamma)\frac{(t+n)!}{n!t!}\right]}$ and the proof is complete. ■

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On a certain subclass of analytic functions involving Sălăgean operator and Ruscheweyh derivative

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Abstract

The main object of this paper is to study some properties of certain subclass of analytic functions in the open unit disc which is defined by the linear operator $L_\alpha^n : \mathcal{A} \to \mathcal{A}$, $L_\alpha^n f(z) = (1-\alpha) R^n f(z) + \alpha S^n f(z)$, $z \in U$, where $R^n f(z)$ is the Ruscheweyh derivative, $S^n f(z)$ the Sălăgean operator and $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1} z^{n+1} + \dots,\ z \in U\}$ is the class of normalized analytic functions with $\mathcal{A}_1 = \mathcal{A}$. These properties include a coefficient inequality, distorsion theorem and extreme points of differential operator. We also discuss the boundedness properties associated with partial sums of functions in the class $\mathcal{TS}_\alpha^n(\beta,\gamma)$.

Keywords: analytic functions, coefficient inequalities, partial sums, starlike functions.
2000 Mathematical Subject Classification: 30C45, 30A20, 34A40.

1 Introduction

Denote by U the unit disc of the complex plane, $U = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}(U)$ the space of holomorphic functions in U.

Let $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1} z^{n+1} + \dots,\ z \in U\}$ with $\mathcal{A}_1 = \mathcal{A}$. Denote by $\mathcal{T}$ the subclass of $\mathcal{A}$ consisting the functions f of the form $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $z \in U$.

Definition 1.1 *(Sălăgean [8]) For $f \in \mathcal{A}$, $n \in \mathbb{N}$, the operator S^n is defined by $S^n : \mathcal{A} \to \mathcal{A}$,*

$$\begin{aligned} S^0 f(z) &= f(z) \\ S^1 f(z) &= z f'(z), \ \dots \\ S^{n+1} f(z) &= z \left(S^n f(z)\right)', \quad z \in U. \end{aligned}$$

Remark 1.1 *If $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $S^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j$, $z \in U$.*
If $f \in \mathcal{T}$, $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, then $S^n f(z) = z - \sum_{j=2}^{\infty} j^n a_j z^j$, $z \in U$.

Definition 1.2 *(Ruscheweyh [7]) For $f \in \mathcal{A}$, $n \in \mathbb{N}$, the operator R^n is defined by $R^n : \mathcal{A} \to \mathcal{A}$,*

$$\begin{aligned} R^0 f(z) &= f(z) \\ R^1 f(z) &= z f'(z), \ \dots \\ (n+1) R^{n+1} f(z) &= z \left(R^n f(z)\right)' + n R^n f(z), \quad z \in U. \end{aligned}$$

Remark 1.2 *If $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $R^n f(z) = z + \sum_{j=2}^{\infty} \frac{(n+j-1)!}{n!(j-1)!} a_j z^j$, $z \in U$.*
If $f \in \mathcal{T}$, $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, then $R^n f(z) = z - \sum_{j=2}^{\infty} \frac{(n+j-1)!}{n!(j-1)!} a_j z^j$, $z \in U$.

Definition 1.3 *[2] Let $\alpha \geq 0$, $n \in \mathbb{N}$. Denote by L_α^n the operator given by $L_\alpha^n : \mathcal{A} \to \mathcal{A}$,*

$$L_\alpha^n f(z) = (1-\alpha) R^n f(z) + \alpha S^n f(z), \qquad z \in U.$$

Remark 1.3 *If $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $L_\alpha^n f(z) = z + \sum_{j=2}^{\infty} \left(\alpha j^n + (1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right) a_j z^j$, $z \in U$.*

If $f \in \mathcal{T}$, $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, then $L_\alpha^n f(z) = z - \sum_{j=2}^{\infty} \left(\alpha j^n + (1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right) a_j z^j$, $z \in U$.

This operator was studied also in [3], [4], [5].

If f and g are analytic functions in U, we say that f is subordinate to g, written $f \prec g$, if there is a function w analytic in U, with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$, such that $f(z) = g(w(z))$ for all $z \in U$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

We follow the works of A. Abubaker and M. Darus [1].

Definition 1.4 *Let $0 < \beta \le 1$, $\alpha, \lambda \ge 0$, $n \in \mathbb{N}$, $\gamma \in \mathbb{C}\backslash\{0\}$. Then, the function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}_\alpha^n(\beta,\gamma)$ if*

$$\left|\frac{1}{\gamma}\left(\frac{z\left(L_\alpha^n f(z)\right)'}{L_\alpha^n f(z)} - 1\right)\right| < \beta, \quad z \in U.$$

We define now the class $\mathcal{TS}_\alpha^n(\beta,\gamma)$ by

$$\mathcal{TS}_\alpha^n(\beta,\gamma) = \mathcal{S}_\alpha^n(\beta,\gamma) \cap \mathcal{T}.$$

2 Coefficient Inequality

Theorem 2.1 *Let the function $f \in \mathcal{T}$. Then $f(z)$ is in the class $\mathcal{TS}_\alpha^n(\beta,\gamma)$ if and only if*

$$\sum_{j=2}^{\infty}(j-1+\beta|\gamma|)\left\{\alpha j^n + (1-\alpha)\frac{(n+j-1)!}{n!\,(j-1)!}\right\} a_j \le \beta|\gamma|, \tag{2.1}$$

where $\gamma \in \mathbb{C} - \{0\}$, $\alpha \ge 0$, $n \in \mathbb{N}$, $z \in U$.

Proof. Let $f(z) \in \mathcal{TS}_\alpha^n(\beta,\gamma)$. Then, we have $Re\{\frac{z(L_\alpha^n f(z))'}{L_\alpha^n f(z)} - 1|\} > -\beta|\gamma|$, where $\frac{z(L_\alpha^n f(z))'}{L_\alpha^n f(z)} = \frac{z - \sum_{j=2}^{\infty} j\left\{\alpha j^n + (1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_j z^j}{z - \sum_{j=2}^{\infty}\left\{\alpha j^n + (1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_j z^j}$ and we obtain $Re\left\{\frac{-\sum_{j=2}^{\infty}(j-1)\left\{\alpha j^n + (1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_j z^j}{z - \sum_{j=2}^{\infty}\left\{\alpha j^n + (1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_j z^j}\right\} > -\beta\,|\gamma|$. By chossing the valkues of z on the real ais and letting $z \to 1^-$ through real values, the above inequality immediately yields the required condition (2.1).

Conversely, by applying the hypothesis (2.1) and letting $|z| = 1$, we obtain $\left|\frac{z(L_\alpha^n f(z))'}{L_\alpha^n f(z)} - 1\right| = \left|\frac{\sum_{j=2}^{\infty}(j-1)\left\{\alpha j^n + (1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_j z^j}{z - \sum_{j=2}^{\infty}\left\{\alpha j^n + (1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_j z^j}\right| \le \frac{\beta|\gamma|\left(1 - \sum_{j=2}^{\infty}\left\{\alpha j^n + (1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_j\right)}{1 - \sum_{j=2}^{\infty}\left\{\alpha j^n + (1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_j} \le \beta|\gamma|$. Hence, by maximum modulus theorem, we have $f \in \mathcal{TS}_\alpha^n(\beta,\gamma)$, which evidently completes the proof of theorem.

Finally, the result is sharp with the extremal functions f_j be in the class $\mathcal{TS}_\alpha^n(\beta,\gamma)$ given by

$$f_j(z) = z - \frac{\beta\,|\gamma|}{(j-1+\beta|\gamma|)\left\{\alpha\left[1+(j-1)\lambda\right]^n + (1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}} z^j, \quad \text{for } j \ge 2. \tag{2.2}$$

■

Corollary 2.2 *Let the function $f \in \mathcal{T}$ be in the class $\mathcal{TS}_{\lambda,\alpha}^n(\beta,\gamma)$. Then, we have*

$$a_j \le \frac{\beta|\gamma|}{(j-1+\beta|\gamma|)\left\{\alpha j^n + (1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}}, \text{ for } j \ge 2. \tag{2.3}$$

The equality is attained for the functions f given by (2.2).

3 Growth and distortion theorems

Theorem 3.1 *Let the function $f \in \mathcal{T}$ be in the class $\mathcal{TS}_\alpha^n(\beta,\gamma)$. Then for $|z| = r$, we have*

$$r - \frac{\beta|\gamma|}{(1+\beta|\gamma|)\left[2^n\alpha + (1-\alpha)(n+1)\right]}r^2 \le |f(z)| \le r + \frac{\beta|\gamma|}{(1+\beta|\gamma|)\left[2^n\alpha + (1-\alpha)(n+1)\right]}r^2.$$

with equality for $f(z) = z - \frac{\beta|\gamma|}{(1+\beta|\gamma|)[2^n\alpha+(1-\alpha)(n+1)]}z^2$.

Proof. In view of Theorem 2.1, we have
$(1+\beta|\gamma|)\left[2^n\alpha+(1-\alpha)(n+1)\right]\sum_{j=2}^{\infty}a_j \le \sum_{j=2}^{\infty}(j-1+\beta|\gamma|)\left[2^n\alpha+(1-\alpha)(n+1)\right]a_j \le \beta|\gamma|$.
Hence $|f(z)| \le r+\sum_{j=2}^{\infty}a_jr^j \le r+r^2\sum_{j=2}^{\infty}a_j \le r+\frac{\beta|\gamma|}{(1+\beta|\gamma|)[2^n\alpha+(1-\alpha)(n+1)]}r^2$ and $|f(z)| \ge r-\sum_{j=2}^{\infty}a_jr^j \ge r-r^2\sum_{j=2}^{\infty}a_j \ge r-\frac{\beta|\gamma|}{(1+\beta|\gamma|)[2^n\alpha+(1-\alpha)(n+1)]}r^2$. This completes the proof. ■

Theorem 3.2 *Let the function $f\in\mathcal{T}$ be in the class $\mathcal{TS}_{\alpha}^{n}(\beta,\gamma)$. Then, for $|z|=r$ we have*

$$r-\frac{2\beta|\gamma|}{(1+\beta|\gamma|)\left[2^n\alpha+(1-\alpha)(n+1)\right]}r^2 \le |f'(z)| \le r+\frac{2\beta|\gamma|}{(1+\beta|\gamma|)\left[2^n\alpha+(1-\alpha)(n+1)\right]}r^2$$

with equality for $f(z)=z-\frac{\beta|\gamma|}{(1+\beta|\gamma|)[2^n\alpha+(1-\alpha)(n+1)]}z^2$.

Proof. We have $|f'(z)| \le r+\sum_{j=2}^{\infty}ja_jr^j \le r+2r^2\sum_{j=2}^{\infty}a_j \le r+\frac{2\beta|\gamma|}{(1+\beta|\gamma|)[2^n\alpha+(1-\alpha)(n+1)]}r^2$ and $|f'(z)| \ge r-\sum_{j=2}^{\infty}ja_jr^j \ge r-2r^2\sum_{j=2}^{\infty}a_j \ge r-\frac{2\beta|\gamma|}{(1+\beta|\gamma|)[2^n\alpha+(1-\alpha)(n+1)]}r^2$, which completes the proof. ■

4 Extreme points

The extreme points of the class $\mathcal{TS}_{\alpha}^{n}(\beta,\gamma)$ will be now determined.

Theorem 4.1 *Let $f_1(z)=z$ and $f_j(z)=z-\frac{\beta|\gamma|}{(j-1+\beta|\gamma|)\left\{\alpha j^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}}z^j$, $j\ge 2$. Assume that f is analytic in U. Then $f\in\mathcal{TS}_{\alpha}^{n}(\beta,\gamma)$ if and only if it can be expressed in the form $f(z)=\sum_{j=1}^{\infty}\mu_jf_j(z)$, where $\mu_j\ge 0$ and $\sum_{j=1}^{\infty}\mu_j=1$.*

Proof. Suppose that $f(z)=\sum_{j=1}^{\infty}\mu_jf_j(z)$ with $\mu_j\ge 0$ and $\sum_{j=1}^{\infty}\mu_j=1$. Then
$f(z)=\sum_{j=1}^{\infty}\mu_jf_j(z)=\mu_1f_1(z)+\sum_{j=2}^{\infty}\mu_jf_j(z)=\mu_1z+\sum_{j=2}^{\infty}\mu_j\left(z-\frac{\beta|\gamma|}{(j-1+\beta|\gamma|)\left\{\alpha j^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}}z^j\right)=z-\sum_{j=2}^{\infty}\mu_j\frac{\beta|\gamma|}{(j-1+\beta|\gamma|)\left\{\alpha j^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}}z^j$. Then $\sum_{j=2}^{\infty}\mu_j\frac{\beta|\gamma|}{(j-1+\beta|\gamma|)\left\{\alpha j^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}}\frac{(j-1+\beta|\gamma|)\left\{\alpha j^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{\beta|\gamma|}=\sum_{j=2}^{\infty}\mu_j=\sum_{j=1}^{\infty}\mu_j-\mu_1=1-\mu_1\le 1$. Thus $f\in\mathcal{TS}_{\alpha}^{n}(\beta,\gamma)$ by Theorem 2.1.

Conversely, suppose that $f\in\mathcal{TS}_{\alpha}^{n}(\beta,\gamma)$. By using (2.3) we may set and $\mu_j=\frac{(j-1+\beta|\gamma|)\left\{\alpha j^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{\beta|\gamma|}a_j$ for $j\ge 2$ and $\mu_1=1-\sum_{j=2}^{\infty}\mu_j$.

Then $f(z)=z-\sum_{j=1}^{\infty}a_jz^j=z-\sum_{j=2}^{\infty}\mu_j\frac{\beta|\gamma|}{(j-1+\beta|\gamma|)\left\{\alpha j^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}}z^j=\mu_1f_1(z)+\sum_{j=2}^{\infty}\mu_jf_j(z)=\sum_{j=1}^{\infty}\mu_jf_j(z)$, with $\mu_j\ge 0$ and $\sum_{j=1}^{\infty}\mu_j=1$, which completes the proof. ■

Corollary 4.2 *The extreme points of $\mathcal{TS}_{\alpha}^{n}(\beta,\gamma)$ are the functions $f_1(z)=z$ and $f_j(z)=\frac{\beta|\gamma|}{(j-1+\beta|\gamma|)\left\{\alpha j^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}}z^j$, for $j\ge 2$.*

5 Partial sums

We investigate in this section the partial sums of functions in the class $\mathcal{TS}_{\alpha}^{n}(\beta,\gamma)$. We shall obtain sharp lower bounds for the real part of its ratios. We shall follow similar works done by Silverman [9] and Khairnar and Moreon [6] about the partial sums of analytic functions. In what follows, we will use the well known result that for an analytic function ω in U, $\mathrm{Re}\left(\frac{1+\omega(z)}{1-\omega(z)}\right)>0$, $z\in U$, if and only if the inequality $|\omega(z)|<1$ is satisfied.

Theorem 5.1 *Let $f(z)=z-\sum_{j=2}^{\infty}a_jz^j\in\mathcal{TS}_{\alpha}^{n}(\beta,\gamma)$ and define its partial sums by $f_1(z)=z$ and $f_m(z)=z-\sum_{j=2}^{m}a_jz^j$, $m\ge 2$. Then*

$$\mathrm{Re}\left(\frac{f(z)}{f_m(z)}\right)\ge 1-\frac{1}{c_{m+1}},\quad z\in U,\ m\in\mathbb{N}, \tag{5.1}$$

and

$$\mathrm{Re}\left(\frac{f_m(z)}{f(z)}\right)\ge\frac{c_{m+1}}{1+c_{m+1}},\quad z\in U,\ m\in\mathbb{N}, \tag{5.2}$$

where

$$c_j = \frac{(j + \beta|\gamma|) \left\{ \alpha j^n + (1-\alpha) \frac{(n+j-1)!}{n!(j-1)!} \right\}}{\beta|\gamma|} \tag{5.3}$$

This estimates in (5.1) and (5.2) are sharp.

Proof. To prove (5.1), it suffices to show that $c_{m+1}\left(\frac{f(z)}{f_m(z)} - \left(1 - \frac{1}{c_{m+1}}\right)\right) \prec \frac{1+z}{1-z}$, $z \in U$. By the subordination property, we can write $\frac{1-\sum_{j=2}^{m} a_j z^{j-1} - c_{m+1}\sum_{j=m+1}^{\infty} a_j z^{j-1}}{1-\sum_{j=2}^{m} a_j z^{j-1}} = \frac{1+\omega(z)}{1-\omega(z)}$ for certain ω analytic in U with $|\omega(z)| \leq 1$. Notice that $\omega(0) = 0$ and $|\omega(z)| \leq \frac{c_{m+1}\sum_{j=m+1}^{\infty} a_j}{2-2\sum_{j=2}^{m} a_j - c_{m+1}\sum_{j=m+1}^{\infty} a_j}$. Now $|\omega(z)| \leq 1$ if and only if $\sum_{j=2}^{m} a_j + c_{m+1}\sum_{j=m+1}^{\infty} a_j \leq \sum_{j=2}^{\infty} c_j a_j \leq 1$. The above inequality holds because c_j is a non-decreasing sequence. This completes the proof of (5.1). Finally, it is observed that equality in (5.1) is attained for the function given by (2.3) when $z = re^{2\frac{\pi i}{m}}$ as $r \to 1^-$.

Similarly, we take $(1 + c_{m+1})\left(\frac{f_m(z)}{f(z)} - \frac{c_{m+1}}{1+c_{m+1}}\right) = \frac{1-\sum_{j=2}^{m} a_j z^{j-1} + c_{m+1}\sum_{j=m+1}^{\infty} a_j z^{j-1}}{1-\sum_{j=2}^{m} a_j z^{j-1}} = \frac{1+\omega(z)}{1-\omega(z)}$, where $|\omega(z)| \leq \frac{(1+c_{m+1})\sum_{j=m+1}^{\infty} a_j}{2-2\sum_{j=2}^{m} a_j + (1-c_{m+1})\sum_{j=m+1}^{\infty} a_j}$. Now $|\omega(z)| \leq 1$ if and only if $\sum_{j=2}^{m} a_j + c_{m+1}\sum_{j=m+1}^{\infty} a_j \leq \sum_{j=2}^{\infty} c_j a_j \leq 1$. This immediately leads to assertion (5.2) of Theorem 5.1. This completes the proof. ■

Using a similar method, we can prove the following theorem.

Theorem 5.2 *If $f \in \mathcal{TS}_\alpha^n(\beta, \gamma)$ and define the partial sums by $f_1(z) = z$ and $f_m(z) = z - \sum_{j=2}^{m} a_j z^j$. Then* $\operatorname{Re}\left(\frac{f'(z)}{f_m'(z)}\right) \geq 1 - \frac{m+1}{c_{m+1}}$, *$z \in U$, $m \in \mathbb{N}$, and* $\operatorname{Re}\left(\frac{f_m'(z)}{f(z)}\right) \geq \frac{c_{m+1}}{m+1+c_{m+1}}$, *where c_j is given by (5.3). The result is sharp for every m, with extremal functions given by (2.2).*

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New classes containing multiplier transformation and Ruscheweyh derivative

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Abstract

In this paper we introduce new classes containig the linear operator $RI_{n,\lambda,l}^{\alpha}:\mathcal{A}\to\mathcal{A}$, $RI_{n,\lambda,l}^{\alpha}f(z)=(1-\alpha)R^nf(z)+\alpha I(n,\lambda,l)f(z)$, $z\in U$, where $R^nf(z)$ is the Ruscheweyh derivative, $I(n,\lambda,l)f(z)$ the multiplier transformation and $\mathcal{A}_n=\{f\in\mathcal{H}(U):f(z)=z+a_{n+1}z^{n+1}+\ldots,\ z\in U\}$ is the class of normalized analytic functions with $\mathcal{A}_1=\mathcal{A}$. Characterization and other properties of these classes are studied.

Keywords: differential operator, distortion theorem.
2000 Mathematical Subject Classification: 30C45, 30A20, 34A40.

1 Introduction

Denote by U the unit disc of the complex plane, $U=\{z\in\mathbb{C}:|z|<1\}$ and $\mathcal{H}(U)$ the space of holomorphic functions in U.

Let $\mathcal{A}_n=\{f\in\mathcal{H}(U):f(z)=z+a_{n+1}z^{n+1}+\ldots,\ z\in U\}$ with $\mathcal{A}_1=\mathcal{A}$.

Definition 1.1 *(Ruscheweyh [20]) For $f\in\mathcal{A}$, $n\in\mathbb{N}$, the operator R^n is defined by $R^n:\mathcal{A}\to\mathcal{A}$,*

$$\begin{aligned} R^0f(z) &= f(z)\\ R^1f(z) &= zf'(z),\ \ldots\\ (n+1)R^{n+1}f(z) &= z\left(R^nf(z)\right)'+nR^nf(z),\quad z\in U. \end{aligned}$$

Remark 1.1 *If $f\in\mathcal{A}$, $f(z)=z+\sum_{j=2}^{\infty}a_jz^j$, then $R^nf(z)=z+\sum_{j=2}^{\infty}\frac{(n+j-1)!}{n!(j-1)!}a_jz^j$, $z\in U$.*

Definition 1.2 *For $f\in\mathcal{A}$, $n\in\mathbb{N}$, $\lambda,l\geq 0$, the operator $I(n,\lambda,l)f(z)$ is defined by the following infinite series*

$$I(n,\lambda,l)f(z)=z+\sum_{j=2}^{\infty}\left(\frac{\lambda(j-1)+l+1}{l+1}\right)^n a_jz^j.$$

Remark 1.2 *It follows from the above definition that*

$$I(0,\lambda,l)f(z)=f(z),$$

$$(l+1)I(n+1,\lambda,l)f(z)=(l+1-\lambda)I(n,\lambda,l)f(z)+\lambda z\left(I(n,\lambda,l)f(z)\right)',\quad z\in U.$$

Remark 1.3 *For $l=0$, $\lambda\geq 0$, the operator $D_\lambda^n=I(n,\lambda,0)$ was introduced and studied by Al-Oboudi [16], which is reduced to the Sălăgean differential operator [21] for $\lambda=1$.*

Definition 1.3 *[7] Let $\alpha,\lambda,l\geq 0$, $n\in\mathbb{N}$. Denote by $RI_{n,\lambda,l}^{\alpha}$ the operator given by $RI_{n,\lambda,l}^{\alpha}:\mathcal{A}\to\mathcal{A}$,*

$$RI_{n,\lambda,l}^{\alpha}f(z)=(1-\alpha)R^nf(z)+\alpha I(n,\lambda,l)f(z),\ z\in U.$$

Remark 1.4 *If $f\in\mathcal{A}$, $f(z)=z+\sum_{j=2}^{\infty}a_jz^j$, then*
$RI_{n,\lambda,l}^{\alpha}f(z)=z+\sum_{j=2}^{\infty}\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_jz^j$, $z\in U$.

This operator was studied also in [13], [14].

Remark 1.5 *For $\alpha = 0$, $RI^0_{m,\lambda,l}f(z) = R^m f(z)$, where $z \in U$ and for $\alpha = 1$, $RI^1_{m,\lambda,l}f(z) = I(m,\lambda,l)f(z)$, where $z \in U$, which was studied in [3], [4], [10], [9]. For $l = 0$, we obtain $RI^\alpha_{m,\lambda,0}f(z) = RD^m_{\lambda,\alpha}f(z)$ which was studied in [5], [6], [11], [12], [17], [18] and for $l = 0$ and $\lambda = 1$, we obtain $RI^\alpha_{m,1,0}f(z) = L^m_\alpha f(z)$ which was studied in [1], [2], [8], [15].*

Definition 1.4 *Let $f \in \mathcal{A}$. Then $f(z)$ is in the class $\mathcal{S}^n_{\lambda,l,\alpha}(\mu)$ if and only if*

$$\operatorname{Re}\left(\frac{z\left(RI^\alpha_{n,\lambda,l}f(z)\right)'}{RI^\alpha_{n,\lambda,l}f(z)}\right) > \mu, \ \ 0 \le \mu < 1, \ \ z \in U.$$

Definition 1.5 *Let $f \in \mathcal{A}$. Then $f(z)$ is in the class $\mathcal{C}^n_{\lambda,l,\alpha}(\mu)$ if and only if*

$$\operatorname{Re}\left(\frac{\left[z\left(RI^\alpha_{n,\lambda,l}f(z)\right)'\right]'}{\left(RI^\alpha_{n,\lambda,l}f(z)\right)'}\right) > \mu, \ \ 0 \le \mu < 1, \ \ z \in U.$$

We study the charaterization and distortion theorems, and other properties of these classes, following the paper of M. Darus and R. Ibrahim [19].

2 General properties of $RD^n_{\lambda,\alpha}$

In this section we study the characterization properties and distortion theorems for the function $f(z) \in \mathcal{A}$ to belong to the classes $\mathcal{S}^n_{\lambda,l,\alpha}(\mu)$ and $\mathcal{C}^n_{\lambda,l,\alpha}(\mu)$ by obtaining the coefficient bounds.

Theorem 2.1 *Let $f \in \mathcal{A}$. If*

$$\sum_{j=2}^{\infty}(j-\mu)\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n + (1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j| \le 1-\mu, \quad 0 \le \mu < 1, \tag{2.1}$$

then $f(z) \in \mathcal{S}^n_{\lambda,l,\alpha}(\mu)$. The result (2.1) is sharp.

Proof. Suppose that (2.1) holds. Since $1-\mu \ge \sum_{j=2}^\infty (j-\mu)\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n + (1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j|$ $\ge \mu\sum_{j=2}^\infty \left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n + (1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j| - \sum_{j=2}^\infty j\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n + (1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j|$, then this implies that $\frac{1+\sum_{j=2}^\infty j\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j|}{1+\sum_{j=2}^\infty\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j|} > \mu$. So, we deduce that $\operatorname{Re}\left(\frac{z\left(RI^\alpha_{n,\lambda,l}f(z)\right)'}{RI^\alpha_{n,\lambda,l}f(z)}\right) > \mu$, $0 \le \mu < 1$, $z \in U$. We have $f(z) \in \mathcal{S}^n_{\lambda,l,\alpha}(\mu)$, which evidently completes the proof.

The assertion (2.1) is sharp and the extremal function is given by $f(z) = z + \sum_{j=2}^\infty \frac{(1-\mu)}{(j-\mu)\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}}z^j$. ∎

Corollary 2.2 *Let the hypotheses of Theorem 2.1 satisfy. Then $|a_j| \le \frac{1-\mu}{(j-\mu)\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}}$, $\forall\, j \ge 2$.*

Theorem 2.3 *Let $f \in \mathcal{A}$. If*

$$\sum_{j=2}^{\infty}j(j-\mu)\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n + (1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j| \le 1-\mu, \quad 0 \le \mu < 1, \tag{2.2}$$

then $f(z) \in \mathcal{C}^n_{\lambda,l,\alpha}(\mu)$. The result (2.2) is sharp.

Proof. Suppose that (2.2) holds. Since $1-\mu \ge \sum_{j=2}^\infty j(j-\mu)\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n + (1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j| \ge$ $\mu\sum_{j=2}^\infty j\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n + (1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j| - \sum_{j=2}^\infty j^2\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n + (1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j|$ then this

implies that $\frac{1+\sum_{j=2}^{\infty} j^2\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j|}{1+\sum_{j=2}^{\infty} j\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j|} > \mu$. So, we deduce that $\operatorname{Re}\left(\frac{\left[z\left(RI_{n,\lambda,l}^{\alpha} f(z)\right)'\right]'}{\left(RI_{n,\lambda,l}^{\alpha} f(z)\right)'}\right) > \mu$, $0 \le \mu < 1$, $z \in U$. We have $f(z) \in \mathcal{C}_{\lambda,l,\alpha}^n(\mu)$, which evidently completes the proof.

The assertion (2.2) is sharp and the extremal function is given by
$f(z) = z + \sum_{j=2}^{\infty} \frac{(1-\mu)}{j(j-\mu)\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}} z^j$. ■

Corollary 2.4 *Let the hypotheses of Theorem 2.3 be satisfied. Then* $|a_j| \le \frac{1-\mu}{j(j-\mu)\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}}$, $\forall$ $j \ge 2$.

Also, we have the following inclusion results:

Theorem 2.5 *Let* $0 \le \mu_1 \le \mu_2 < 1$. *Then* $\mathcal{S}_{\lambda,l,\alpha}^n(\mu_1) \supseteq \mathcal{S}_{\lambda,l,\alpha}^n(\mu_2)$.

Proof. By Theorem 2.1. ■

Theorem 2.6 *Let* $0 \le \mu_1 \le \mu_2 < 1$. *Then* $\mathcal{C}_{\lambda,l,\alpha}^n(\mu_1) \supseteq \mathcal{C}_{\lambda,l,\alpha}^n(\mu_2)$.

Proof. By Theorem 2.3. ■

We introduce the following distortion theorems.

Theorem 2.7 *Let the function* $f \in \mathcal{A}$ *and* $\sum_{j=2}^{\infty}(j-\mu)\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j| \le 1-\mu$, $0 \le \mu < 1$. *Then for* $z \in U$ *and* $0 \le \mu < 1$, $\left|RI_{n,\lambda,l}^{\alpha} f(z)\right| \ge |z| - \frac{1-\mu}{2-\mu}|z|^2$ *and* $\left|RI_{n,\lambda,l}^{\alpha} f(z)\right| \le |z| + \frac{1-\mu}{2-\mu}|z|^2$.

Proof. By using Theorem 2.1, one can verify that $(2-\mu)\sum_{j=2}^{\infty}\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j| \le \sum_{j=2}^{\infty}(j-\mu)\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j| \le 1-\mu$.

Hence, $\sum_{j=2}^{\infty}\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j| \le \frac{1-\mu}{2-\mu}$. We obtain
$\left|RI_{n,\lambda,l}^{\alpha} f(z)\right| = \left|z+\sum_{j=2}^{\infty}\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_j z^j\right| \le$
$|z|+\sum_{j=2}^{\infty}\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j||z|^j \le |z|+\sum_{j=2}^{\infty}\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j||z|^2$
$\le |z| + \frac{1-\mu}{2-\mu}|z|^2$. The other assertion can be proved as follows
$\left|RI_{n,\lambda,l}^{\alpha} f(z)\right| = \left|z+\sum_{j=2}^{\infty}\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_j z^j\right| \ge$
$|z|-\sum_{j=2}^{\infty}\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j||z|^j \ge |z|-\sum_{j=2}^{\infty}\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j||z|^2$
$\ge |z| - \frac{1-\mu}{2-\mu}|z|^2$. This completes the proof. ■

Theorem 2.8 *Let the function* $f \in \mathcal{A}$ *and* $\sum_{j=2}^{\infty} j(j-\mu)\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j| \le 1-\mu$, $0 \le \mu < 1$. *Then for* $z \in U$ *and* $0 \le \mu < 1$, $\left|RI_{n,\lambda,l}^{\alpha} f(z)\right| \ge |z| - \frac{1-\mu}{2(2-\mu)}|z|^2$ *and* $\left|RI_{n,\lambda,l}^{\alpha} f(z)\right| \le |z| + \frac{1-\mu}{2(2-\mu)}|z|^2$.

Proof. By using Theorem 2.3, one can verify that $2(2-\mu)\sum_{j=2}^{\infty}\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j| \le \sum_{j=2}^{\infty} j(j-\mu)\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j| \le 1-\mu$.

Hence, $\sum_{j=2}^{\infty}\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j| \le \frac{1-\mu}{2(2-\mu)}$. We obtain
$\left|RI_{n,\lambda,l}^{\alpha} f(z)\right| = \left|z+\sum_{j=2}^{\infty}\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_j z^j\right| \le$
$|z|+\sum_{j=2}^{\infty}\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j||z|^j \le |z|+\sum_{j=2}^{\infty}\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j||z|^2$
$\le |z| + \frac{1-\mu}{2(2-\mu)}|z|^2$. The other assertion can be proved as follows
$\left|RI_{n,\lambda,l}^{\alpha} f(z)\right| = \left|z+\sum_{j=2}^{\infty}\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_j z^j\right| \ge$
$|z|-\sum_{j=2}^{\infty}\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j||z|^j \ge |z|-\sum_{j=2}^{\infty}\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j||z|^2$
$\ge |z| - \frac{1-\mu}{2(2-\mu)}|z|^2$. This completes the proof. ■

Also, we have the following distortion results.

Theorem 2.9 *Let the hypotheses of Theorem 2.1 be satisfied. Then* $|f(z)| \geq |z| - \frac{1-\mu}{(2-\mu)\left[\alpha\left(\frac{1+\lambda+l}{l+1}\right)^n+(1-\alpha)(n+1)\right]}|z|^2$ *and* $|f(z)| \leq |z| + \frac{1-\mu}{(2-\mu)\left[\alpha\left(\frac{1+\lambda+l}{l+1}\right)^n+(1-\alpha)(n+1)\right]}|z|^2$.

Proof. In virtue of Theorem 2.1, we have $(2-\mu)\left[\alpha\left(\frac{1+\lambda+l}{l+1}\right)^n+(1-\alpha)(n+1)\right]\sum_{j=2}^{\infty}|a_j| \leq \sum_{j=2}^{\infty}(j-\mu)\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j| \leq 1-\mu$, thus, $\sum_{j=2}^{\infty}|a_j| \leq \frac{1-\mu}{(2-\mu)\left[\alpha\left(\frac{1+\lambda+l}{l+1}\right)^n+(1-\alpha)(n+1)\right]}$. We obtain $|f(z)| = \left|z+\sum_{j=2}^{\infty}a_j z^j\right| \leq |z| + \sum_{j=2}^{\infty}|a_j||z|^2 \leq |z| + \frac{1-\mu}{(2-\mu)\left[\alpha\left(\frac{1+\lambda+l}{l+1}\right)^n+(1-\alpha)(n+1)\right]}|z|^2$. The other assertion can be proved as follows $|f(z)| \geq |z| - \sum_{j=2}^{\infty}|a_j||z|^2 \geq |z| - \frac{1-\mu}{(2-\mu)\left[\alpha\left(\frac{1+\lambda+l}{l+1}\right)^n+(1-\alpha)(n+1)\right]}|z|^2$. This completes the proof. ■

In the same way we can get the following result.

Theorem 2.10 *Let the hypotheses of Theorem 2.3 be satisfied. Then* $(j-\mu)\left\{\alpha\left[1+(j-1)\lambda\right]^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\} \geq 0$ *and* $0 \leq \mu < 1$ *poses* $|f(z)| \geq |z| - \frac{1-\mu}{2(2-\mu)\left[\alpha\left(\frac{1+\lambda+l}{l+1}\right)^n+(1-\alpha)(n+1)\right]}|z|^2$ *and* $|f(z)| \leq |z| + \frac{1-\mu}{2(2-\mu)\left[\alpha\left(\frac{1+\lambda+l}{l+1}\right)^n+(1-\alpha)(n+1)\right]}|z|^2$.

Proof. In virtue of Theorem 2.3, we have $2(2-\mu)\left[\alpha\left(\frac{1+\lambda+l}{l+1}\right)^n+(1-\alpha)(n+1)\right]\sum_{j=2}^{\infty}|a_j| \leq \sum_{j=2}^{\infty}j(j-\mu)\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j| \leq 1-\mu$, thus, $\sum_{j=2}^{\infty}|a_j| \leq \frac{1-\mu}{2(2-\mu)\left[\alpha\left(\frac{1+\lambda+l}{l+1}\right)^n+(1-\alpha)(n+1)\right]}$. We obtain $|f(z)| = \left|z+\sum_{j=2}^{\infty}a_j z^j\right| \leq |z| + \sum_{j=2}^{\infty}|a_j||z|^2 \leq |z| + \frac{1-\mu}{2(2-\mu)\left[\alpha\left(\frac{1+\lambda+l}{l+1}\right)^n+(1-\alpha)(n+1)\right]}|z|^2$. The other assertion can be proved as follows $/|f(z)| = \left|z+\sum_{j=2}^{\infty}a_j z^j\right| \geq |z| - \sum_{j=2}^{\infty}|a_j||z|^2 \geq |z| - \frac{1-\mu}{2(2-\mu)\left[\alpha\left(\frac{1+\lambda+l}{l+1}\right)^n+(1-\alpha)(n+1)\right]}|z|^2$. This completes the proof. ■

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NEW ITERATIVE SCHEME FOR THE APPROXIMATION OF FIXED POINTS OF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN UNIFORMLY CONVEX BANACH SPACES

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Abstract. We prove the existence of a fixed point for asymtotically nonexpensive mappings defined on a unifromly convex metric space. An modified two-stepIshikawa type iterative scheme is constructed which converges to the fixed point.

2010 Mathematics Subject Classification: 47J25.

Key words and phrases: Iteration process, asymptotically nonexpansive mappings, unifromly convex metric spaces.

1. Introduction

Browder [6], Göhde [11] and Kirk [16] extended Banach contraction principle to nonexpansive mappings on their own. A valuable fixed point theory for the class of nonexpensive mappings exists (e.g., see [8], [11]). Kirk [16] pointed out that there exists a L-Lipschitzian mapping which has no fixed point. Later Goebel and Kirk [10] introduced the notion of asymptotically nonexpensive

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mappings and obtained a generalization of the Browder [6], Göhde [11] and Kirk [16] results.

The iterative approximation problems for fixed points of asymptotically nonexpansive mappings types were widely studied by Bose [5], Ǵornicki [12], Jung et al. [14], Lim and Xu [17] and Xu [29].

In his important paper Takahahashi [26] introduced the notion of convexity in metric spaces. Afterwards Beg et al. [2]-[4], Ćirić [8], Gajić and Sojakovic [9], Guay et al. [13], Shimizu and Takahashi [22] and many other authors have studied fixed point theorems on convex metric spaces.

Furthermore Shimizu and Takahashi [23] introduced the concept of uniform convexity in convex metric spaces and studied its properties.

Definition 1.1. ([26]) Let (X, d) be a metric space. A mapping $W : X \times X \times [0, 1] \to X$ is said to be a *convex structure* on X if for each $(x, y, \lambda) \in X \times X \times [0, 1]$ and $u \in X$,

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y).$$

The metric space X together with W is called a *convex metric space.*

Definition 1.2. Let X be a convex metric space. A nonempty subset A of X is said to be *convex* if $W(x, y, \lambda) \in A$ whenever $(x, y, \lambda) \in A \times A \times [0, 1]$.

Takahashi [26] has shown that open spheres $B(x, r) = \{y \in X : d(x, y) < r\}$ and closed spheres $B(x, r) = \{y \in X : d(x, y) \leq r\}$ are convex. All normed spaces and their convex subsets are convex metric spaces. But there are many examples of convex metric spaces which are not embedded in any normed space (see Takahashi [26]).

Recently, Beg [2] introduced and studied the notion of 2-uniformly convex metric spaces.

Definition 1.3. A convex metric space X is said to have *property* (B) if it satisfies $d(W(x, a, \alpha),\, W(y, a, \alpha)) = \alpha d(x, y)$. Taking $x = a$, property (B) implies $\alpha d(x, y) = W(y, a, \alpha)$.

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Definition 1.4. A convex complete metric space X is said to be *uniformly convex* if for all $x, y, a \in X$,

$$\left[d\left(a, W\left(x, y, \frac{1}{2}\right)\right)\right]^2 \leq \frac{1}{2}\left(1-\delta\left(\frac{d(x,y)}{\max\{d(a,x), d(a,y)\}}\right)\right)\left([d(a,x)]^2+[d(a,y)]^2\right),$$

where the function δ is a strictly increasing function on the set of strictly positive numbers and $\delta(0) = 0$.

Remark 1.5. Uniformly convex Banach spaces are uniformly convex metric spaces.

Definition 1.6. A uniformly convex metric space X is said to be *2-uniformly convex* if there exists a constant $c > 0$ such that $\delta(\epsilon) \geq c\epsilon^2$.

Definition 1.7. (1) Let A be a nonempty subset of a metric space X. A mapping $T : A \to A$ is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\sum_{n=1}^{\infty}(k_n - 1) < +\infty$ such that

$$d(T^n x, T^n y) \leq k_n d(x, y)$$

for all $x, y \in A$ and $n \geq 1$.

(2) T is said to be *uniformly L-Lipschitzian* with a Lipschitzian constant $L \geq 1$, i.e., there exists a constant $L \geq 1$ such that

$$d(T^n x, T^n y) \leq L d(x, y)$$

for all $x, y \in A$ and $n \geq 1$.

This is a class of mapping introduced by Goebel and Kirk [10], where it is shown that if A is a nonempty bounded closed convex subset of a uniformly convex Banach space and $T : A \to A$ is asymptotically nonexpensive, then T has a fixed point and, moreover, the set $Fix(T)$ of fixed points of T is closed and convex.

Remark 1.8. As an application of the Lagrange mean value theorem, we can see that

$$t^q - 1 \leq q t^q (t - 1)$$

for $t \geq 1$ and $q > 1$. This together with $\sum_{n=1}^{\infty}(k_n - 1) < +\infty$ implies that $\sum_{n=1}^{\infty}(k_n^q - 1) < +\infty$.

Theorem 1.9. ([23]) *If (X, d) is uniformly convex complete metric space, then every decreasing sequence of nonempty closed bounded convex subsets of X has nonempty intersection.*

Definition 1.10. Let (X, d) be a metric space and Y a topological space. A mapping $T : X \to X$ is said to be *completely continuous* if the image of each bounded set in X is contained in a compact subset of Y.

In [3], Beg proved the following results.

Theorem 1.11. *Let A be a nonempty closed bounded convex subset of a uniformly convex complete metric space (X, d) and $T : A \to A$ be an asymptotically nonexpansive mapping. Then T has a fixed point.*

Theorem 1.12. *Let (X, d) be a convex metric space and A be a nonempty convex subset of X. Let $L > 0$ and $T : A \to A$ be uniformly L-Lipschitzian. Let $x_1 \in A$. Define $y_n = W\big(T^n x_n, x_n, \frac{1}{2}\big)$, $x_{n+1} = W\big(T^n y_n, x_n, \frac{1}{2}\big)$ and set $c_n = d(T^n x_n, x_n)$ for all $n \in \mathbb{N}$. Then*

$$d(x_n, Tx_n) \leq c_n + c_{n-1}(L + 3L^2 + 2L^3)$$

for all $n \in \mathbb{N}$.

Theorem 1.13. *Let (X, d) be a 2-uniformly convex metric space having property (B), A be a nonempty closed bounded convex subset of X and $T : A \to A$ be asymptotically nonexpensive with sequence $\{k_n\} \in [1, +\infty)^{\mathbb{N}}$ and $\sum_{n=1}^{\infty} (k_n^2 - 1) < +\infty$. Let $x_1 \in A$. Define $x_{n+1} = W\big(T^n x_n, x_n, \frac{1}{2}\big)$ for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.*

Theorem 1.14. *Let (X, d) be 2-uniformly convex metric space having property (B), A be a nonempty closed bounded convex subset of X and $T : A \to A$ completely continuous asymptotically nonexpensive mapping with sequence $\{k_n\} \in [1, +\infty)^{\mathbb{N}}$ and $\sum_{n=1}^{\infty} (k_n^2 - 1) < +\infty$. Let $x_1 \in A$. Define $x_{n+1} = W\big(T^n x_n, x_n, \frac{1}{2}\big)$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges to some fixed point of T.*

Recently, Agrawal et al. [1] introduced a new iteration process namely S-iteration process and studied the iterative approximation problems for fixed points of nearly asymptotically nonexpansive mappings.

The purpose of this paper is to construct a two-step type iteration scheme, convergent to the fixed point, for asymptotically nonexpensive mappings defined on a 2-uniformly convex metric space. The results established in this

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paper improve the results due to Agrawal et al. [1], Chang [7], Khan and Takahashi [15], Liu and Kang [19], Osilike and Aniagbosor [20], Rhoades [21], Schu [24], [25], Tan and Xu [27], [28] and others.

2. Main Results

In the sequel, we will need the following results. The following lemma is now well known.

Lemma 2.1. *Let $\{a_n\}$ and $\{b_n\}$ be sequences of non-negative real numbers such that $a_{n+1} \le (1+b_n)\, a_n$ for all $n \ge 1$ and $\sum_{n=1}^{\infty} b_n < \infty$. Then $\lim_{n\to\infty} a_n$ exists.*

Theorem 2.2. ([2]) *Let (X,d) be a uniformly convex metric space having property (B). Then X is 2-unformly convex if and only if there exists a number $c > 0$ such that $2\big[d\big(a, W\big(x, y, \frac{1}{2}\big)\big)\big]^2 + c\,[d(x,y)]^2 \le [d(a,x)]^2 + [d(a,y)]^2$ for all $a, x, y \in X$.*

Now we prove our main results.

Lemma 2.3. *Let (X,d) be a convex metric space and A be a nonempty convex subset of X. Let $L > 0$ and $T : A \to A$ be uniformly L-Lipschitzian. Let $x_1 \in A$. Define $y_n = W\big(T^n x_n, x_n, \frac{1}{2}\big)$, $x_{n+1} = W\big(T^n y_n, T^n x_n, \frac{1}{2}\big)$ and set $\delta_n = d(T^n x_n, x_n)$ for all $n \in \mathbb{N}$. Then*

$$d(x_n, Tx_n) \le \delta_n + \frac{L^2}{4}(L+5)\delta_{n-1}$$

for all $n \in \mathbb{N}$.

Proof. For all $n \in \mathbb{N}$, set $\delta_n = d(x_n, T^n x_n)$. Consider

$$\begin{aligned} d(Tx_n, x_n) &\le d(Tx_n, T^n x_n) + d(T^n x_n, x_n) \\ &\le Ld(T^{n-1}x_n, x_n) + \delta_n, \end{aligned} \tag{2.1}$$

$$\begin{aligned} d(T^{n-1}x_n, x_n) &= d\left(T^{n-1}x_n, W\left(T^{n-1}y_{n-1}, T^{n-1}x_{n-1,}\frac{1}{2}\right)\right) \\ &\le \frac{1}{2}d(T^{n-1}x_n, T^{n-1}y_{n-1}) + \frac{1}{2}d(T^{n-1}x_n, T^{n-1}x_{n-1}) \\ &\le \frac{1}{2}Ld(x_n, y_{n-1}) + \frac{1}{2}Ld(x_{n-1}, x_n), \end{aligned} \tag{2.2}$$

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$$\begin{aligned} d(x_{n-1}, x_n) &= d\left(x_{n-1}, W\left(T^{n-1}y_{n-1}, T^{n-1}x_{n-1,}\frac{1}{2}\right)\right) \\ &\leq \frac{1}{2}d(x_{n-1}, T^{n-1}y_{n-1}) + \frac{1}{2}d(x_{n-1}, T^{n-1}x_{n-1}) \\ &\leq \frac{1}{2}d(x_{n-1}, T^{n-1}y_{n-1}) + \frac{1}{2}\delta_{n-1}, \end{aligned} \tag{2.3}$$

$$\begin{aligned} d(x_{n-1}, T^{n-1}y_{n-1}) &\leq d(x_{n-1}, T^{n-1}x_{n-1}) + d(T^{n-1}x_{n-1}, T^{n-1}y_{n-1}) \\ &\leq \delta_{n-1} + Ld(x_{n-1}, y_{n-1}) \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} d(x_{n-1}, y_{n-1}) &= d\left(x_{n-1}, W\left(T^{n-1}x_{n-1}, x_{n-1}, \frac{1}{2}\right)\right) \\ &\leq \frac{1}{2}d(x_{n-1}, T^{n-1}x_n) \\ &= \frac{1}{2}\delta_{n-1}. \end{aligned} \tag{2.5}$$

(2.4) implies

$$d(x_{n-1}, T^{n-1}y_{n-1}) \leq \left(\frac{1}{2}L + 1\right)\delta_{n-1}. \tag{2.6}$$

Substituting (2.6) in (2.3), we get

$$d(x_{n-1}, x_n) \leq \left(\frac{1}{4}L + 1\right)\delta_{n-1}. \tag{2.7}$$

Also

$$\begin{aligned} d(x_n, y_{n-1}) &\leq d(x_n, x_{n-1}) + d(x_{n-1}, y_{n-1}) \\ &\leq \frac{1}{2}\left(\frac{1}{2}L + 3\right)\delta_{n-1}. \end{aligned} \tag{2.8}$$

Substitution of (2.7) and (2.8) in (2.2) yields

$$d(T^{n-1}x_n, x_n) \leq \frac{1}{4}L(L+5)\delta_{n-1}.$$

Finally from (2.1) we obtain

$$d(Tx_n, x_n) \leq \delta_n + \frac{L^2}{4}(L+5)\delta_{n-1}.$$

This completes the proof. □

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Theorem 2.4. *Let (X, d) be 2-uniformly convex metric space having property (B), A be a nonempty closed convex subset of X and $T : A \to A$ be asymptotically nonexpensive with sequence $\{k_n\} \in [1, +\infty)^{\mathbb{N}}$ and $\sum_{n=1}^{\infty} (k_n - 1) < +\infty$ satisfying*

$$(C) \qquad d(x, T^n y) \leq d(T^n x, T^n y)$$

for all $x, y \in A$. Assume that, for $x_1 \in A$, we define $x_{n+1} = W\big(T^n y_n, T^n x_n, \frac{1}{2}\big)$, $y_n = W\big(T^n x_n, x_n, \frac{1}{2}\big)$ for all $n \in \mathbb{N}$. Then we have $\lim_{n\to\infty} d(T^n y_n, x_n) = 0 = \lim_{n\to\infty} d(T^n x_n, x_n)$.

Proof. Since T is asymptotically nonexpensive, it possesses a fixed point $p \in A$ by Theorem 1.11.

Claim. $\{x_n\}$ is bounded. For this claim, we compute as follows

$$\begin{aligned} d(p, x_{n+1}) &= d\left(p, W\left(T^n y_n, T^n x_n, \frac{1}{2}\right)\right) \leq \frac{1}{2} d(p, T^n y_n) + \frac{1}{2} d(p, T^n x_n) \\ &= \frac{1}{2} d(T^n p, T^n y_n) + \frac{1}{2} d(T^n p, T^n x_n) \\ &\leq \frac{k_n}{2} \left(d(p, y_n) + d(p, x_n)\right) \end{aligned}$$

and

$$\begin{aligned} d(p, y_n) &= d\left(p, W\left(T^n x_n, x_n, \frac{1}{2}\right)\right) \leq \frac{1}{2} d(p, T^n x_n) + \frac{1}{2} d(p, x_n) \\ &= \frac{1}{2} d(T^n p, T^n x_n) + \frac{1}{2} d(p, x_n) \leq \frac{k_n}{2} d(p, x_n) + \frac{1}{2} d(p, x_n) \\ &= \frac{1}{2} (k_n + 1)\, d(p, x_n). \end{aligned}$$

Hence

$$\begin{aligned} d(p, x_{n+1}) &\leq \frac{1}{2}\left[\frac{k_n}{2}(k_n + 1) + k_n\right] d(p, x_n) \\ &\leq (k_n (k_n + 1) + k_n) d(p, x_n) \\ &\leq \left[1 + (k_n - 1) + \left(k_n^3 - 1\right)\right] d(p, x_n). \end{aligned}$$

Using Remark 1.8 and Lemma 2.1, it follows that $\lim_{n\to\infty} d(p, x_n)$ exists and the sequence $\{x_n\}$ is bounded. Let $M = \sup_{n\geq 1} d(p, x_n)$.

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With the help of Theorem 2.2, we have

$$\begin{aligned} &2\left[d\left(p, x_{n+1}\right)\right]^2+c\left[d(T^n y_n, T^n x_n)\right]^2 \\ &=2\left[d\left(p, W\left(T^n y_n, T^n x_n, \frac{1}{2}\right)\right)\right]^2+c\left[d(T^n y_n, T^n x_n)\right]^2 \\ &\leq\left[d(p, T^n y_n)\right]^2+\left[d(p, T^n x_n)\right]^2=\left[d(T^n p, T^n y_n)\right]^2+\left[d(T^n p, T^n x_n)\right]^2 \\ &\leq k_n^2\left(\left[d(p, y_n)\right]^2+\left[d(p, x_n)\right]^2\right), \end{aligned} \tag{2.9}$$

where

$$\begin{aligned} &2\left[d\left(p, y_n\right)\right]^2+c\left[d(T^n x_n, x_n)\right]^2 \\ &=2\left[d\left(p, W\left(T^n x_n, x_n ; \frac{1}{2}\right)\right)\right]^2+c\left[d(T^n x_n, x_n)\right]^2 \\ &\leq\left[d(p, T^n x_n)\right]^2+\left[d(p, x_n)\right]^2=\left[d(T^n p, T^n x_n)\right]^2+\left[d(p, x_n)\right]^2 \\ &\leq k_n^2\left[d(p, x_n)\right]^2+\left[d(p, x_n)\right]^2 \end{aligned}$$

implies

$$\begin{aligned} \left[d\left(p, y_n\right)\right]^2 &\leq \frac{1}{2}(1+k_n^2)\left[d(p, x_n)\right]^2-\frac{c}{2}\left[d(T^n x_n, x_n)\right]^2 \\ &\leq \frac{1}{2}(1+k_n^2)\left[d(p, x_n)\right]^2 . \end{aligned} \tag{2.10}$$

Substituting (2.10) in (2.9), we get

$$2\left[d\left(p, x_{n+1}\right)\right]^2+c\left[d(T^n y_n, T^n x_n)\right]^2 \leq k_n^2\left(\frac{1}{2}(1+k_n^2)+1\right)\left[d(p, x_n)\right]^2$$

implies

$$\begin{aligned} &\frac{c}{2}\left[d(T^n y_n, T^n x_n)\right]^2 \\ &\leq\left(k_n^2(1+k_n^2)+k_n^2\right)\left[d(p, x_n)\right]^2-\left[d(p, x_{n+1})\right]^2 \\ &\leq\left(1+\left(k_n^2-1\right)+\left(k_n^5-1\right)\right)\left[d(p, x_n)\right]^2-\left[d(p, x_{n+1})\right]^2 \\ &\leq\left(\left(k_n^2-1\right)+\left(k_n^5-1\right)\right) M^2+\left[d(p, x_n)\right]^2-\left[d(p, x_{n+1})\right]^2 . \end{aligned}$$

Thus

$$\begin{aligned} &\frac{c}{2} \sum_{j=1}^m\left[d(T^j y_j, T^j x_j)\right]^2 \\ &\leq M^2 \sum_{j=1}^m\left(\left(k_j^2-1\right)+\left(k_j^5-1\right)\right)+\sum_{j=1}^m\left(\left[d(p, x_j)\right]^2-\left[d(p, x_{j+1})\right]^2\right). \end{aligned}$$

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Hence

$$\sum_{j=1}^{\infty}\left[d(T^j y_j, T^j x_j)\right]^2 < +\infty.$$

It implies that

$$\lim_{n\to\infty} d(T^n y_n, T^n x_n) = 0.$$

By using condition (C), we have

$$d(T^n y_n, x_n) \leq d(T^n y_n, T^n x_n) \to 0 \quad \text{as } n \to \infty.$$

Now

$$\begin{aligned} d(T^n x_n, x_n) &\leq d(T^n x_n, T^n y_n) + d(T^n y_n, x_n) \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{2.11}$$

This completes the proof. □

Theorem 2.5. *Let (X, d) be a 2-uniformly convex metric space having property (B), A be a nonempty closed bounded convex subset of X and $T : A \to A$ be completely continuous asymptotically nonexpensive mapping with sequence $\{k_n\} \in [1, +\infty)^{\mathbb{N}}$ and $\sum_{n=1}^{\infty}(k_n - 1) < +\infty$ satisfying the condition (C). Let $x_1 \in A$. Define $x_{n+1} = W\big(T^n y_n, T^n x_n; \frac{1}{2}\big)$, $y_n = W\big(T^n x_n, x_n; \frac{1}{2}\big)$ for all $n \in N$. Then $\{x_n\}$ converges to the fixed point of T.*

Proof. With the help of Lemma 2.3 and (2.11), we have

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

Since T is completely continuous and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{Tx_{n_k}\}$ converges. Therefore $\{x_{n_k}\}$ converges from $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Let $\lim_{k\to\infty} x_{n_k} = p$. It follows from the continuity of T and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ that $p = Tp$. We know that $\lim_{n\to\infty} d(p, x_n)$ exists. But $\lim_{k\to\infty} d(p, x_{n_k}) = 0$. This implies $\lim_{n\to\infty} d(p, x_n) = 0$, i.e., $\lim_{n\to\infty} x_n = p$. This compleets the proof. □

Remark 2.6. 1. Theorem 1.12 is proved for 2-step iteration process, while Theorems 1.13 and 1.14 are proved for 1-step iteration process.

2. In Theorem 2.4, we do not need the boundedness assumption, so the boundedness assumption in Theorem 1.13 is superfluous.

3. In Theorems 2.4 and 2.5, the condition (C) is not new; it is due to Liu et al. [18].

Acknowledgment

This study was supported by research funds from Dong-A University.

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Global Behavior and Periodicity of Some Difference Equations

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Abstract

In this paper we study the boundedness, investigate the global convergence and the periodicity of the solutions to the following recursive sequence

$$x_{n+1} = ax_n + \frac{bx_{n-1}^2 + cx_{n-2}x_{n-3}}{dx_{n-1}^2 + ex_{n-2}x_{n-3}}, \quad n = 0, 1, \ldots,$$

where the parameters a, b, c, d and e are positive real numbers and the initial conditions x_{-3}, x_{-2}, x_{-1} and $x_0 \in (0, \infty)$. Also we give some numerical examples of some special cases of considered equation and presented some rleated graphs and figures using Matlab.

Keywords: recursive sequence, boundedness, global stability, periodic solutions.
Mathematics Subject Classification: 39A10

1 Introduction

Our goal in this paper is to investigate the global stability character and the periodicity of the solutions of the recursive sequence

$$x_{n+1} = ax_n + \tfrac{bx_{n-1}^2 + cx_{n-2}x_{n-3}}{dx_{n-1}^2 + ex_{n-2}x_{n-3}}, \tag{1}$$

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where the parameters a, b, c, d and e are positive real numbers and the initial conditions x_{-3}, x_{-2}, x_{-1}, and $x_0 \in (0, \infty)$. Also we give some numerical examples of some special cases of Eq. (1) and presented some related graphs and figures using Matlab.

The study of the nonlinear rational difference equations of a higher order is quite challenging and rewarding, and the results about these equations offer prototypes towards the development of the basic theory of the global behavior of nonlinear difference equations of a big order.

Recently there has been a lot of interest in studying the global attractivity, the boundedness character and the periodicity nature of nonlinear difference equations where for example: DeVault et al. [3] have studied the global stability and the periodic character of solutions of the equation

$$y_{n+1} = \frac{p+y_{n-k}}{qy_n+y_{n-k}}.$$

Elabbasy et al. [4] investigated the global stability, periodicity character and gave the solution of special case of the following recursive sequence

$$x_{n+1} = ax_n - \frac{bx_n}{cx_n-dx_{n-1}}.$$

Elabbasy et al. [5] investigated the global stability, periodicity character and gave the solution of some special cases of the difference equation

$$x_{n+1} = \frac{dx_{n-l}x_{n-k}}{cx_{n-s}-b} + a.$$

Yang et al. [15] investigated the invariant intervals, the global attractivity of equilibrium points, and the asymptotic behavior of the solutions of the recursive sequence

$$x_{n+1} = \frac{ax_{n-1}+bx_{n-2}}{c+dx_{n-1}x_{n-2}},$$

For some related work see [1–25].

Let I be an interval of real numbers and let $F : I^{k+1} \to I$ be a continuously differentiable function. Consider the difference equation

$$y_{n+1} = F\left(y_n, y_{n-1}, ..., y_{n-k}\right), \quad n = 0, 1, ..., \tag{2}$$

with $y_{-k}, ..., y_0 \in I$.
Recall that the point $\overline{y} \in I$ is called an equilibrium point of Eq.(2) if

$$F(\overline{y}, \overline{y}, ..., \overline{y}) = \overline{y}.$$

That is, $y_n = \overline{y}$ for $n \geq 0$, is a solution of Eq.(2), or equivalently, $\overline{y}$ is a fixed point of F.

Let $\overline{y}$ be an equilibrium point of Eq.(2). Then the linearized equation of Eq.(2) about $\overline{y}$ is given by

$$w_{n+1} = \sum_{i=0}^{k} p_i w_{n-i}, \quad n = 0, 1, ..., \tag{3}$$

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where $p_i = \frac{\partial f}{\partial y_{n-i}}(\overline{y}, ..., \overline{y}), i = 0, 1, 2..., k$ and the characteristic equation of Eq.(3) is

$$\lambda^{(k+1)} - p_1\lambda^k - p_2\lambda^{(k-1)} - ... - p_k\lambda - p_{(k+1)} = 0.$$

Theorem A [9]: Assume that $p, q \in R$ and $k \in \{0, 1, 2, ...\}$. Then $|p| + |q| < 1$ is a sufficient condition for the asymptotic stability of the difference equation

$$u_{n+1} + pu_n + qu_{n-k} = 0, \quad n = 0, 1, ...$$

Remark: Theorem A can be easily extended to a general linear equations of the form

$$u_{n+k} + p_1u_{n+k-1} + ... + p_ku_n = 0, \quad n = 0, 1, ... \tag{4}$$

where $p_1, p_2, ..., p_k$ and $k \in \{1, 2, ...\}$. Then Eq.(4) is asymptotically stable provided that $\sum_{i=1}^{k} |p_i| < 1$.

Theorem B [6]: Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of Eq.(2), and suppose that there exist constants $A \in I$ and $B \in I$ such that $A \leq y_n \leq B$ for all $n \geq -k$. Let ℓ_0 be a limit point of the sequence $\{y_n\}_{n=-k}^{\infty}$. Then the following statements are true:

(i) There exists a solution $\{L_n\}_{n=-\infty}^{\infty}$ of Eq.(2), called a full limiting sequence of $\{y_n\}_{n=-k}^{\infty}$, such that $L_0 = \ell_0$, and such that for every $N \in \{...,-1,0,1,...\}$ L_N is a limit point of $\{y_n\}_{n=-k}^{\infty}$.

(ii) For every $i_0 \leq -k$, there exists a subsequence $\{y_{r_i}\}_{i=0}^{\infty}$ of $\{y_n\}_{n=-k}^{\infty}$ such that $L_N = \lim_{i\to\infty} y_{r_i+N}$ for every $N \geq i_0$.

Theorem C [10]: Assume for the difference equation (2) that $F(Y) > 0$ for all $0 \neq Y \in I^{k+1}$ where $Y = (y_0, y_1, ..., y_k)$.If $\sum_{j=0}^{k} y_j \left|\frac{\partial F}{\partial y_j}(Y)\right| \leq F(Y)$ for all $Y \in I^{k+1}$, then Eq.(2) has stability trichotomy, that is exactly one of the following three cases holds for all solutions of Eq.(2):

(i) $\lim_{n\to\infty} y_n = \infty$ for all $(y_{-k}, y_{-k+1}, ..., y_{-1}, y_0) \neq (0, 0, ...0)$.

(ii) $\lim_{n\to\infty} y_n = 0$ for all initial points and $\overline{y} = 0$ is the only equilibrium point of Eq.(2).

(iii) $\lim_{n\to\infty} y_n = \overline{y}$ for all $(y_{-k}, y_{-k+1}, ..., y_{-1}, y_0) \neq (0, 0, ...0)$ and $\overline{y}$ is the only positive equilibrium of Eq.(2).

2 Local Stability of the Equilibrium Point of Eq.(1)

This section deals with study the local stability character of the equilibrium point of Eq.(1)

The equilibrium points of Eq.(1) given by $\overline{x} = a\overline{x} + \frac{b+c}{d+e}$. Then if $a < 1$, Eq.(1) has the unique positive equilibrium point $\overline{x} = \frac{b+c}{(1-a)(d+e)}$. Let $f : (0, \infty)^4 \longrightarrow (0, \infty)$ be a continuous function defined by $f(u, v, w, t) = au + \frac{bv^2+cwt}{dv^2+ewt}$.

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Therefore it follows that

$$\begin{aligned}
\tfrac{\partial f(u,v,w,t)}{\partial u} &= a,\ \tfrac{\partial f(u,v,w,t)}{\partial v} = \tfrac{2(be-dc)vwt}{(dv^2+ewt)^2},\\
\tfrac{\partial f(u,v,w,t)}{\partial w} &= \tfrac{(cd-be)v^2t}{(dv^2+ewt)^2},\ \tfrac{\partial f(u,v,w,t)}{\partial t} = \tfrac{(cd-be)v^2w}{(dv^2+ewt)^2}.
\end{aligned}$$

Then we see that

$$\begin{aligned}
\tfrac{\partial f(\overline{x},\overline{x},\overline{x},\overline{x})}{\partial u} &= a = -a_3,\ \tfrac{\partial f(\overline{x},\overline{x},\overline{x},\overline{x})}{\partial v} = \tfrac{2(be-cd)}{(d+e)^2\overline{x}} = \tfrac{2(be-cd)(1-a)}{(d+e)(b+c)} = -a_2,\\
\tfrac{\partial f(\overline{x},\overline{x},\overline{x},\overline{x})}{\partial w} &= \tfrac{(cd-be)}{(d+e)^2\overline{x}} = \tfrac{(cd-be)(1-a)}{(d+e)(b+c)} = -a_1,\\
\tfrac{\partial f(\overline{x},\overline{x},\overline{x},\overline{x})}{\partial t} &= \tfrac{(cd-be)}{(d+e)^2\overline{x}} = \tfrac{(cd-be)(1-a)}{(d+e)(b+c)} = -a_0.
\end{aligned}$$

Then the linearized equation of Eq.(1) about $\overline{x}$ is

$$y_{n+1} + a_3 y_n + a_2 y_{n-1} + a_1 y_{n-2} + a_0 y_{n-3} = 0, \tag{5}$$

whose characteristic equation is

$$\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0. \tag{6}$$

Theorem 1 *Assume that* $4\left|(be-dc)\right| < (d+e)(b+c)$. *Then the positive equilibrium point of Eq.(1) is locally asymptotically stable.*

Proof: It is follows by Theorem A that, Eq.(5) is asymptotically stable if all roots of Eq.(6) lie in the open disc $|\lambda| < 1$ that is if

$$\begin{aligned}
|a_3| + |a_2| + |a_1| + |a_0| &< 1.\\
|a| + \left|\tfrac{2(be-cd)(1-a)}{(d+e)(b+c)}\right| + \left|\tfrac{2(cd-be)(1-a)}{(d+e)(b+c)}\right| &< 1,
\end{aligned}$$

and so

$$4\left|\tfrac{(be-dc)(1-a)}{(d+e)(b+c)}\right| < (1-a), \qquad a < 1,$$

or $4\left|be-dc\right| < (d+e)(b+c)$. The proof is complete.

3 Boundedness and Existence of Unbounded Solutions of Eq.(1)

Here we study the boundedness nature and persistence of solutions of Eq.(1).

Theorem 2 *Assume* $a < 1$. *Then every solution of Eq.(1) is bounded and persists.*

Proof: Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Eq.(1). It follows from Eq.(1) that

$$x_{n+1} = ax_n + \tfrac{bx_{n-1}^2 + cx_{n-2}x_{n-3}}{dx_{n-1}^2 + ex_{n-2}x_{n-3}} = ax_n + \tfrac{bx_{n-1}^2}{dx_{n-1}^2 + ex_{n-2}x_{n-3}} + \tfrac{cx_{n-2}x_{n-3}}{dx_{n-1}^2 + ex_{n-2}x_{n-3}}.$$

Then

$$x_{n+1} \leq ax_n + \frac{bx_{n-1}^2}{dx_{n-1}^2} + \frac{cx_{n-2}x_{n-3}}{ex_{n-2}x_{n-3}} = ax_n + \frac{b}{d} + \frac{c}{e} \text{ for all } \quad n \geq 1.$$

By using a comparison, we see that

$$\limsup_{n\to\infty} x_n \leq \frac{be+cd}{ed(1-a)} = M. \tag{7}$$

Thus the solution is bounded from above.
Now we wish to show that there exists $m > 0$ such that $x_n \geq m$ for all $n \geq 1$.

The transformation $x_n = \frac{1}{y_n}$, will reduce Eq.(1) to the equivalent form

$$y_{n+1} = \frac{y_n\left(dy_{n-2}y_{n-3}+ey_{n-1}^2\right)}{a\left(dy_{n-2}y_{n-3}+ey_{n-1}^2\right)+y_n\left(by_{n-2}y_{n-3}+cy_{n-1}^2\right)}.$$

It follows that

$$\begin{aligned} y_{n+1} &\leq \frac{y_n\left(dy_{n-2}y_{n-3}+ey_{n-1}^2\right)}{y_n\left(by_{n-2}y_{n-3}+cy_{n-1}^2\right)} = \frac{dy_{n-2}y_{n-3}}{by_{n-2}y_{n-3}+cy_{n-1}^2} + \frac{ey_{n-1}^2}{by_{n-2}y_{n-3}+cy_{n-1}^2} \\ &\leq \frac{dy_{n-2}y_{n-3}}{by_{n-2}y_{n-3}} + \frac{ey_{n-1}^2}{cy_{n-1}^2} \leq \frac{d}{b} + \frac{e}{c} = \frac{be+cd}{bc} = H \quad \text{for all} \quad n \geq 1. \end{aligned}$$

Thus we obtain

$$x_n = \frac{1}{y_n} \geq \frac{1}{H} = \frac{bc}{be+cd} = m \quad \text{for all} \quad n \geq 1. \tag{8}$$

From (7) and (8) we see that

$$m \leq x_n \leq M \qquad \text{for all} \quad n \geq 1.$$

Therefore every solution of Eq.(1) is bounded and persists.

Theorem 3 *Assume that $a \geq 1$. Then Eq.(1) possesses unbounded solutions.*

Proof: Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Eq.(1) and set $A = \frac{bx_{n-1}^2+cx_{n-2}x_{n-3}}{dx_{n-1}^2+ex_{n-2}x_{n-3}}$. Then it is easy to show that $\frac{\min\{b,c\}}{\max\{d,e\}} \leq A < \frac{b}{d} + \frac{c}{e}$. Therefore $x_{n+1} = ax_n + A$. Then

$$x_n = \begin{cases} a^n x_0 + \frac{A(a^n-1)}{a-1}, & if \quad a \neq 1, \\ x_0 + A\, n, & if \quad a = 1. \end{cases} \quad \text{for all} \quad n \geq 1.$$

Then, for $a \geq 1$, we obtain $\lim_{n\to\infty} x_n = \infty$, and this completes the proof.

4 Existence of Periodic Solutions

In this section we study the existence of periodic solutions of Eq.(1). The following theorem states the necessary and sufficient conditions that this equation has periodic solutions of prime period two.

Theorem 4 *Eq.(1) has positive prime period two solutions if and only if*

$$(i)\ (b-c)(d-e)(1+a)+4(bae+cd)>0,\ d>e, \quad b>c.$$

Proof: First suppose that there exists a prime period two solution

$$...,p,q,p,q,...$$

of Eq.(1). We will prove that Condition (i) holds.
We see from Eq.(1) that

$$\begin{aligned} p &= aq+\tfrac{bp^2+cpq}{dp^2+epq}=aq+\tfrac{bp+cq}{dp+eq}, \\ q &= ap+\tfrac{bq^2+cqp}{dq^2+eqp}=ap+\tfrac{bq+cp}{dq+ep}. \end{aligned}$$

Then

$$dp^2+epq = adpq+aeq^2+bp+cq, \tag{9}$$
$$dq^2+epq = adpq+aep^2+bq+cp. \tag{10}$$

Subtracting (9) from (10) gives

$$d(p^2-q^2)=-ae(p^2-q^2)+(b-c)(p-q).$$

Since $p\neq q$, it follows that

$$p+q=\tfrac{(b-c)}{(d+ae)}. \tag{11}$$

Again, adding (9) and (10) yields

$$d(p^2+q^2)+2epq = 2adpq+ae(p^2+q^2)+(b+c)(p+q),$$
$$(d-ae)(p^2+q^2)+2(e-ad)pq = (b+c)(p+q). \tag{12}$$

It follows by (11), (12) and the relation

$$p^2+q^2=(p+q)^2-2pq \quad \text{for all } \ p,q\in R,$$

that

$$2(e-d)(1+a)pq = \tfrac{2(bae+cd)(b-c)}{(d+ae)^2}.$$
$$pq = \tfrac{(bae+cd)(b-c)}{(d+ae)^2(e-d)(1+a)}. \tag{13}$$

Now it is clear from Eq.(11) and Eq.(13) that p and q are the two distinct roots of the quadratic equation

$$t^2-\left(\tfrac{(b-c)}{(d+ae)}\right)t+\left(\tfrac{(bae+cd)(b-c)}{(d+ae)^2(e-d)(1+a)}\right) = 0,$$
$$(d+ae)t^2-(b-c)t+\left(\tfrac{(bae+cd)(b-c)}{(d+ae)(e-d)(1+a)}\right) = 0, \tag{14}$$

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and so

$$[b-c]^2 + \frac{4(bae+cd)(b-c)}{(d-e)(1+a)} > 0.$$

or

$$(b-c)(d-e)(1+a) + 4(bae+cd) > 0.$$

Therefore Inequality (i) holds.
Second suppose that Inequality (i) is true. We will show that Eq.(1) has a prime period two solution. Assume that

$$p = \frac{b-c+\zeta}{2(d+ae)}, \quad q = \frac{b-c-\zeta}{2(d+ae)}, \text{ where } \zeta = \sqrt{[b-c]^2 - \frac{4(bae+cd)(b-c)}{(e-d)(1+a)}}.$$

We see from Inequality (i) that

$$(b-c)(d-e)(1+a) + 4(bae+cd) > 0, \ b > c, \quad d > e,$$

which equivalents to $(b-c)^2 > \frac{4(bae+cd)(b-c)}{(e-d)(1+a)}$. Therefore p and q are distinct real numbers.
Set $x_{-3} = p$, $x_{-2} = q$, $x_{-1} = p$ and $x_0 = q$. We wish to show that $x_1 = x_{-1} = p$ and $x_2 = x_0 = q$. It follows from Eq.(1) that

$$x_1 = aq + \frac{bp^2+cqp}{dp^2+eqp} = aq + \frac{bp+cq}{dp+eq} = a\left(\frac{b-c-\zeta}{2(d+ae)}\right) + \frac{b\left(\frac{b-c+\zeta}{2(d+ae)}\right)+c\left(\frac{b-c-\zeta}{2(d+ae)}\right)}{d\left(\frac{b-c+\zeta}{2(d+ae)}\right)+e\left(\frac{b-c-\zeta}{2(d+ae)}\right)}.$$

Dividing the denominator and numerator by $2(d+ae)$ gives

$$x_1 = \frac{ab-ac-a\zeta}{2(d+ae)} + \frac{b(b-c+\zeta)+c(b-c-\zeta)}{d(b-c+\zeta)+e(b-c-\zeta)} = \frac{ab-ac-a\zeta}{2(d+ae)} + \frac{(b-c)[(b+c)+\zeta]}{(d+e)(b-c)+(d-e)\zeta}.$$

Multiplying the denominator and numerator of the right side by $(d+e)(b-c) - (d-e)\zeta$ gives

$$\begin{aligned}
x_1 &= \frac{ab-ac-a\zeta}{2(d+ae)} + \frac{(b-c)[(b+c)+\zeta][(d+e)(b-c)-(d-e)\zeta]}{[(d+e)(b-c)+(d-e)\zeta][(d+e)(b-c)-(d-e)\zeta]} = \frac{ab-ac-a\zeta}{2(d+ae)} \\
&\quad + \frac{(b-c)\left\{(d+e)\left(b^2-c^2\right)+\zeta[(d+e)(b-c)-(d-e)(b+c)]-(d-e)\zeta^2\right\}}{(d+e)^2(b-c)^2-(d-e)^2\zeta^2} \\
&= \frac{ab-ac-a\zeta}{2(d+ae)} + \frac{(b-c)\left\{(d+e)\left(b^2-c^2\right)+2\zeta(eb-cd)-(d-e)(b-c)^2-\frac{4(bae+cd)(b-c)}{(1+a)}\right\}}{(d+e)^2(b-c)^2-(d-e)^2\left([b-c]^2-\frac{4(bae+cd)(b-c)}{(e-d)(1+a)}\right)} \\
&= \frac{ab-ac-a\zeta}{2(d+ae)} + \frac{(b-c)\left\{2(b-c)\left[dc+eb-\frac{2(bae+cd)}{(1+a)}\right]+2\zeta(eb-cd)\right\}}{4(b-c)\left[ed(b-c)+\frac{(e-d)(bae+cd)}{(1+a)}\right]}
\end{aligned}$$

Multiplying the denominator and numerator of the right side by $(1+a)$ we obtain

$$\begin{aligned}
x_1 &= \frac{ab-ac-a\zeta}{2(d+ae)} + \frac{(b-c)[(dc+eb)(1+a)-2(bae+cd)]+\zeta(1+a)(eb-cd)}{2[ed(b-c)(1+a)+(e-d)(bae+cd)]} \\
&= \frac{ab-ac-a\zeta}{2(d+ae)} + \frac{(eb-dc)\{(b-c)(1-a)+\zeta(1+a)\}}{2(eb-cd)(d+ae)}
\end{aligned}$$

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$$= \frac{ab-ac-a\zeta+(b-c)(1-a)+\zeta(1+a)}{2(d+ae)} = \frac{b-c+\zeta}{2(d+ae)} = p.$$

Similarly as before one can easily show that $x_2 = q$.
Then it follows by induction that

$$x_{2n} = q \quad \text{and} \quad x_{2n+1} = p \quad \text{for all} \quad n \geq -1.$$

Thus Eq.(1) has the prime period two solution ...,p,q,p,q,... where p and q are the distinct roots of the quadratic equation (14) and the proof is complete.

5 Global Attractor of the Equilibrium Point of Eq.(1)

In this section we investigate the global asymptotic stability of Eq.(1).

Lemma 5 *For any values of the quotient* $\frac{b}{d}$ *and* $\frac{c}{e}$*, the function* $f(u,v,w,t)$ *defined by Eq.(6) has the monotonicity behavior in its arguments.*

Proof: The proof follows by some simple computations and it will be omitted.

Theorem 6 *The equilibrium point* $\overline{x}$ *is a global attractor of Eq.(1) if one of the following statements holds*

$$(1)\ be \geq dc \ \text{ and } \ 2(2fc-be) \geq e(b-c)\left[\tfrac{be}{cf}\right]^2, \ \text{ where } f = d(1-a). \tag{15}$$

$$(2)\ be \leq dc \ \text{ and } 2(2gb-cd) \geq e(c-b)\left[\tfrac{cd}{gb}\right]^2, \ \text{ where } g = e(1-a). \tag{16}$$

Proof: Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Eq.(1) and again let f be a function defined by Eq.(4)
We will prove the theorem when Case (1) is true and the proof of Case (2) is similar and will be omitted.
Assume that (15) is true, then it results from the calculations after formula (4) that the function $f(u,v,w,t)$ is non-decreasing in u,v and non-increasing in w,t. Thus from Eq.(1), we see that

$$x_{n+1} = ax_n + \tfrac{bx_{n-1}^2+cx_{n-2}x_{n-3}}{dx_{n-1}^2+ex_{n-2}x_{n-3}} \leq ax_n + \tfrac{bx_{n-1}^2+c(0)}{dx_{n-1}^2+e(0)} = ax_n + \tfrac{b}{d}.$$

Then

$$\begin{aligned} x_n &\leq \tfrac{b}{d(1-a)} = \tfrac{b}{f} = H \quad \text{for all} \quad n \geq 1. && (17)\\ x_{n+1} &= ax_n + \tfrac{bx_{n-1}^2+cx_{n-2}x_{n-3}}{dx_{n-1}^2+ex_{n-2}x_{n-3}} \geq a(0) + \tfrac{b(0)+cx_{n-2}x_{n-3}}{d(0)+ex_{n-2}x_{n-3}} \\ &\geq \tfrac{cx_{n-2}x_{n-3}}{ex_{n-2}x_{n-3}} = \tfrac{c}{e} = h \quad for\ all \quad n \geq 1. && (18) \end{aligned}$$

Then from Eqs.(17) and (18), we see that

$$0 < h = \tfrac{c}{e} \leq x_n \leq \tfrac{b}{f} = H \quad \text{for all } \ n \geq 1.$$

Let $\{x_n\}_{n=0}^{\infty}$ solution of Eq.(1) with

$$I := \lim_{n\to\infty} \inf x_n \text{ and } S := \lim_{n\to\infty} \sup x_n.$$

It suffices to show that $I = S$. Now it follows from Eq.(1) that

$$I \geq f(I, I, S, S) \quad \Rightarrow \quad I \geq aI + \tfrac{bI^2+cS^2}{dI^2+eS^2},$$

and so

$$bI^2 + cS^2 - fI^3 \leq e(1-a)S^2 I. \tag{19}$$

Similarly, we see from Eq.(1) that

$$S \leq f(S, S, I, I) \quad \Rightarrow \quad S \leq aS + \tfrac{bS^2+cI^2}{dS^2+eI^2},$$

and so

$$bS^2 + cI^2 - fS^3 \geq e(1-a)SI^2. \tag{20}$$

Therefore it follows from Eqs.(19) and (20) that

$$bI^3 + cS^2 I - fI^4 \leq e(1-a)S^2 I^2 \leq bS^3 + cSI^2 - fS^4.$$

$$f(I^4 - S^4) + cSI(I - S) - b(I^3 - S^3) \geq 0,$$

if and only if

$$(I - S)\left[(I^2 + S^2)\{f(I + S) - b\} + SI\{c - b\}\right] \geq 0,$$

and so

$$I \geq S \quad \text{ if } \quad (I^2 + S^2)\{f(I + S) - b\} + SI\{c - b\} \geq 0.$$

Now, we know by (15) that

$$2(2fc - be) \geq e(b - c)(\tfrac{be}{cf})^2 \ \Rightarrow \ 2\left(\tfrac{c}{e}\right)^2\left(2f\left(\tfrac{c}{e}\right) - b\right) \geq (b - c)(\tfrac{b}{f})^2.$$

$$\begin{aligned} (I^2 + S^2)\{f(I + S) - b\} &\geq \left[\left(\tfrac{c}{e}\right)^2 + \left(\tfrac{c}{e}\right)^2\right]\left[f\left(\tfrac{c}{e} + \tfrac{c}{e}\right) - b\right] \\ &\geq (b - c)(\tfrac{b}{f})(\tfrac{b}{f}) \geq (b - c)IS. \\ (I^2 + S^2)\{f(I + S) - b\} + (c - b)IS &\geq 0, \end{aligned}$$

and so it follows that $I \geq S$. Therefore $I = S$. This completes the proof.

Theorem 7 *The equilibrium point $\overline{x}$ is a global attractor of Eq.(1) if one of the following statements holds*

$$(1)\ be \geq dc \ \text{ and } \ 2(2fc - be) \geq e(b - c)\left[\frac{be}{cf}\right]^2, \ \text{ where } f = d(1 - a). \tag{15}$$

$$(2)\ be \leq dc \ \text{ and } 2(2gb - cd) \geq e(c - b)\left[\frac{cd}{gb}\right]^2, \ \text{ where } g = e(1 - a). \tag{16}$$

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Proof: Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Eq.(1) and rewrite Eq.(1) in the following form

$$x_{n+1} = f(x_n, x_{n-1}, x_{n-2}, x_{n-3}) = ax_n + \frac{bx_{n-1}^2 + cx_{n-2}x_{n-3}}{dx_{n-1}^2 + ex_{n-2}x_{n-3}}.$$

Then it is easy to obtain that

$$\frac{\partial f}{\partial x_n} = a, \quad \frac{\partial f}{\partial x_{n-1}} = \frac{2x_{n-1}x_{n-2}x_{n-3}(be-cd)}{(dx_{n-1}^2+ex_{n-2}x_{n-3})^2}, \quad \frac{\partial f}{\partial x_{n-2}} = \frac{x_{n-1}^2 x_{n-3}(be-cd)}{(dx_{n-1}^2+ex_{n-2}x_{n-3})^2},$$

$$\frac{\partial f}{\partial x_{n-3}} = \frac{x_{n-1}^2 x_{n-2}(be-cd)}{(dx_{n-1}^2+ex_{n-2}x_{n-3})^2}.$$

Therefore

$$\begin{aligned} &\sum_{j=0}^{3} x_{n-j}\left|\frac{\partial f}{\partial x_{n-j}}(x_n, x_{n-1}, x_{n-2}, x_{n-3})\right| \\ =\ & ax_{n-1} + \frac{2x_{n-1}^2 x_{n-2}x_{n-3}|be-cd| + 2x_{n-1}^2 x_{n-2}x_{n-3}|cd-be|}{(dx_{n-1}^2+ex_{n-2}x_{n-3})^2}. \end{aligned}$$

Thus it follows for any values of b, c, d, and e, that

$$\begin{aligned} \sum_{j=0}^{3} x_{n-j}\left|\frac{\partial f}{\partial x_{n-j}}(x_n, x_{n-1}, x_{n-2}, x_{n-3})\right| &= ax_{n-1} < ax_{n-1} + \frac{bx_{n-1}^2 + cx_{n-2}x_{n-3}}{dx_{n-1}^2 + ex_{n-2}x_{n-3}} \\ &= f(x_n, x_{n-1}, x_{n-2}, x_{n-3}). \end{aligned}$$

Then the result follows by Theorem C.

Remark It follows from Eq.(1), when $\frac{b}{d} = \frac{c}{e}$, that $x_{n+1} = ax_n + \lambda$ for all $n \geq -3$ and for some constant λ. It is easy to solve this linear difference equation of the first order.

Numerical examples

For confirming the results of this paper, we consider numerical examples which represent different types of solutions to Eq. (1).

Example 1. See Fig. 2, since $x_{-3} = 11,\ x_{-2} = 8,\ x_{-1} = 6,\ x_0 = 5,\ a = 0.1,\ b = 6,\ c = 1.3,\ d = 4,\ e = 6.$

Example 2. We consider $x_{-3} = 11,\ x_{-2} = 8,\ x_{-1} = 6,\ x_0 = 5,\ a = 1,\ b = 1.6,\ c = 3,\ d = 7,\ e = 2.$ See Fig. 2.

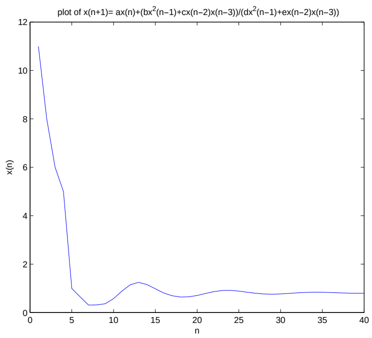

Figure 1.

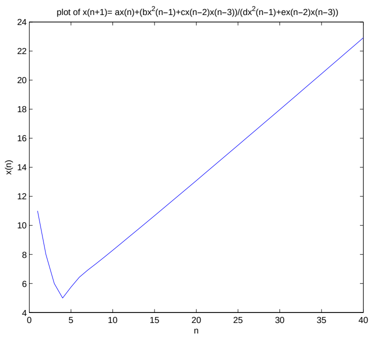

Figure 2.

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Example 3. See Fig. 3, since $x_{-3} = 11,\ x_{-2} = 3,\ x_{-1} = 4,\ x_0 = 15,\ a = 0.9,\ b = 3,\ c = 5,\ d = 4,\ e = 6.$
Example 4. Fig. 4. shows the solutions when $a = 0.3,\ b = 0.2,\ c = 0.1,\ d = 2.5,\ e = 1.6,\ x_{-3} = p,\ x_{-2} = q,\ x_{-1} = p,\ x_0 = q.$

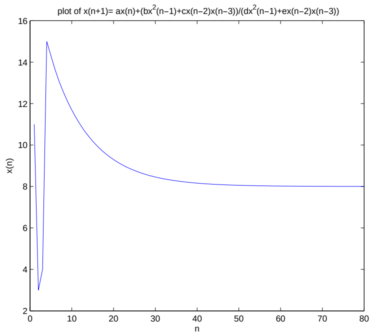

Figure 3.

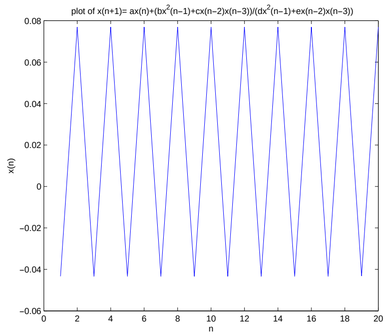

Figure 4.

Acknowledgements

This Project was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, Saudi Arabia under grant no. (142/130/1434). The authors, therefore, acknowledge with thanks DSR technical and financial support. Last, but not least, sincere appreciations are dedicated to all our colleagues in the Faculty of Science, Rabigh branch for their nice wishes.

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Further results on normal families of meromorphic functions concerning shared values

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Abstract

In this paper, we prove two normality criteria for families of some functions concerning shared values, the results generalize those given by Hu and Meng. Some examples are given to show the sharpness of our results.

Keywords and phrases: Normal family; meromorphic function; shared value.

1 Introduction and main results

Let $\mathbb{C}$ denote the complex plane and $f(z)$ be a non-constant meromorphic function in $\mathbb{C}$. It is assumed that the reader is familiar with the standard notation used in the Nevanlinna value distribution theory such as the characteristic function $T(r,f)$, the proximity function $m(r,f)$, the counting function $N(r,f)$ (see, e.g. [6, 16, 17]), and $S(r,f)$ denotes any quantity that satisfies the condition $S(r,f) = o(T(r,f))$ as $r \to \infty$ outside of a possible exceptional set of finite linear measure.

Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, a be a finite complex number, if $f - a$ and $g - a$ have the same zeros (Ignoring multiplicities), then we say that f and g share a.

Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain $D \subset \mathbb{C}$. $\mathcal{F}$ is said to be normal in D, in the sense of Montel, if for any sequence $f_n \in \mathcal{F}$, there exists a subsequence f_{n_j} such that f_{n_j} converges spherically locally uniformly in D, to a meromorphic function or ∞ (see [6, 17]).

According to Bloch's principle, every condition which reduces a meromorphic function in $\mathbb{C}$ to a constant, makes a family of meromorphic functions in a domain D normal. Although the principle is false in general, many authors proved normality criteria for families of meromorphic functions by starting from Picard type theorems. For instance,

Theorem A. [5] *Let $n \geq 5$ be an integer, a, $b \in \mathbb{C}$ and $a \neq 0$. If, for a meromorphic function f , $f' + af^n \neq b$ for all $z \in \mathbb{C}$, then f must be a constant.*

Theorem B. [10, 11] *Let $n \geq 3$ be an integer, a, $b \in \mathbb{C}$, $a \neq 0$ and $\mathcal{F}$ be a family of meromorphic functions in a domain D. If $f' + af^n \neq b$ for all $f \in \mathcal{F}$, then $\mathcal{F}$ is a normal family.*

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In 2008, Zhang [18] improved theorem B by the idea of shared values, he got

Theorem C. *Let $\mathcal{F}$ be a family of meromorphic functions in D, n be a positive integer and a, b be two constants such that $a \neq 0, \infty$ and $b \neq \infty$. If $n \geq 4$ and for each pair of functions f and $g \in \mathcal{F}$, $f' - af^n$ and $g' - ag^n$ share the value b, then $\mathcal{F}$ is normal in D.*

In 1998, Wang and Fang [13] proved

Theorem D. *Let k, $n \geq k+1$ be positive integers and f be a transcendental meromorphic function, then $(f^n)^{(k)}$ assumes every finite nonzero value infinitely often.*

Using the idea of shared values, Li and Gu [9] obtained a corresponding normality criteria.

Theorem E. *Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D. Let k, $n \geq k+2$ be positive integers and $a \neq 0$ be a finite complex number. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share a in D for every pair of functions f, $g \in \mathcal{F}$, then $\mathcal{F}$ is normal in D.*

In 2004, Alotaibi [1] got

Theorem F. *Suppose that f is a transcendental meromorphic function in the plane. Let $a \not\equiv 0$ be a small function of f, then $af(f^{(k)})^n - 1$ has infinitely many zeros.*

Using the idea of shared values, Hu and Meng [8] obtained a corresponding normality criteria.

Theorem G. *Take positive integers n and k with n, $k \geq 2$ and take a nonzero complex number a. Let $\mathcal{F}$ be a family of meromorphic functions in the plane domain D such that each $f \in \mathcal{F}$ has only zeros of multiplicity at least k. For each pair $(f, g) \in \mathcal{F}$, if $f(f^{(k)})^n$ and $g(g^{(k)})^n$ share a, then $\mathcal{F}$ is normal in D.*

In 1996, Yang and Hu [15] got

Theorem H. *Take nonnegative integers $n, n_1, \cdots, n_k$ with $n \geq 1$, $n_1 + n_2 + \cdots + n_k \geq 1$ and define $d = n + n_1 + n_2 + \cdots + n_k$. Let f be a transcendental meromorphic function with the deficiency $\delta(0, f) > 3/(3d+1)$. Then for any nonzero value c, the function $f^n(f')^{n_1} \cdots (f^{(k)})^{n_k} - c$ has infinitely many zeros. Moreover, if $n \geq 2$, the deficient condition can be omitted.*

It's natural to ask whether there exists normality criteria corresponding to Theorem H. We consider this problem and obtain

Theorem 1.1. *Let $a \neq 0$ be a constant, $n \geq 2$, $k \geq 1$, $n_k \geq 1$, n_j $(j = 1, 2, \cdots, k-1)$ be nonnegative integers. Let $\mathcal{F}$ be a family of meromorphic functions in the plane domain D such that each $f \in \mathcal{F}$ has only zeros of multiplicity at least k. For each pair $(f, g) \in \mathcal{F}$, if $f^n(f')^{n_1} \cdots (f^{(k)})^{n_k}$ and $g^n(g')^{n_1} \cdots (g^{(k)})^{n_k}$ share a, then $\mathcal{F}$ is normal in D.*

Example 1.1. *Let $D = \{z : |z| < 1\}$ and $\mathcal{F} = \{f_m\}$ where $f_m := e^{mz}$, and for every pair of functions $f, g \in \mathcal{F}$, $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 0 in D, it is easy to verify that $\mathcal{F}$ is not normal at the point $z = 0$.*

Example 1.2. *Let $D = \{z : |z| < 1\}$ and $\mathcal{F} = \{f_m\}$ where $f_m := mz + \frac{1}{m^{k}(k+1)!}$. Then $(f_m^{k+1})^{(k)} = m^{k+1}(k+1)!z + 1$, and for every pair of functions $f, g \in \mathcal{F}$, $(f^{k+1})^{(k)}$ and $(g^{k+1})^{(k)}$ share 1 in D, it is easy to verify that $\mathcal{F}$ is not normal at the point $z = 0$.*

Remark 1.2. In Theorem 1.1, let $k = n_k = 1$, then $f^n(f')^{n_1} \cdots (f^{(k)})^{n_k} = \frac{(f^{n+1})'}{n+1}$ and $g^n(g')^{n_1} \cdots (g^{(k)})^{n_k} = \frac{(g^{n+1})'}{n+1}$, this case is a corollary of Theorem E. Examples 1.1 and 1.2

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given by Li and Gu show that the condition $a \neq 0$ in Theorem E is inevitable and $n \geq k+2$ in Theorem E is sharp. The examples also show that the conditions in Theorem 1.1 are sharp, at least for the case $k = n_k = 1$.

If $n = 1$, we have

Theorem 1.3. *Let $a \neq 0$ be a constant, $k > 2$, $n_k \geq 1$, n_j $(j = 1, 2, \cdots, k-1)$ be nonnegative integers such that $n_1 + \cdots + n_{k-1} \geq 1$. Let $\mathcal{F}$ be a family of meromorphic functions in the plane domain D such that each $f \in \mathcal{F}$ has only zeros of multiplicity at least k. For each pair $(f, g) \in \mathcal{F}$, if $f(f')^{n_1} \cdots (f^{(k)})^{n_k}$ and $g(g')^{n_1} \cdots (g^{(k)})^{n_k}$ share a, then $\mathcal{F}$ is normal in D.*

Remark 1.4. In Theorem 1.3, if $n_1 \geq 2$, the theorem still holds for $k = 2$.

Corollary 1.5. *Let $a \neq 0$ be a constant, n, k, n_k be positive integers such that $nk \geq 2$ and n_j $(j = 1, 2, \cdots, k-1)$ be nonnegative integers. Let $\mathcal{F}$ be a family of holomorphic functions in the plane domain D such that each $f \in \mathcal{F}$ has only zeros of multiplicity at least k. For each pair $(f, g) \in \mathcal{F}$, if $f^n(f')^{n_1} \cdots (f^{(k)})^{n_k}$ and $g^n(g')^{n_1} \cdots (g^{(k)})^{n_k}$ share a, then $\mathcal{F}$ is normal in D.*

Remark 1.6. Examples 1.2 shows that Corollary 1.5 fails if $n = k = 1$ and thus the condition $nk \geq 2$ in Corollary 1.5 is inevitable.

2 Preliminary lemmas

Lemma 2.1 ([12])**.** *Let $\mathcal{F}$ be a family of meromorphic functions on the unit disc Δ, all of whose zeros have the multiplicity at least k, and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ wherever $f(z) = 0, f \in \mathcal{F}$. Then if $\mathcal{F}$ is not normal, there exist, for each $0 \leq \alpha \leq k$:*
(a) a number r, $0 < r < 1$,
(b) points $z_n, |z_n| < r$,
(c) functions $f_n \in \mathcal{F}$, and
(d) positive numbers $\rho_n \to 0$
such that $\rho_n^{-\alpha} f_n(z_n + \rho_n \xi) = g_n(\xi) \to g(\xi)$ locally uniformly with respect to the spherical metric, where $g(\xi)$ is a non-constant meromorphic function on $\mathbb{C}$, all of whose zeros have multiplicity at least k, such that $g^\sharp(\xi) \leq g^\sharp(0) = kA + 1$. In particular, if g is an entire function, it is of exponential type. Here, as usual, $g^\sharp(z) = |g'(z)|/(1+|g(z)|^2)$ is the spherical derivative.

Lemma 2.2 ([4])**.** *Let f be an entire function and M a positive integer. If $f^\sharp(z) \leq M$ for all $z \in \mathbb{C}$, then f has the order at most one.*

Lemma 2.3 ([19])**.** *Take nonnegative integers $n, n_1, \cdots, n_k$ with $n \geq 1$, $n_k \geq 1$ and define $d = n + n_1 + n_2 + \cdots + n_k$. Let f be a transcendental meromorphic function whose zeros have multiplicity at least k. Then for any nonzero value c, the function $f^n(f')^{n_1} \cdots (f^{(k)})^{n_k} - c$ has infinitely many zeros, provided that $n_1 + n_2 + \cdots + n_{k-1} \geq 1$ and $k > 2$ if $n = 1$. Specially, if f is transcendental entire, the function $f^n(f')^{n_1} \cdots (f^{(k)})^{n_k} - c$ has infinitely many zeros.*

Lemma 2.4. *Take nonnegative integers $n_1, \cdots, n_k, k$ with $n_k \geq 1$, $k \geq 2$, $n_1 + n_2 + \cdots + n_{k-1} \geq 1$ and define $d = 1 + n_1 + n_2 + \cdots + n_k$. Let f be a non-constant rational function whose zeros have multiplicity at least k. Then for any nonzero value c, the function $f(f')^{n_1} \cdots (f^{(k)})^{n_k} - c$ has at least two distinct zeros.*

Proof. We shall divide our argument into two cases.
Case 1. Suppose that $f(f')^{n_1} \cdots (f^{(k)})^{n_k} - c$ has exactly one zero.
Case 1.1. If f is a non-constant polynomial, since the zeros of f have multiplicity at least k, we know that $f^n(f')^{n_1} \cdots (f^{(k)})^{n_k}$ is also a non-constant polynomial, so $f(f')^{n_1} \cdots (f^{(k)})^{n_k} - c$ has at least one zero, suppose that

$$f(f')^{n_1} \cdots (f^{(k)})^{n_k} = c + B(z - z_0)^l, \tag{2.1}$$

where B is a nonzero constant and $l > 1$ is an integer. Then (2.5) implies that $f(f')^{n_1} \cdots (f^{(k)})^{n_k}$ has only simple zeros, which contradicts the assumption that the zeros of f have multiplicity at least $k \geq 2$.

Case 1.2. If f is a non-constant rational function but not a polynomial. Set

$$f(z) = A\frac{(z-a_1)^{m_1}(z-a_2)^{m_2}\cdots(z-a_s)^{m_s}}{(z-b_1)^{l_1}(z-b_2)^{l_2}\cdots(z-b_t)^{l_t}}, \tag{2.2}$$

where A is a nonzero constant and $m_i \geq k\,(i = 1, 2, \cdots, s), l_j \geq 1\,(j = 1, 2, \cdots, t)$.
Then

$$f^{(k)}(z) = A\frac{(z-a_1)^{m_1-k}(z-a_2)^{m_2-k}\cdots(z-a_s)^{m_s-k}g_k(z)}{(z-b_1)^{l_1+k}(z-b_2)^{l_2+k}\cdots(z-b_t)^{l_t+k}}. \tag{2.3}$$

For simplicity, we denote

$$m_1 + m_2 + \cdots + m_s = M \geq ks, \tag{2.4}$$

$$l_1 + l_2 + \cdots + l_t = N \geq t. \tag{2.5}$$

It is easily obtained that

$$\deg(g_k) \leq k(s+t-1), \tag{2.6}$$

moreover, $g_k(z) = (M-N)(M-N-1)\cdots(M-N-k+1)z^{k(s+t-1)} + c_m z^{k(s+t-1)-1} + \cdots + c_0$.
Combining (2.6) and (2.7) yields

$$f(f')^{n_1}\cdots(f^{(k)})^{n_k} = A^d\frac{(z-a_1)^{dm_1-\sum_{j=1}^k jn_j}\cdots(z-a_s)^{dm_s-\sum_{j=1}^k jn_j}g(z)}{(z-b_1)^{dl_1+\sum_{j=1}^k jn_j}\cdots(z-b_t)^{dl_t+\sum_{j=1}^k jn_j}} = \frac{P(z)}{Q(z)}, \tag{2.7}$$

where $P(z)$, $Q(z)$, $g(z)$ are polynomials and $g(z) = \prod_{j=1}^k g_j^{n_j}(z)$ with $\deg(g) \leq \sum_{j=1}^k jn_j(s+t-1)$. Moreover $g(z) = (M-N)^{d-1}(M-N-1)^{d-1-n_1}\cdots(M-N-k+1)^{n_k}z^{\sum_{j=1}^k jn_j(s+t-1)} +$

$d_m z^{\sum_{j=1}^{k} jn_j(s+t-1)-1} + \cdots + d_0$.
Then from (2.10) we obtain

$$(f(f')^{n_1}\cdots(f^{(k)})^{n_k})' = A^d \frac{(z-a_1)^{dm_1-\sum_{j=1}^{k} jn_j-1}\cdots(z-a_s)^{dm_s-\sum_{j=1}^{k} jn_j-1}h(z)}{(z-b_1)^{dl_1+\sum_{j=1}^{k} jn_j+1}\cdots(z-b_t)^{dl_t+\sum_{j=1}^{k} jn_j+1}}, \quad (2.8)$$

where $h(z)$ is a polynomial with $\deg(h) \leq (\sum_{j=1}^{k} jn_j+1)(s+t-1)$.

Since $f(f')^{n_1}\cdots(f^{(k)})^{n_k}-c$ has exactly one zero, we obtain from (2.11) that

$$f(f')^{n_1}\cdots(f^{(k)})^{n_k} = c + \frac{B(z-z_0)^l}{(z-b_1)^{dl_1+\sum_{j=1}^{k} jn_j}\cdots(z-b_t)^{dl_t+\sum_{j=1}^{k} jn_j}}, \quad (2.9)$$

where B is a nonzero constant. Then

$$(f(f')^{n_1}\cdots(f^{(k)})^{n_k})' = \frac{B(z-z_0)^{l-1}H(z)}{(z-b_1)^{dl_1+\sum_{j=1}^{k} jn_j+1}\cdots(z-b_t)^{dl_t+\sum_{j=1}^{k} jn_j+1}}, \quad (2.10)$$

where $H(z)$ is a polynomial of the form

$$H(z) = B(l-dN-\sum_{j=1}^{k} jn_j t)z^t + B_{t-1}z^{t-1} + \cdots + b_0. \quad (2.11)$$

Case 1.2.1. If $l \neq dN + \sum_{j=1}^{k} jn_j t$.
From (2.11) we get $\deg(p) \geq \deg(Q)$, namely

$$dM - \sum_{j=1}^{k} jn_j s + \deg(g) \geq dN + \sum_{j=1}^{k} jn_j t. \quad (2.12)$$

In view of $\deg(g) \leq \sum_{j=1}^{k} jn_j(s+t-1)$, we deduce from (2.16) that

$$d(M-N) \geq \sum_{j=1}^{k} jn_j, \quad (2.13)$$

which implies $M > N$.
Combining (2.12), (2.14) and (2.15) yields

$$dM - (\sum_{j=1}^{k} jn_j+1)s \leq \deg(H) = t \leq N < M, \quad (2.14)$$

which implies that

$$\sum_{j=1}^{k} n_j M - \sum_{j=1}^{k} jn_j s < s, \quad (2.15)$$

this together with (2.8) implies that

$$\sum_{j=1}^{k-1}(k-j)n_j < 1. \tag{2.16}$$

which is a contradiction since $n_1 + n_2 + \cdots + n_{k-1} \geq 1$.

Case 1.2.2. If $l = dN + \sum_{j=1}^{k} jn_j t$.
If $M > N$, with similar discussion as above, we get the same contradiction.
If $M \leq N$, combining (2.12) and (2.14) yields

$$l - 1 \leq \deg(h) \leq (\sum_{j=1}^{k} jn_j + 1)(s + t - 1), \tag{2.17}$$

we deduce from (2.21) that

$$dN \leq \sum_{j=1}^{k} jn_j s + s + t - \sum_{j=1}^{k} jn_j < \sum_{j=1}^{k} jn_j s + s + N. \tag{2.18}$$

Note that $M \leq N$, from (2.22) we get

$$(d-1)M - \sum_{j=1}^{k} jn_j s < s. \tag{2.19}$$

Similar to the end of the proof in Case 1.2.1, we get a contradiction. Case 1 has been ruled out.

Case 2. Suppose that $f(f')^{n_1} \cdots (f^{(k)})^{n_k} - c$ has no zero.
Similar to the proof of Case 1.1, we deduce that f is not a polynomial. If f is a non-constant rational function but not a polynomial. Similar to the proceeding of proof in Case 1.2.1, we get a contradiction.
Hence $f^n(f')^{n_1} \cdots (f^{(k)})^{n_k} - c$ has at least two distinct zeros.
This proves Lemma 2.4.

Using the similar proof of Lemma 2.4, we get

Lemma 2.5. *Take nonnegative integers n, $n_1, \cdots, n_k, k$ with $n \geq 2$, $n_k \geq 1$, $k \geq 1$. Let f be a non-constant rational function whose zeros have multiplicity at least k. Then for any nonzero value c, the function $f^n(f')^{n_1} \cdots (f^{(k)})^{n_k} - c$ has at least two distinct zeros.*

Lemma 2.6. *Take nonnegative integers n, $n_1, \cdots, n_k, k$ with $n \geq 1$, $n_k \geq 1$, $k \geq 2$. Let f be a non-constant polynomial whose zeros have multiplicity at least k. Then for any nonzero value c, the function $f^n(f')^{n_1} \cdots (f^{(k)})^{n_k} - c$ has at least two distinct zeros.*

3 Proof of Theorem 1.3

Without loss of generality, we may assume $D = \Delta = \{z : |z| < 1\}$. For simplicity, we denote $f_j(z_j + \rho_j\xi)$ by f_j and set

$$\gamma_M = 1 + n_1 + \cdots + n_k, \quad \Gamma_M = \sum_{j=1}^{k} jn_j$$

Suppose that $\mathcal{F}$ is not normal in D. By Lemma 2.1, for $0 \leq \alpha < k$, there exist: $r < 1$, $z_j \to 0$ $(j \to \infty)$, $f_j \in \mathcal{F}$ and $\rho_j \to 0^+$ such that $g_j(\xi) = \rho_j^{-\Gamma_M/\gamma_M} f_j(z_j + \rho_j\xi) \to g(\xi)$ locally uniformly with respect to the spherical metric, where $g(\xi)$ is a non-constant meromorphic function on $\mathbb{C}$, all of whose zeros have multiplicity at least k, such that $g^\sharp(\xi) \leq g^\sharp(0) = kA + 1$.
On every compact subset of $\mathbb{C}$ which contains no poles of g, we have uniformly

$$\begin{aligned} f_j(f_j')^{n_1} \cdots (f_j^{(k)})^{n_k} - a &= g_j(\xi)(g_j'(\xi))^{n_1} \cdots (g_j^{(k)}(\xi))^{n_k} - a \\ &\to g(g')^{n_1} \cdots (g^{(k)})^{n_k} - a. \end{aligned} \tag{3.1}$$

If $g^n(g')^{n_1} \cdots (g^{(k)})^{n_k} \equiv a$, then g has no zeros and no poles, thus g is an entire function. By Lemma 2.2, g is of order at most 1. Moreover, $g(\xi) = e^{c_1\xi + c_0}$, where $c_1(\neq 0)$ and c_0 are constants. Thereby, we get

$$g(\xi)(g'(\xi))^{n_1} \cdots (g^{(k)}(\xi))^{n_k} = c_1^{\Gamma_M} e^{\gamma_M(c_1\xi + c_0)}, \tag{3.2}$$

which contradicts the case $g^n(g')^{n_1} \cdots (g^{(k)})^{n_k} \equiv a$.
Since g is a non-constant meromorphic function, by Lemmas 2.3 and 2.4, we deduce that $g(\xi)(g'(\xi))^{n_1} \cdots (g^{(k)}(\xi))^{n_k} - a$ has at least two distinct zeros.
We claim that $g(\xi)(g'(\xi))^{n_1} \cdots (g^{(k)}(\xi))^{n_k} - a$ has just a unique zero.
Let ξ_0 and $\xi_0^\star$ be two distinct zeros of $g(\xi)(g'(\xi))^{n_1} \cdots (g^{(k)}(\xi))^{n_k} - a$. We choose a positive number δ small enough such that $D_1 \bigcap D_2 = \emptyset$, and such that $g(g')^{n_1} \cdots (g^{(k)})^{n_k} - a$ has no other zeros in $D_1 \bigcup D_2$ except for ξ_0 and $\xi_0^\star$, where

$$D_1 = \{\xi \in \mathbb{C} \mid |\xi - \xi_0| < \delta\}, \quad D_2 = \{\xi \in \mathbb{C} \mid |\xi - \xi_0^\star| < \delta\}.$$

By (3.1) and Hurwitz's theorem, for sufficiently large j there exist points $\xi_j \in D_1$, $\xi_j^\star \in D_2$ such that

$$f_j(z_j + \rho_j\xi_j)(f_j'(z_j + \rho_j\xi_j))^{n_1} \cdots (f_j^{(k)}(z_j + \rho_j\xi_j))^{n_k} - a = 0,$$

$$f_j(z_j + \rho_j\xi_j^\star)(f_j'(z_j + \rho_j\xi_j^\star))^{n_1} \cdots (f_j^{(k)}(z_j + \rho_j\xi_j^\star))^{n_k} - a = 0.$$

By the assumption in Theorem 1.1, $f_1(f_1')^{n_1} \cdots (f_1^{(k)})^{n_k}$ and $f_j(f_j')^{n_1} \cdots (f_j^{(k)})^{n_k}$ share a for each j, it follows that

$$f_1(z_j + \rho_j\xi_j)(f_1'(z_j + \rho_j\xi_j))^{n_1} \cdots (f_1^{(k)}(z_j + \rho_j\xi_j))^{n_k} - a = 0,$$

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$$f_1(z_j+\rho_j\xi_j^\star)(f_1'(z_j+\rho_j\xi_j^\star))^{n_1}\cdots(f_1^{(k)}(z_j+\rho_j\xi_j^\star))^{n_k}-a=0.$$

Let $j\to\infty$, and note that $z_j+\rho_j\xi_j\to 0$, $z_j+\rho_j\xi_j^\star\to 0$, we get

$$f_1(0)(f_1'(0))^{n_1}\cdots(f_1^{(k)}(0))^{n_k}-a=0.$$

Since the zeros of $f_1(f_1')^{n_1}\cdots(f_1^{(k)})^{n_k}-a$ have no accumulation points, in fact, for sufficiently large j, we have

$$z_j+\rho_j\xi_j=0,\quad z_j+\rho_j\xi_j^\star=0.$$

Thus

$$\xi_j=-\frac{z_j}{\rho_j},\quad \xi_j^\star=-\frac{z_j}{\rho_j}.$$

This contradicts the fact that
$\xi_j\in D(\xi_0,\delta)$, $\xi_j^\star\in D(\xi_0^\star,\delta)$ and $D(\xi_0,\delta)\bigcap D(\xi_0^\star,\delta)=\emptyset$.
So $g(\xi)(g'(\xi))^{n_1}\cdots(g^{(k)}(\xi))^{n_k}-a$ has just a unique zero, which contradicts the fact that $g(\xi)(g'(\xi))^{n_1}\cdots(g^{(k)}(\xi))^{n_k}-a$ has at least two distinct zeros.
This completes the proof of Theorem 1.3.

By Theorem H, Lemmas 2.3, 2.5 and 2.6, the proofs of Theorem 1.1 and Corollary 1.5 can be carried out in the line of Theorem 1.3, we omit the process here.

4 Discussion

In Theorem 1.3, if $n_1=n_2=\cdots=n_{k-1}=0$, then $f(f')^{n_1}\cdots(f^{(k)})^{n_k}=f(f^{(k)})^{n_k}$, $g(g')^{n_1}\cdots(g^{(k)})^{n_k}=g(g^{(k)})^{n_k}$. This case is the same as the case in Theorem G. Hu and Meng proved that Theorem G holds for the case $k\geq 2, n_k\geq 2$. They also gave an example [8, Example 1.3] to show that Theorem G is not valid for the case $k=1$. It's natural to ask whether Theorem 1.3 holds for the case $n_k=1$, $n_1+\cdots+n_{k-1}=0$. Actually, this is an open problem as follows.

Problem 4.1. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, let k be a positive integer and b be a finite nonzero value. If, for every $f\in\mathcal{F}$, all zeros of f have multiplicity at least k, and $f(z)f^{(k)}(z)\neq b$, is $\mathcal{F}$ normal in D?*

Xu and Cao [14] gave a partial answer to Problem 4.1.

Theorem 4.1. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, let k be a positive integer and b be a finite nonzero value. If, for every $f\in\mathcal{F}$, all zeros of f have multiplicity at least $k+1$, and $f(z)f^{(k)}(z)\neq b$, then $\mathcal{F}$ is normal in D.*

Acknowledgements This research was supported by the National Natural Science Foundation of China (Grant No. 11171184, 11401574) and the Fundamental Research Funds for the Central Universities (Grant No. 3122013k008), Training plan for the Outstanding Young Teachers in Higher Education of Guangdong (no. Yq 2013159). Moreover, The authors found a similar job ("Some Normality Criteria" arXiv:1404.1465, by Gopal Datt and Sanjay Kumar) after submitting this paper a month. Their paper can also be used as a reference.

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Positive solutions of difference-summation boundary value problem for a second-order difference equation

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Abstract

In this paper, we study the existence of positive solutions to the difference-summation boundary value problem

$$\Delta^2 u(t-1) + a(t)f(u) = 0, \qquad t \in \{1, 2, ..., T\},$$

$$u(0) = \beta\Delta u(0), \qquad u(T+1) = \alpha\sum_{s=1}^{\eta} u(s),$$

where f is continuous, $T \geq 3$ is a fixed positive integer, $\eta \in \{1, 2, ..., T-1\}$, $0 < \alpha < \frac{2(T+1)}{\eta(\eta+1)}$, $0 < \beta < \frac{2(T+1)-\alpha\eta(\eta+1)}{2(\alpha\eta-1)}$ and $\Delta u(t-1) = u(t) - u(t-1)$. We show the existence of at least one positive solution if f is either superlinear or sublinear by applying the fixed point theorem due to Krasnoselskii in cones.

Keywords : Positive solution; Boundary value problem; Fixed point theorem; Cone

2010 Mathematics Subject Classification: 39A15, 34B15

1 Introduction

The study of the existence of solutions of multipoint boundary value problems for linear second-order ordinary differential and difference equations was initiated by Ilin and Moiseev [1]. Then Gupta [2] studied three-point boundary value problems

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 19, NO.2, 2015, COPYRIGHT 2015 EUDOXUS PRESS, LLC

for nonlinear second-order ordinary differential equations. Since then, nonlinear second-order three-point boundary value problems have also been studied by many authors, one may see the text books [3-4] and the papers [6-11]. However, all these papers are concerned with problems with three-point boundary condition restrictions on the difference of the solutions and the solutions themselves, for example,

$$
\begin{aligned}
u(0) &= 0, & u(T+1) &= 0 \\
u(0) &= 0, & au(s) &= u(T+1), \\
u(0) &= 0, & u(T+1) - au(s) &= b. \\
u(0) - \alpha\Delta u(0) &= 0, & u(T+1) &= \beta u(s). \\
u(0) - \alpha\Delta u(0) &= 0, & \Delta u(T+1) &= 0 \\
u(0) &= 0, & u(T+1) &= \alpha\sum_{s=1}^{\eta} u(s) \\
u(0) &= \beta\sum_{s=1}^{\eta} u(s), & u(T+1) &= \alpha\sum_{s=1}^{\eta} u(s)
\end{aligned}
$$

and so forth.

In [6], Leggett-Williams developed a fixed point theorem to prove the existence of three positive solutions for Hammerstein integral equations. Since then, this theorem has been reported to be a successful technique for dealing with the existence of three solutions for the two-point boundary value problems of differential and difference equations; see [7,8]. In [9], X. Lin and W. Liu using the properties of the associate Green's function and Leggett-Williams fixed point theorem, studied the existence of positive solutions of the problem.

G. Zhang and R. Medina [10], T. Sitthiwirattham and J.Tariboon [11], studied the existence of positive solutions for second order boundary value problems of difference equations by applying the Krasnoselskii's fixed point theorem. In [12], J. Henderson and H.B. Thompson used lower and upper solution methods.

In this paper, we consider the existence of positive solutions to the equation

$$\Delta^2 u(t-1) + a(t)f(u) = 0, \qquad t \in \{1, 2, ..., T\}, \tag{1.1}$$

with difference-summation boundary condition

$$u(0) = \beta\Delta u(0), \qquad u(T+1) = \alpha\sum_{s=1}^{\eta} u(s), \tag{1.2}$$

where f is continuous.

The aim of this paper is to give some results for existence of positive solutions to (1.1)-(1.2).

Let $\mathbb{N}$ be the nonnegative integer, we let $\mathbb{N}_{i,j} = \{k \in \mathbb{N} | \quad i \leq k \leq j\}$ and $\mathbb{N}_p = \mathbb{N}_{0,p}$. By the positive solution of (1.1)-(1.2) we mean that a function $u(t) : \mathbb{N}_{T+1} \to [0, \infty)$ and satisfies the problem (1.1)-(1.2).

Throughout this paper, we suppose the following conditions hold:
$(H1)$ $T \geq 3$ is a fixed positive integer, $\eta \in \{1, 2, ..., T-1\}$, constant $\alpha, \beta > 0$ such that $0 < \alpha < \frac{2(T+1)}{\eta(\eta+1)}$ and $0 < \beta < \frac{2(T+1)-\alpha\eta(\eta+1)}{2(\alpha\eta-1)}$.
$(H2)$ $f \in C([0, \infty), [0, \infty))$, f is either superlinear or sublinear. Set

$$f_0 = \lim_{u \to 0^+} \frac{f(u)}{u}, \qquad f_\infty = \lim_{u \to \infty} \frac{f(u)}{u}.$$

Then $f_0 = 0$ and $f_\infty = \infty$ correspond to the superlinear case, and $f_0 = \infty$ and $f_\infty = 0$ correspond to the sublinear case.
$(H3)$ $a \in C(\mathbb{N}_{T+1}, [0, \infty))$ and there exists $t_0 \in \mathbb{N}_{\eta,T+1}$ such that $a(t_0) > 0$.

The proof of the main theorem is based upon an application of the following Krasnoselskii's fixed point theorem in a cone.

Theorem 1.1. ([5]). *Let E be a Banach space, and let $K \subset E$ be a cone. Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let*

$$A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \longrightarrow K$$

be a completely continuous operator such that
(i) $\|Au\| \leqslant \|u\|, \quad u \in K \cap \partial\Omega_1$, and $\|Au\| \geqslant \|u\|, \quad u \in K \cap \partial\Omega_2$; or
(ii) $\|Au\| \geqslant \|u\|, \quad u \in K \cap \partial\Omega_1$, and $\|Au\| \leqslant \|u\|, \quad u \in K \cap \partial\Omega_2$.
Then A has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

2 Preliminaries

We now state and prove several lemmas before stating our main results.

Lemma 2.1. *The problem*

$$\Delta^2 u(t-1) + y(t) = 0, \qquad t \in \mathbb{N}_{1,T}, \tag{2.1}$$

$$u(0) = \beta\Delta u(0), \qquad u(T+1) = \alpha \sum_{s=1}^{\eta} u(s), \tag{2.2}$$

has a unique solution

$$\begin{aligned} u(t) =& \frac{2(t+\beta)}{2(T+1)-\alpha\eta(\eta+1)-2\beta(\alpha\eta-1)}\sum_{s=1}^{T}(T-s+1)y(s) \\ &- \frac{\alpha(t+\beta)}{2(T+1)-\alpha\eta(\eta+1)-2\beta(\alpha\eta-1)}\sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1)y(s) \\ &- \sum_{s=1}^{t-1}(t-s)y(s), \quad t\in\mathbb{N}_{T+1}. \end{aligned}$$

Proof. From $\Delta^2 u(t-1) = \Delta u(t) - \Delta u(t-1)$ and the first equation of (2.1), we get

$$\begin{aligned} \Delta u(t) - \Delta u(t-1) &= -y(t), \\ \Delta u(t-1) - \Delta u(t-2) &= -y(t-1), \\ &\vdots \\ \Delta u(1) - \Delta u(0) &= -y(1). \end{aligned}$$

We sum the above equations to obtain

$$\Delta u(t) = \Delta u(0) - \sum_{s=1}^{t} y(s), t\in\mathbb{N}_T. \tag{2.3}$$

We define $\sum_{s=p}^{q} y(s) = 0$; if $p > q$. Similarly, we sum (2.3) from $t = 0$ to $t = h$, and by using the boundary condition $u(0) = \beta\Delta u(0)$ in (2.2), we obtain

$$u(h+1) = (h+1+\beta)\Delta u(0) - \sum_{s=1}^{h}(h+1-s)y(s), h\in\mathbb{N}_T,$$

by changing the variable from $h+1$ to t, we have

$$u(t) = (t+\beta)\Delta u(0) - \sum_{s=1}^{t-1}(t-s)y(s), t\in\mathbb{N}_{T+1}. \tag{2.4}$$

From (2.4),

$$\begin{aligned} \sum_{s=1}^{\eta} u(s) &= \left(\frac{1}{2}\eta(\eta+1)+\beta\eta\right)\Delta u(0) - \sum_{s=1}^{\eta-1}\sum_{l=1}^{\eta-s} ly(s) \\ &= \left(\frac{1}{2}\eta(\eta+1)+\beta\eta\right)\Delta u(0) - \frac{1}{2}\sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1)y(s) \end{aligned}$$

Again using the boundary condition $u(T+1)=\alpha\sum_{s=1}^{\eta}u(s)$ in (2.2), we obtain

$$(T+1+\beta)\Delta u(0)-\sum_{s=1}^{T}(T-s+1)y(s)=\alpha\left(\frac{1}{2}\eta(\eta+1)+\beta\eta\right)\Delta u(0)-\frac{\alpha}{2}\sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1)y(s)$$

Thus,

$$\begin{aligned}\Delta u(0)=&\frac{2}{2(T+1)-\alpha\eta(\eta+1)-2\beta(\alpha\eta-1)}\sum_{s=1}^{T}(T-s+1)y(s)\\&-\frac{\alpha}{2(T+1)-\alpha\eta(\eta+1)-2\beta(\alpha\eta-1)}\sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1)y(s).\end{aligned}$$

Therefore, (2.1)-(2.2) has a unique solution

$$\begin{aligned}u(t)=&\frac{2(t+\beta)}{2(T+1)-\alpha\eta(\eta+1)-2\beta(\alpha\eta-1)}\sum_{s=1}^{T}(T-s+1)y(s)\\&-\frac{\alpha(t+\beta)}{2(T+1)-\alpha\eta(\eta+1)-2\beta(\alpha\eta-1)}\sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1)y(s)\\&-\sum_{s=1}^{t-1}(t-s)y(s),\quad t\in\mathbb{N}_{T+1}.\end{aligned}$$

□

Lemma 2.2. *The function*

$$G(t,s)=\frac{1}{\Lambda}\begin{cases}(s+\beta)[2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)]+\alpha s(t+\beta)(1-s), & s\in\mathbb{N}_{1,t-1}\cap\mathbb{N}_{1,\eta-1}\\ 2(s+\beta)(T+1-t)+\alpha\eta(t-s)(\eta+1+2\beta), & s\in\mathbb{N}_{\eta,t-1}\\ (t+\beta)[2(T+1)-\alpha\eta(\eta+1)+2s(\alpha\eta-1)+\alpha s(1-s)], & s\in\mathbb{N}_{t,\eta-1}\\ 2(t+\beta)(T+1-s), & s\in\mathbb{N}_{t,T}\cap\mathbb{N}_{\eta,T}\end{cases}\tag{2.5}$$

where

$$\Lambda=2(T+1)-\alpha\eta(\eta+1)-2\beta(\alpha\eta-1)>0,$$

is the Green's function of the problem

$$\begin{aligned}&-\Delta^2u(t-1)=0,\qquad t\in\mathbb{N}_{1,T},\\&u(0)=\beta\Delta u(0),\qquad u(T+1)=\alpha\sum_{s=1}^{\eta}u(s).\end{aligned}\tag{2.6}$$

Proof. Suppose $t < \eta$. The unique solution of problem (2.1)-(2.2) can be written

$$\begin{aligned}
u(t) =& -\sum_{s=1}^{t-1}(t-s)y(s) \\
&+ \frac{2(t+\beta)}{\Lambda}\left[\sum_{s=1}^{t-1}(T-s+1)y(s) + \sum_{s=t}^{\eta-1}(T-s+1)y(s) + \sum_{s=\eta}^{T}(T-s+1)y(s)\right] \\
&- \frac{\alpha(t+\beta)}{\Lambda}\left[\sum_{s=1}^{t-1}(\eta-s)(\eta-s+1)y(s) + \sum_{s=t}^{\eta-1}(\eta-s)(\eta-s+1)y(s)\right] \\
=&\frac{1}{\Lambda}\sum_{s=1}^{t-1}\left[(s+\beta)[2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)] + \alpha s(t+\beta)(1-s)\right]y(s) \\
&+ \frac{1}{\Lambda}\sum_{s=t}^{\eta-1}\left[(t+\beta)[2(T+1)-\alpha\eta(\eta+1)+2s(\alpha\eta-1)+\alpha s-\alpha s^2]\right]y(s) \\
&+ \frac{1}{\Lambda}\sum_{s=\eta}^{T}2(t+\beta)(T+1-s)y(s) \\
=&\sum_{s=1}^{T}G(t,s)y(s).
\end{aligned}$$

Suppose $t \geq \eta$. The unique solution of problem (2.1)-(2.2) can be written

$$\begin{aligned}
u(t) =& -\sum_{s=1}^{\eta-1}(t-s)y(s) - \sum_{s=\eta}^{t-1}(t-s)y(s) \\
&+ \frac{2(t+\beta)}{\Lambda}\left[\sum_{s=1}^{\eta-1}(T-s+1)y(s) + \sum_{s=\eta}^{t-1}(T-s+1)y(s) + \sum_{s=t}^{T}(T-s+1)y(s)\right] \\
&- \frac{\alpha(t+\beta)}{\Lambda}\sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1)y(s) \\
=&\frac{1}{\Lambda}\sum_{s=1}^{\eta-1}\left[(s+\beta)[2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)] + \alpha s(t+\beta)(1-s)\right]y(s) \\
&+ \frac{1}{\Lambda}\sum_{s=\eta}^{t-1}\left[2(s+\beta)(T+1-t)+\alpha\eta(t-s)(\eta+1+2\beta)\right]y(s) \\
&+ \frac{1}{\Lambda}\sum_{s=t}^{T}2(t+\beta)(T+1-s)y(s) \\
=&\sum_{s=1}^{T}G(t,s)y(s).
\end{aligned}$$

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Then the unique solution of problem (2.1)-(2.2) can be written as $u(t) = \sum_{s=1}^{T} G(t,s)y(s)$. The proof is complete. $\square$

We observe that the condition $0 < \alpha < \frac{2(T+1)}{\eta(\eta+1)}$ and $0 < \beta < \frac{2(T+1)-\alpha\eta(\eta+1)}{2(\alpha\eta-1)}$. implies $G(t,s)$ is positive on $\mathbb{N}_{1,T} \times \mathbb{N}_{1,T}$, which mean that the finite set

$$\left\{ \frac{G(t,s)}{G(t,t)} : t \in \mathbb{N}_{T+1}, s \in \mathbb{N}_{1,T} \right\},$$

take positive values. Then we let

$$M_1 = \min \left\{ \frac{G(t,s)}{G(t,t)} : t \in \mathbb{N}_{1,T}, s \in \mathbb{N}_{1,T} \right\} \tag{2.7}$$

$$M_2 = \max \left\{ \frac{G(t,s)}{G(t,t)} : t \in \mathbb{N}_{T+1}, s \in \mathbb{N}_{1,T} \right\} \tag{2.8}$$

Lemma 2.3. *Let $(t,s) \in \mathbb{N}_{1,T} \times \mathbb{N}_{1,T}$. Then we have*

$$G(t,s) \geq M_1 G(t,t.) \tag{2.9}$$

where $0 < M_1 < 1$ is a constant given by

$$M_1 = \begin{cases} \min & \left\{ \frac{(1+\beta)[2T-\alpha\eta(\eta-1)]-\alpha(\eta-1+\beta)(\eta-2)^2}{(\eta+\beta-1)[2(T+2-\eta)+\alpha\eta(\eta-3)]}, \frac{2[T-\alpha-\alpha\eta(\eta-2)]}{2(T+2-\eta)+\alpha\eta(\eta-3)}, \right. \\ & \frac{2}{2(T+2-\eta)+\alpha\eta(\eta-3)}, \frac{2(\eta+\beta)+\alpha\eta(\eta+1+2\beta)}{2(T+\beta)(T+1-\eta)}, \frac{1}{T+1-\eta}, \\ & \left. \frac{(1+\beta)[2(T+1-\eta)+\alpha\eta(\eta-1)]-\alpha(T+\beta)(\eta-1)^2+\alpha(\eta-1)(\eta+\beta)}{2(T+\beta)(T+1-\eta)} \right\}; \quad \text{if } \alpha > \frac{1}{\eta} \\ \min & \left\{ \frac{(1+\beta)[2(T+2-\eta)+\alpha\eta(\eta-3)]-\alpha(\eta-1+\beta)(\eta-2)^2}{(\eta+\beta-1)[2T-\alpha\eta(\eta-1)]}, \frac{2(T+2-\eta-\alpha)}{2T-\alpha\eta(\eta-1)}, \right. \\ & \frac{2}{2T-\alpha\eta(\eta-1)}, \frac{2(\eta+\beta)+\alpha\eta(\eta+1+2\beta)}{2(T+\beta)(T+1-\eta)}, \frac{1}{T+1-\eta}, \\ & \left. \frac{(1+\beta)[2(T+2-\eta)+\alpha\eta(\eta-3)]-\alpha(\eta-1+\beta)(\eta-2)^2}{(\eta+\beta-1)[2T-\alpha\eta(\eta-1)]}, \frac{2(T+2-\eta-\alpha)}{2T-\alpha\eta(\eta-1)} \right\}; \quad \text{if } 0 < \alpha < \frac{1}{\eta} \end{cases} \tag{2.10}$$

Proof. In order that (2.9) holds, it is sufficient that M_1 satisfies

$$M_1 \leq \min_{(t,s)\in\mathbb{N}_{1,T}\times\mathbb{N}_{1,T}} \frac{G(t,s)}{G(t,t)}. \tag{2.11}$$

Then we may choose

$$M_1 \leq \min \left\{ \min_{(t,s)\in\mathbb{N}_{1,\eta-1}\times\mathbb{N}_{1,T}} \frac{G(t,s)}{G(t,t)}, \min_{(t,s)\in\mathbb{N}_{\eta,T}\times\mathbb{N}_{1,T}} \frac{G(t,s)}{G(t,t)} \right\}. \tag{2.12}$$

since

$$\min_{(t,s)\in\mathbb{N}_{1,\eta-1}\times\mathbb{N}_{1,T}} \frac{G(t,s)}{G(t,t)}$$
$$= \min_{t\in\mathbb{N}_{1,\eta-1}} \left\{ \min_{s\in\mathbb{N}_{1,t-1}} \frac{(s+\beta)[2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)]+\alpha s(t+\beta)(1-s)}{(t+\beta)[2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)+\alpha t(1-t)]}, \right.$$
$$\min_{s\in\mathbb{N}_{t,\eta-1}} \frac{(t+\beta)[2(T+1)-\alpha\eta(\eta+1)+2s(\alpha\eta-1)+\alpha s(1-s)]}{(t+\beta)[2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)+\alpha t(1-t)]},$$
$$\left. \min_{s\in\mathbb{N}_{\eta,T}} \frac{2(t+\beta)(T+1-s)}{(t+\beta)[2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)+\alpha t(1-t)]} \right\}$$

$$\geq \min_{t\in\mathbb{N}_{1,\eta-1}} \left\{ \frac{(1+\beta)[2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)]+\alpha(t-1)(t+\beta)(2-t)}{(t+\beta)[2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)+\alpha t(1-t)]}, \right.$$
$$\min_{s\in\mathbb{N}_{t,\eta-1}} \frac{2(T+1)-\alpha\eta(\eta+1)+2s(\alpha\eta-1)+\alpha s(1-s)}{2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)+\alpha t(1-t)},$$
$$\left. \frac{2}{2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)+\alpha t(1-t)} \right\}$$

$$\geq \begin{cases} \displaystyle\min_{t\in\mathbb{N}_{1,\eta-1}} & \left\{ \dfrac{(1+\beta)[2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)]-\alpha(t+\beta)(t-1)^2+\alpha(t+\beta)(t-1)}{(t+\beta)[2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)+\alpha t(1-t)]}, \right. \\ & \dfrac{2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)+\alpha(\eta-1)(2-\eta)}{2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)+\alpha t(1-t)}, \\ & \left. \dfrac{2}{2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)+\alpha t(1-t)} \right\}; \qquad \text{if } \alpha > \dfrac{1}{\eta} \\ \displaystyle\min_{t\in\mathbb{N}_{1,\eta-1}} & \left\{ \dfrac{(1+\beta)[2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)]-\alpha(t+\beta)(t-1)^2+\alpha(t+\beta)(t-1)}{(t+\beta)[2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)+\alpha t(1-t)]}, \right. \\ & \dfrac{2(T+1)-\alpha\eta(\eta+1)+2(\eta-1)(\alpha\eta-1)+\alpha(2-\eta)(\eta-1)}{2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)+\alpha t(1-t)}, \\ & \left. \dfrac{2}{2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)+\alpha t(1-t)} \right\}; \qquad \text{if } 0 < \alpha < \dfrac{1}{\eta} \end{cases}$$

$$\geq \begin{cases} \min & \left\{ \dfrac{(1+\beta)[2T-\alpha\eta(\eta-1)]-\alpha(\eta-1+\beta)(\eta-2)^2}{(\eta+\beta-1)[2(T+2-\eta)+\alpha\eta(\eta-3)]}, \dfrac{2[T-\alpha-\alpha\eta(\eta-2)]}{2(T+2-\eta)+\alpha\eta(\eta-3)}, \right. \\ & \left. \dfrac{2}{2(T+2-\eta)+\alpha\eta(\eta-3)} \right\}; \quad \text{if } \alpha > \dfrac{1}{\eta} \\ \min & \left\{ \dfrac{(1+\beta)[2(T+2-\eta)+\alpha\eta(\eta-3)]-\alpha(\eta-1+\beta)(\eta-2)^2}{(\eta+\beta-1)[2T-\alpha\eta(\eta-1)]}, \dfrac{2(T+2-\eta-\alpha)}{2T-\alpha\eta(\eta-1)}, \right. \\ & \left. \dfrac{2}{2T-\alpha\eta(\eta-1)} \right\}; \qquad \text{if } 0 < \alpha < \dfrac{1}{\eta} \end{cases}$$

Similarly, we get

$$\min_{(t,s)\in\mathbb{N}_{\eta,T}\times\mathbb{N}_{1,T}} \frac{G(t,s)}{G(t,t)}$$

$$\geq \begin{cases} \min & \left\{ \frac{(1+\beta)[2(T+1-\eta)+\alpha\eta(\eta-1)]-\alpha(T+\beta)(\eta-1)^2+\alpha(\eta-1)(\eta+\beta)}{2(T+\beta)(T+1-\eta)}, \right. \\ & \left. \frac{2(\eta+\beta)+\alpha\eta(\eta+1+2\beta)}{2(T+\beta)(T+1-\eta)},\ \frac{1}{T+1-\eta} \right\}; \quad \text{if } \ \alpha > \frac{1}{\eta} \\ \min & \left\{ \frac{(1+\beta)[2(1+\alpha\eta T)-\alpha\eta(\eta+1)]-\alpha(T+\beta)(\eta-1)^2+\alpha(\eta-1)(\eta+\beta)}{2(T+\beta)(T+1-\eta)}, \right. \\ & \left. \frac{2(\eta+\beta)+\alpha\eta(\eta+1+2\beta)}{2(T+\beta)(T+1-\eta)},\ \frac{1}{T+1-\eta} \right\}; \quad \text{if } \ 0<\alpha<\frac{1}{\eta} \end{cases} \tag{2.13}$$

The (2.10) is immediate from (2.13)-(2.14) □

Lemma 2.4. *Let* $(t,s)\in\mathbb{N}_{T+1}\times\mathbb{N}_{1,T}$. *Then we have*

$$G(t,s)\leq M_2G(t,t.) \tag{2.14}$$

where $M_2\geq 1$ *is a constant given by*

$$M_2 = \begin{cases} \max & \left\{ \frac{(\eta-2+\beta)[2(T+2-\eta)+\alpha\eta(\eta-3)}{2(1+\beta)[T-\alpha(\eta-1)^2]}, \frac{2(T+2-\eta)+\alpha\eta(\eta-3)}{2[T-\alpha(\eta-1)^2]}, \frac{T+1-\eta}{T-\alpha(\eta-1)^2} \right. \\ & \left. \frac{(\eta-1+\beta)[\alpha\eta(2T-\eta-1)+2]}{2(\eta+\beta)}, \frac{2(T-1+\beta)(T+1-\eta)+\alpha\eta(T-\eta)(\eta+1+2\beta)}{2(\eta+\beta)} \right\}; \quad \textit{if} \ \ \alpha>\frac{1}{\eta} \\ \max & \left\{ \frac{(\eta-2+\beta)[2T-\alpha\eta(\eta-1)}{2(1+\beta)(T+2-\eta-\alpha)}, \frac{T+1-\eta}{T+2-\eta-\alpha}, \frac{(\eta-1+\beta)[2(T+1-\eta)+\alpha\eta(\eta-1)}{2(\eta+\beta)}, \right. \\ & \left. \frac{2(T-1+\beta)(T+1-\eta)+\alpha\eta(T-\eta)(\eta+1+2\beta)}{2(\eta+\beta)} \right\}; \quad \textit{if} \ \ 0<\alpha<\frac{1}{\eta} \end{cases}$$

Proof. For $k=0$, from (2.5) we get

$$G(0,s)=2\beta(T+1-s)<2\beta(T+1)=G(0,0).$$

Then we may choose $M_2=1$. For $k\in\mathbb{N}_{1,T}$, if (2.14) holds, it is sufficient that M_2 satisfies

$$M_2\geq \max_{(t,s)\in\mathbb{N}_{1,T}\times\mathbb{N}_{1,T}} \frac{G(t,s)}{G(t,t)}. \tag{2.15}$$

Then we may choose

$$M_2\geq \max\left\{ \max_{(t,s)\in\mathbb{N}_{1,\eta-1}\times\mathbb{N}_{1,T}} \frac{G(t,s)}{G(t,t)}, \max_{(t,s)\in\mathbb{N}_{\eta,T}\times\mathbb{N}_{1,T}} \frac{G(t,s)}{G(t,t)} \right\}. \tag{2.16}$$

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since

$$\begin{aligned}
&\max_{(t,s)\in\mathbb{N}_{1,\eta-1}\times\mathbb{N}_{1,T}} \frac{G(t,s)}{G(t,t)}\\
&= \max_{t\in\mathbb{N}_{1,\eta-1}} \Bigg\{ \max_{s\in\mathbb{N}_{1,t-1}} \frac{(s+\beta)[2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)]+\alpha s(t+\beta)(1-s)}{(t+\beta)[2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)+\alpha t(1-t)]},\\
&\qquad \max_{s\in\mathbb{N}_{t,\eta-1}} \frac{(t+\beta)[2(T+1)-\alpha\eta(\eta+1)+2s(\alpha\eta-1)+\alpha s(1-s)]}{(t+\beta)[2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)+\alpha t(1-t)]},\\
&\qquad \max_{s\in\mathbb{N}_{\eta,T}} \frac{2(t+\beta)(T+1-s)}{(t+\beta)[2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)+\alpha t(1-t)]}\Bigg\}
\end{aligned}$$

$$\begin{aligned}
&\le \max_{t\in\mathbb{N}_{1,\eta-1}} \Bigg\{ \frac{(t-1+\beta)[2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)]}{(t+\beta)[2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)+\alpha t(1-t)]},\\
&\qquad \max_{s\in\mathbb{N}_{t,\eta-1}} \frac{2(T+1)-\alpha\eta(\eta+1)+2s(\alpha\eta-1)+\alpha s(1-s)}{2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)+\alpha t(1-t)},\\
&\qquad \frac{2(T+1-\eta)}{2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)+\alpha t(1-t)}\Bigg\}
\end{aligned}$$

$$\le \begin{cases}
\max\limits_{t\in\mathbb{N}_{1,\eta-1}} \Bigg\{ \dfrac{(t-1+\beta)[2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)]}{(t+\beta)[2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)+\alpha t(1-t)]},\\
\quad \dfrac{2(T+1)-\alpha\eta(\eta+1)+2(\eta-1)(\alpha\eta-1)+\alpha t(1-t)}{2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)+\alpha t(1-t)},\\
\quad \dfrac{2(T+1-\eta)}{2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)+\alpha t(1-t)}\Bigg\}; & \text{if } \alpha > \dfrac{1}{\eta}\\
\max\limits_{t\in\mathbb{N}_{1,\eta-1}} \Bigg\{ \dfrac{(t-1+\beta)[2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)]}{(t+\beta)[2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)+\alpha t(1-t)]},\\
\quad \dfrac{2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)+\alpha t(1-t)}{2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)+\alpha t(1-t)},\\
\quad \dfrac{2(T+1-\eta)}{2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)+\alpha t(1-t)}\Bigg\}; & \text{if } 0<\alpha < \dfrac{1}{\eta}
\end{cases}$$

$$\le \begin{cases}
\max \Bigg\{ \dfrac{(\eta-2+\beta)[2(T+2-\eta)+\alpha\eta(\eta-3)}{2(1+\beta)[T-\alpha(\eta-1)^2]}, \dfrac{2(T+2-\eta)+\alpha\eta(\eta-3)}{2[T-\alpha(\eta-1)^2]},\\
\quad \dfrac{T+1-\eta}{T-\alpha(\eta-1)^2}\Bigg\}; & \text{if } \alpha > \dfrac{1}{\eta}\\
\max \Bigg\{ \dfrac{(\eta-2+\beta)[2T-\alpha\eta(\eta-1)}{2(1+\beta)(T+2-\eta-\alpha)}, 1, \dfrac{T+1-\eta}{T+2-\eta-\alpha}\Bigg\}; & \text{if } 0<\alpha < \dfrac{1}{\eta}
\end{cases}$$

Similarly, we get

$$\max_{(t,s)\in\mathbb{N}_{\eta,T}\times\mathbb{N}_{1,T}}\frac{G(t,s)}{G(t,t)}$$
$$\leq\begin{cases}\max\left\{\dfrac{(\eta-1+\beta)[\alpha\eta(2T-\eta-1)+2]}{2(\eta+\beta)},\right.\\ \left.\dfrac{2(T-1+\beta)(T+1-\eta)+\alpha\eta(T-\eta)(\eta+1+2\beta)}{2(\eta+\beta)},\ 1\right\}; & \text{if } \alpha>\dfrac{1}{\eta}\\ \max\left\{\dfrac{(\eta-1+\beta)[2(T+1-\eta)+\alpha\eta(\eta-1)}{2(\eta+\beta)},\right.\\ \left.\dfrac{2(T-1+\beta)(T+1-\eta)+\alpha\eta(T-\eta)(\eta+1+2\beta)}{2(\eta+\beta)},\ 1\right\}; & \text{if } 0<\alpha<\dfrac{1}{\eta}\end{cases}$$

For $t=T+1$ from (2.5) we get,

$$\begin{aligned}G(T+1,s)&=\alpha\eta(s+\beta)[2(T+1)-(\eta+1)]-\alpha s(T+1+\beta)(s-1)\\&<\alpha\eta(T+1+\beta)[2(T+1)-(\eta+1)]-\alpha T(T+1+\beta)(T+1)\\&=G(T+1,T+1).\end{aligned}$$

Then we choose $M_2=1$. So (2.16) is immediate from (2.19)-(2.20). $\square$

3 Main Results

Now we are in the position to establish the main result.

Theorem 3.1. *Assume* $(H1)$ - $(H3)$ *hold. Then the problem* (1.1)-(1.2) *has at least one positive solution.*

Proof. In the following, we denote

$$m=\min_{t\in\mathbb{N}_{\eta,T}}G(t,t),\quad M=\max_{t\in\mathbb{N}_{T+1}}G(t,t).$$

Then $0<m<M$.

Let Ebe the Banach's space defined by $E=\{u:\mathbb{N}_{T+1}\to R\}$. Define

$$K=\{u\in E: u\geqslant 0, t\in\mathbb{N}_{T+1}\ and\ \min_{t\in\mathbb{N}_{1,T}}u(t)\geq\sigma\parallel u\parallel\}.$$

where $\sigma=\frac{M_1m}{M_2M}\in(0,1)$, $\parallel u\parallel=\max_{t\in\mathbb{N}_{T+1}}\mid u(t)\mid$. It is obvious that K is a cone in E.

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We define the operator $F : K \to E$ by

$$(Fu)(t) = \sum_{s=1}^{T} G(t,s)a(s)f(u(s)), t \in \mathbb{N}_{T+1}.$$

It is clear that problem (1.1)-(1.2) has a solution u if and only if $u \in K$ is a fixed point of operator F. We shall now show that the operator F maps K to itself. For this, let $u \in K$, from $(H_2) - (H_3)$, we get

$$(Fu)(t) = \sum_{s=1}^{T} G(t,s)a(s)f(u(s)) \geq 0, t \in \mathbb{N}_{T+1}. \tag{3.1}$$

from (2.8), we obtain

$$\begin{aligned}(Fu)(t) =& \sum_{s=1}^{T} G(t,s)a(s)f(u(s)) \leq M_2 \sum_{s=1}^{T} G(t,t)a(s)f(u(s)) \\ \leq& M_2 M \sum_{s=1}^{T} a(s)f(u(s)), \quad t \in \mathbb{N}_{T+1}.\end{aligned}$$

Therefore

$$\| Fu \| \leq M_2 M \sum_{s=1}^{T} a(s)f(u(s)). \tag{3.2}$$

Now from $(H_2), (H_3)$, (2.7) and (3.2), for $t \in \mathbb{N}_{\eta,T}$, we have

$$\begin{aligned}(Fu)(t) \geq& M_1 \sum_{s=1}^{T} G(t,t)a(s)f(u(s)) \geq M_1 m \sum_{s=1}^{T} a(s)f(u(s)) \\ \geq& \frac{M_1 m}{M_2 M} \| Fu \| = \sigma \| u \| .\end{aligned}$$

Then

$$\min_{t \in \mathbb{N}_{\eta,T}} (Fu)(t) \geq \sigma \| u \| . \tag{3.3}$$

From (3.1)-(3.2), we obtain $Fu \in K$, Hence $F(K) \subseteq K$. So $F : k \to K$ is completely continuous.

Superlinear case. $f_0 = 0$ and $f_\infty = \infty$. Since $f_0 = 0$, we may choose $H_1 > 0$ so that $f(u) \leqslant \epsilon_1 u$, for $0 < u \leqslant H_1$, where $\epsilon_1 > 0$ satisfies

$$\epsilon_1 M_2 M \sum_{s=1}^{T} a(s) \leq 1. \tag{3.4}$$

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Thus, if we let

$$\Omega_1 = \{u \in E : \|u\| < H_1\},$$

then for $u \in K \cap \partial\Omega_1$, we get

$$\begin{aligned}(Fu)(t) \leq & M_2 \sum_{s=1}^{T} G(t,t)a(s)f(u(s)) \leq \epsilon_1 M_2 M \sum_{s=1}^{T} a(s)u(s) \\ \leq & \epsilon_1 M_2 M \sum_{s=1}^{T} a(s)\|u\| \leq \|u\|.\end{aligned}$$

Thus $\|Fu\| \leq \|u\|$, $u \in K \cap \partial\Omega_1$.

Further, since $f_\infty = \infty$, there exists $\widehat{H}_2 > 0$ such that $f(u) \geq \epsilon_2 u$, for $u \geq \widehat{H}_2$, where $\epsilon_2 > 0$ satisfies

$$\epsilon_2 M_1 \sigma \sum_{s=\eta}^{T} G(\eta,\eta)a(s) \geq 1. \tag{3.5}$$

Let $H_2 = \max\{2H_1, \frac{\widehat{H}_2}{\sigma}\}$ and $\Omega_2 = \{u \in E : \|u\| < H_2\}$. Then $u \in K \cap \partial\Omega_2$ implies

$$\min_{t\in\mathbb{N}_{\eta,T}} u(t) \geq \sigma\|u\| \geq \widehat{H}_2.$$

Applying (2.7) and (3.5), we get

$$\begin{aligned}(Fu)(\eta) = & M_1 \sum_{s=1}^{T} G(\eta,s)a(s)f(u(s)) \geq M_1 \sum_{s=\eta}^{T} G(\eta,\eta)a(s)f(u(s)) \\ \geq & \varepsilon_2 M_1 \sum_{s=\eta}^{T} G(\eta,\eta)a(s)y(s) \geq \varepsilon_2 M_1 \sigma \sum_{s=\eta}^{T} G(\eta,\eta)a(s)\|u\| \\ \geq & \|u\|.\end{aligned}$$

Hence, $\|Fu\| \geq \|u\|$, $u \in K \cap \partial\Omega_2$. By the first part of Theorem 1.1, F has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$ such that $H_1 \leqslant \|u\| \leqslant H_2$.

Sublinear case. $f_0 = \infty$ and $f_\infty = 0$. Since $f_0 = \infty$, choose $H_3 > 0$ such that $f(u) \geqslant \epsilon_3 u$ for $0 < u \leqslant H_3$, where $\varepsilon_3 > 0$ satisfies

$$\epsilon_3 M_1 \sigma \sum_{s=\eta}^{T} G(\eta,\eta)a(s) \geqslant 1. \tag{3.6}$$

Let

$$\Omega_3 = \{u \in E : \|u\| < H_3\},$$

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then for $u \in K \cap \partial\Omega_3$, we get

$$\begin{aligned}(Fu)(\eta) \geq & M_1 \sum_{s=\eta}^{T} G(\eta,\eta)a(s)f(u(s)) \geq \epsilon_3 M_1 \sum_{s=\eta}^{T} G(\eta,\eta)a(s)y(s) \\ \geq & \varepsilon_3 M_1 \sigma \sum_{s=\eta}^{T} G(\eta,\eta)a(s)\|u\| \geq \|u\|.\end{aligned}$$

Thus, $\|Fu\| \geqslant \|u\|$, $u \in K \cap \partial\Omega_3$.

Now, since $f_\infty = 0$, there exists $\widehat{H}_4 > 0$ so that $f(u) \leqslant \epsilon_4 u$ for $u \geqslant \widehat{H}_4$, where $\epsilon_4 > 0$ satisfies

$$\epsilon_4 M_2 M \sum_{s=\eta}^{T} a(s) \geqslant 1. \tag{3.7}$$

Subcase 1. Suppose f is bounded, $f(u) \leq L$ for all $u \in [0,\infty)$ for some $L > 0$. Let $H_4 = \max\{2H_3, LM_2M \sum_{s=1}^{T} a(s)\}$.

Then for $u \in K$ and $\|u\| = H_4$, we get

$$\begin{aligned}(Fu)(\eta) \leq & M_2 \sum_{s=1}^{T} G(t,t)a(s)f(u(s)) \leq LM_2M \sum_{s=1}^{T} a(s) \\ \leq & H_4 = \|u\|\end{aligned}$$

Thus $(Fu)(t) \leq \|u\|$.

Subcase 2. Suppose f is unbounded, there exist $H_4 > \max\{2H_3, \frac{\widehat{H}_4}{\sigma}\}$ such that $f(u) \leq f(H_4)$ for all $0 < u \leq H_4$. Then for $u \in K$ with $\|u\| = H_4$ from (2.8) and (3.7), we have

$$\begin{aligned}(Fu)(t) \leq & M_2 \sum_{s=1}^{T} G(t,t)a(s)f(u(s)) \leq M_2M \sum_{s=1}^{T} a(s)f(H_4) \\ \leq & \epsilon_4 M_2 M \sum_{s=1}^{T} a(s)H_4 \leq H_4 = \|u\|.\end{aligned}$$

Thus in both cases, we may put $Omega_4 = \{u \in E : \|u\| < H_4\}$. Then

$$\|Fu\| \leqslant \|u\|, u \in K \cap \partial\Omega_4.$$

By the second part of Theorem 1.1, A has a fixed point u in $K \cap (\overline{\Omega}_4 \setminus \Omega_3)$, such that $H_3 \leqslant \|u\| \leqslant H_4$. This completes the sublinear part of the theorem. Therefore, the problem (1.1)-(1.2) has at least one positive solution. □

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4 Some examples

In this section, in order to illustrate our result, we consider some examples.

Example 4.1 Consider the BVP

$$\Delta^2 u(t-1) + t^2 u^k = 0, \qquad t \in N_{1,4}, \tag{4.1}$$

$$u(0) = \frac{1}{4}\Delta u(0), \qquad u(5) = \frac{2}{3}\sum_{s=1}^{2} u(s). \tag{4.2}$$

Set $\alpha = \frac{2}{3},\ \beta = \frac{1}{4},\ \eta = 2,\ T = 4,\ a(t) = t^2,\ f(u) = u^k$.
We can show that

$$2(T+1) - \alpha\eta(\eta+1) - 2\beta(\alpha\eta - 1) = \frac{40}{6} > 0.$$

Case I : $k \in (1, \infty)$. In this case, $f_0 = 0,\ f_\infty = \infty$ and (i) of theorem 3.1 holds. Then BVP (4.1)-(4.2) has at least one positive solution.

Case II : $k \in (0, 1)$. In this case, $f_0 = \infty,\ f_\infty = 0$ and (ii) of theorem 3.1 holds. Then BVP (4.1)-(4.2) has at least one positive solution.

Example 4.2 Consider the BVP

$$\Delta^2 u(t-1) + e^t t^e (\frac{\pi \sin u + 2\cos u}{u^2}) = 0, \qquad t \in N_{1,4}, \tag{4.3}$$

$$u(0) = \frac{2}{5}\Delta u(0), \qquad u(5) = \frac{1}{3}\sum_{s=1}^{3} u(s), \tag{4.4}$$

Set $\alpha = \frac{1}{3},\ \beta = \frac{2}{5},\ \eta = 3,\ T = 4,\ a(t) = e^t t^e,\ f(u) = \frac{\pi \sin u + 2\cos u}{u^2}$.
We can show that

$$\Lambda = 2(T+1) - \alpha\eta(\eta+1) - 2\beta(\alpha\eta - 1) = 6 > 0,$$

Through a simple calculation we can get $f_0 = \infty,\ f_\infty = 0$. Thus, by (ii) of theorem 3.1, we can get BVP (4.3)-(4.4) has at least one positive solution.

Acknowledgement(s) : This research (KMUTNB-GEN-56-12) is supported by King Mongkut's University of Technology North Bangkok, Thailand.

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Essential norm of extended Cesàro operators from $F(p,q,s)$ space to Bloch-type space

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Abstract: Let T_g be the extended Cesàro operator with holomorphic symbol g on the unit ball of $\mathbb{C}^n$. In this paper, we characterize the essential norm of T_g as an operator from the $F(p,q,s)$ space to the Bloch-type space $\mathcal{B}_\mu$, where μ is a given normal weight.

Keywords: $F(p,q,s)$ space, Bloch-type space, Extended Cesàro operator, Essential norm.

2010 MR Subject Classification: Primary 32AB37, Secondary 47B38.

1 Introduction

Let $\mathbb{C}^n$ be the Euclidean space of complex dimension n. For $z=(z_1,\cdots,z_n)$ and $w=(w_1,\cdots,w_n)$ in $\mathbb{C}^n$, we define

$$\langle z,w\rangle=\sum_{j=1}^{n} z_j\overline{w_j},\quad |z|=\sqrt{\langle z,z\rangle}.$$

Let $\mathbb{B}=\{z\in\mathbb{C}^n:|z|<1\}$ be the open unit ball in $\mathbb{C}^n$. When $n=1$, the open unit ball is just the open unit disc $\mathbb{D}$.

Let $H(\mathbb{B})$ be the family of all holomorphic functions on $\mathbb{B}$. For $a\in\mathbb{B}$, let $h(z,a)=\log\frac{1}{|\varphi_a(z)|}$ be the Green function with logarithmic singularity at a, where φ_a is the holomorphic automorphism of $\mathbb{B}$ which interchanges 0 and a. Denote by $\Re f(z)=\sum_{j=1}^{n} z_j\frac{\partial f}{\partial z_j}(z)$ the radial derivative of $f\in H(\mathbb{B})$.

For $0<p,s<\infty$, $-n-1<q<\infty$, the $F(p,q,s)$ space consists of all functions $f\in H(\mathbb{B})$ for which

$$\|f\|^p_{F(p,q,s)}=|f(0)|^p+\sup_{a\in\mathbb{B}}\int_{\mathbb{B}}|\Re f(z)|^p(1-|z|^2)^q h^s(z,a)dv(z)<\infty,$$

where dv is the normalized volume measure on $\mathbb{B}$. In one complex variable setting, the $F(p,q,s)$ space was first introduced by Zhao in [1]. Many people call the $F(p,q,s)$ space

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general function space because we can get many function spaces, such as Hardy space, Bergman space, Q_p space, BMOA space, Besov space and Bolch-type space, if we take some special parameters of p, q and s (see [2]). Notice that the $F(p, q, s)$ space is just the space of constant functions for $q + s \leq -1$.

A positive continuous function μ on $[0, 1)$ is said to be normal if there are two constants $b > a > 0$ such that

$$\frac{\mu(r)}{(1-r)^a} \downarrow 0, \qquad \frac{\mu(r)}{(1-r)^b} \uparrow \infty$$

as $r \to 1^-$. For example, $\omega(r) = (1-r^2)^p$ with $0 < p < \infty$, $\omega(r) = (1-r^2)^\alpha \left(\ln \frac{e^{\frac{\beta}{\alpha}}}{1-r^2}\right)^\beta$ with $0 < \alpha < \infty$ and $0 \leq \beta < \infty$, and $\omega(r) = 1/\{\log\log e^2(1-r^2)^{-1}\}$ are all normal weights. From now on if we say that a function $\mu : \mathbb{B} \longrightarrow (0, \infty)$ is normal we also assume that it is radial on $\mathbb{B}$, that is $\mu(z) = \mu(|z|)$ for $z \in \mathbb{B}$.

A function $f \in H(\mathbb{B})$ is said to belong to the Bloch-type space $\mathcal{B}_\mu$ if

$$\|f\|_{\mathcal{B},\mu} = \sup_{z\in\mathbb{B}} \mu(z)|\nabla f(z)| < \infty,$$

and it is said to belong to the little Bloch-type space $\mathcal{B}_{\mu,0}$ if

$$\lim_{|z|\to 1} \mu(z)|\nabla f(z)| = 0.$$

Here $\nabla f(z) = \left(\frac{\partial f}{\partial z_1}(z), \cdots, \frac{\partial f}{\partial z_n}(z)\right)$ is the complex gradient of f. It is easy to check that both of $\mathcal{B}_\mu$ and $\mathcal{B}_{\mu,0}$ are Banach spaces under the norm

$$\|f\|_\mu = |f(0)| + \|f\|_{\mathcal{B},\mu}$$

and that $\mathcal{B}_{\mu,0}$ is a closed subspace of $\mathcal{B}_\mu$. When $\mu(r) = 1 - r^2$ and $\mu(r) = (1 - r^2)^{1-\alpha}$ with $\alpha \in (0, 1)$, two typical normal weights, the induced spaces $\mathcal{B}_\mu$ are the Bloch space and the Lipschitz space, respectively. And also, the space $\mathcal{B}_{(1-r^2)\log 1/(1-r^2)}$ is the logarithmic Bloch space which was studied in [3]. It follows from [4] that $f \in \mathcal{B}_\mu$ if and only if $\sup_{z\in\mathbb{B}} \mu(z)|\Re f(z)| < \infty$, and $f \in \mathcal{B}_{\mu,0}$ if and only if $\lim_{|z|\to 1} \mu(z)|\Re f(z)| = 0$. Furthermore,

$$\|f\|_\mu \simeq |f(0)| + \sup_{z\in\mathbb{B}} \mu(z)|\Re f(z)|.$$

For $g \in H(\mathbb{B})$, the extended Cesàro operator T_g on $H(\mathbb{B})$ is defined by

$$T_g f(z) = \int_0^1 f(tz)\Re g(tz)\frac{dt}{t}, \qquad f \in H(\mathbb{B}), \qquad z \in \mathbb{B}.$$

This operator is also called the Riemann-Stieltjes operator, which was first introduced in [5]. For related results see also [4, 6-9] and the references therein.

Let X and Y be two Banach (or Fréchet) spaces. And let T be a bounded linear operator from X to Y with the operator norm $\|T\|_{X\to Y}$. Denote by $\mathcal{K}$ the set of all compact operators

from X to Y. The essential norm $\|T\|_{e,X\to Y}$ is defined to be the distance from T to $\mathcal{K}$, that is,

$$\|T\|_{e,X\to Y} = \inf_{Q\in\mathcal{K}} \|T-Q\|_{X\to Y}.$$

Clearly, $T : X \to Y$ is compact if and only if $\|T\|_{e,X\to Y} = 0$. And if $T : X \to Y$ is not bounded, then $\|T\|_{e,X\to Y} = \infty$.

The essential norm for some linear operators, such as composition operator, weighted composition operator and Toeplitz operator, has attracted many mathematician's attention; for example, see [10-14] and the references therein. In the unit disc setting, Liu, Lou and X-iong gave the essential norm of the integral operators I_g and T_g acting on several holomorphic function spaces in [15]. Here,

$$I_g f(z) = \int_0^z f'(\xi)g(\xi)d\xi, \quad T_g f(z) = \int_0^z f(\xi)g'(\xi)d\xi, z \in \mathbb{D}.$$

At the same time, Yu and Liu considered the similar problem between Bloch-type space and Q_K type space on the unit disc in [16]. Meanwhile, in the case of the unit ball, Hu obtained the essential norm of the extended Cesàro operator T_g from the Bergman space $A^p(\mu)$ to the Bergman space $A^q(\mu)$ with $0 < p, q < \infty$ in [17]. Motivated by these results, the purpose of this paper is to study the essential norm of T_g from the $F(p,q,s)$ space to the Bloch-type space $\mathcal{B}_\mu$ on the unit ball.

In what follows, C stands for positive constants whose value may change from line to line but does not depend on the functions in $H(\mathbb{B})$. The expression $A \simeq B$ means $C^{-1}A \le B \le CA$.

2 Main results

For $g \in H(\mathbb{B})$, set $M_\infty(g,r) = \sup_{|z|=r} |g(z)|$. It is well known that $M_\infty(g,r)$ is increasing with $r \in [0,1)$. In the proof of our main theorems, we need the following lemmas.

Lemma 2.1([17]) Let ψ be a positive continuous function on $[0,1)$ with

$$0 < \limsup_{r\to 1} \psi(r) \le \infty.$$

Then there is a constant C such that for any $g \in H(\mathbb{B})$,

$$\sup_{z\in\mathbb{B}} |g(z)|\psi(|z|) \le C \limsup_{|z|\to 1} |g(z)|\psi(|z|).$$

Lemma 2.2 ([18]) Let μ be normal and $g \in H(\mathbb{B})$. Then for $n+1+q > p$,

$$\|T_g\|_{F(p,q,s)\to\mathcal{B}_\mu} \simeq \sup_{z\in\mathbb{B}} \mu(z)|\Re g(z)|(1-|z|^2)^{1-\frac{n+1+q}{p}};$$

and for $n+1+q=p$,

$$\|T_g\|_{F(p,q,s)\to\mathcal{B}_\mu} \simeq \sup_{z\in\mathbb{B}} \mu(z)|\Re g(z)| \log\frac{2}{1-|z|^2}.$$

Theorem 2.1 Suppose μ is a normal weight, $g\in H(\mathbb{B})$ and $n+1+q>p$. Then

$$\|T_g\|_{e,F(p,q,s)\to\mathcal{B}_\mu} \simeq \limsup_{|z|\to 1} \mu(z)|\Re g(z)|(1-|z|^2)^{1-\frac{n+1+q}{p}}.$$

Proof. Given $\zeta\in\mathbb{B}$, take

$$f_\zeta(z)=\frac{1-|\zeta|^2}{(1-\langle z,\zeta\rangle)^{\frac{n+1+q}{p}}}, \quad z\in\mathbb{B}.$$

Then by [19],

$$\|f_\zeta\|_{F(p,q,s)}\le C \quad\text{and}\quad f_\zeta(\zeta)=(1-|\zeta|^2)^{1-\frac{n+1+q}{p}}.$$

Moreover, it is easy to see that $f_\zeta(z)$ converges to 0 uniformly on any compact subset of $\mathbb{B}$ as $|\zeta|\to 1$. Hence, for any $Q\in\mathcal{K}$, we obtain

$$\lim_{|\zeta|\to 1}\|Qf_\zeta\|_\mu=0.$$

Suppose $\{\zeta_j\}\subset\mathbb{B}$ such that $\lim_{j\to\infty}|\zeta_j|=1$ and

$$\lim_{j\to\infty}\mu(\zeta_j)|\Re g(\zeta_j)|(1-|\zeta_j|^2)^{1-\frac{n+1+q}{p}} = \limsup_{|z|\to 1}\mu(z)|\Re g(z)|(1-|z|^2)^{1-\frac{n+1+q}{p}}.$$

Notice that $\Re(T_g f)=f\Re g$ and $\Re g(0)=0$. Then for each $Q\in\mathcal{K}$,

$$\begin{aligned}
\|T_g-Q\|_{F(p,q,s)\to\mathcal{B}_\mu} &\ge C\limsup_{j\to\infty}\|(T_g-Q)f_{\zeta_j}\|_\mu\\
&\ge C\left\{\limsup_{j\to\infty}\|T_g f_{\zeta_j}\|_\mu-\lim_{j\to\infty}\|Qf_{\zeta_j}\|_\mu\right\}\\
&= C\limsup_{j\to\infty}\|T_g f_{\zeta_j}\|_\mu\\
&\ge C\limsup_{j\to\infty}\sup_{z\in\mathbb{B}}\mu(z)|f_{\zeta_j}(z)\Re g(z)|\\
&\ge C\limsup_{j\to\infty}\mu(\zeta_j)|f_{\zeta_j}(\zeta_j)\Re g(\zeta_j)|\\
&= C\limsup_{j\to\infty}\mu(\zeta_j)|\Re g(\zeta_j)|(1-|\zeta_j|^2)^{1-\frac{n+1+q}{p}}.
\end{aligned}$$

By the definition of essential norm and the estimate above, we get

$$\|T_g\|_{e,F(p,q,s)\to\mathcal{B}_\mu}\ge C\limsup_{|z|\to 1}\mu(z)|\Re g(z)|(1-|z|^2)^{1-\frac{n+1+q}{p}}. \tag{2.1}$$

Now, we prove the reverse inequality. This will be split into two cases.

Case 1. Suppose

$$\limsup_{|z|\to 1} \mu(z)(1-|z|^2)^{1-\frac{n+1+q}{p}} = 0. \tag{2.2}$$

For $\rho \in (0,1)$, set $g_\rho(z) = g(\rho z)$. Then

$$\Re g_\rho(z) = \sum_{j=1}^n z_j \frac{\partial g_\rho}{\partial z_j}(z) = \sum_{j=1}^n z_j \frac{\partial g}{\partial z_j}(\rho z)\rho = \Re g(\rho z). \tag{2.3}$$

It is clear that $g_\rho(z)$ is holomorphic on the closed unit ball $\overline{\mathbb{B}}$. This, with (2.2) in mind, shows

$$\lim_{|z|\to 1} \mu(z)|\Re g_\rho(z)|(1-|z|^2)^{1-\frac{n+1+q}{p}} = 0.$$

Thus, Theorem 3.2 in [18] implies that $T_{g_\rho} : F(p,q,s) \to \mathcal{B}_\mu$ is compact. Since

$$\lim_{\delta\to 1} \sup_{\delta<|z|<1} \mu(z)|\Re g(z)|(1-|z|^2)^{1-\frac{n+1+q}{p}} = \limsup_{|z|\to 1} \mu(z)|\Re g(z)|(1-|z|^2)^{1-\frac{n+1+q}{p}},$$

we know that for any $\varepsilon > 0$, there exists some $\delta \in (0,1)$ such that

$$\sup_{\delta<|z|<1} \mu(z)|\Re g(z)|(1-|z|^2)^{1-\frac{n+1+q}{p}} < \limsup_{|z|\to 1} \mu(z)|\Re g(z)|(1-|z|^2)^{1-\frac{n+1+q}{p}} + \varepsilon. \tag{2.4}$$

This, together with the fact that $M_\infty(\Re g, r)$ is increasing with r, yields

$$\begin{aligned}
\sup_{\delta<|z|<1} \mu(z)|\Re g(\rho z)|(1-|z|^2)^{1-\frac{n+1+q}{p}} &= \sup_{\delta<r<1} \mu(r)M_\infty(\Re g, \rho r)(1-r^2)^{1-\frac{n+1+q}{p}} \\
&\le \sup_{\delta<r<1} \mu(r)M_\infty(\Re g, r)(1-r^2)^{1-\frac{n+1+q}{p}} \\
&< \limsup_{|z|\to 1} \mu(z)|\Re g(z)|(1-|z|^2)^{1-\frac{n+1+q}{p}} + \varepsilon.
\end{aligned} \tag{2.5}$$

On the other hand, by (2.2), there is a constant C_1 such that

$$\sup_{|z|\le\delta} \mu(z)(1-|z|^2)^{1-\frac{n+1+q}{p}} \le C_1.$$

For the fixed $\delta \in (0,1)$ above, it follows from $\Re g \in H(\overline{B(0,\delta)})$ that we have some ρ being close to 1 enough such that

$$|\Re g(z) - \Re g(\rho z)| < \frac{\varepsilon}{C_1},$$

where $\overline{B(0,\delta)} = \{z \in \mathbb{C}^n : |z| \le \delta\}$. Hence,

$$\sup_{|z|\le\delta} |\Re g(z) - \Re g(\rho z)|\mu(z)(1-|z|^2)^{1-\frac{n+1+q}{p}} < \varepsilon. \tag{2.6}$$

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Therefore, by Lemma 2.2 and (2.3)-(2.6), we obtain

$$\begin{aligned}
\|T_g\|_{e,F(p,q,s)\to\mathcal{B}_\mu} &\leq \|T_g - T_{g_\rho}\|_{F(p,q,s)\to\mathcal{B}_\mu} \\
&\simeq \sup_{z\in\mathbb{B}} |\Re(g-g_\rho)(z)|\mu(z)(1-|z|^2)^{1-\frac{n+1+q}{p}} \\
&= \sup_{z\in\mathbb{B}} |\Re g(z) - \Re g(\rho z)|\mu(z)(1-|z|^2)^{1-\frac{n+1+q}{p}} \\
&\leq \sup_{|z|\leq\delta} |\Re g(z) - \Re g(\rho z)|\mu(z)(1-|z|^2)^{1-\frac{n+1+q}{p}} \\
&+ \sup_{\delta<|z|<1} |\Re g(z) - \Re g(\rho z)|\mu(z)(1-|z|^2)^{1-\frac{n+1+q}{p}} \\
&\leq \sup_{|z|\leq\delta} |\Re g(z) - \Re g(\rho z)|\mu(z)(1-|z|^2)^{1-\frac{n+1+q}{p}} \\
&+ \sup_{\delta<|z|<1} |\Re g(z)|\mu(z)(1-|z|^2)^{1-\frac{n+1+q}{p}} + \sup_{\delta<|z|<1} |\Re g(\rho z)|\mu(z)(1-|z|^2)^{1-\frac{n+1+q}{p}} \\
&< 3\varepsilon + 2\limsup_{|z|\to 1} \mu(z)|\Re g(z)|(1-|z|^2)^{1-\frac{n+1+q}{p}}.
\end{aligned}$$

This gives

$$\|T_g\|_{e,F(p,q,s)\to\mathcal{B}_\mu} \leq C\limsup_{|z|\to 1} \mu(z)|\Re g(z)|(1-|z|^2)^{1-\frac{n+1+q}{p}}. \tag{2.7}$$

Case 2. Suppose

$$\limsup_{|z|\to 1} \mu(z)(1-|z|^2)^{1-\frac{n+1+q}{p}} \neq 0.$$

Then by Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned}
\|T_g\|_{e,F(p,q,s)\to\mathcal{B}_\mu} &\leq \|T_g\|_{F(p,q,s)\to\mathcal{B}_\mu} \\
&\simeq \sup_{0\leq r<1} M_\infty(\Re g, r)\mu(r)(1-r^2)^{1-\frac{n+1+q}{p}} \\
&\leq C\limsup_{r\to 1} M_\infty(\Re g, r)\mu(r)(1-r^2)^{1-\frac{n+1+q}{p}}.
\end{aligned}$$

This, together with (2.1) and (2.7), implies

$$\|T_g\|_{e,F(p,q,s)\to\mathcal{B}_\mu} \simeq \limsup_{|z|\to 1} \mu(z)|\Re g(z)|(1-|z|^2)^{1-\frac{n+1+q}{p}}.$$

This gives the desired result.

Theorem 2.2 Suppose μ is a normal weight, $g \in H(\mathbb{B})$ and $n+1+q = p$. Then

$$\|T_g\|_{e,F(p,q,s)\to\mathcal{B}_\mu} \simeq \limsup_{|z|\to 1} \mu(z)|\Re g(z)|\log\frac{2}{1-|z|^2}.$$

Proof. Given $\zeta \in \mathbb{B}$, set

$$f_\zeta(z) = \frac{\left(\log\frac{2}{1-\langle z,\zeta\rangle}\right)^2}{\log\frac{2}{1-|\zeta|^2}}, \quad z \in \mathbb{B}.$$

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Then by Lemma 2.5 in [18],

$$\|f_\zeta\|_{F(p,q,s)} \le C.$$

Furthermore, it is easy to check that $f_\zeta(\zeta) = \log \frac{2}{1-|\zeta|^2}$, and $f_\zeta(z)$ converges to 0 uniformly on any compact subset of $\mathbb{B}$ as $|\zeta| \to 1$. This shows that for any $Q \in \mathcal{K}$,

$$\lim_{|\zeta|\to 1} \|Qf_\zeta\|_\mu = 0.$$

Suppose $\{\zeta_j\} \subset \mathbb{B}$ such that $\lim\limits_{j\to\infty} |\zeta_j| = 1$ and

$$\lim_{j\to\infty} \mu(\zeta_j)|\Re g(\zeta_j)| \log \frac{2}{1-|\zeta_j|^2} = \limsup_{|z|\to 1} \mu(z)|\Re g(z)| \log \frac{2}{1-|z|^2}.$$

Because $\Re(T_g f) = f\Re g$ and $\Re g(0) = 0$, for any $Q \in \mathcal{K}$, we obtain

$$\begin{aligned}
\|T_g - Q\|_{F(p,q,s)\to\mathcal{B}_\mu} &\ge C \limsup_{j\to\infty} \|(T_g - Q)f_{\zeta_j}\|_\mu \\
&\ge C \left\{ \limsup_{j\to\infty} \|T_g f_{\zeta_j}\|_\mu - \lim_{j\to\infty} \|Qf_{\zeta_j}\|_\mu \right\} \\
&= C \limsup_{j\to\infty} \|T_g f_{\zeta_j}\|_\mu \\
&\ge C \limsup_{j\to\infty} \sup_{z\in\mathbb{B}} \mu(z)|f_{\zeta_j}(z)\Re g(z)| \\
&\ge C \limsup_{j\to\infty} \mu(\zeta_j)|f_{\zeta_j}(\zeta_j)\Re g(\zeta_j)| \\
&= C \limsup_{j\to\infty} \mu(\zeta_j)|\Re g(\zeta_j)| \log \frac{2}{1-|\zeta_j|^2}.
\end{aligned}$$

That means

$$\|T_g\|_{e,F(p,q,s)\to\mathcal{B}_\mu} \ge C \limsup_{|z|\to 1} \mu(z)|\Re g(z)| \log \frac{2}{1-|z|^2}. \tag{2.8}$$

Next, we prove the reverse inequality. As in (1) above, this proof is divided into two steps.

Step 1. Suppose

$$\limsup_{|z|\to 1} \mu(z) \log \frac{2}{1-|z|^2} = 0. \tag{2.9}$$

For $g \in H(\mathbb{B})$, set $g_\rho(z)$ as in (1). Then (2.9) shows

$$\lim_{|z|\to 1} \mu(z)|\Re g_\rho(z)| \log \frac{2}{1-|z|^2} = 0.$$

It follows from Theorem 3.2 in [18] that $T_{g_\rho} : F(p,q,s) \to \mathcal{B}_\mu$ is compact. Notice that

$$\lim_{\delta\to 1} \sup_{\delta<|z|<1} \mu(z)|\Re g(z)| \log \frac{2}{1-|z|^2} = \limsup_{|z|\to 1} \mu(z)|\Re g(z)| \log \frac{2}{1-|z|^2}.$$

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Then for any $\varepsilon > 0$, there exists $\delta \in (0,1)$ such that

$$\sup_{\delta<|z|<1} \mu(z)|\Re g(z)| \log \frac{2}{1-|z|^2} < \limsup_{|z|\to 1} \mu(z)|\Re g(z)| \log \frac{2}{1-|z|^2} + \varepsilon. \tag{2.10}$$

Since $M_\infty(\Re g, r)$ is increasing with r, we have

$$\begin{aligned}
\sup_{\delta<|z|<1} \mu(z)|\Re g(\rho z)| \log \frac{2}{1-|z|^2} &= \sup_{\delta<r<1} \mu(r) M_\infty(\Re g, \rho r) \log \frac{2}{1-r^2} \\
&\le \sup_{\delta<r<1} \mu(r) M_\infty(\Re g, r) \log \frac{2}{1-r^2} \\
&< \limsup_{|z|\to 1} \mu(z)|\Re g(z)| \log \frac{2}{1-|z|^2} + \varepsilon. \qquad (2.11)
\end{aligned}$$

On the other hand, by (2.9) we get

$$\sup_{|z|\le\delta} \mu(z) \log \frac{2}{1-|z|^2} \le C_2.$$

For the given $\delta \in (0,1)$ above, $\Re g \in H(\overline{B(0,\delta)})$ implies that there exists some ρ being close to 1 enough such that

$$|\Re g(z) - \Re g(\rho z)| < \frac{\varepsilon}{C_2}.$$

Thus,

$$\sup_{|z|\le\delta} |\Re g(z) - \Re g(\rho z)| \mu(z) \log \frac{2}{1-|z|^2} < \varepsilon. \tag{2.12}$$

It follows from Lemma 2.2 and (2.10)-(2.12) that

$$\begin{aligned}
\|T_g\|_{e,F(p,q,s)\to\mathcal{B}_\mu} &\le \|T_g - T_{g_\rho}\|_{F(p,q,s)\to\mathcal{B}_\mu} \\
&\simeq \sup_{z\in\mathbb{B}} |\Re(g-g_\rho)(z)| \mu(z) \log \frac{2}{1-|z|^2} \\
&= \sup_{z\in\mathbb{B}} |\Re g(z) - \Re g(\rho z)| \mu(z) \log \frac{2}{1-|z|^2} \\
&\le \sup_{|z|\le\delta} |\Re g(z) - \Re g(\rho z)| \mu(z) \log \frac{2}{1-|z|^2} \\
&\quad + \sup_{\delta<|z|<1} |\Re g(z) - \Re g(\rho z)| \mu(z) \log \frac{2}{1-|z|^2} \\
&\le \sup_{|z|\le\delta} |\Re g(z) - \Re g(\rho z)| \mu(z) \log \frac{2}{1-|z|^2} \\
&\quad + \sup_{\delta<|z|<1} |\Re g(z)| \mu(z) \log \frac{2}{1-|z|^2} + \sup_{\delta<|z|<1} |\Re g(\rho z)| \mu(z) \log \frac{2}{1-|z|^2} \\
&< 3\varepsilon + 2 \limsup_{|z|\to 1} \mu(z)|\Re g(z)| \log \frac{2}{1-|z|^2}.
\end{aligned}$$

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That means

$$\|T_g\|_{e,F(p,q,s)\to\mathcal{B}_\mu} \le C\limsup_{|z|\to1}\mu(z)|\Re g(z)|\log\frac{2}{1-|z|^2}. \tag{2.13}$$

Step 2. Suppose

$$\limsup_{|z|\to1}\mu(z)\log\frac{2}{1-|z|^2}\neq 0.$$

Then by Lemma 2.1 and Lemma 2.2,

$$\begin{aligned}\|T_g\|_{e,F(p,q,s)\to\mathcal{B}_\mu} &\le \|T_g\|_{F(p,q,s)\to\mathcal{B}_\mu}\\ &\simeq \sup_{0\le r<1} M_\infty(\Re g,r)\mu(r)\log\frac{2}{1-r^2}\\ &\le C\limsup_{r\to1} M_\infty(\Re g,r)\mu(r)\log\frac{2}{1-r^2}.\end{aligned}$$

This, together with (2.8) and (2.13), implies

$$\|T_g\|_{e,F(p,q,s)\to\mathcal{B}_\mu} \simeq \limsup_{|z|\to1}\mu(z)|\Re g(z)|\log\frac{2}{1-|z|^2}.$$

This gives the desired result.

Theorem 2.3 Suppose μ is a normal weight, $g\in H(\mathbb{B})$ and $n+1+q<p$. Then

$$\|T_g\|_{e,F(p,q,s)\to\mathcal{B}_\mu} = \begin{cases} 0, & \text{if } g\in\mathcal{B}_\mu;\\ \infty, & \text{if } g\notin\mathcal{B}_\mu.\end{cases}$$

Proof. If $\|g-g(0)\|_\mu<\infty$, then by Theorem 3.3 in [18], we know that $T_g: F(p,q,s)\to\mathcal{B}_\mu$ is bounded if and only if $T_g: F(p,q,s)\to\mathcal{B}_\mu$ is compact. Hence, the definition of essential norm implies

$$\|T_g\|_{e,F(p,q,s)\to\mathcal{B}_\mu}=0.$$

On the other hand, if $\|g-g(0)\|_\mu=\infty$, then $g\notin\mathcal{B}_\mu$. So Theorem 3.3 in [18] shows that $T_g: F(p,q,s)\to\mathcal{B}_\mu$ is not bounded. Thus,

$$\|T_g\|_{e,F(p,q,s)\to\mathcal{B}_\mu}=\infty.$$

This gives the desired result.

Finally, if we take

$$\mu(r)=(1-r)^\alpha\left(\ln\frac{e^{\frac{\beta}{\alpha}}}{1-r}\right)^\beta$$

with $0<\alpha<\infty$ and $0\le\beta<\infty$, then the Bloch-type space is called the logarithmic Bloch-type space $\mathcal{B}^\alpha_{\log^\beta}$ in [20]. For $\beta=0$, it becomes the α-Bloch space. And when $\alpha=\beta=1$, it is just the logarithmic Bloch space.

By Theorem 2.1-2.3, we can characterize the essential norm of $T_g: F(p,q,s)\longrightarrow\mathcal{B}^\alpha_{\log^\beta}$. And then we have these following corollaries.

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Corollary 2.1 Suppose μ is a normal weight, $g \in H(\mathbb{B})$ and $n+1+q > p$. Then

$$\|T_g\|_{e,F(p,q,s)\to\mathcal{B}^\alpha_{\log^\beta}} \simeq \limsup_{|z|\to 1}(1-|z|)^\alpha \left(\ln \frac{e^{\frac{\beta}{\alpha}}}{1-|z|}\right)^\beta |\Re g(z)|(1-|z|^2)^{1-\frac{n+1+q}{p}}.$$

Corollary 2.2 Suppose μ is a normal weight, $g \in H(\mathbb{B})$ and $n+1+q = p$. Then

$$\|T_g\|_{e,F(p,q,s)\to\mathcal{B}^\alpha_{\log^\beta}} \simeq \limsup_{|z|\to 1}(1-|z|)^\alpha \left(\ln \frac{e^{\frac{\beta}{\alpha}}}{1-|z|}\right)^\beta |\Re g(z)| \log \frac{2}{1-|z|^2}.$$

Corollary 2.3 Suppose μ is a normal weight, $g \in H(\mathbb{B})$ and $n+1+q < p$. Then

$$\|T_g\|_{e,F(p,q,s)\to\mathcal{B}^\alpha_{\log^\beta}} = \begin{cases} 0, & \text{if } g \in \mathcal{B}^\alpha_{\log^\beta}; \\ \infty, & \text{if } g \notin \mathcal{B}^\alpha_{\log^\beta}. \end{cases}$$

Acknowledgement This research was supported by the NNSF of China (Nos. 11471111, 11101139, 11271124) and the NSF of Zhejiang Province (No. LY14A010017).

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STABILITY IN VOLTERRA INTEGRAL EQUATIONS OF FRACTIONAL ORDER WITH CONTROL VARIABLE

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ABSTRACT. In this paper, we study the stability of solutions of a nonlinear functional integral equation of fractional order with control variable. The equation is considered in the Banach space of real functions defined, continuous and bounded on an unbounded interval. Existence of a control variable provides the extension of some previous results obtains in other studies. We will also include an example in order to indicate the validity of the assumptions.

1. INTRODUCTION

The theory of differential and integral equations of fractional order play a very pivotal role in describing some real world problems appearing in physics, mechanics, engineering, among others [1-6]. This theory has recently received a lot of attention and now constitutes a significant branch of nonlinear analysis. Recently, Banas and O'Regan [7] studied the existence and attractivity of the solutions of the following quadratic Volterra integral equation of the fractional order

$$x(t) = p(t) + \frac{f(t, x(t))}{\Gamma(\alpha)} \int_0^t \frac{\phi(t, s, x(s))}{(t-s)^{1-\alpha}} ds. \tag{1.1}$$

On the other hands, in the more realistic situation, a physical system may be continuously perturbed via unpredictable forces. These perturbations are generally results of the change in the system's parameters. In the language of the control theory, these perturbation functions may be regarded as control variables [8, 9]. In particular, the integral equations of fractional order should be considered along with a control variable. In this paper we consider the following quadratic Volterra integral equation of the fractional order with control variable as a generalization of equation (1.1)

2010 *Mathematics Subject Classification.* 54H99.

Key words and phrases. Volterra integral equation, Integral equation of fractional order, Measure of noncompactness, Stability, Control variable .

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$$\begin{aligned} x(t) &= p(t) + \frac{f(t,x(t))}{\Gamma(\alpha)} \int_0^t \frac{\phi(t,s,x(s),v(s))}{(t-s)^{1-\alpha}} ds, \\ \frac{dv}{dt} &= -\eta(t)v(t) + g(t,x(t)), \quad t \in \mathbb{R}_+, \end{aligned} \tag{1.2}$$

where $\Gamma(\alpha)$ is the gamma function, $p = p(t)$, $\phi = \phi(t,s,x,v)$, $f = f(t,x)$, $\eta = \eta(t)$ and $g = g(t,x)$ are given, while $x = x(t)$ and $v = v(t)$ are unknown functions. It is clear that Eq. (1.2) includs Eq. (1.1) as a special case.

2. **Preliminaries**

In this section, we present some definitions and results which will be needed further on. Let $\mathbb{R}_+ = [0,\infty)$ and $(E, \|.\|)$ be an infinite dimensional Banach space with zero element θ. We write $B(x,r)$ to denote the closed ball centered at x with radius r and $\overline{X}$, $ConvX$ to denote the closer and closed convex hall of X. Let m_E denote the family of all nonempty bounded subsets of E and n_E indicate the family of all relatively compact sets. We use the following definition of a measure of noncompactness [11].

Definition 2.1. A mapping $\mu : m_E \to \mathbb{R}_+$ is said to be a measure of noncompactness in E if it satisfies the following conditions
(1) The family $\ker\mu = \{X \in m_E : \mu(X) = 0\}$ is nonempty and $\ker\mu \subseteq n_E$.
(2) $X \subseteq Y \Rightarrow \mu(X) \leq \mu(Y)$.
(3) $\mu(\bar{X}) = \mu(X)$.
(4) $\mu(convX) = \mu(X)$.
(6) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y) for \lambda \in [0,1]$. If (X_n) is a sequence of close sets from m_E such that $X_{n+1} \subset X_n$, $(n = 1, 2, ...)$ and if $\lim_{n\to\infty} \mu(X_n) = 0$, then the intersection set $X_\infty = \bigcap_{n=1}^{\infty}$ is nonempty.

We will need the following fixed point theorem of Darbo type [11].

Theorem 2.2. *Let Λ be a nonempty bounded closed convex subset of the space E and let $F : \Lambda \to \Lambda$ be a continuous operator such that $\mu(FA) \leq k\mu(A)$ for each each nonempty subset A of Λ, where $k \in [0,1)$ is a constant. Then F has at least one fixed point in Λ.*

We will work in the Banach space $BC(\mathbb{R}_+)$ consisting of all bounded and continuous functions on $\mathbb{R}_+$. The space $BC(\mathbb{R}_+)$ is equipped with the standard norm $\|x\| = \sup\{|x(t)| : t \in \mathbb{R}_+\}$.
Let X be a nonempty bounded subset of $BC(\mathbb{R}_+)$ and T be a positive number. For $x \in X$ and $\varepsilon \geq 0$, define

$$\omega^T(x,\varepsilon) = \sup\{|x(t) - x(s)| : s,t \in [0,T], \ \ |t-s| \leq \varepsilon\},$$

$$\omega^T(X,\varepsilon) = \sup\{\omega^T(x,\varepsilon) : \ x \in X\}, \qquad \omega_0^T(X) = \lim_{\varepsilon\to 0} \omega^T(X,\varepsilon)$$

$$\omega_0(X) = \lim_{T\to\infty} \omega_0^T(X), \qquad X(t) = \{x(t) : x \in X\},$$

$$\mathrm{diam}X(t) = \sup\{|x(t) - y(t)| : x, y \in X\}$$

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and

$$\mu(X) = \omega_0(X) + \limsup_{t\to\infty} \operatorname{diam} X(t). \tag{2.1}$$

Banaś has shown in [12] that μ is a measure of noncompactness in the space $BC(\mathbb{R}_+)$.
Let for the function x belong to $BC(\mathbb{R}_+)$, $\nu_x(t)$ be the solution of the second equation in (1.2). In this way, any solution of system (1.2) should be represented by $X(t) = (x(t), \nu_x(t))$.

Definition 2.3. We say that solutions of (1.2) are locally attractive if there exist balls B_r and B_R in the space $BC(\mathbb{R}^+)$ such that for arbitrary solutions $X(t) = (x(t), \nu_x(t))$ and $Y(t) = (y(t), \nu_y(t))$ of (1.2) belonging to $B_r \times B_R$, we have that

$$\begin{aligned} &\lim_{t\to\infty} (x(t) - y(t)) = 0, \\ &\lim_{t\to\infty} (\nu_x(t) - \nu_y(t)) = 0. \end{aligned} \tag{2.2}$$

In the case when the limits (2.2) are uniform with respect to $B_r \times B_R$ i.e. when for each $\varepsilon > 0$ there exists $T > 0$ such that

$$\begin{aligned} &|(x(t) - y(t))| \leq \varepsilon, \\ &|\nu_x(t) - \nu_y(t)| \leq \varepsilon, \end{aligned} \tag{2.3}$$

for all solutions $X(t) = (x(t), \nu_x(t))$ and $Y(t) = (y(t), \nu_y(t))$ of (1.2) belong to $B_r \times B_R$ and for any $t \geq T$, we will say that solutions of system (1.2) are asymptotically stable (uniformly locally attractive).

Definition 2.4. The solution $X(t) = (x(t), \nu_x(t))$ of Eq (1.2) is said to be globally attractive if (2.2) holds for each solution $Y(t) = (y(t), \nu_y(t))$ of (1.2).

3. Main results

First we will consider system (1.2) under the following assumptions:
(H_1) the function $p : \mathbb{R}_+ \to \mathbb{R}$ is continuous an bounded.
$(H_2)$the function $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists continuous function $n : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|f(t, x) - f(t, y)| \leq n(t)|x(t) - y(t)|, \tag{3.1}$$

(H_3) the function $\phi : \mathbb{R}_+^2 \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and there exist functions $m_i : \mathbb{R}_+ \to \mathbb{R}_+$ and nondecreasing functions $\chi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, 2$ with $\chi_i(0) = 0$ such that

$$|\phi(t, s, x_1, y_1) - \phi(t, s, x_2, y_2)| \leq m_1(t)\chi_1(|x_1 - x_2|) + m_2(t)\chi_2(|y_1 - y_2|),$$

for all $t, s \in \mathbb{R}_+$ and $x_i, y_i \in \mathbb{R}$.
(H_4) the functions a_i, b_i, c, $d : \mathbb{R}_+ \to \mathbb{R}_+$, i=1, 2 defined by

$$\begin{aligned} a_i(t) &= n(t)m_i(t)t^\alpha, \\ b_i(t) &= m_i(t)|f(t, 0)|t^\alpha, \\ c(t) &= n(t)\phi_1(t)t^\alpha \\ d(t) &= \phi_1(t)|f(t, 0)|t^\alpha \end{aligned}$$

are bounded and $\lim_{t\to\infty} a_i = 0$ and $\lim_{t\to\infty} b_i = 0$.

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(H_5) the functions $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ and $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ are continuous positive functions such that

$$\eta_L = \inf\{\eta(t) : t \in \mathbb{R}_+\} > 0,$$
$$\eta_M = \sup\{\eta(t) : t \in \mathbb{R}_+\} < \infty,$$
$$g_L = \inf\{g(t, x(t)) : t \in \mathbb{R}_+, , x \in BC(\mathbb{R}_+)\} > 0,$$
$$g_M = \sup\{g(t, x(t)) : t \in \mathbb{R}_+, , x \in BC(\mathbb{R}_+)\} < \infty,$$

and there exists constant $k > 0$ such that

$$|g(t, x) - g(t, y)| \le k|x - y|, \qquad t \in \mathbb{R}_+.$$

Since for any x belong to $BC(\mathbb{R}_+)$ function $g_x(t) = g(t, x(t))$ depends on x, the dynamical behavior of the solution of second equation in (1.2) may be depend on the dynamical behavior of x. The following lemma enable us to study the relation between asymptotic behavior of the function $x \in BC(\mathbb{R}_+)$ and asymptotic behavior of the solution $\nu_x(t)$ of the second equation in (1.2).

Lemma 3.1. *Under assumption (H_5) suppose that $x, y \in BC(\mathbb{R}_+)$, then for the solutions of the second equation in (1.2) the following are true:*
(I) if $\lim_{t\to\infty} |x(t) - y(t)| = 0$, then $\lim_{t\to\infty} |\nu_x(t) - \nu_y(t)| = 0$.

(II) Suppose that $\varepsilon > 0$ and $|x(t) - y(t)| \le \varepsilon$ for any $t \in [T, +\infty]$ and

$$t - T > \frac{-1}{\eta_L} \ln \frac{\varepsilon \eta_L}{|\eta_L|\nu_x(T) - \nu_y(T)| - k\varepsilon|}, \tag{3.2}$$

then there exists $T' > T$ such that $|\nu_x(t) - \nu_y(t)| \le (1 + \frac{k}{\eta_L})\varepsilon$ for any $t \ge T'$.

(III) For any $x \in BC(\mathbb{R}_+)$, the solution $\nu_x(t)$ with positive initial value, $\nu_x(T) \ge 0$, bounded above and below by positive constants, i.e., there exist real numbers Q and R such that

$$0 < Q \le \nu_x(t) \le R,$$

for all $x \in BC(\mathbb{R}_+)$ and $t \ge 0$.

Proof: (I) Let $\varepsilon > 0$ and $x, y \in BC(\mathbb{R}_+)$ such that $|x(t) - y(t)| \le \varepsilon$ for any $t \ge T$. Let $\xi(t) = \nu_x(t) - \nu_y(t)$, then

$$\begin{aligned}
\dot{\xi}(t) &= \dot{\nu}_x(t) - \dot{\nu}_y(t) = (-\eta(t)\nu_x(t) + g(t, x(t))) - (-\eta(t)\nu_y(t) + g(t, y(t))) \\
&= -\nu(t)(\nu_x(t) - \nu_y(t)) + g(t, x(t)) - g(t, y(t))) \\
&= -\eta(t)\xi(t) + (g(t, x(t)) - g(t, y(t))).
\end{aligned}$$

The solutions of this equation on $[T, +\infty]$ is given by [10],

$$\begin{aligned}
\xi(t) &= \xi(T) \exp\{-\int_T^t \eta(s) ds\} \\
&+ \int_T^t [g(s, x(s)) - g(s, y(s))] \exp\{-\int_s^t \eta(\tau) d\tau\} ds, \quad t \ge T.
\end{aligned} \tag{3.3}$$

Noticing $\eta_L > 0$ and in view of Lipschitz condition of the function g, we have

$$\begin{aligned}
|\xi(t)| &\le |\xi(T)| \exp\{-\eta_L(t - T)\} + \int_T^t k|x(s) - y(s)| \exp\{-\eta_L(t - s)\} ds \\
&\le |\xi(T)| \exp\{-\eta_L(t - T)\} + \frac{k\varepsilon}{\eta_L}[1 - \exp\{-\eta_L(t - T)\}].
\end{aligned} \tag{3.4}$$

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Therefore $\lim_{t\to\infty} |\nu_x(t) - \nu_y(t)| \le \frac{k\varepsilon}{\eta_L}$. Since $\varepsilon > 0$ is arbitrary, this indicates that $\lim_{t\to\infty} |\nu_x(t) - \nu_y(t)| = 0$.
(II) From the inequality (3.4) we obtain

$$\begin{aligned} |\nu_x(t) - \nu_y(t)| &\le \frac{1}{\eta_L} |\eta_L|\nu_x(T) - \nu_y(T)| - k\varepsilon| \times \exp\{-\eta_L(t-T)\} \\ &+ \frac{k\varepsilon}{\eta_L}, \qquad t \ge T. \end{aligned} \tag{3.5}$$

Substituting inequality (3.2) into (3.5) we have $|\nu_x(t) - \nu_y(t)| \le (1 + \frac{k}{\eta_L})\varepsilon$ for all $t \ge T'$ where

$$T' = T + \frac{-1}{\eta_L} \ln \frac{\varepsilon \eta_L}{|\eta_L|\nu_x(T) - \nu_y(T)| - k\varepsilon|}, \tag{3.6}$$

(III) For any $x \in BC(\mathbb{R}_+)$ the solution $\nu_x(t)$ of the second equation in (1.2) with positive initial value $v_x(T) > 0$ can be expressed as follow

$$\nu_x(t) = \nu_x(T) \exp\{-\int_T^t \eta(s)ds\} + \int_T^t [g(s, x(s))] \exp\{-\int_s^t \eta(\tau)d\tau\}ds, \quad t \ge T. \tag{3.7}$$

In view of the fact, $v_x(T) > 0, g(t, x(t)) > 0, \eta(t) > 0$ for all $t \ge T$, it follows $v_x(T) > 0$ for all $t \ge T$ and $x \in BC(\mathbb{R}_+)$. Moreover, we have

$$\frac{d\nu_x}{dt} = -\eta(t)\nu_x(t) + g(t, x(t)) \le -\eta_L \nu(t) + g_M. \tag{3.8}$$

Based on inequality (3.8) we obtain,

$$\nu_x(t) \le \nu_x(T) \exp\{-\eta_L(t-T)\} + \frac{g_M}{\eta_L}[1 - \exp\{-\eta_L(t-T)\}],$$

for all $t \ge T$. Thus

$$\limsup_{t\to\infty} \nu_x(t) \le \frac{g_M}{\eta_L}. \tag{3.9}$$

Also, we have

$$\frac{d\nu_x}{dt} = -\eta(t)\nu_x(t) + g(t, x(t)) \ge -\eta_M \nu(t) + g_L. \tag{3.10}$$

Based on inequality (3.10) we obtain,

$$\nu_x(t) \ge \nu_x(T) \exp\{-\eta_L(t-T)\} + \frac{g_L}{\eta_M}[1 - \exp\{-\eta_L(t-T)\}],$$

for all $t \ge T$. Thus

$$\liminf_{t\to\infty} \nu_x(t) \ge \frac{g_L}{\eta_M}. \tag{3.11}$$

According to inequalities (3.9) and (3.11), solution $\nu_x(t)$ bounded above and belove by positive constants. ■

Now, we consider the following assumption:
(H_6) there exists a positive solution r_0 of inequality

$$\|p\| + \frac{1}{\Gamma(\alpha+1)}\{(A_1\chi_1(r) + A_2\chi_2(R) + C)r + B_1\chi_1(r) + B_2\chi_2(R) + D\} \le r, \tag{3.12}$$

where R is defined in part (III) of Lemma (3.1) and

$$A_i = \sup\{a_i(t) : t \in \mathbb{R}_+\}, \qquad i = 1, 2$$

$$\begin{aligned} B_i &= \sup\{b_i(t) : t \in \mathbb{R}_+\}, \qquad i = 1, 2 \\ C &= \sup\{c(t) : t \in \mathbb{R}_+\}, \\ D &= \sup\{d(t) : t \in \mathbb{R}_+\}. \end{aligned}$$

Remark 3.2.

The inequality introduced in (H_6) is equivalence with the following inequality

$$0 < \|p\| + \frac{1}{\Gamma(\alpha+1)}\{B_1\chi_1(r) + B_2\chi_2(R) + D\} \le r\{1 - \frac{A_1\chi_1(r) + A_2\chi_2(R) + C}{\Gamma(\alpha+1)}\},$$

that demonstrate

$$\lambda = \frac{A_1\chi_1(r) + A_2\chi_2(R) + C}{\Gamma(\alpha+1)} < 1.$$

Considering the operators F, Φ, G on the space $BC(\mathbb{R}_+)$ by the formulas

$$\begin{aligned} (Fx)(t) &= f(t, x(t)), \\ (\Phi x)(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\phi(t, s, x(s), v_x(s))}{(t-s)^{1-\alpha}} ds, \\ (Gx)(t) &= p(t) + (Fx)(t)(\Phi x)(t), \end{aligned}$$

we have the following lemma

Lemma 3.3. *Under assumptions (H_1)-(H_6), operator G transforms the ball B_{r_0} in the space $BC(\mathbb{R}_+)$ into itself, where r_0 satisfies the inequality introduced by (H_6).*

Proof. In view of assumption (H_2), for any $x \in BC(\mathbb{R}_+)$ the function Fx is continuous on $\mathbb{R}_+$. We indicate that the function Φx is continuous for any $x \in BC(\mathbb{R}_+)$ as well. To do this let x be an arbitrary function belong to $BC(\mathbb{R}_+)$ and take a fix $T > 0$ and $\varepsilon > 0$. Assume the $t_1, t_2 \in [0, T]$ and $|t_1 - t_2| \le \varepsilon$. With loss of generality we assume that $t_1 < t_2$. Taking these assumptions into account we have

$$\begin{aligned} |(\Phi x)(t_2) - (\Phi x)(t_1)| &= \frac{1}{\Gamma(\alpha)} | \int_0^{t_2} \frac{\phi(t_2, s, x(s), v_x(s))}{(t_2-s)^{1-\alpha}} ds \\ &+ \int_{t_1}^{t_2} \frac{\phi(t_2, s, x(s), v_x(s))}{(t_2-s)^{1-\alpha}} ds - \int_0^{t_1} \frac{\phi(t_1, s, x(s), v_x(s))}{(t_1-s)^{1-\alpha}} ds | \\ &\le \frac{1}{\Gamma(\alpha)} \int_0^{t_1} | \frac{\phi(t_2, s, x(s), v_x(s))}{(t_2-s)^{1-\alpha}} - \frac{\phi(t_1, s, x(s), v_x(s))}{(t_2-s)^{1-\alpha}} | ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} | \frac{\phi(t_2, s, x(s), v_x(s))}{(t_2-s)^{1-\alpha}} | ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^{t_1} | \frac{\phi(t_1, s, x(s), v_x(s))}{(t_2-s)^{1-\alpha}} - \frac{\phi(t_1, s, x(s), v_x(s))}{(t_2-s)^{1-\alpha}} | ds \\ &\le \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \frac{|\phi(t_2, s, x(s), v_x(s)) - \phi(t_1, s, x(s), v_x(s))|}{(t_2-s)^{1-\alpha}} ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \frac{|\phi(t_2, s, x(s), v_x(s))|}{(t_2-s)^{1-\alpha}} ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^{t_1} |\phi(t_1, s, x(s), v_x(s))| [\frac{1}{(t_1-s)^{1-\alpha}} - \frac{1}{(t_2-s)^{1-\alpha}}] ds \end{aligned}$$

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$$
\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\frac{W^T(\phi,\varepsilon,x)}{(t_2-s)^{1-\alpha}}ds\\
&+ \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\frac{|\phi(t_2,s,x(s),v_x(s))-\phi(t_2,s,0,0)|+|\phi(t_2,s,0,0)|}{(t_2-s)^{1-\alpha}}ds\\
&+ \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(|\phi(t_1,s,x(s),v_x(s))-\phi(t_1,s,0,0)|+|\phi(t_1,s,0,0)|)\\
&\times [\frac{1}{(t_1-s)^{1-\alpha}}-\frac{1}{(t_2-s)^{1-\alpha}}]ds\\
&\leq \frac{W^T(\phi,\varepsilon,x)}{\Gamma(\alpha+1)}(t_2^\alpha-(t_2-t_1)^\alpha)\\
&+ \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\frac{m_1(t_2)\chi_1(|x(s)|)+m_2(t_2)\chi_2(|v_x(s)|)+\phi_1(t_2)}{(t_2-s)^{1-\alpha}}ds\\
&+ \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(m_1(t_1)\chi_1(|x(s)|)+m_2(t_1)\chi_2(|v_x(s)|)+\phi_1(t_1))\\
&\times [\frac{1}{(t_1-s)^{1-\alpha}}-\frac{1}{(t_2-s)^{1-\alpha}}]ds\\
&\leq \frac{W^T(\phi,\varepsilon,x)}{\Gamma(\alpha+1)}t_2^\alpha\\
&+ \frac{m_1(t_2)\chi_1(\|x\|)+m_2(t_2)\chi_2(\|v_x\|)+\phi_1(t_2)}{\Gamma(\alpha+1)}(t_2-t_1)^\alpha\\
&+ \frac{m_1(t_1)\chi_1(\|x\|)+m_2(t_1)\chi_2(\|v_x\|)+\phi_1(t_1)}{\Gamma(\alpha+1)}(t_1^\alpha-t_2^\alpha+(t_2-t_1)^\alpha)\\
&\leq \frac{1}{\Gamma(\alpha+1)}[T^\alpha W^T(\phi,\varepsilon,x)\\
&+ 2\varepsilon^\alpha\{m_1(T)\chi_1(\|x\|)+m_2(T)\chi_2(\|v_x\|)+\overline{\phi_1}(T)\}] \qquad (3.13)
\end{aligned}
$$

where

$$\phi_1(t)=\sup\{\phi(t,s,0,0):\ 0\leq s\leq t\},$$

$W^T(\phi,\varepsilon,x)=\sup\{|\phi(t_2,s,y,z)-\phi(t_1,s,y,z)|:0\leq s\leq t_1<t_2\leq T,\ |t_2-t_1|\leq\varepsilon,|y|\leq\|x\|,z\in[l,L]\}$,
and for any function $h:\mathbb{R}_+\to\mathbb{R}$ the following notation has been used

$$\overline{h}(T)=\sup\{h(t):t\in[0,T]\}. \qquad (3.14)$$

Based on inequality (3.13) we obtain

$$\omega^T(\Phi x,\varepsilon)\leq\frac{1}{\Gamma(\alpha+1)}[T^\alpha W^T(\phi,\varepsilon,x)+2\varepsilon^\alpha\{m_1(T)\chi_1(\|x\|)+m_2(T)\chi_2(\|v_x\|)+\overline{\phi_1}(T)\}].$$

With due attention to the uniform continuity of the function $\phi(t,s,x,v)$ on the set $[0,T]\times[0,T]\times[-\|x\|,\|x\|]\times[Q,R]$, we have that $W^T(\phi,\varepsilon,x)\to 0$ as $t\to 0$. In this way and keeping in mind the above inequality we infer that the function Φx is continuous on the interval $[0,T]$ for any $T>0$ and consequently Φx is continuous on $\mathbb{R}_+$. In view of continuity of functions p, Fx and Φx on $\mathbb{R}_+$ we deduce that Gx is continuous on $\mathbb{R}_+$.

Now we show that the function Gx is bounded and operator G transforms the ball B_{r_0} into itself. Let us take an arbitrary function $x\in BC(\mathbb{R}_+)$. For any fixed

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$t \in \mathbb{R}_+$ we have

$$\begin{aligned}
|(Gx)(t)| &\leq |p(t)| + \frac{1}{\Gamma(\alpha)}(|f(t,x(t)) - f(t,0)| + |f(t,0)|) \\
&\times \int_0^t \frac{|\phi(t,s,x(s),v_x(s)) - \phi(t,s,0,0)| + |\phi(t,s,0,0)|}{(t-s)^{1-\alpha}} ds \\
&\leq |p(t)| + \frac{1}{\Gamma(\alpha)}(n(t)|x(t)| + |f(t,0)|) \int_0^t \frac{m_1(t)\chi_1(|x(s)|) + m_2(t)\chi_2(|v_x(s)|) + \phi_1(t)}{(t-s)^{1-\alpha}} ds \\
&\leq |p(t)| + \frac{(n(t)\|x\| + |f(t,0)|)}{\Gamma(\alpha+1)}[m_1(t)\chi_1(\|x\|) + m_2(t)\chi_2(\|v_x\|) + \phi_1(t)]t^\alpha \\
&\leq |p(t)| + \frac{1}{\Gamma(\alpha+1)}\{a_1(t)\|x\|\chi_1(\|x\|) + a_2(t)\|x\|\chi_2(\|v_x\|) \\
&+ c(t)\|x\| + b_1(t)\chi_1(\|x\|) + b_2(t)\chi_2(\|v_x\|) + d(t)\}.
\end{aligned}$$

Thus in view of assumptions (H_1) and (H_4) the function Gx is bounded. Consequently,

$$\|Gx\| \leq \|p\| + \frac{1}{\Gamma(\alpha+1)}\{(A_1\chi_1(\|x\|) + A_2\chi_2(R) + C(t))\|x\| + B_1\chi_1(\|x\|) + B_2\chi_2(R) + D\}$$

In accordance with the assumption (H_6) we deduce that there is $r_0 > 0$ such that operator G transforms the Ball B_{r_0} into itself. □

Theorem 3.4. *Under assumptions (H_1)-(H_6), system (1.2) has at least one solution $(x(t), v(t))$ with positive condition $v(0) > 0$ which belongs to space $BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$. Moreover, solutions of system (1.2) are uniformly locally attractive.*

Proof. Let X be an arbitrary subset of the bell B_{r_0} in the space $BC(\mathbb{R}_+)$ describe in Lemma 3.3. Then, keeping in mind assumptions (H_2)-(H_4), for x, y belong to X and for an arbitrary fixed $t \in \mathbb{R}_+$ we have

$$\begin{aligned}
|(Gx)(t) - (Gy)(t)| &= |\frac{f(t,x(t))}{\Gamma(\alpha)} \int_0^t \frac{\phi(t,s,x(s),v_x(s))}{(t-s)^{1-\alpha}} ds \\
&- \frac{f(t,y(t))}{\Gamma(\alpha)} \int_0^t \frac{\phi(t,s,y(s),v_y(s))}{(t-s)^{1-\alpha}} ds| \\
&\leq |\frac{f(t,x(t)) - f(t,y(t))}{\Gamma(\alpha)} \int_0^t \frac{\phi(t,s,x(s),v_x(s))}{(t-s)^{1-\alpha}} ds| \\
&+ |\frac{f(t,y(t))}{\Gamma(\alpha)} \int_0^t \frac{\phi(t,s,x(s),v_x(s)) - \phi(t,s,y(s),v_y(s))}{(t-s)^{1-\alpha}} ds
\end{aligned}$$

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$$
\begin{aligned}
&\leq \frac{n(t)|x(t)-y(t)|}{\Gamma(\alpha)}\int_0^t \frac{|\phi(t,s,x(s),v_x(s))-\phi(t,s,0,0)|+|\phi(t,s,0,0)|}{(t-s)^{1-\alpha}}ds\\
&+ \frac{|f(t,y(t))-f(t,0)|+|f(t,0)|}{\Gamma(\alpha)}\\
&\times \int_0^t \frac{m_1(t)\chi_1(|x(s)-y(s)|)+m_2(t)\chi_2(|v_x(s)-v_y(s)|)}{(t-s)^{1-\alpha}}ds\\
&\leq \frac{n(t)|x(t)-y(t)|}{\Gamma(\alpha)}\int_0^t \frac{m_1(t)\chi_1(|x(s)|)+m_2(t)\chi_2(|v_x(s)|)+\phi_1(t)}{(t-s)^{1-\alpha}}ds\\
&+ \frac{n(t)|y(t)|+|f(t,0)|}{\Gamma(\alpha)}\int_0^t \frac{m_1(t)\chi_1(|x(s)-y(s)|)+m_2(t)\chi_2(|v_x(s)-v_y(s)|)}{(t-s)^{1-\alpha}}ds\\
&\leq \frac{n(t)(|x(t)|+|y(t)|)m_1(t)}{\Gamma(\alpha)}\int_0^t \frac{\chi_1(|x(s)|)}{(t-s)^{1-\alpha}}ds\\
&+ \frac{n(t)(|x(t)|+|y(t)|)m_2(t)}{\Gamma(\alpha)}\int_0^t \frac{\chi_2(|v_x(s)|)}{(t-s)^{1-\alpha}}ds\\
&+ \frac{n(t)|x(t)-y(t)|\phi_1(t)}{\Gamma(\alpha)}\int_0^t \frac{ds}{(t-s)^{1-\alpha}}\\
&+ \frac{n(t)|y(t)|m_1(t)}{\Gamma(\alpha)}\int_0^t \frac{\chi_1(|x(s)|+|y(s)|)}{(t-s)^{1-\alpha}}ds\\
&+ \frac{n(t)|y(t)|m_2(t)}{\Gamma(\alpha)}\int_0^t \frac{\chi_2(|v_x(s)|+|v_y(s)|)}{(t-s)^{1-\alpha}}ds\\
&+ \frac{|f(t,0)|m_1(t)}{\Gamma(\alpha)}\int_0^t \frac{\chi_1(|x(s)|+|y(s)|)}{(t-s)^{1-\alpha}}ds+\frac{|f(t,0)|m_2(t)}{\Gamma(\alpha)}\int_0^t \frac{\chi_2(|v_x(s)|+|v_y(s)|)}{(t-s)^{1-\alpha}}ds\\
&\leq \frac{2n(t)r_0m_1(t)\chi_1(r_0)}{\Gamma(\alpha)}\int_0^t \frac{ds}{(t-s)^{1-\alpha}}+\frac{2n(t)r_0m_2(t)\chi_2(L)}{\Gamma(\alpha)}\int_0^t \frac{ds}{(t-s)^{1-\alpha}}\\
&+ \frac{n(t)\phi_1(t)}{\Gamma(\alpha)}diamX(t)\int_0^t \frac{ds}{(t-s)^{1-\alpha}}+\frac{n(t)r_0m_1(t)\chi_1(2r_0)}{\Gamma(\alpha)}\int_0^t \frac{ds}{(t-s)^{1-\alpha}}\\
&+ \frac{n(t)r_0m_2(t)\chi_2(2L)}{\Gamma(\alpha)}\int_0^t \frac{ds}{(t-s)^{1-\alpha}}+\frac{|f(t,0)|m_1(t)\chi_1(2r_0)}{\Gamma(\alpha)}\int_0^t \frac{ds}{(t-s)^{1-\alpha}}\\
&+ \frac{|f(t,0)|m_2(t)\chi_2(2L)}{\Gamma(\alpha)}\int_0^t \frac{ds}{(t-s)^{1-\alpha}}\\
&\leq \frac{2a_1(t)r_0\chi_1(r_0)}{\Gamma(\alpha+1)}+\frac{2a_2(t)r_0\chi_2(R)}{\Gamma(\alpha+1)}+\frac{c(t)}{\Gamma(\alpha+1)}diamX(t)+\frac{a_1(t)r_0\chi_1(2r_0)}{\Gamma(\alpha+1)}\\
&+ \frac{a_2(t)r_0\chi_2(2R)}{\Gamma(\alpha+1)}+\frac{b_1(t)\chi_1(2r_0)}{\Gamma(\alpha+1)}+\frac{b_2(t)\chi_2(2R)}{\Gamma(\alpha+1)}. \qquad (3.15)
\end{aligned}
$$

Consequently, based on assumption H_4 we obtain

$$\limsup_{t\to\infty} diam(GX)(t) \leq \lambda \limsup_{t\to\infty} diamX(t), \qquad (3.16)$$

where $\lambda < 1$ is defined in Remark 3.2.

Now, we show that $\mu(GX) \leq \lambda\mu(X)$, where the measure of noncompactness μ defined by formula (2.1). To do this let us take arbitrary numbers $\varepsilon > 0$, $T > 0$

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and $t_1, t_2 \in [0, T]$ such that $|t_1 - t_2| \le \varepsilon$. With loss of generality we assume that $t_1 < t_2$. By (H_1)-(H_4) and inequality (3.13), for a fixed function $x \in X$ we have.

$$\begin{aligned}
|(Gx)(t_2) - (Gx)(t_1)| &\le |p(t_2) - p(t_1)| + |(Fx)(t_2)(\Phi x)(t_2) - (Fx)(t_1)(\Phi x)(t_2)| \\
&+ |(Fx)(t_1)(\Phi x)(t_2) - (Fx)(t_1)(\Phi x)(t_1)| \\
&\le \omega^T(p, \varepsilon) + \frac{|f(t_2, x(t_2)) - f(t_1, x(t_1))|}{\Gamma(\alpha)} \int_0^{t_2} \frac{\phi(t_2, s, x(s), v_x(s))}{(t_2 - s)^{1-\alpha}} ds \\
&+ \frac{|f(t_1, x(t_1))|}{\Gamma(\alpha + 1)} [T^\alpha W^T(\phi, \varepsilon, x) + 2\varepsilon^\alpha (m_1(T)\chi_1(r_0) + m_2(T)\chi_2(R) + \phi_1(T))] \\
&\le \omega^T(p, \varepsilon) + \frac{|f(t_2, x(t_2)) - f(t_2, x(t_1))| - |f(t_2, x(t_1)) - f(t_1, x(t_1))|}{\Gamma(\alpha)} \\
&\times \int_0^{t_2} \frac{|\phi(t_2, s, x(s), v_x(s)) - \phi(t_2, s, 0, 0)| + |\phi(t_2, s, 0, 0)|}{(t_2 - s)^{1-\alpha}} ds \\
&+ \frac{|f(t_1, x(t_1)) - f(t_1, 0)| + |f(t_1, 0)|}{\Gamma(\alpha + 1)} \\
&\times [T^\alpha W^T(\phi, \varepsilon, x) + 2\varepsilon^\alpha (m_1(T)\chi_1(r_0) + m_2(T)\chi_2(R) + \phi_1(T))] \\
&\le \omega^T(p, \varepsilon) + \frac{n(t_2)|x(t_2) - x(t_1))| + \omega_1^T(f, \varepsilon)}{\Gamma(\alpha)} \\
&\times \int_0^{t_2} \frac{m_1(t_2)\chi_1(|x(s)|) + m_2(t_2)\chi_2(|v_x(s)|) + \phi_1(t_2)}{(t_2 - s)^{1-\alpha}} ds \\
&+ \frac{n(t_1)|x(t_1)| + |f(t_1, 0)|}{\Gamma(\alpha + 1)} \\
&\times [T^\alpha W^T(\phi, \varepsilon, x) + 2\varepsilon^\alpha (m_1(T)\chi_1(r_0) + m_2(T)\chi_2(R) + \phi_1(T))] \\
&\le \omega^T(p, \varepsilon) + \frac{n(t_2)\omega^T(x, \varepsilon) + \omega_1^T(f, \varepsilon)}{\Gamma(\alpha + 1)} \\
&\times [m_1(t_2)\chi_1(r_0) + m_2(t_2)\chi_2(R) + \phi_1(t_2)] t_2^\alpha \\
&+ \frac{n(t_1)r_0 + |f(t_1, 0)|}{\Gamma(\alpha + 1)} \\
&\times [T^\alpha W^T(\phi, \varepsilon, x) + 2\varepsilon^\alpha (m_1(T)\chi_1(r_0) + m_2(T)\chi_2(R) + \phi_1(T))]
\end{aligned}$$

$$\begin{aligned}
&\leq \quad \omega^T(p,\varepsilon) + \frac{a_1(t)\chi_1(r_0) + a_2(t)\chi_2(L) + c(t)}{\Gamma(\alpha+1)}\omega^T(x,\varepsilon)\\
&+ \quad \frac{\omega_1^T(f,\varepsilon)}{\Gamma(\alpha+1)}[m_1(t_2)\chi_1(r_0) + m_2(t_2)\chi_2(R) + \phi_1(t_2)]T^\alpha\\
&+ \quad \frac{n(t_1)r_0 + |f(t_1,0)|}{\Gamma(\alpha+1)}\\
&\times \quad [T^\alpha W^T(\phi,\varepsilon,x) + 2\varepsilon^\alpha(m_1(T)\chi_1(r_0) + m_2(T)\chi_2(R) + \phi_1(T))]\\
&\leq \quad \omega^T(p,\varepsilon) + \frac{A_1\chi_1(r_0) + A_2\chi_2(R) + C}{\Gamma(\alpha+1)}\omega^T(x,\varepsilon)\\
&+ \quad \frac{\omega_1^T(f,\varepsilon)}{\Gamma(\alpha+1)}[\overline{m_1}(T)\chi_1(r_0) + \overline{m_2}(T)\chi_2(R) + \overline{\phi_1}(T)]T^\alpha\\
&+ \quad \frac{\overline{n}(T)r_0 + \overline{f}(T)}{\Gamma(\alpha+1)}\\
&\times \quad [T^\alpha W^T(\phi,\varepsilon,x) + 2\varepsilon^\alpha(\overline{m}_1(T)\chi_1(r_0) + \overline{m}_2(T)\chi_2(R) + \overline{\phi_1}(T))],
\end{aligned}$$

wherein

$$\omega_1^T(f,\varepsilon) = \sup\{|f(t_2,x) - f(t_1,x)| : t_1,t_2 \in [0,T], |t_1 - t_2| \leq \varepsilon, x \in [-r_0,r_0]\},$$

and $\overline{\phi_1}(T))$, $\overline{m}_i(T)$, $i = 1,2$, are defined according to (3.14).

From the continuity of f and ϕ on the sets $[0,T] \times [-r_0,r_0]$ and $[0,T] \times [0,T] \times [-r_0,r_0] \times [Q,R]$, respectively, we have,

$$\omega_0^T(GX) \leq \lambda\omega_0^T(X).$$

Consequently, by taking $T \to \infty$ we have

$$\omega^T(GX) \leq \lambda\omega^T(X).$$

Thus based on (3.16) we obtain

$$\mu(GX) \leq \lambda\mu(X).$$

Thus, based on Theorem 2.2 operator G has at least one fixed point $x \in B_r$.
Finally, we show that solutions of system (1.2) are uniformly locally attractive. Let x and y be the fixed points of the operator G. In view of inequality (3.15) we have

$$|x(t) - y(t)| \quad = \quad |(Gx)(t) - (Gy)(t)| \leq \frac{c(t)}{\Gamma(\alpha+1)}|x(t) - y(t)| + p(t) \leq \lambda|x(t) - y(t)| + p(t),$$

where $\lambda < 1$ is defined in Remark 3.2 and

$$\begin{aligned}
p(t) \quad &= \quad \frac{2a_1(t)r_0\chi_1(r_0)}{\Gamma(\alpha+1)} + \frac{2a_2(t)r_0\chi_2(R)}{\Gamma(\alpha+1)} + \frac{a_1(t)r_0\chi_1(2r_0)}{\Gamma(\alpha+1)}\\
&+ \quad \frac{a_2(t)r_0\chi_2(2R)}{\Gamma(\alpha+1)} + \frac{b_1(t)\chi_1(2r_0)}{\Gamma(\alpha+1)} + \frac{b_2(t)\chi_2(2R)}{\Gamma(\alpha+1)}.
\end{aligned}$$

Consequently

$$\lim_{t\to\infty} |x(t) - y(t)| \leq \lim_{t\to\infty} \frac{1}{1-\lambda}p(t) = 0$$

In view of assumption (H_4) there exists $T \geq 0$ such that for $t \geq T$ we have

$$|x(t) - y(t)| \leq \frac{\eta_L\varepsilon}{k + \eta_L} \leq \varepsilon \tag{3.17}$$

According to relation above and appealing to Lemma 3.1 there exists $T' > T$ such that

$$|v_x(t) - v_y(t)| \leq \varepsilon, \qquad t \geq T'.$$

Where $v_x(t)$ and $v_y(t)$ belong to ball B_R.. This compleat the proof of theorem. ∎

4. An example

Consider the following quadratic Volterra integral equations of fractional order with control variable

$$\begin{aligned} x(t) &= \frac{t^2}{1+4t^2} + \frac{t+tx(t))}{\Gamma(3/2)} \int_0^t \frac{\frac{\exp(s-t)}{10}(\sqrt[3]{x^2(s)} + \arctan(v(s))) + \frac{1}{1+3t^2}}{\sqrt{t-s}} ds \\ \frac{dv}{dt} &= -\frac{1+t^2}{4+\cos t + t^2} v + \frac{2+\cos t + x^2(t)}{5+\cos t + 2x^2(t)}, \qquad t \geq 0. \end{aligned} \tag{4.1}$$

Note that above equation is a special case of Eq. (1.2) where,

$$p(t) = \frac{t^2}{1+4t^2}, \qquad f(t,x) = t + tx(t),$$

$$\phi(t,s,x,v) = \frac{\exp(s-t)}{10}\{\sqrt[3]{x^2(s)} + \arctan(v(s))\} + \frac{1}{1+3t^2},$$

$$\eta(t) = \frac{1+t^2}{4+\cos t + t^2}, \qquad g(t,x) = \frac{2+\cos t + x^2}{5+\cos t + 2x^2}.$$

The function p is continuous and bounded on $\mathbb{R}_+$ and $\|p\| = 0.25$. Moreover, the function f satisfies assumption (H_2) with $n(t) = t$ and $|f(t,0)| = f(t,0) = t$. Further, observe that the function $\phi(t,s,x,v)$ satisfies assumption (H_3), where $m_1(t) = m_2(t) = e^{-t}$, $\chi_1(r) = \sqrt[3]{r^2}$, $\chi_2(r) = \arctan(r)$ and $\phi(t,s,0,0) = 1/(1+3t^2) = \phi_1(t)$.

Also, we have

$$a_1(t) = a_2(t) = b_1(t) = b_2(t) = t^{\frac{3}{2}} e^{-t},$$

$$c(t) = d(t) = \frac{t^{\frac{3}{2}}}{1+3t^2}.$$

Thus, it easily seen that functions a_i, b_i, c and d are bounded as well as $\lim_{t\to\infty} a_i = 0$ and $\lim_{t\to\infty} b_i = 0$, $i = 1,2$. In addition, $A_1 = a_1(\frac{3}{2}) = A_2 = B_1 = B_2 = 0.04101...$ and $C = c(1) = D = 0.25$.

Since $|\partial g/\partial x| < 1$, function g satisfies the Lipschitz condition with respect to the second variable. Moreover, simple calculation shows that $\alpha_L \geq \frac{1}{5}$, $\alpha_M \leq 1$, $g_L \geq \frac{1}{4}$ $g_M \leq 1$.

Finally, to check that assumption (H_6) is satisfied let us note that inequality (3.12) has the form

$$\begin{aligned} 0.25 &+ \frac{1}{\Gamma(3/2)}(0.04101\sqrt[3]{r^2} + 0.04101\arctan(L) + 0.25)r \\ &+ 0.04101\sqrt[3]{r^2} + 0.04101\arctan(L) + 0.25\} \leq r \end{aligned}$$

Let us rewrite the above inequality in the following form

$$I(r) \leq r\Gamma(3/2),$$

wherein

$$I(r) = \Gamma(3/2)\times 0.25 + 0.04101\sqrt[3]{r^2} + 0.04101\arctan(L) + 0.25)r + 0.04101\sqrt[3]{r^2} + 0.04101\arctan(L) + 0.25.$$

Since $\arctan(L) < \pi/2 = 1.5707...$ and $\Gamma(3/2) \leq 0.880$, for $r = 1$ we have

$$\begin{aligned} I(r) &\leq 0.886 \times 0.25 + 0.04101 + 0.04101 \times 1.5707 \\ &+ 0.25 + 0.04101 + 0.04101 \times 1.5707 + 0.25 = 0.545 < 0.880 \leq r\Gamma(3/2). \end{aligned}$$

Thus, Eq. (4.1) satisfies assumptions (H_1)-(H_6). Now, based on Theorem 3.4 we infer that this equation has a solution in the space$BC(\mathbb{R}_+)$ belonging to the ball B_1 and solutions of this system are uniformly locally attractive.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 19, NO.2, 2015, COPYRIGHT 2015 EUDOXUS PRESS, LLC

Some properties of the Hausdorff fuzzy metric on finite sets

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September 30, 2014

In the paper, some properties of the Hausdorff fuzzy metric on the family of nonempty finite sets, as precompactness, completeness, compactness and F-boundedness are explored. Also, an illustrative example of a complete fuzzy metric space that does not induce the complete Hausdorff fuzzy metric on the family of nonempty finite sets is given.

Keywords: Fuzzy metric, The Hausdorff fuzzy metric, Precompact, Complete, F-bounded.

AMS Subject Classifications: 54A40, 54B20, 54E35

1 Introduction

Fuzzy metric, which is an important notion in Fuzzy Topology, have been introduced by many authors from different points of view [2, 3, 9, 11]. In particular, George and Veeramani [3] gave a definition of fuzzy metric with the help of continuous t-norms and proved that the topology induced by the fuzzy metric is first countable and Hausdorff. Later, Gregori and Romaguera [7] proved that the topological space induced by the fuzzy metric is metrizable. This version of fuzzy metric is more restrictive, but it determines the class of spaces that are tightly connected with the class of metrizable topological spaces. Hence it is interesting to study the version of fuzzy metric. Gregori and

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This work was supported by Nation Natural Science Foundation of China (No. 11471153, No. 11471202), the foundation of Fujian Province (No. JA14200, No. 2013J01029, No. JA13198), the foundation of Minnan Normal University (No. SJ1118).

Romaguera [8] presented a characterization of the given fuzzy metric spaces that are completable. A fuzzy analogue of the Prokhorov metric defined on the set of all probability measures of a compact fuzzy metric space was considered by Repovš and Savchenko [15]. Kočinac [10] gave some Selection properties of fuzzy metric spaces. Other more contributions to the study of fuzzy metric spaces can be found in [4, 5, 12, 13, 14, 17, 18].

In order to study the hyperspaces in a fuzzy metric space, Rodríguez-López and Romaguera [16] gave a definition of Hausdorff fuzzy metric on the family of nonempty compact sets. In the paper, we study the Hausdorff fuzzy metric on the family of nonempty finite subsets of a given fuzzy metric space and explore some properties of the Hausdorff fuzzy metric, as precompactness, completeness, compactness and F-boundedness. Also, we give an example of a complete fuzzy metric space that does not induce the complete Hausdorff fuzzy metric on the family of nonempty finite subsets of a given fuzzy metric space.

2 Preliminaries

Throughout the paper the letter **N** shall denote the set of all nature numbers. Our basic reference for general topology is [1].

Definition 2.1 [3] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a *continuous t-norm* if it satisfies the following conditions:
(i) $*$ is associative and commutative;
(ii) $*$ is continuous;
(iii) $a * 1 = a$ for all $a \in [0,1]$;
(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0,1]$.

The following are examples of t-norms: $a * b = \min\{a, b\}$; $a * b = a \cdot b$; $a * b = \max\{a + b - 1, 0\}$.

Definition 2.2 [3] A 3-tuple $(X, M, *)$ is said to be a *fuzzy metric space* if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times (0, \infty)$ satisfying the following conditions for all $x, y, z \in X$ and $s, t \in (0, \infty)$:
(i) $M(x, y, t) > 0$;
(ii) $M(x, y, t) = 1$ if and only if $x = y$;
(iii) $M(x, y, t) = M(y, x, t)$;
(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(v) the function $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

If $(X, M, *)$ is a fuzzy metric space, we will say that $(M, *)$ is a fuzzy metric on X. It was proved in [3] that every fuzzy metric $(M, *)$ on X generates a topology τ_M on X which has a base the family of open sets of the form $\{B_M(x, r, t) | x \in X, t > 0, r \in (0, 1)\}$, where $B_M(x, r, t) = \{y \in X | M(x, y, t) > 1 - r\}$ for all $r \in (0, 1)$ and $t > 0$.

Definition 2.3 [7] A fuzzy metric space $(X, M, *)$ is called *precompact* if for each $r \in (0, 1)$ and $t > 0$, there is a finite subset A of X such that $X = \bigcup_{a \in A} B_M(a, r, t)$.

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Obviously, each subset of a precompact fuzzy metric space is precompact.

Definition 2.4 [5] Let $(X, M, *)$ be a fuzzy metric space.
(a) A sequence $\{x_n\}_{n\in\mathbf{N}}$ in X is called *Cauchy* if for each $r \in (0,1)$ and $t > 0$, there exists an $N \in \mathbf{N}$ such that $M(x_n, x_m, t) > 1 - r$ for all $n, m \geq N$.
(b) $(X, M, *)$ is called *complete* if every Cauchy sequence in X is convergent with respect to τ_M.

Definition 2.5 [7] A fuzzy metric space $(X, M, *)$ is called *compact* if (X, τ_M) is a compact topological space.

Definition 2.6 [3] A fuzzy metric space $(X, M, *)$ is said to be *F-bounded* if there exist $t > 0$ and $0 < r < 1$ such that $M(x, y, t) > 1 - r$ for all $x, y \in X$.

Clearly, each subset of an F-bounded fuzzy metric space is F-bounded.

3 Main results

Given a fuzzy metric space $(X, M, *)$, we will denote by $\mathcal{P}(X)$, Comp(X) and Fin(X), the set of nonempty subsets, the set of nonempty compact subsets and the set of nonempty finite subsets of (X, τ_M), respectively. For every $B \in \mathcal{P}(X)$, $a \in X$ and $t > 0$, let $M(a, B, t) := \sup_{b\in B} M(a, b, t)$, $M(B, a, t) := \sup_{b\in B} M(b, a, t)$ (see Definition 2.4 of [18]). By condition (iii) in Definition 2.2, we observe that $M(a, B, t) = M(B, a, t)$.

Definition 3.1 [16] Let $(X, M, *)$ be a fuzzy metric space. For every $A, B \in$ Comp(X) and $t > 0$, define H_M: Comp$(X)\times$ Comp$(X) \times (0, \infty) \to [0, 1]$ by

$$H_M(A, B, t) = \min\{\inf_{a\in A} M(a, B, t), \inf_{b\in B} M(A, b, t)\}.$$

It was proved in [16] that (Comp(X),$H_M, *$) is a fuzzy metric space. $(H_M, *)$ is called *the Hausdorff fuzzy metric on Comp(X)*.

Lemma 3.2 [16] *Let $(X, M, *)$ be a fuzzy metric space. Then (Comp(X),$H_M, *$) is precompact if and only if $(X, M, *)$ is precompact.*

Lemma 3.3 [16] *Let $(X, M, *)$ be a fuzzy metric space. Then (Comp(X),$H_M, *$) is complete if and only if $(X, M, *)$ is complete.*

Lemma 3.4 [7] *A fuzzy metric space is compact if and only if it is precompact and complete.*

From the three preceding lemmas we immediately conclude the following.

Corollary 3.5 *Let $(X, M, *)$ be a fuzzy metric space. Then (Comp(X),$H_M, *$) is compact if and only if $(X, M, *)$ is compact.*

In a fuzzy metric space $(X, M, *)$, it it clear that $H_M(\{x\}, \{y\}, t) = M(x, y, t)$ for all $x, y \in X$ and $t > 0$. So we can regard X as a subset of Fin(X).

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Theorem 3.6 *Let $(X, M, *)$ be a fuzzy metric space. Then $(Fin(X), H_M, *)$ is precompact if and only if $(X, M, *)$ is precompact.*

Proof Suppose that $(\mathrm{Fin}(X), H_M, *)$ is precompact. Since $X \subseteq \mathrm{Fin}(X)$, we conclude that $(X, M, *)$ is precompact.

Conversely, suppose that $(X, M, *)$ is precompact. Then, by Lemma 3.2, we obtain that $\mathrm{Comp}(X)$ is precompact. Since $\mathrm{Fin}(X) \subseteq \mathrm{Comp}(X)$, we deduce that $(\mathrm{Fin}(X), H_M, *)$ is precompact.

Lemma 3.7 [3] *A sequence $\{x_n\}_{n\in\mathbf{N}}$ in a fuzzy metric space $(X, M, *)$ converges to x with respect to τ_M if and only if $M(x_n, x, t) \to 1$ as $n \to \infty$.*

Lemma 3.8 *Let $(X, M, *)$ be a complete fuzzy metric space and A be a closed subset of (X, τ_M). Then $(A, M, *)$ is complete.*

Proof Let $\{x_n\}_{n\in\mathbf{N}}$ be a Cauchy sequence in $(A, M, *)$. Then $\{x_n\}_{n\in\mathbf{N}}$ is also a Cauchy sequence in $(X, M, *)$. Since $(X, M, *)$ is complete, there exists an $x \in X$ such that $M(x_n, x, t) \to 1$ as $n \to \infty$. Moreover, notice that A is a closed subset of (X, τ_M). It follows that $x \in A$. Thus, $(A, M, *)$ is complete.

Theorem 3.9 *In a fuzzy metric space $(X, M, *)$, if $(Fin(X), H_M, *)$ is complete, then $(X, M, *)$ is complete.*

Proof Suppose that $(\mathrm{Fin}(X), H_M, *)$ is complete. Now we Take $F \in \mathrm{Fin}(X)$, where F contains at least two points. Let $a, b \in F$ and $t > 0$, with $a \neq b$. Put $M(a, b, 2t) = \varepsilon$. Then there exists a $\varepsilon_1 \in (\varepsilon, 1)$ such that $\varepsilon_1 * \varepsilon_1 > \varepsilon$. Take $\varepsilon_2 \in (\varepsilon_1, 1)$. We show that

$$B_M(a, 1 - \varepsilon_1, t) \cap B_M(b, 1 - \varepsilon_1, t) = \emptyset.$$

In fact, otherwise, we can choose a $c \in B_M(a, 1 - \varepsilon_1, t) \cap B_M(b, 1 - \varepsilon_1, t)$. Hence

$$M(a, b, 2t) \geq M(a, c, t) * M(c, b, t) \geq \varepsilon_1 * \varepsilon_1 > \varepsilon = M(a, b, 2t),$$

which is a contradict. Let $x \in X$. If $x \in \overline{B_M(a, 1 - \varepsilon_1, t)}$, where $\overline{B_M(a, 1 - \varepsilon_1, t)}$ is the closure of $B_M(a, 1 - \varepsilon_1, t)$, then $x \notin B_M(b, 1 - \varepsilon_1, t)$. So $M(b, x, t) \leq \varepsilon_1$. Hence

$$H_M(F, \{x\}, t) \leq \inf_{y\in F} M(y, \{x\}, t) = \inf_{y\in F} M(y, x, t) \leq M(b, x, t) \leq \varepsilon_1.$$

If $x \notin \overline{B_M(a, 1 - \varepsilon_1, t)}$, then $x \notin B_M(a, 1 - \varepsilon_1, t)$. So $M(a, x, t) \leq \varepsilon_1$. Hence $H_M(F, \{x\}, t) \leq \varepsilon_1$. Whence

$$\{\{x\} | x \in X\} \cap H_M(F, 1 - \varepsilon_2, t) = \emptyset,$$

which means that X is a closed subset of $(\mathrm{Fin}(X), \tau_{H_M})$. Consequently, by Lemma 3.8, $(X, M, *)$ is complete.

The converse of the preceding theorem is false. We illustrate this fact with an example.

4

Example 3.10 Let $X = \{1, \frac{1}{2}, \cdots, \frac{1}{n}, \cdots, 0\}$ and d be Euclidian metric of X. Denote $a * b = a \cdot b$ for all $a, b \in [0,1]$. Define the function M by

$$M(x, y, t) = \frac{1}{1 + d(x, y)}$$

for all $x, y \in X$ and $t > 0$. Then $(X, M, *)$ is a compact fuzzy metric space (see [6]). Therefore, according to Corollary 3.5, (Comp(X),$H_M, *$) is compact. Also, since $(X, M, *)$ is complete, we have that (Comp(X),$H_M, *$) is complete by Lemma 3.3. Put $A = \{1, \frac{1}{3}, \cdots, \frac{1}{2n-1}, \ldots, 0\}$. Then $A \in \mathrm{Comp}(X) \backslash \mathrm{Fin}(X)$. Let $r \in (0,1)$ and $t > 0$. Then there exists a $k \in \mathbf{N}$ such that if $n \geq k$, we get $\frac{1}{2n-1} \in B_M(0, r, t)$. Since $B = \{1, \frac{1}{3}, \cdots, \frac{1}{2n-3}, 0\} \in \mathrm{Fin}(X)$, we have $H_M(A, B, t) > 1 - r$. Hence $B \in B_{H_M}(A, r, t)$. So Fin(X) is not a closed subset of (Comp$(X), \tau_{H_M}$), which implies that (Fin$(X), H_M, *$) fails to be complete.

By Lemma 3.4, Theorem 3.6 and Theorem 3.9, we immediately deduce the following.

Corollary 3.11 *In a fuzzy metric space $(X, M, *)$, if (Fin$(X), H_M, *$) is compact, then $(X, M, *)$ is compact.*

Due to Example 3.10, the converse of the above corollary is false obviously. Now, a question arise naturally.

Question 3.12 *In a complete fuzzy metric space $(X, M, *)$, under what condition is (Fin$(X), H_M, *$) complete?*

Theorem 3.13 *Let $(X, M, *)$ be a fuzzy metric space. Then (Fin$(X), H_M, *$) is F-bounded if and only if $(X, M, *)$ is F-bounded.*

Proof Assume that (Fin$(X), H_M, *$) is F-bounded. Since $X \subseteq \mathrm{Fin}(X)$, we get that $(X, M, *)$ is F-bounded.

Conversely, assume that $(X, M, *)$ is F-bounded. Then there exist $r \in (0,1)$ and $t > 0$ such that $M(x, y, t) > 1 - r$ whenever $x, y \in X$. Let $A, B \in \mathrm{Fin}(X)$. Then, for each $a \in A$, there exists a $b_a \in B$ such that

$$M(a, B, t) = \sup\{M(a, b, t) | b \in B\} = \max\{M(a, b, t) | b \in B\} = M(a, b_a, t).$$

So there exists an $a_0 \in A$ such that

$$\inf_{a \in A} M(a, B, t) = \inf_{a \in A} M(a, b_a, t) = \min\{M(a, b_a, t) | a \in A\} = M(a_0, b_{a_0}, t) > 1 - r.$$

Similarly, we have that $\inf\limits_{b \in B} M(A, b, t) > 1 - r$. Hence $H_M(A, B, t) > 1 - r$. We finish the proof.

4 Conclusion

We have studied some properties of the Hausdorff fuzzy metric on the family of nonempty finite subsets of a given fuzzy metric space, as precompactness,

completeness, compactness and F-boundedness. Also, we have given an example to illustrate that a complete fuzzy metric space does not induce the complete Hausdorff fuzzy metric on the family of nonempty finite subsets of a given fuzzy metric space.

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LOCALLY BOUNDEDNESS AND CONTINUITY OF SUPERPOSITION OPERATORS ON DOUBLE SEQUENCE SPACES C_{r0}

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Abstract. Let $\mathbb{R}$ be set of all real numbers, $\mathbb{N}$ be the set of all natural numbers and $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$. In this paper, we define the superposition operator P_g where $g : \mathbb{N}^2 \times \mathbb{R} \to \mathbb{R}$ by $P_g((x_{ks})) = g(k, s, x_{ks})$ for all real double sequence (x_{ks}). Chew & Lee [4] and Petranuarat & Kemprasit [11] have characterized $P_g : c_0 \to l_1$ and $P_g : c_0 \to l_q$ where $1 \leq q < \infty$, respectively. The main aim of this paper is to construct the necessary and sufficient conditions for the boundedness and continuity of $P_g : C_{r0} \to \mathcal{L}_1$ and $P_g : C_{r0} \to \mathcal{L}_p$ where $1 \leq p < \infty$.

1. INTRODUCTION

Let Ω be a double sequence spaces which are the vector spaces with coordinate-wise addition and scalar multiplication. Let any sequence $x = (x_{ks}) \in \Omega$. If for any $\varepsilon > 0$ there exist $N \in \mathbb{N}$ and $l \in \mathbb{R}$ such that $|x_{ks} - l| < \varepsilon$ for all $k, s \geq N$, then we call that the double sequence $x = (x_{ks})$ is convergent in Pringsheim's sense and denoted by $p - \lim x_{ks} = l$. If the double sequence $x = (x_{ks})$ converges in Pringsheim's sense and, in addition, the limits that $\lim\limits_{k} x_{ks}$ and $\lim\limits_{s} x_{ks}$ exist, then it is called regularly convergent and denoted by $r - \lim x_{ks}$. The space C_{r0} is defined by

$$C_{r0} = \{x = (x_{ks}) \in \Omega : r - \lim x_{ks} = 0\}$$

and it is a Banach space with the norm $\|x\|_{C_{r0}} = \sup\limits_{k,s\in\mathbb{N}} |x_{ks}|$. The space of all bounded double sequences is denoted by M_u and defined by

$$M_u := \left\{ x = (x_{ks}) \in \Omega : \|x\|_\infty = \sup_{k,s\in\mathbb{N}} |x_{ks}| < \infty \right\},$$

and it is a Banach space with the norm $\|.\|_\infty$. The spaces $\mathcal{L}_p$, $1 \leq p < \infty$ are defined by

$$\mathcal{L}_p := \left\{ x = (x_{ks}) \in \Omega : \sum_{k,s=1}^{\infty} |x_{ks}|^p < \infty \right\}$$

where $\sum\limits_{k,s=1}^{\infty} = \sum\limits_{k=1}^{\infty}\sum\limits_{s=1}^{\infty}$. Also it is a Banach space with the norm $\|x\|_p = \left(\sum\limits_{k,s=1}^{\infty} |x_{ks}|^p \right)^{\frac{1}{p}}$. Also, the spaces C_{r0} and $\mathcal{L}_p$ are shown different symbols as ${}_0c_2^R$ and l_2^p, recpectively.

2000 *Mathematics Subject Classification.* 47H30; 46A45.

Key words and phrases. Superposition Operators, Continuity, Locally Bounded, Double Sequence Spaces, Regularly Convergent.

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It is known that $\mathcal{L}_p \subset \mathcal{L}_q \subset C_{r0} \subset M_u$ where $1 \leq p < q < \infty$. The sequence e^{ks} is defined as $e^{ks}_{ij} = \begin{cases} 1 & ,k=i \text{ and } s=j \\ 0 & ,\text{otherwise} \end{cases}$. If we consider the sequence s_{nm} defined by $s_{nm} = \sum\limits_{k=1}^{n}\sum\limits_{s=1}^{m} x_{ks}$ $(n, m \in \mathbb{N})$, then the pair of $((x_{ks}), (s_{nm}))$ is called a double series. Also (x_{ks}) is called general term of series and (s_{nm}) is called partial sums sequence. Let v be convergence notions, i.e., in Pringsheim's sense or regularly convergent. If the partial sums sequence (s_{nm}) is convergent to a real number s in v-sense, i.e.

$$v - \lim_{n,m} \sum_{k=1}^{n}\sum_{s=1}^{m} x_{ks} = s,$$

then the series $((x_{ks}), (s_{nm}))$ is called convergent to s in the $v-$sense ,or, $v-$convergent and the sum of series equals to s. It's denoted by

$$\sum_{k,s=1}^{\infty} x_{ks} = s.$$

It is known that if the series is $v-$convergent, then the $v-$limit of the general term of the series equals to zero. The remaining term of the series $\sum\limits_{k=1}^{\infty}\sum\limits_{s=1}^{\infty} x_{ks}$ is defined by

$$R_{nm} = \sum_{k=1}^{n-1}\sum_{s=m}^{\infty} x_{ks} + \sum_{k=n}^{\infty}\sum_{s=1}^{m-1} x_{ks} + \sum_{k=n}^{\infty}\sum_{s=m}^{\infty} x_{ks}. \tag{1.1}$$

We will demonstrate the formula (1.1) briefly with

$$\sum_{\max\{k,s\} \geq N} x_{ks}$$

for $n = m = N$. It is known that if the series is $v-$convergent, then the $v-$limit of the remaining term of series is zero. For more details on double sequence and series, one can referee [1],[2],[3],[7],[9],[10],[13],[16] and the references therein.

Locally boundedness and continuity of superposition operators on sequence spaces are discussed by some authors [4],[5],[6],[8],[11],[12],[15]. In [4], Chew Tuan Seng and Lee Peng Yee have given necessary and sufficient conditions for the continuity of the superposition operator acting from sequence space c_0 into l_1. In [11], Somkit Petranuarat and Yupaporn Kemprasit have given necessary and sufficient conditions for continuity of the superposition operator acting from sequence space c_0 into l_q with $1 \leq q < \infty$.

We extend the definition of superposition operators for double sequence spaces as follows. Let X, Y be two double sequence spaces. A superposition operator P_g on X is a mapping from X into Ω defined by $P_g(x) = (g(k, s, x_{ks}))_{k,s=1}^{\infty}$ where the function $g : \mathbb{N}^2 \times \mathbb{R} \to \mathbb{R}$ satisfies
(1) $g(k, s, 0) = 0$ for all $k, s \in \mathbb{N}$.
If $P_g(x) \in Y$ for all $x \in X$, we say that P_g acts from X into Y and write $P_g : X \to Y$. Moreover, we shall assume the additionally some of the following conditions:
(2) $g(k, s, .)$ is continuous for all $k, s \in \mathbb{N}$
(2′) $g(k, s, .)$ is bounded on every bounded subset of $\mathbb{R}$ for all $k, s \in \mathbb{N}$.
It is obvious that if the function $g(k, s, .)$ satisfies the propety (2), then g satisfies

$(2')$ from [15]. Also, it is not hard to see that if the function $g(k,s,.)$ is locally bounded on $\mathbb{R}$, then g satisfies $(2')$ from [15].

In this paper, we characterize the superposition operator acting from the double sequence space C_{r0} into $\mathcal{L}_1$ under the hypothesis that the function $g(k,s,.)$ satisfies $(2')$. We discuss the continuity and locally boundedness of the superposition operator P_g by using the methods in [4],[15]. Then we generalize our works as the superposition operator acting from the space C_{r0} into $\mathcal{L}_p$ where $1 \le p < \infty$ without assuming that the function $g(k,s,.)$ satisfies $(2')$ by using the methods in [11],[15].

2. SUPERPOSITION OPERATORS OF C_{r0} INTO $\mathcal{L}_1$

Theorem 1. *Let $g : \mathbb{N}^2 \times \mathbb{R} \to \mathbb{R}$ satisfies $(2')$. Then $P_g : C_{r0} \to \mathcal{L}_1$ if and only if there exist $\alpha > 0$ and $(c_{ks})_{k,s=1}^{\infty} \in \mathcal{L}_1$ such that*

$$|g(k,s,t)| \le c_{ks} \text{ whenever } |t| \le \alpha$$

for all $k,s \in \mathbb{N}$.

Proof. Assume that there exist $\alpha > 0$ and $(c_{ks})_{k,s=1}^{\infty} \in \mathcal{L}_1$ such that $|g(k,s,t)| \le c_{ks}$ whenever $|t| \le \alpha$ for all $k,s \in \mathbb{N}$. Let $x = (x_{ks}) \in C_{r0}$. Hence $p - \lim x_{ks} = 0$ and the limits that $\lim_{k\to\infty} x_{ks}$ and $\lim_{s\to\infty} x_{ks}$ exist. Therefore there exists $N \in \mathbb{N}$ such that $|x_{ks}| \le \alpha$ for all $k,s \in \mathbb{N}$ with $\max\{k,s\} \ge N$. Then, we find

$$\sum_{\max\{k,s\}\ge N} |g(k,s,x_{ks})| \le \sum_{\max\{k,s\}\ge N} c_{ks} \le \sum_{k,s=1}^{\infty} c_{ks} < \infty.$$

So, we get $P_g(x) = g(k,s,x_{ks}) \in \mathcal{L}_1$.

Conversely, suppose that $P_g : C_{r0} \to \mathcal{L}_1$. The sets $A(\alpha)$ and $B(k,s,\alpha)$ is defined as

$$A(\alpha) = \{t \in \mathbb{R} : |t| \le \alpha\}$$

and

$$B(k,s,\alpha) = \sup\{|g(k,s,t)| : t \in A(\alpha)\}$$

for all $k,s \in \mathbb{N}$ and $\alpha > 0$. So, we see that $|g(k,s,t)| \le B(k,s,\alpha)$ whenever $|t| \le \alpha$. We will show that there is $\alpha_1 > 0$ such that $(B(k,s,\alpha_1))_{k,s=1}^{\infty} \in \mathcal{L}_1$. Assume the contrary,that is, $\sum_{k,s=1}^{\infty} B(k,s,\alpha) = \infty$ for all $\alpha > 0$. Therefore $\sum_{k,s=1}^{\infty} B\left(k,s,\frac{1}{i}+\frac{1}{j}\right) = \infty$ for each $i,j \in \mathbb{N}$. Then there exist sequence of positive integers $n_0 = 0 < n_1 < n_2 < \cdots < n_i < \cdots$ and $m_0 = 0 < m_1 < m_2 < \cdots < m_j < \cdots$ such that

$$\sum_{k=n_{i-1}+1}^{n_i} \sum_{s=m_{j-1}+1}^{m_j} B\left(k,s,\frac{1}{i}+\frac{1}{j}\right) > 1$$

for each $i,j \in \mathbb{N}$ and also there exists $\varepsilon_{ij} > 0$ such that

$$\sum_{k=n_{i-1}+1}^{n_i} \sum_{s=m_{j-1}+1}^{m_j} B\left(k,s,\frac{1}{i}+\frac{1}{j}\right) - \varepsilon_{ij}(n_i - n_{i-1})(m_j - m_{j-1}) > 1. \tag{2.1}$$

Let $i,j \in \mathbb{N}$ be fixed. Since g satisfies $(2')$, we see that $B\left(k,s,\frac{1}{i}+\frac{1}{j}\right) < \infty$ for all $i,j \in \mathbb{N}$ with $n_{i-1}+1 \le k \le n_i$ and $m_{j-1}+1 \le s \le m_j$. Then, there exists

$x_{ks} \in A\left(\frac{1}{i}+\frac{1}{j}\right)$ such that

$$|g(k,s,x_{ks})| > B\left(k,s,\frac{1}{i}+\frac{1}{j}\right) - \varepsilon_{ij} \tag{2.2}$$

for each $k,s \in \mathbb{N}$ satisfying $n_{i-1}+1 \le k \le n_i$ and $m_{j-1}+1 \le s \le m_j$. So, we find

$$\begin{aligned}\sum_{k=n_{i-1}+1}^{n_i}\sum_{s=m_{j-1}+1}^{m_j}|g(k,s,x_{ks})| &> \sum_{k=n_{i-1}+1}^{n_i}\sum_{s=m_{j-1}+1}^{m_j}B\left(k,s,\frac{1}{i}+\frac{1}{j}\right) - \sum_{k=n_{i-1}+1}^{n_i}\sum_{s=m_{j-1}+1}^{m_j}\varepsilon_{ij}\\ &> \sum_{k=n_{i-1}+1}^{n_i}\sum_{s=m_{j-1}+1}^{m_j}B\left(k,s,\frac{1}{i}+\frac{1}{j}\right) -\\ &\quad -\varepsilon_{ij}(n_i-n_{i-1})(m_j-m_{j-1})\\ &> 1\end{aligned}$$

by using (2.1) and (2.2). Therefore we obtain that

$$\sum_{k=1}^{\infty}\sum_{s=1}^{\infty}|g(k,s,x_{ks})| = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\sum_{k=n_{i-1}+1}^{n_i}\sum_{s=m_{j-1}+1}^{m_j}|g(k,s,x_{ks})|\right) = \infty.$$

Hence we get $g(k,s,x_{ks}) \notin \mathcal{L}_1$. Since $x_{ks} \in A\left(\frac{1}{i}+\frac{1}{j}\right)$ whenever $n_{i-1}+1 \le k \le n_i$ and $m_{j-1}+1 \le s \le m_j$, we find $|x_{ks}| \le \frac{1}{i}+\frac{1}{j}$. Hence, we obtain $x=(x_{ks}) \in C_{r0}$. This contradicts the assumption that $P_g : C_{r0} \to \mathcal{L}_1$. Then there exists $\alpha_1 > 0$ such that $(B(k,s,\alpha_1))_{k,s=1}^{\infty} \in \mathcal{L}_1$. If we put $c_{ks} = B(k,s,\alpha_1)$ for all $k,s \in \mathbb{N}$, the proof is completed. □

Theorem 2. *If $P_g : C_{r0} \to \mathcal{L}_1$, then P_g is continuous on C_{r0} if and only if $g(k,s,.)$ is continuous on $\mathbb{R}$ for all $k,s \in \mathbb{N}$.*

Proof. Suppose that P_g is continuous on C_{r0}. Let $k,s \in \mathbb{N}$, $t_0 \in \mathbb{R}$ and $\varepsilon > 0$. Since P_g is continuous at $t_0e^{(ks)} \in C_{r0}$, there exists $\delta > 0$ such that

$$\left\|z - t_0e^{(ks)}\right\|_{C_{r0}} < \delta \text{ implies } \left\|P_g(z) - P_g\left(t_0e^{(ks)}\right)\right\|_1 < \varepsilon \tag{2.3}$$

for all $z=(z_{ks}) \in C_{r0}$. Let $t \in \mathbb{R}$ such that $|t-t_0| < \delta$ and defined $y=(y_{nm})$ by $y_{nm} = \begin{cases} t, & k=n \text{ and } s=m \\ 0, & \text{others}\end{cases}$. So $y=(y_{nm}) \in C_{r0}$ and we have $\|y - t_0e^{(ks)}\|_{C_{r0}} = |t-t_0| < \delta$. From (2.3), we find

$$|g(k,s,t) - g(k,s,t_0)| = \left\|P_g(y) - P_g\left(t_0e^{(ks)}\right)\right\|_1 < \varepsilon.$$

Therefore, the function $g(k,s,.)$ is continuous on $\mathbb{R}$ for each $k,s \in \mathbb{N}$.

Conversely, assume that the function $g(k,s,.)$ is continuous on $\mathbb{R}$ for each $k,s \in \mathbb{N}$. We will show that P_g is continuous on C_{r0}. Let $x=(x_{ks}) \in C_{r0}$ and $\varepsilon > 0$. Since g satisfies $(2')$, then P_g acts from C_{r0} to $\mathcal{L}_1$ by Theorem1. Hence, there is $\alpha > 0$ and $(c_{ks}) \in \mathcal{L}_1$ such that

$$|g(k,s,t)| \le c_{ks} \text{ whenever } |t| \le \alpha \tag{2.4}$$

for all $k,s \in \mathbb{N}$. Since $(x_{ks}) \in C_{r0} \subset M_u$ and $(c_{ks}) \in \mathcal{L}_1$, there exists $N \in \mathbb{N}$ such that

$$|x_{ks}| \le \frac{\alpha}{2} \text{ for all } k,s \in \mathbb{N} \text{ with } max\{k,s\} \ge N$$

and

$$\sum_{\max\{k,s\}\geq N} c_{ks} < \frac{\varepsilon}{3}.$$

So, $|x_{ks}| \leq \alpha$ for all $k, s \in \mathbb{N}$ with $max\{k,s\} \geq N$. From (2.4), we write $|g(k,s,x_{ks})| \leq c_{ks}$ for all $k,s \in \mathbb{N}$ with $max\{k,s\} \geq N$. Hence, we have

$$\sum_{\max\{k,s\}\geq N} |g(k,s,x_{ks})| \leq \sum_{\max\{k,s\}\geq N} c_{ks} < \frac{\varepsilon}{3}. \tag{2.5}$$

Since $g(k,s,.)$ is continuous at x_{ks} for all $k,s \in \{1,2,\ldots,N-1\}$, there exists $\delta > 0$ with $\delta = \min\left\{1, \frac{\alpha}{2}\right\}$ such that

$$|t - x_{ks}| < \delta \text{ implies } |g(k,s,t) - g(k,s,x_{ks})| < \frac{\varepsilon}{3(N-1)} \tag{2.6}$$

for any $t \in \mathbb{R}$. Let $z = (z_{ks}) \in C_{r0}$ satisfying $\|z - x\|_{C_{r0}} < \delta$. Thus,

$$|z_{ks} - x_{ks}| \leq \sup_{k,s\in\mathbb{N}} |z_{ks} - x_{ks}| = \|z - x\|_{C_{r0}} < \delta$$

for each $k, s \in \mathbb{N}$. By using (2.6), we find

$$|g(k,s,z_{ks}) - g(k,s,x_{ks})| < \frac{\varepsilon}{3(N-1)}$$

for all $k,s \in \{1,2,\ldots,N-1\}$. Hence, we have

$$\sum_{k,s=1}^{N-1} |g(k,s,z_{ks}) - g(k,s,x_{ks})| < \frac{\varepsilon}{3}. \tag{2.7}$$

Since $|z_{ks}| \leq |z_{ks} - x_{ks}| + |x_{ks}| < \delta + \frac{\alpha}{2} \leq \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha$ for all $k,s \in \mathbb{N}$ with $max\{k,s\} \geq N$, it follows from (2.4) that $|g(k,s,z_{ks})| \leq c_{ks}$ for all $k,s \in \mathbb{N}$ with $max\{k,s\} \geq N$.We obtain

$$\begin{aligned} \|P_g(z) - P_g(x)\| &= \sum_{k,s=1}^{\infty} |g(k,s,z_{ks}) - g(k,s,x_{ks})| \\ &\leq \sum_{k,s=1}^{N-1} |g(k,s,z_{ks}) - g(k,s,x_{ks})| + \sum_{\max\{k,s\}\geq N} |g(k,s,z_{ks})| + \\ &\quad + \sum_{\max\{k,s\}\geq N} |g(k,s,x_{ks})| \\ &< \varepsilon \end{aligned}$$

by using (2.5) and (2.7). This is completed the proof. $\square$

Theorem 3. *If $P_g : C_{r0} \to \mathcal{L}_1$, then P_g is locally bounded on C_{r0} if and only if g satisfies* $(2')$.

Proof. Assume that g satisfies $(2')$ and let $z = (z_{ks}) \in C_{r0}$. By Theorem1, there exist $(c_{ks}) \in \mathcal{L}_1$ and $\alpha > 0$ such that

$$|g(k,s,t)| \leq c_{ks} \text{ whenever } |t| \leq \alpha \tag{2.8}$$

for all $k,s \in \mathbb{N}$ with $\max\{k,s\} \geq N$. Let $x = (x_{ks}) \in C_{r0}$ satisfying

$$\|z - x\|_{C_{r0}} \leq \frac{\alpha}{2}.$$

So, we have that

$$\sup_{k,s\in\mathbb{N}} |z_{ks} - x_{ks}| \leq \frac{\alpha}{2}. \tag{2.9}$$

Since $r - \lim z_{ks} = 0$, there exists $N' \in \mathbb{N}$ such that $|z_{ks}| \leq \frac{\alpha}{2}$ for all $k,s \in \mathbb{N}$ with $\max\{k,s\} \geq N'$. Hence,

$$\sup_{\max\{k,s\}\geq N} |z_{ks}| \leq \frac{\alpha}{2}. \tag{2.10}$$

By using (2.9) and (2.10), we find

$$|x_{ks}| \leq \sup_{\max\{k,s\}\geq N} |x_{ks}| \leq \sup_{k,s\in\mathbb{N}} |z_{ks} - x_{ks}| + \sup_{\max\{k,s\}\geq N} |z_{ks}| < \alpha$$

for all $k,s \in \mathbb{N}$ with $\max\{k,s\} \geq N$. From (2.8), we have that

$$|g(k,s,x_{ks})| \leq c_{ks}$$

for all $k,s \in \mathbb{N}$ with $\max\{k,s\} \geq N$. Therefore,

$$\sum_{\max\{k,s\}\geq N} |g(k,s,x_{ks})| \leq \sum_{\max\{k,s\}\geq N} c_{ks} \leq \sum_{k,s=1}^{\infty} c_{ks} = \|c_{ks}\|_1 . \tag{2.11}$$

If we put $m_{ks} = \sup\limits_{|t-z_{ks}|\leq\frac{\alpha}{2}} |g(k,s,t)|$, since g satisfies $(2')$ we see that $m_{ks} < \infty$ for all $k,s \in \mathbb{N}$. We have

$$|g(k,s,x_{ks})| \leq m_{ks} \tag{2.12}$$

for each $k,s \in \mathbb{N}$. By using (2.11) and (2.12), we obtain

$$\begin{aligned} \|P_g(x)\|_1 &= \sum_{k,s=1}^{\infty} |g(k,s,x_{ks})| = \sum_{k,s=1}^{N-1} |g(k,s,x_{ks})| + \sum_{\max\{k,s\}\geq N} |g(k,s,x_{ks})| \\ &\leq \sum_{k,s=1}^{N-1} m_{ks} + \sum_{k,s=1}^{\infty} c_{ks} = \sum_{k,s=1}^{N-1} m_{ks} + \|c_{ks}\|_1 . \end{aligned}$$

Therefore,

$$\begin{aligned} \|P_g(x) - P_g(z)\|_1 &\leq \|P_g(x)\|_1 + \|P_g(z)\|_1 \\ &\leq \|P_g(z)\|_1 + \sum_{k,s=1}^{N-1} m_{ks} + \|c_{ks}\|_1 . \end{aligned}$$

Let $\gamma = \|P_g(z)\|_1 + \sum\limits_{k,s=1}^{N-1} m_{ks} + \|c_{ks}\|_1$, then we write $\|P_g(x) - P_g(z)\|_1 \leq \gamma$. Hence, P_g is locally bounded on C_{r0}.

Conversely, assume that P_g is locally bounded on C_{r0}. To complete the proof, it is sufficient that g is locally bounded on $\mathbb{R}$. The sequence $y = (y_{ks})$ is defined as

$$y_{ks} = \begin{cases} a & ,k = n \text{ and } s = m \\ \frac{1}{k} + \frac{1}{s} & , \text{others} \end{cases}.$$

for all $k,s \in \mathbb{N}$ and $a \in \mathbb{R}$. So, it is obvious that $y = (y_{ks}) \in C_{r0}$. From the hypothesis, there exists $\alpha, \beta > 0$ such that

$$\|P_g(x) - P_g(y)\| \leq \beta \text{ whenever } \|x - y\|_{C_{r0}} \leq \alpha. \tag{2.13}$$

Also, the sequence $x = (x_{ks})$ is defined as $x_{ks} = \begin{cases} b & ,k= n \text{ and } s=m \\ \frac{1}{k}+\frac{1}{s} & , \text{others} \end{cases}$ for all $k, s \in \mathbb{N}$ and $b \in \mathbb{R}$ with $|b-a| \leq \alpha$. So, it is obvious that $x = (x_{ks}) \in C_{r0}$. Thus, we find

$$\|x-y\|_{C_{r0}} = \sup_{k,s\in\mathbb{N}} |x_{ks} - y_{ks}| = |b-a| \leq \alpha.$$

Therefore, $\|P_g(x) - P_g(y)\| \leq \beta$ from (2.13). Then, we obtain

$$|g(k,s,b) - g(k,s,a)| \leq \sum_{k,s=1}^{\infty} |g(k,s,x_{ks}) - g(k,s,y_{ks})| = \|P_g(x) - P_g(y)\| \leq \beta.$$

Since $b \in \mathbb{R}$ is arbitrary, $g(k,s,.)$ is locally bounded on $\mathbb{R}$. □

Example 1. *Let $g : \mathbb{N}^2 \times \mathbb{R} \to \mathbb{R}$ defined by*

$$g(k,s,t) = \frac{|t|}{4^{k+s}}$$

for all $k, s \in \mathbb{N}$ and for all $t \in \mathbb{R}$. Since $g(k,s,.)$ is continuous on $\mathbb{R}$ for all $k,s \in \mathbb{N}$, g satisfies $(2')$. Let $\alpha = 1$ and $|t| \leq 1$. Then for all $k,s \in \mathbb{N}$,

$$\begin{aligned} |g(k,s,t)| &= \frac{|t|}{4^{k+s}} \\ &\leq \frac{1}{4^{k+s}} \end{aligned}$$

Since $\sum\limits_{k,s=1}^{\infty} \frac{1}{4^{k+s}} < \infty$, we put $c_{ks} = \frac{1}{4^{k+s}}$ for all $k,s \in \mathbb{N}$. By Theorem1, we find that $P_g : C_{r0} \to \mathcal{L}_1$. Since $g(k,s,.)$ is continuous on $\mathbb{R}$ for all $k,s \in \mathbb{N}$, then the superposition operator P_g is continuous on C_{r0} by Theorem 2. Also, since $g(k,s,.)$ is bounded on $\mathbb{R}$ for all $k,s \in \mathbb{N}$, then the superposition operator P_g is locally bounded on C_{r0} by Theorem 3.

3. SUPERPOSITION OPERATORS OF C_{r0} INTO $\mathcal{L}_p$ $(1 \leq p < \infty)$

In this section, we extend theorems that proved in section 2 as the superposition operator acting from the space C_{r0} into $\mathcal{L}_p$ where $1 \leq p < \infty$ by using the methods in [11],[15]. For characterization of the superposition operator $P_g : C_{r0} \to \mathcal{L}_p$, we will use the following proposition that it can be expanded with same method as Proposition 3.1 in [14].

Proposition 1. *Let Ω be a double sequence space. If $\mathcal{L}_1 \subseteq \Omega$ and $P_g : \Omega \to M_u$, then there exist $N \in \mathbb{N}$ and $\alpha > 0$ such that $(g(k,s,.))_{\max\{k,s\}\geq N}^{\infty}$ is uniformly bounded on $[-\alpha, \alpha]$.*

Theorem 4. *Let $g : \mathbb{N}^2 \times \mathbb{R} \to \mathbb{R}$. Then $P_g : C_{r0} \to \mathcal{L}_p$ if and only if there exist $N \in \mathbb{N}$ and $\alpha > 0$ such that*

$$\sum_{\max\{k,s\}\geq N} \sup_{|t|\leq\alpha} |g(k,s,t)|^p < \infty. \tag{3.1}$$

Proof. Suppose that P_g acts from C_{r0} to $\mathcal{L}_p$. Since $\mathcal{L}_1 \subseteq C_{r0}$ and $\mathcal{L}_p \subseteq M_u$, we see that there exist $N \in \mathbb{N}$ and $\alpha > 0$ such that $(g(k,s,.))_{\max\{k,s\}\geq N}^{\infty}$ is uniformly bounded on $[-\alpha, \alpha]$ by Proposition1. Therefore, $\sup\limits_{|t|\leq\alpha} |g(k,s,t)|^p < \infty$ for all $k,s \in \mathbb{N}$ with $\max\{k,s\} \geq N$. We define $B(k,s,\beta)$ by

$$B(k,s,\beta) = \sup_{|t|\leq\alpha} |g(k,s,t)|^p \tag{3.2}$$

for all $\beta \in \mathbb{R}$ with $0 < \beta \leq \alpha$. We assert that $\sum_{\max\{k,s\} \geq N} B(k,s,\beta) < \infty$ for some $\beta \in \mathbb{R}$ with $0 < \beta \leq \alpha$. To show that the case is true, we assume the contrary. Therefore, $\sum_{\max\{k,s\} \geq N} B\left(k,s,\alpha\left(\frac{1}{i}+\frac{1}{j}\right)\right) = \infty$ for all $i,j \in \mathbb{N}$. Hence, there exist $n' > n$ and $m' > m$ such that

$$\sum_{k=n}^{n'}\sum_{s=1}^{m'-1} B\left(k,s,\alpha\left(\frac{1}{i}+\frac{1}{j}\right)\right)+\sum_{k=1}^{n'-1}\sum_{s=m}^{m'} B\left(k,s,\alpha\left(\frac{1}{i}+\frac{1}{j}\right)\right)+\sum_{k=n}^{n'}\sum_{s=m}^{m'} B\left(k,s,\alpha\left(\frac{1}{i}+\frac{1}{j}\right)\right) > 1$$

for all $i,j \in \mathbb{N}$ and $n,m \geq N$. Then, there exist subsequences $(n_k)_{k=1}^{\infty}$ of $(n)_{n=1}^{\infty}$ and $(m_k)_{k=1}^{\infty}$ of $(m)_{m=1}^{\infty}$ such that

$$\begin{aligned} & \sum_{k=n_i+1}^{n_{i+1}}\sum_{s=1}^{m_{j+1}-1} B\left(k,s,\alpha\left(\frac{1}{i}+\frac{1}{j}\right)\right)+\sum_{k=1}^{n_{i+1}-1}\sum_{s=m_j+1}^{m_{j+1}} B\left(k,s,\alpha\left(\frac{1}{i}+\frac{1}{j}\right)\right)+ \\ & +\sum_{k=n_i+1}^{n_{i+1}}\sum_{s=m_j+1}^{m_{j+1}} B\left(k,s,\alpha\left(\frac{1}{i}+\frac{1}{j}\right)\right) \\ > {} & 1 \end{aligned}$$

for all $i,j \in \mathbb{N}$ and $n \geq n_1$, $m \geq m_1$. We set $\mathcal{F} = \{(k,s) : k \leq n_1 \text{ and } s \leq m_1\}$. If $(k,s) \in \mathcal{F}$, we take $x_{ks} = 0$. If $k > n_1$ and $s > m_1$, then there exist $i \in \mathbb{N}$ and $j \in \mathbb{N}$ such that $n_i < k \leq n_{i+1}$ and $m_j < s \leq m_{j+1}$. Hence, there exists $x_{ks} \in \left[-\alpha\left(\frac{1}{i}+\frac{1}{j}\right), \alpha\left(\frac{1}{i}+\frac{1}{j}\right)\right]$ such that

$$0 \leq B\left(k,s,\alpha\left(\frac{1}{i}+\frac{1}{j}\right)\right) < |g(k,s,x_{ks})|^p + 2^{-(k+s)} \tag{3.3}$$

from (3.1). Therefore, it is obvious that $x_{ks} \in C_{r0}$. By using (3.2), we write

$$\begin{aligned} r^2 & < \sum_{i=1}^{r}\sum_{j=1}^{r}\left(\sum_{k=n_i+1}^{n_{i+1}}\sum_{s=1}^{m_{j+1}-1} B\left(k,s,\alpha\left(\frac{1}{i}+\frac{1}{j}\right)\right)+\sum_{k=1}^{n_{i+1}-1}\sum_{s=m_j+1}^{m_{j+1}} B\left(k,s,\alpha\left(\frac{1}{i}+\frac{1}{j}\right)\right)+\right. \\ & \left. +\sum_{k=n_i+1}^{n_{i+1}}\sum_{s=m_j+1}^{m_{j+1}} B\left(k,s,\alpha\left(\frac{1}{i}+\frac{1}{j}\right)\right)\right) \\ & = \sum_{k=n_1+1}^{n_{r+1}}\sum_{s=1}^{m_{r+1}-1} B\left(k,s,\alpha\left(\frac{1}{i}+\frac{1}{j}\right)\right)+\sum_{k=1}^{n_{r+1}-1}\sum_{s=m_1+1}^{m_{r+1}} B\left(k,s,\alpha\left(\frac{1}{i}+\frac{1}{j}\right)\right) \\ & +\sum_{k=n_1+1}^{n_{r+1}}\sum_{s=m_1+1}^{m_{r+1}} B\left(k,s,\alpha\left(\frac{1}{i}+\frac{1}{j}\right)\right) \\ & < \sum_{k=n_1+1}^{n_{r+1}}\sum_{s=1}^{m_{r+1}-1} |g(k,s,x_{ks})|^p+\sum_{k=1}^{n_{r+1}-1}\sum_{s=m_1+1}^{m_{r+1}} |g(k,s,x_{ks})|^p+\sum_{k=n_1+1}^{n_{r+1}}\sum_{s=m_1+1}^{m_{r+1}} |g(k,s,x_{ks})|^p+ \\ & +\sum_{k=1}^{\infty}\sum_{s=1}^{\infty} 2^{-(k+s)}. \end{aligned}$$

for all $r \in \mathbb{N}$. Hence, $(g(k,s,x_{ks}))_{k,s=1}^{\infty} \notin \mathcal{L}_p$. This is a contradiction, because of $P_g : C_{r0} \to \mathcal{L}_p$.

Conversely, suppose that there exist $N \in \mathbb{N}$ and $\alpha > 0$ such that

$$\sum_{\max\{k,s\} \geq N} \sup_{|t| \leq \alpha} |g(k,s,t)|^p < \infty.$$

To show that $P_g : C_{r0} \to \mathcal{L}_p$, let $x = (x_{ks}) \in C_{r0}$. Since $r - \lim x_{ks} = 0$, there exists $N' > N$ such that $|x_{ks}| \leq \alpha$ for all $k, s \in \mathbb{N}$ with $\max\{k,s\} \geq N'$. Therefore, we find

$$\sum_{\max\{k,s\} \geq N'} |g(k,s,x_{ks})|^p \leq \sum_{\max\{k,s\} \geq N} \sup_{|t| \leq \alpha} |g(k,s,t)|^p < \infty.$$

Thus, we get $P_g(x) = g(k,s,x_{ks}) \in \mathcal{L}_p$. □

We need the following proposition that proved in [14] to show the continuity of the superposition operator $P_g : C_{r0} \to \mathcal{L}_p$.

Proposition 2. *Let X be a normed double sequence space containing all finite double sequences and Y be a normed double sequence space such that $Y \subseteq M_u$. Suppose that*
(i) $P_g : X \to Y$,
(ii) there exists $\alpha > 0$ such that $\|e^{mn}\|_X \leq \alpha$ for all $m, n \in \mathbb{N}$,
(iii) $\|.\|_{M_u} \leq \beta \|.\|_Y$ on Y for some $\beta > 0$.
If P_g is continuous at x, then for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|t - x_{ks}| < \delta \text{ implies } |g(k,s,t) - g(k,s,x_{ks})| < \varepsilon$$

for all $k, s \in \mathbb{N}$ and $t \in \mathbb{R}$.

Theorem 5. *If $P_g : C_{r0} \to \mathcal{L}_p$, then P_g is continuous on C_{r0} if and only if $g(k,s,.)$ is continuous on $\mathbb{R}$ for all $k, s \in \mathbb{N}$.*

Proof. Since the conditions in Proposition 2 provided, it's not hard to see the necessary condition.

Conversely, let any $x = (x_{ks}) \in C_{r0}$ and assume that $g(k,s,.)$ is continuous at x_{ks} for all $k, s \in \mathbb{N}$. Then, there exist $N_1 \in \mathbb{N}$ and $\alpha > 0$ such that

$$\sum_{\max\{k,s\} \geq N_1} \sup_{|t| \leq \alpha} |g(k,s,t)|^p < \infty \tag{3.4}$$

by Theorem 4. Since $x = (x_{ks}) \in C_{r0}$, there exist $N_2 \geq N_1$ such that $|x_{ks}| \leq \frac{\alpha}{2}$ for all $k, s \in \mathbb{N}$ with $\max\{k,s\} \geq N_2$. Let $\varepsilon > 0$. From (3.4), we see that

$$\sum_{k=1}^{N_1-1} \sum_{s=N_1}^{\infty} \sup_{|t| \leq \alpha} |g(k,s,t)|^p < \infty, \sum_{k=N_1}^{\infty} \sum_{s=1}^{N_1-1} \sup_{|t| \leq \alpha} |g(k,s,t)|^p < \infty, \sum_{k=N_1}^{\infty} \sum_{s=N_1}^{\infty} \sup_{|t| \leq \alpha} |g(k,s,t)|^p < \infty.$$

Therefore, there exists $N \in \mathbb{N}$ with $N \geq N_2$ such that

$$\sum_{k=1}^{N_1-1} \sum_{s=N}^{\infty} \sup_{|t| \leq \alpha} |g(k,s,t)|^p < \frac{\varepsilon^p}{3.2^{p+1}}$$

$$\sum_{k=N}^{\infty} \sum_{s=1}^{N_1-1} \sup_{|t| \leq \alpha} |g(k,s,t)|^p < \frac{\varepsilon^p}{3.2^{p+1}}$$

$$\sum_{k=N_1}^{N-1} \sum_{s=N}^{\infty} \sup_{|t| \leq \alpha} |g(k,s,t)|^p + \sum_{k=N}^{\infty} \sum_{s=N_1}^{N-1} \sup_{|t| \leq \alpha} |g(k,s,t)|^p + \sum_{k=N}^{\infty} \sum_{s=N}^{\infty} \sup_{|t| \leq \alpha} |g(k,s,t)|^p < \frac{\varepsilon^p}{3.2^{p+1}}.$$

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Consequently, we obtain that there exists $N \in \mathbb{N}$ with $N \geq N_2$ such that

$$\sum_{\max\{k,s\} \geq N} |g(k,s,t)|^p < \frac{\varepsilon^p}{2^{p+1}}. \tag{3.5}$$

Since $g(k,s,.)$ is continuous at x_{ks} for all $k,s \in \{1,2,...,N-1\}$, there is $\delta \in \mathbb{R}$ with $0 < \delta \leq \frac{\beta}{2}$ such that

$$|g(k,s,t) - g(k,s,x_{ks})| < \left(\frac{\varepsilon^p}{2(N-1)}\right)^{\frac{1}{p}} \text{ whenever } |t - x_{ks}| < \delta. \tag{3.6}$$

Let $z = (z_{ks}) \in C_{r0}$ satisfying $\|z-x\|_{C_{r0}} < \delta$. Thus, $|z_{ks} - x_{ks}| \leq \|z-x\|_{C_{r0}} < \delta$. From (3.5), we find $|g(k,s,z_{ks}) - g(k,s,x_{ks})|^p < \frac{\varepsilon^p}{2(N-1)}$ for all $k,s \in \{1,2,...,N-1\}$. We write $|z_{ks}| \leq |z_{ks} - x_{ks}| + |x_{ks}| < \delta + \frac{\beta}{2} \leq \frac{\beta}{2} + \frac{\beta}{2} = \beta$ for all $k,s \in \mathbb{N}$ with $\max\{k,s\} \geq N$. We have

$$\begin{aligned} |g(k,s,z_{ks}) - g(k,s,x_{ks})|^p &\leq 2^p \max\{|g(k,s,z_{ks})|^p, |g(k,s,x_{ks})|^p\} \\ &\leq 2^p \sup_{|t| \leq \alpha} |g(k,s,t)|^p \end{aligned}$$

for all $k,s \in \mathbb{N}$ with $\max\{k,s\} \geq N$. By using (3.4), we obtain

$$\sum_{\max\{k,s\} \geq N} |g(k,s,z_{ks}) - g(k,s,x_{ks})|^p \leq 2^p \sum_{\max\{k,s\} \geq N} \sup_{|t| \leq \alpha} |g(k,s,t)|^p < \frac{\varepsilon^p}{2}.$$

Therefore,

$$\begin{aligned} \sum_{k,s=1}^{\infty} |g(k,s,z_{ks}) - g(k,s,x_{ks})|^p &= \sum_{k,s=1}^{N-1} |g(k,s,z_{ks}) - g(k,s,x_{ks})|^p + \\ &\quad + \sum_{\max\{k,s\} \geq N} |g(k,s,z_{ks}) - g(k,s,x_{ks})|^p \\ &< (N-1)\frac{\varepsilon^p}{2(N-1)} + \frac{\varepsilon^p}{2} < \varepsilon^p. \end{aligned}$$

Hence, we get $\|P_g(z) - P_g(x)\| = \left(\sum_{k,s=1}^{\infty} |g(k,s,z_{ks}) - g(k,s,x_{ks})|^p\right)^{\frac{1}{p}} < \varepsilon$. It is completed the proof. □

Theorem 6. *If $P_g : C_{r0} \to \mathcal{L}_p$, then P_g is locally bounded on C_{r0} if and only if g satisfies $(2')$.*

Proof. Assume that g satisfies $(2')$ and let $z = (z_{ks}) \in C_{r0}$. By Theorem4, there exist $N \in \mathbb{N}$ and $\alpha > 0$ such that

$$\sum_{\max\{k,s\} \geq N} \sup_{|t| \leq \alpha} |g(k,s,t)|^p < \infty. \tag{3.7}$$

Let $x = (x_{ks}) \in C_{r0}$ satisfying $\|z-x\|_{C_{r0}} \leq \frac{\alpha}{2}$. So, we have that

$$\sup_{k,s \in \mathbb{N}} |z_{ks} - x_{ks}| \leq \frac{\alpha}{2}. \tag{3.8}$$

Since $r-\lim z_{ks}=0$, there exists $N'\in\mathbb{N}$ such that $|z_{ks}|\leq\frac{\alpha}{2}$ for all $k,s\in\mathbb{N}$ with $\max\{k,s\}\geq N$. Hence,

$$\sup_{\max\{k,s\}\geq N}|z_{ks}|\leq\frac{\alpha}{2}. \tag{3.9}$$

By using (3.8) and (3.9), we find

$$|x_{ks}|\leq\sup_{\max\{k,s\}\geq N}|x_{ks}|\leq\sup_{k,s\in\mathbb{N}}|z_{ks}-x_{ks}|+\sup_{\max\{k,s\}\geq N}|z_{ks}|<\alpha$$

for all $k,s\in\mathbb{N}$ with $\max\{k,s\}\geq N$. From (3.7), we have that

$$\sum_{\max\{k,s\}\geq N}|g(k,s,x_{ks})|^p\leq\sum_{\max\{k,s\}\geq N}\sup_{|t|\leq\alpha}|g(k,s,t)|^p<\infty \tag{3.10}$$

for all $k,s\in\mathbb{N}$ with $\max\{k,s\}\geq N$. If we put $m_{ks}=\sup_{|t-z_{ks}|\leq\frac{\alpha}{2}}|g(k,s,t)|^p$, since g satisfies $(2')$ we see that $m_{ks}<\infty$ for all $k,s\in\mathbb{N}$. We have

$$|g(k,s,x_{ks})|^p\leq m_{ks} \tag{3.11}$$

for each $k,s\in\mathbb{N}$. By using (3.10) and (3.11), we obtain

$$\begin{aligned}\|P_g(x)\|_p^p &= \sum_{k,s=1}^{\infty}|g(k,s,x_{ks})|^p=\sum_{k,s=1}^{N-1}|g(k,s,x_{ks})|^p+\sum_{\max\{k,s\}\geq N}|g(k,s,x_{ks})|^p\\ &\leq \sum_{k,s=1}^{N-1}m_{ks}+\sum_{\max\{k,s\}\geq N}\sup_{|t|\leq\alpha}|g(k,s,t)|^p<\infty.\end{aligned}$$

If we take $A=\sum_{k,s=1}^{N-1}m_{ks}+\sum_{\max\{k,s\}\geq N}\sup_{|t|\leq\alpha}|g(k,s,t)|^p<\infty$, then

$$\begin{aligned}\|P_g(x)-P_g(z)\|_p &\leq \|P_g(x)\|_p+\|P_g(z)\|_p\\ &\leq \|P_g(z)\|_p+A^{\frac{1}{p}}.\end{aligned}$$

Let $\gamma=\|P_g(z)\|_p+A^{\frac{1}{p}}$, then we write $\|P_g(x)-P_g(z)\|_p\leq\gamma$. Hence, P_g is locally bounded on C_{r0}.

Conversely, assume that P_g is locally bounded on C_{r0}. To complete the proof, it's sufficient that g is locally bounded on $\mathbb{R}$. The sequence $y=(y_{ks})$ is defined as

$$y_{ks}=\begin{cases} a & ,k=\ n \text{ and } s=m\\ \frac{1}{k}+\frac{1}{s} & ,\text{ others}\end{cases}$$

for all $k,s\in\mathbb{N}$ and $a\in\mathbb{R}$. So, it's obvious that $y=(y_{ks})\in C_{r0}$. From the hypothesis, there exists $\alpha,\beta>0$ such that

$$\|P_g(x)-P_g(y)\|_p\leq\beta \text{ whenever } \|x-y\|_{C_{r0}}\leq\alpha. \tag{3.12}$$

Also, the sequence $x=(x_{ks})$ is defined as

$$x_{ks}=\begin{cases} b & ,k=\ n \text{ and } s=m\\ \frac{1}{k}+\frac{1}{s} & ,\text{ others}\end{cases}$$

for all $k,s\in\mathbb{N}$ and $b\in\mathbb{R}$ with $|b-a|\leq\alpha$. So, it's obvious that $x=(x_{ks})\in C_{r0}$. Thus, we find

$$\|x-y\|_{C_{r0}}=\sup_{k,s\in\mathbb{N}}|x_{ks}-y_{ks}|=|b-a|\leq\alpha.$$

Therefore, $\|P_g(x) - P_g(y)\|_p \leq \beta$ from (3.11). Then, we obtain

$$\begin{aligned} |g(k,s,b) - g(k,s,a)| &\leq \left(\sum_{k,s=1}^{\infty} |g(k,s,x_{ks}) - g(k,s,y_{ks})|^p \right)^{\frac{1}{p}} \\ &= \|P_g(x) - P_g(y)\|_p \leq \beta. \end{aligned}$$

Since $b \in \mathbb{R}$ is arbitrary, $g(k,s,.)$ is locally bounded on $\mathbb{R}$. □

Example 2. *Let $g : \mathbb{N}^2 \times \mathbb{R} \to \mathbb{R}$ defined by*

$$g(k,s,t) = \left(\frac{|t(t-1)|}{2^{k+s}} \right)^{\frac{1}{p}}$$

for all $k,s \in \mathbb{N}$ and for all $t \in \mathbb{R}$. Let $\alpha = 2$ and $|t| \leq 2$. Then for all $k,s \in \mathbb{N}$,

$$\sum_{\max\{k,s\} \geq N} \sup_{|t| \leq 2} |g(k,s,t)| = \sum_{\max\{k,s\} \geq N} \sup_{|t| \leq 2} \frac{\left(|t(t-1)|^{\frac{1}{p}}\right)^p}{2^{k+s}} \leq \sum_{\max\{k,s\} \geq N} \frac{2}{2^{k+s}} \leq \sum_{k,s=1}^{\infty} \frac{2}{2^{k+s}} < \infty.$$

By Theorem 4, we find that $P_g : C_{r0} \to \mathcal{L}_p$. Since $g(k,s,.)$ is continuous and bounded on $\mathbb{R}$ for all $k,s \in \mathbb{N}$, then the superposition operator P_g is continuous and locally bounded on C_{r0} by Theorem 5 and Theorem 6, respectively.

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A Fixed Point Approach to Stability of Cubic Lie Derivatives in Banach Algebras

Seong Sik Kim, John Michael Rassias, Afrah A.N. Abdou and Yeol Je Cho

Abstract: In this paper, we investigate the new stability, superstability and hyperstability of the cubic Lie derivations associated with the system of general cubic functional equations:

$$\begin{cases} f(xy) = x^3 f(y) + f(x)y^3, \\ f(x+ky) - kf(x+y) + kf(x-y) - f(x-ky) = 2k(k^2-1)f(y) \end{cases}$$

in Banach algebras.

1. **Introduction**

The stability theory of functional equations mainly deals with the following question:

Is it true that the solution of a given equation differing slightly from an another given one must necessarily be close to the solution of the equation in the question?

In the case of a positive answer, we say that the functional equation in question is *stable*. The functional equation is called the *superstability* (see [12]) if every approximately solution is an exact solution of it. Also, it can happen that there is no such alternative, that is, all the solutions of the stability inequality are exactly the solutions of the functional equation. In this case, we say that the functional equation is the *hyperstability* (see [3, 13, 18]). For more details on stability of functional equations, refer to [6].

In 1940, the stability problem concerning the stability of group homomorphisms of functional equations was originally introduced by Ulam [22]. The famous Ulam stability problem was partially solved by Hyers [14] for the linear functional equation of Banach spaces. Subsequently, the result of Hyers was generalized by Aoki [1] for additive mappings. In 1978, Rassias [21] provided a generalization of Hyers' theorem which allows the Cauchy difference to be unbounded as follows:

Theorem R. ([21]) *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then the limit $L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leq \frac{2\epsilon}{2-2^p}\|x\|^p$$

for all $x \in E$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in E$, then the mapping T is linear.

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[0]**2000 Mathematics Subject Classification**: 39B52, 39B72, 46H25, 47H47.

[0]**Keywords**: cubic Lie derivation, stability, superstability, hyperstability, Banach algebra

Later, Cădariu and Radu [4] applied the fixed point method to investigation of the Jensen functional equation. They could present a short and a simple proof, which is different from the direct method initiated by Hyers in 1941 for the generalized Hyers-Ulam stability of Jensen functional equation and for quadratic functional equation.

The following fixed point theorem proved by Diaz and Margolis [7] plays an important role in proving our theorem:

Theorem 1.1. ([7]) *Suppose that (Ω, d) is a complete generalized metric space and $T : \Omega \to \Omega$ is a strictly contractive mapping with the Lipshitz constant L. Then, for any $x \in \Omega$, either $d(T^n x, T^{n+1}x) = \infty$ for all nonnegative integers $n \geq 0$ or there exists a natural number n_o such that*

(1) *$d(T^n x, T^{n+1}x) < \infty$ for all $n \geq n_o$;*
(2) *the sequence $\{T^n x\}$ is convergent to a fixed point $y*$ of T;*
(3) *$y*$ is the unique fixed point of T in the set $\Lambda = \{y \in \Omega : d(T^{n_o}x, y) < \infty\}$;*
(4) *$d(y, y*) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in \Lambda$.*

Let $\mathcal{A}$ be a complex Banach algebra and $\mathcal{M}$ be a Banach $\mathcal{A}$-bimodule. For all $x, y \in \mathcal{A}$, let a Banach algebra $\mathcal{A}$ endowed with $[x, y] = xy - yx$. A mapping $f : \mathcal{A} \to \mathcal{M}$ is called a *cubic homogeneous mapping* if

$$f(\lambda x) = \lambda^3 f(x)$$

for all $x \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. A cubic homogeneous mapping $D : \mathcal{A} \to \mathcal{M}$ is called a *cubic derivation* ([12]) if

$$D(xy) = D(x)y^3 + x^3 D(y)$$

for all $x, y \in \mathcal{A}$. A cubic homogeneous mapping $\delta : \mathcal{A} \to \mathcal{M}$ is called a *cubic Lie derivation* ([9]) if

$$\delta([x, y]) = [\delta(x), y^3] + [x^3, \delta(y)]$$

for all $x, y \in \mathcal{A}$.

Jun and Kim [16] introduced the following functional equation:

$$f(2x + y) + f(2x - y) - 2f(x + y) - 2f(x - y) = 12f(x) \tag{1.1}$$

and gave a general solution and the generalized Hyers-Ulam stability problem of the functional equation (1.1). It is easy to see that $f(x) = cx^3$ is a solution of the equation (1.1). Thus it is natural that the equation (1.1) is called a *cubic functional equation* and every solution of the cubic functional equation is called a *cubic mapping.*

Now, we consider a mapping $f : X \to Y$ satisfying the following functional equation:

$$f(x + ky) - kf(x + y) + kf(x - y) - f(x - ky) = 2k(k^2 - 1)f(y) \tag{1.2}$$

for some $k \in \mathbb{Z}^+$ with $k \geq 3$.

Note that (1) $f(0) = 0$, (2) f is an odd mapping and (3) $f(kx) = k^3 f(x)$ and $f(k^n x) = k^{3n} f(x)$ for all $n \in \mathbb{Z}^+$. Also, it is easy to see that a function $f(x) = x^3$ is a solution of the equation (1.2). Thus the equation (1.2) is called the *general cubic functional equation.* Eskandani et al. [8] and Javdian et al. [15] gave the stability of the general cubic functional equation (1.2) in quasi-β-normed spaces and fuzzy normed spaces, respectively. In the last few decades, the stability of some type for cubic functional equations have been proved by [2, 11, 17, 19, 20].

Motivated by these results, we investigate the stability and hyperstability of the cubic Lie derivations associated with the system of general cubic functional equations:

$$\begin{cases} f(xy) = x^3 f(y) + f(x)y^3, \\ f(x+ky) - kf(x+y) + kf(x-y) - f(x-ky) = 2k(k^2-1)f(y) \end{cases}$$

in Banach algebras. Also, we investigate the superstability of (1.2) by suitable control functions.

Throughout this paper, let $\mathcal{A}$ be a Banach algebra, $\mathcal{M}$ be a Banach $\mathcal{A}$-bimodule and let $\Lambda = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. For any mapping $f : \mathcal{A} \to \mathcal{M}$, we define

$$\begin{aligned} \Delta_\mu f(x,y) &= f(\mu x + \mu ky) - \mu^3 kf(x+y) + \mu^3 kf(x-y) \\ &\quad - f(\mu x - \mu ky) - 2\mu^3 k(k^2-1)f(y) \end{aligned}$$

for all $x, y \in \mathcal{A}$ and $\mu \in \Lambda$.

2. Stability of cubic Lie derivations

Now, we give the main results in this paper.

Theorem 2.1. *Let $\varphi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ be a mapping such that there exists $L < 1$ with*

$$\varphi(kx, ky) \le k^3 L\varphi(x, y)$$

for all $x, y \in \mathcal{A}$. Suppose that a mapping $f : \mathcal{A} \to \mathcal{M}$ with $f(0) = 0$ satisfies the following conditions:

$$\big\|f([x,y]) - [f(x), y^3] - [x^3, f(y)]\big\| \le \varphi(x,y), \tag{2.1}$$

$$\|\Delta_\mu f(x,y)\| \le \varphi(x,y), \tag{2.2}$$

for all $x, y \in \mathcal{A}$ and $\mu \in \Lambda$. If, for each fixed $x \in \mathcal{A}$, the mapping $r \mapsto f(rx)$ from $\mathbb{R}$ to $\mathcal{M}$ is continuous, then there exists a unique cubic Lie derivation mapping $\delta : \mathcal{A} \to \mathcal{M}$ which satisfies the equation (1.2) and the following inequality:

$$\|f(x) - f(-x) - \delta(x)\| \le \frac{1}{2k^3(1-L)}\widetilde{\varphi}(0,x) \tag{2.3}$$

for all $x \in \mathcal{A}$ and $k \in \mathbb{Z}^+$ with $k \ge 3$, where $\widetilde{\varphi}(0,x) = \varphi(0,x) + \varphi(0,-x)$ and the mapping δ is defined by

$$\delta(x) = \lim_{m\to\infty} \frac{1}{k^{3m}}\big(f(k^m x) - f(-k^m x)\big).$$

Proof. It follows form $\varphi(kx, ky) \le k^3 L\varphi(x,y)$ that

$$\lim_{m\to\infty} \frac{1}{k^{3m}}\varphi(k^m x, k^m y) = 0 \tag{2.4}$$

for all $y \in \mathcal{A}$. Substituting $x = 0$ and $\mu = 1$ in (2.2), we have

$$\big\|p(ky) - kp(y) - 2k(k^2-1)f(y)\big\| \le \varphi(0,y) \tag{2.5}$$

for all $y \in \mathcal{A}$, where $p(y) = f(x) - f(-y)$. Replacing y by $-y$ in (2.5), we have

$$\big\|p(ky) - kp(y) - 2k(k^2-1)f(-y)\big\| \le \varphi(0,-y) \tag{2.6}$$

for all $y \in \mathcal{A}$. Thus it follows from (2.5) and (2.6) that

$$\left\|p(y) - \frac{1}{k^3}p(ky)\right\| \le \frac{1}{2k^3}\widetilde{\varphi}(0,y) \tag{2.7}$$

for all $y \in \mathcal{A}$, where $\widetilde{\varphi}(0,y) = \varphi(0,y) + \varphi(0,-y)$.

Let $\Omega = \{g | g : \mathcal{A} \to \mathcal{M},\ g(0) = 0\}$ and introduce a generalized metric d on Ω as follows:

$$d(g,h) = \inf\{c \in \mathbb{R}^+ : \|g(x) - h(x)\| \le c\widetilde{\varphi}(0,x) \text{ for all } x \in \mathcal{A}\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to see that (Ω, d) is a generalized complete metric space ([5]).

Now, we define a mapping $T : \Omega \to \Omega$ by

$$Tg(x) = \frac{1}{k^3} g(kx)$$

for all $x \in \mathcal{A}$. Let $g, h \in \Omega$ and $c \in [0,\infty)$ be an arbitrary constant with $d(g,h) < c$. Then we have

$$\|g(x) - h(x)\| \le c\widetilde{\varphi}(0,x)$$

for all $x \in \mathcal{A}$ and so

$$\|Tg(x) - Th(x)\| \le \frac{c}{k^3}\widetilde{\varphi}(0,kx) \le Lc\widetilde{\varphi}(0,x)$$

for all $x \in \mathcal{A}$. This means that

$$d(Tg, Th) \le Ld(g,h)$$

for all $g, h \in \Omega$. Thus T is a strictly contractive self-mapping on Ω with the Lipschitz constant L and so, by Theorem 1.1, there exists a unique mapping $\delta : \mathcal{A} \to \mathcal{M}$, which is a unique fixed point of T in the set $\Omega_1 = \{g \in \Omega : d(f,g) < \infty\}$, such that

$$\delta(kx) = k^3\delta(x)$$

and

$$\delta(x) = \lim_{m\to\infty} \frac{1}{k^{3m}} p(k^m x) = \lim_{m\to\infty} \frac{1}{k^{3m}} \big(f(k^m x) - f(-k^m x)\big) \tag{2.8}$$

for all $x \in \mathcal{A}$ since $\lim_{m\to\infty} d(T^m p, \delta) = 0$. It follows from (2.7) that $d(p, Tp) \le \frac{1}{2k^3}$. Again, it follows from Theorem 1.1 that

$$d(p,\delta) \le \frac{1}{1-L} d(p,Tp) \le \frac{1}{2k^3(1-L)}.$$

Thus we obtain

$$\|p(x) - \delta(x)\| = \|f(x) - f(-x) - \delta(x)\| \le \frac{1}{2k^3(1-L)}\widetilde{\varphi}(0,x)$$

for all $x \in X$.

Further, it follows from (2.2), (2.4) and (2.8) that

$$\begin{aligned} \|\Delta_\mu \delta(x,y)\| &\le \lim_{m\to\infty} \frac{1}{k^{3m}} \|\Delta_\mu p(k^m x, k^m y)\| \\ &\le \lim_{m\to\infty} \frac{1}{k^{3m}} \widetilde{\varphi}(k^m x, k^m y) \\ &\le \lim_{m\to\infty} L^m(\varphi(x,y) - \varphi(-x,-y)) \\ &= 0, \end{aligned} \tag{2.9}$$

which gives $\Delta_\mu \delta(x,y) = 0$ for all $x, y \in \mathcal{A}$ and $\mu \in \Lambda$. If we put $\mu = 1$ in (2.9), then

$$\delta(x+ky) - k\delta(x+y) + k\delta(x-y) - \delta(x-ky) - 2k(k^2-1)\delta(y) = 0.$$

Thus the mapping $\delta : \mathcal{A} \to \mathcal{A}$ is cubic (see [8]). Therefore, there exists a unique cubic mapping δ on $\mathcal{A}$ satisfying (2.3).

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Next, it follows from (2.9) that $\Delta_\mu \delta(0, x) = 0$ and $\delta(\mu k x) = (\mu k)^3 \delta(x)$ for all $x \in \mathcal{A}$ and $\mu \in \Lambda$. Let the mapping $f(tx)$ be continuous in $t \in \mathbb{R}$ for any fixed $x_0 \in \mathcal{A}$. For any continuous linear functional ρ on $\mathcal{M}$, now, we can define a function $\psi : \mathbb{R} \to \mathbb{R}$ by

$$\psi(t) = \rho(\delta(tx_0))$$

for all $t \in \mathbb{R}$, where $x_0 \in \mathcal{A}$ is fixed. Set

$$\psi_n(t) = \rho\Big(\frac{f(k^n t x_o)}{k^{3n}}\Big)$$

for all $t \in \mathbb{R}$ and $n \in \mathbb{Z}^+$. Since $\psi(t)$ is the pointwise limit of the sequence of $\psi_n(t)$, ψ is a continuous cubic mapping and $\psi(t) = t^3\psi(1)$ for all $t \in \mathbb{R}$ (see Theorem 2.3 in [16]). Thus we have

$$\rho(\delta(tx_0)) = \psi(t) = t^3\psi(1) = t^3\rho(\delta(x_0)) = \rho(t^3(\delta(x_0)),$$

which gives $\delta(tx_0) = t^3\delta(x_0)$ for all $t \in \mathbb{R}$. Let $\lambda \in \mathbb{C}$. Then $\frac{\lambda}{|\lambda|} \in \Lambda$ and $\frac{|\lambda|}{k} \in \mathbb{R}$ and so

$$\delta(\lambda x_0) = \delta\Big(k\frac{\lambda}{|\lambda|}\Big(\frac{|\lambda|}{k}x_0\Big)\Big) = k^3\frac{\lambda^3}{|\lambda|^3}\delta\Big(\Big(\frac{|\lambda|}{k}\Big)x_0\Big) = \lambda^3\delta(x_0).$$

Since $x_o \in \mathcal{A}$ is arbitrary, δ is cubic homogeneous in $\mathcal{A}$.

Finally, it follows from (2.1) and (2.8) that

$$\begin{aligned}
&\|\delta([x, y]) - [\delta(x), y^3] - [x^3, \delta(y)]\| \\
&\le \lim_{m\to\infty} \frac{1}{k^{3m}}\|\delta(k^m[x, y]) - [\delta(k^m x), y^3] - [x^3, \delta(k^m y)]\| \\
&\le \lim_{m\to\infty} \frac{1}{(k^{3m})^2}\Big(\varphi(k^m x, k^m y) - \varphi(-k^m x, -k^m y)\Big) \\
&= 0
\end{aligned}$$

for all $x, y \in \mathcal{A}$. Then we have

$$\delta([x, y]) = [\delta(x), y^3] + [x^3, \delta(y)]$$

for all $x, y \in \mathcal{A}$. Thus, δ is a unique cubic Lie derivation on $\mathcal{A}$ satisfying (2.3). This complete the proof. □

From Theorem 2.1, we have the following:

Corollary 2.2. *Let θ be positive real numbers. Suppose that a mapping $f : \mathcal{A} \to \mathcal{M}$ with $f(0) = 0$ satisfies the following conditions:*

$$\Big\|f([x, y]) - [f(x), y^3] - [x^3, f(y)]\Big\| \le \theta,$$

$$\Big\|\mathcal{D}_\mu f(x, y)\Big\| \le \theta$$

for all $x, y \in \mathcal{A}$ and $\mu \in \Lambda$. Then there exists a unique cubic Lie derivation $\delta : \mathcal{A} \to \mathcal{M}$ such that

$$\|f(x) - f(-x) - \delta(x)\| \le \frac{\theta}{k^3(1 - L)}$$

for all $x \in \mathcal{A}$.

Theorem 2.3. *Let $\varphi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ be a mapping such that there exists $L < 1$ with*

$$\varphi\Big(\frac{x}{k}, \frac{y}{k}\Big) \leq \frac{L}{k^3}\varphi(x, y)$$

for all $x, y \in \mathcal{A}$. Suppose that a mapping $f : \mathcal{A} \to \mathcal{M}$ with $f(0) = 0$ satisfies (2.1) and (2.2). If, for any fixed $x \in \mathcal{A}$, the mapping $r \mapsto f(rx)$ from $\mathbb{R}$ to $\mathcal{M}$ is continuous, then there exists a unique cubic Lie derivation $\delta : \mathcal{A} \to \mathcal{M}$ which satisfies the equation (1.2) and the following inequality:

$$\|f(x) - f(-x) - \delta(x)\| \leq \frac{L}{2k^3(1-L)}\widetilde{\varphi}(0, x) \tag{2.10}$$

for all $x \in \mathcal{A}$ and $k \in \mathbb{Z}^+$ with $k \geq 3$, where $\widetilde{\varphi}(0, x) = \varphi(0, x) + \varphi(0, -x)$ and the mapping δ is defined by

$$\delta(x) = \lim_{m\to\infty} k^{3m}\Big(f\big(\frac{x}{k^m}\big) - f\big(-\frac{x}{k^m}\big)\Big)$$

for all $x \in \mathcal{A}$.

Proof. Let Ω and d be as in the proof of Theorem 2.1. Then (Ω, d) is a generalized complete metric space and we consider a mapping $T : \Omega \to \Omega$ defined by

$$Tg(x) = k^3 g\Big(\frac{x}{k}\Big)$$

for all $x \in \mathcal{A}$. Then we have

$$d(Tg, Th) \leq Ld(g, h)$$

for all $g, h \in \Omega$. It follows from (2.7) that

$$\Big\|p(x) - k^3 p\big(\frac{x}{k}\big)\Big\| \leq \frac{L}{2k^3}\widetilde{\varphi}(0, x)$$

for all $x \in \mathcal{A}$ and $k \in \mathbb{Z}^+$ with $k \geq 3$. Thus we have $d(Tp, p) \leq \frac{L}{2k^3} < \infty$. Therefore, by Theorem 1.1, it follows that there exists a unique mapping δ, which is a unique fixed point of T in the set $\Omega_1 = \{g \in \Omega : d(f, g) < \infty\}$, such that

$$\delta(x) = \lim_{m\to\infty} \frac{1}{k^{3m}} p(k^m x) = \lim_{m\to\infty} k^{3m}\Big(f\big(\frac{x}{k^m}\big) - f\big(-\frac{x}{k^m}\big)\Big)$$

for all $x \in \mathcal{A}$. Thus we have

$$d(p, \delta) \leq \frac{1}{1-L} d(p, Tp) \leq \frac{L}{2k^3(1-L)},$$

which implies that (2.10) holds. The remaining assertion goes through in the similar method to the corresponding proof of Theorem 2.1. This complete the proof. □

Remark 2.1. Using the idea of Gavtuta [10], we can obtain other results for the stability of the general cubic functional equation $\Delta_\mu f(x, y) = 0$.

Let $\ell \in \{-1, 1\}$ be fixed and $\varphi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ be a mapping such that

$$\lim_{m\to\infty} \frac{1}{k^{3\ell m}} \varphi(k^{\ell m}x, k^{\ell m}y) = 0$$

for all $x, y \in \mathcal{A}$ and

$$\sum_{j=\frac{1-\ell}{2}}^{\infty} \frac{1}{k^{3\ell j}} \widetilde{\varphi}(0, k^{\ell j}x) < \infty$$

for all $x \in \mathcal{A}$, where $\widetilde{\varphi}(0, x) = \varphi(0, x) + \varphi(0, -x)$. Suppose that a mapping $f : \mathcal{A} \to \mathcal{M}$ with $f(0) = 0$ satisfies (2.1) and (2.2). If, for any fixed $x \in \mathcal{A}$, the mapping $r \mapsto f(rx)$ from $\mathbb{R}$ to $\mathcal{M}$

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is continuous, then there exists a unique cubic Lie derivation mapping $\delta : \mathcal{A} \to \mathcal{M}$ which satisfies the equation (1.2) and the following inequality:

$$\|f(x) - f(-x) - \delta(x)\| \le \frac{1}{2k^3} \sum_{j=\frac{1-\ell}{2}}^{\infty} \frac{1}{k^{3\ell j}} \widetilde{\varphi}(0, k^{\ell j} x) \tag{2.11}$$

for all $x \in \mathcal{A}$ and $k \in \mathbb{Z}^+$ with $k \ge 3$.

(1) Let $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ and θ, r be positive real numbers with $r \neq 3$. Then there exists a unique cubic Lie derivation $\delta : \mathcal{A} \to \mathcal{M}$ such that

$$\|f(x) - f(-x) - \delta(x)\| \le \frac{\theta}{|k^3 - k^r|} \|x\|^r$$

for all $x \in \mathcal{A}$.

(2) The stability problem for $r = 3$ is singular in $\|\mathcal{D}_\mu f(x, y)\| \le \theta(\|x\|^r + \|y\|^r)$ (see [8]).

Now, from Theorem 2.1, we can consider the superstability of cubic derivation as follows:

Theorem 2.4. *Let θ and r be positive real numbers with $0 < r < 3$. Suppose that there exists a mapping $\varphi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ such that*

$$\lim_{m\to\infty} \frac{1}{(k^{3m})^2} \varphi(k^m x, k^m y) = 0$$

for all $x, y \in \mathcal{A}$. If a mapping $f : \mathcal{A} \to \mathcal{M}$ with $f(0) = 0$ satisfies the following conditions:

$$\left\| f([x, y]) - [f(x), y^3] - [x^3, f(y)] \right\| \le \varphi(x, y), \tag{2.12}$$

$$\|\mathcal{D}_\mu f(x, y)\| \le \theta \|x\|^r \tag{2.13}$$

for all $x, y \in \mathcal{A}$ and $\mu \in \Lambda$, then f is a cubic Lie derivation.

Proof. Letting $\mu = 1$ and putting $x = y = 0$ in (2.13), we have $f(0) = 0$. Also, letting $x = 0$, $y = x$ in (2.13) and using the oddness of f, we obtain $f(kx) = k^3 f(x)$ for all $x \in \mathcal{A}$. By induction,

$$f(k^m x) = k^{3m} f(x)$$

for all $x \in \mathcal{A}$ and $m \in \mathbb{Z}^+$. On the other hand, it follows from Theorem 2.1 that the mapping $\delta : \mathcal{A} \to \mathcal{M}$ defined by

$$\delta(x) = \lim_{m\to\infty} \frac{1}{k^{3m}} p(k^m x) = \lim_{m\to\infty} \frac{1}{k^{3m}} \big(f(k^m x) - f(-k^m x)\big)$$

is a unique cubic Lie derivation. Then $\delta(x) = f(x) - f(-x)$ for all $x \in \mathcal{A}$. So the mapping f is a cubic Lie derivation. This complete the proof. □

From Theorem 2.4, we have the following:

Corollary 2.5. *Let θ and r, s be positive real numbers with $r + s \neq 3$. Suppose that there exists a mapping $\varphi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ such that*

$$\lim_{m\to\infty} \frac{1}{(k^{3m})^2} \varphi(k^m x, k^m y) = 0$$

for all $x, y \in \mathcal{A}$. If a mapping $f : \mathcal{A} \to \mathcal{M}$ with $f(0) = 0$ satisfies the following conditions:

$$\left\| f([x, y]) - [f(x), y^3] - [x^3, f(y)] \right\| \le \varphi(x, y),$$

$$\|\mathcal{D}_\mu f(x, y)\| \le \theta(\|x\|^r + \|y\|^s + \|x\|^r \|y\|^s)$$

for all $x, y \in \mathcal{A}$. Then f is a cubic Lie derivation.

Corollary 2.6. *Let θ and $r \in \mathbb{R}^+$ be positive real numbers with $r > 3$. Suppose that a mappings $f : \mathcal{A} \to \mathcal{M}$ with $f(0) = 0$ satisfies the following conditions:*

$$\big\|f([x, y]) - [f(x), y^3] - [x^3, f(y)]\big\| \leq \theta\|x\|^r,$$

$$\|\mathcal{D}_\mu f(x, y)\| \leq \theta\|x\|^r$$

for all $x, y \in \mathcal{A}$. Then f is a cubic Lie derivation.

Next, we show the hyperstability of cubic derivation as follows:

Theorem 2.7. *Let $\mathcal{A}$ be a Banach algebra with an element e which is not a zero divisor. Suppose that $f : \mathcal{A} \to \mathcal{A}$ is a mapping for which there exists a function $\varphi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ such that*

$$\lim_{n\to\infty} \frac{\varphi(x, ny)}{n^3} = 0, \tag{2.14}$$

$$\|f(x)y^3 - x^3 f(y)\| \leq \varphi(x, y), \tag{2.15}$$

$$\|f(z)([x, y])^3 - z^3\big([f(x), y^3] - [x^3, f(y)]\big)\| \leq \varphi([x, y], z) \tag{2.16}$$

for all $x, y, z \in \mathcal{A}$. Then f is a cubic Lie derivation.

Proof. Let $x, y, z \in \mathcal{A}$. Then we have

$$\begin{aligned}
&\big\|n^3 z^3\big(f(x + ky) - kf(x + y) + kf(x - y) - f(x - ky) - 2k(k^2 - 1)f(y)\big)\big\| \\
&\leq \big\|n^3 z^3 f(x + ky) - f(nz)(x + ky)^3\big\| + k\big\|n^3 z^3 f(x + y) - f(nz)(x + y)^3\big\| \\
&\quad + k\big\|n^3 z^3 f(x - y) - f(nz)(x - y)^3\big\| + \big\|n^3 z^3 f(x - ky) - f(nz)(x - ky)^3\big\| \\
&\quad + 2k(k^2 - 1)\big\|n^3 z^3 f(y) - f(nz)y^3\big\|
\end{aligned}$$

for all $n \in \mathbb{Z}^+$, which gives, by (2.15),

$$\begin{aligned}
&\big\|z^3\big(f(x + ky) - kf(x + y) + kf(x - y) - f(x - ky) - 2k(k^2 - 1)f(y)\big)\big\| \\
&\leq n^{-3}\big(\varphi(x + ky, nz) + k\varphi(x + y, nz) + k\varphi(x - y, nz) \\
&\qquad + \varphi(x - ky, nz) + 2k(k^2 - 1)\varphi(y, nz)\big).
\end{aligned}$$

Letting $n \to \infty$ and using (2.14), we have

$$z^3\big(f(x + ky) - kf(x + y) + kf(x - y) - f(x - ky) - 2k(k^2 - 1)f(y)\big) = 0 \tag{2.17}$$

for all $x, y, z \in \mathcal{A}$. Putting $z = e$ in (2.17), f satisfies a cubic functional equation (1.2).

Next, let $\lambda \in \mathbb{C}$ and $x, y \in \mathcal{A}$. Then we have

$$\begin{aligned}
\big\|n^3 y^3\big(f(\lambda x) - \lambda^3 f(x)\big)\big\| &\leq \big\|n^3 y^3 f(\lambda x) - f(\lambda y)\lambda^3 x^3\big\| + |\lambda|^3\big\|f(ny)x^3 - n^3 y^3 f(x)\big\| \\
&\leq \varphi(\lambda x, ny) + |\lambda|^3\varphi(x, ny),
\end{aligned}$$

which gives

$$\big\|y^3\big(f(\lambda x) - \lambda^3 f(x)\big)\big\| \leq n^{-3}\big(\varphi(\lambda, ny) + |\lambda|^3\varphi(x, ny)\big).$$

Taking $n \to \infty$, we obtain

$$y^3\big(f(\lambda x) - \lambda^3 f(x)\big) = 0$$

for all $x \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. Letting $y = e$, we have $f(\lambda x) = \lambda^3 f(x)$ for all $x \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. Thus the mapping f is cubic homogeneous.

Finally, let $x, y, z \in \mathcal{A}$. Then we have

$$\begin{aligned}
&\big\|n^3 z^3\big(f([x,y]) - [f(x), y^3] - [x^3, f(y)]\big)\big\| \\
&\le \big\|n^3 z^3 f([x,y]) - ([x,y])^3 f(nz)\big\| \\
&\quad + \big\|([x,y])^3 f(nz) - n^3 z^3\big([f(x), y^3] + [x^3, f(y)]\big)\big\| \\
&\le 2\varphi([x,y], nz)
\end{aligned}$$

for all $n \in \mathbb{Z}^+$ and so

$$\big\|z^3\big(f([x,y]) - [f(x), y^3] - [x^3, f(y)]\big)\big\| \le n^{-3} \cdot 2\varphi([x,y], nz),$$

which implies

$$f([x,y]) = [f(x), y^3] + [x^3, f(y)]$$

for all $x, y \in \mathcal{A}$. Therefore, the mapping f is a cubic Lie derivation in $\mathcal{A}$. This completes the proof. □

From Theorem 2.7, we have the following:

Corollary 2.8. *Let θ and r be positive real numbers with $r < 3$. Suppose that $\mathcal{A}$ is a Banach algebra with an element e which is not a zero divisor and $f : \mathcal{A} \to \mathcal{A}$ is a mapping such that*

$$\big\|f(z)([x,y])^3 - z^3\big([f(x), y^3] - [x^3, f(y)]\big)\big\| \le \theta\|[x,y]\|^r \|z\|^r,$$

$$\|f(x)y^3 - x^3 f(y)\| \le \theta\|x\|^r \|y\|^r$$

for all $x, y, z \in \mathcal{A}$. Then f is a cubic Lie derivation in $\mathcal{A}$.

Proof. The proof follows from Theorem 2.7 by taking $\varphi(x, y) = \theta\|x\|^r\|y\|^r$ for all $x, y \in \mathcal{A}$. □

3. **Acknowledgements**

The first author was supported by Dongeui University (2015AA071). Also, this project was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, under Grant No. (31-130-35-HiCi). The authors, therefore, acknowledge with thanks DSR technical and financial support.

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Volume 19, Number 3 September 2015
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

An international publication of Eudoxus Press, LLC
(twelve times annually)

Editor in Chief: George Anastassiou
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On the shadowing property of functional equations

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Abstract. Using the shadowing property of the dynamical system, we prove the generalized Hyers-Ulam stability of quintic functional equations.

Keywords: generalized Hyers-Ulam stability, pseudo-orbit, shadowing property, quintic mapping.

1. Introduction and preliminaries

Let us introduce some notations which will be used throughout this paper (see [16, 18]). We denote $\mathbb{N}$ the set of all non-negative integers, X a complete normed space, $B(x, s)$ the closed ball centered at x with radius s. Let $\phi : X \to X$ be given.

For $\delta \geq 0$ a sequence $(x_k)_{k\in\mathbb{N}}$ in X is a *δ-pseudo-orbit* for ϕ if

$$d(x_{k+1}, \phi(x_k)) \leq \delta \text{ for } k \in \mathbb{N}.$$

A 0-pseudo-orbit is called an orbit, that is, a sequence $(x_k)_{k\in\mathbb{N}}$ is an orbit if $x_{k+1} = \phi(x_k)$ for $k \in \mathbb{N}$.

The notion of pseudo-orbit very often appears in several areas of the dynamical systems, especially in the case of numerical simulations of dynamical systems.It is natural to suggest a question whether a behavior tracing along the pseudo-orbit is closed to the real behavior of systems. This property is called *shadowing property.* The shadowing property is very useful to figure the stable dynamical system.

Now we proceed to the notion of local invertibility and shadowing.

Let $s, R > 0$ be given. A function $\phi : X \to X$ is *locally (s, R)-invertible* at $x_0 \in X$ if for any point y in $B(\phi(x_0), R)$, there exists a unique element x in $B(x_0, s)$ such that $\phi(x) = y$. If ϕ is locally (s, R)-invertible at each $x \in X$, then we say that ϕ is locally (s, R)-invertible.

For a locally (s, R)-invertible function ϕ, we define a function $\phi^{-1}_{x_0} : B(\phi(x_0), R) \to B(x_0, s)$ in such a way that $\phi^{-1}_{x_0}(y)$ denote the unique element x from the above definition which satisfies $\phi(x) = y$. Moreover, we put

$$\mathrm{lip}_R\phi^{-1} := \sup_{x_0\in X} \mathrm{lip}(\phi^{-1}_{x_0}).$$

We will need the following result in [23].

Theorem 1.1. *Let $l \in (0, 1), R \in (0, \infty)$ be fixed and let $\phi : X \to X$ be locally (lR, R)-invertible. We assume additionally that $lip_R(\phi^{-1}) \leq l$. Let $\delta \leq (1-l)R$ and let $(x_k)_{k\in\mathbb{N}}$ be an arbitrary δ-pseudo-orbit. Then there exists a unique element $y \in X$ such that*

$$d(x_{k+1}, \phi(y)) \leq lR$$

[0]2000 Mathematics Subject Classification: Primary 39B52, 37C75, 34D30.

for $k \in \mathbb{N}$.

Moreover,

$$d(x_{k+1}, \phi(y) \leq \frac{l\delta}{1-l}$$

for $k \in \mathbb{N}$.

Let X be a semigroup. Then the function $|| \cdot || : X \to \mathbb{R}$ is called a (semigroup) norm if it satisfies the following properties:

(1) for all $x \in X,\ ||x|| \geq 0$.

(2) for all $x \in X,\ k \in \mathbb{N},\ ||kx|| = |k| \cdot ||x||$.

(3) for all $x, y \in X,\ ||x|| + ||y|| \geq ||x * y||$ and also the equality holds when $x = y$, where $*$ is the binary operation on X.

Note that $(X, *, || \cdot ||)$ is called a *normed group* if X is a group with an identity e, and it additionally satisfies that $||x|| = 0$ if and only if $x = e$.

We say that $(X, *, || \cdot ||)$ is a *normed (semi-) group* if X is a (semi-)group with a norm $|| \ ||$. We say that $(X, +, d)$ is a metric group if X is an Abelian group with a translation invariant metric, that is

$$d(a + x, b + y) = d(a, b)$$

for $a, b, x \in X$. Conveniently, we write $||x||$ instead of $d(x, 0)$. Thus we have

$$d(x, y) = ||x - y||$$

$x, y \in X$. Given an Abelian group X and $n \in \mathbb{Z}$, we define the mapping $[n_X] : X \to X$ by the formula

$$[n_X](x) := nx$$

for $x \in X$.

Since X is a normed group, it is clear that $[n_X](x)$ is locally $(\frac{R}{n}, R)$-invertible at 0, and $\mathrm{lip}_R[n_X]^{-1} = \frac{1}{n}$.

The stability problem of functional equations originated from a question of Ulam [24] concerning the stability of group homomorphisms. Hyers [11] gave a first affirmative partial answer to the question of Ulam in the case of Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Th.M. Rassias [19] for linear mappings by considering an unbounded Cauchy difference. The paper of Th.M. Rassias [19] has provided a lot of influence in the development of what we call *generalized Hyers-Ulam stability* of functional equations. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [10] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach.

The functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y)$$

is called a *quadratic functional equation.* In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping.* A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [20] for mappings $f : X \to Y$, where X is a normed

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space and Y is a Banach space. Cholewa [3] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [5] proved the generalized Hyers-Ulam stability of the quadratic functional equation.

In [14], Jun and Kim considered the following cubic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x). \tag{1.1}$$

It is easy to show that the function $f(x)=x^3$ satisfies the functional equation (1.1), which is called a *cubic functional equation* and every solution of the cubic functional equation is said to be a *cubic mapping.*

In [15], Lee et al. considered the following quartic functional equation

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y). \tag{1.2}$$

It is easy to show that the function $f(x)=x^4$ satisfies the functional equation (1.2), which is called a *quartic functional equation* and every solution of the quartic functional equation is said to be a *quartic mapping.*

The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1], [4], [6]–[9], [12]).

In this paper, we investigate the generalized Hyers-Ulam stability of the following quintic functional equations by using shadowing property:

$$\begin{aligned}
Df(x,y) &:= f(3x+y)-f(3x)+f(2x-y)+3f(2x)-5f(x-y)-5f(2x+y)\\
&\quad +10f(x+y)+27f(x)-10f(y)=0,\\
Ef(x,y) &:= f(3x+y)+f(2x-y)-5f(2x+y)+5f(3x)-5f(x-y)\\
&\quad -42f(2x)-10f(y)+10f(x+y)+9f(x)=0.
\end{aligned}$$

2. Stability using the shadowing property

First, we are going to introduce the result of the stability of the homogeneity equation worked by Tabor et al. [22]. Let X be a complete metric space and G be a metric space. Let $\psi: G \to G$ and $\phi: X \to X$ be given. A mapping $f: G \to X$ is called *homogeneous* if

$$f(\psi(x))=\phi(f(x))$$

for $x \in G$. From the stability of the homogeneity equation and the shadowing property Tabor et al. obtained the following lemma [22].

Lemma 2.1. [22] *Let $l \in (0,1), R \in (0,\infty), \delta \in (0,(1-l)R), \varepsilon > 0, m \in \mathbb{N}$, $n \in \mathbb{Z}$. Let G be a commutative semigroup and X a complete Abelian metric group. We assume that the mapping $[n_X]$ is locally (lR,R)-invertible and $lip_R([n_x]^{-1}) \leq l$.*

Let $f : G \to X$ satisfy the following two inequalities

$$\left\|\sum_{i=1}^{N} a_i f(b_i x + c_i y)\right\| \le \varepsilon$$

$$\|f(mx) - nf(x)\| \le \delta$$

for $x, y \in G$, where a_i are endomorphisms in X and b_i, c_i are endomorphisms in G. We assume additionally that there exists a $K \in \{1, \ldots, N\}$ such that

$$\sum_{i=1}^{K} lip(a_i)\delta \le (1-l)R, \qquad \varepsilon + \sum_{i=K+1}^{N} lip(a_i)\frac{l\delta}{1-l} \le lR.$$

Then there exists a unique homogeneous mapping $F : G \to X$ such that

$$F(mx) = nF(x)$$

and

$$||f(x) - F(x)|| \le \frac{l\delta}{1-l}$$

for $x \in G$. Moreover, F satisfies

$$\sum_{i=1}^{N} a_i F(b_i x + c_i y) = 0$$

for $x, y \in G$.

Let $R > 0$ be a real number, G an Abelian group, and X a complete normed Abelian group.

Theorem 2.2. *Let $\varepsilon \le \frac{1}{10^2}R$ be arbitrary and let $f : G \to X$ be a mapping with $f(0) = 0$ such that*

$$||Df(x, y)|| \le \varepsilon \tag{2.1}$$

for all $x, y \in G$. Then there exists a unique quintic mapping $F : G \to X$ such that

$$\begin{aligned} F(2x) &= 2^5 F(x), \\ DF(x, y) &= 0, \\ ||f(x) - F(x)|| &\le \frac{1}{3}\varepsilon \end{aligned}$$

for all $x, y \in G$.

Proof. Letting $y = 0$ in (2.1), we have

$$|| - f(2x) + 32f(x)|| \le \varepsilon.$$

To apply Lemma 2.1 for the mapping f, we may let $l = \frac{1}{4}$, $\delta = \varepsilon$, $K = 2$, $a_1 = a_2 = a_3 =$

$(-1)^{i-1}id_X$, $a_{k+2i} = (-1)^{i+1}(k+(2i-1))id_X$ (i=1,2), $a_{2k+2i-1} = (-1)(5i)id_X$ (i= 1,2), $a_{4k} = 3(4k+1)id_X$. Then we have

$$\begin{aligned}
\delta = \varepsilon \leq \frac{1}{100}R &\leq \frac{3}{4}R = (1-l)R,\\
\sum_{i=1}^{K} \mathrm{lip}(a_i)\delta = 2\varepsilon &\leq \frac{1}{50}R \leq \frac{3}{4}R = (1-l)R,\\
\varepsilon + \sum_{i=K+1}^{N} \mathrm{lip}(a_i)\frac{l\delta}{1-l} &\leq \varepsilon + 60 \cdot \frac{\delta}{3} = 21\varepsilon \leq \frac{1}{4}R = lR.
\end{aligned}$$

Thus we also obtain $\mathrm{lip}([n_X]^{-1}) \leq l$ and so all conditions of Lemma 2.1 are satisfied. Hence we conclude that there exists a unique mapping $F : G \to X$ such that

$$F(2x) = 2^5F(x), \qquad DF(x,y) = 0,$$

for all $x, y \in G$ and also we have

$$||f(x) - F(x)|| \leq \frac{1}{3}\varepsilon$$

for all $x \in G$. □

We prove the generalized Hyers-Ulam stability of another quintic functional equation $Ef(x,y) = 0$.

Theorem 2.3. *Let $\varepsilon \leq \frac{1}{200}R$ be arbitrary and let $f : G \to X$ be a mapping with $f(0) = 0$ such that*

$$||Ef(x,y)|| \leq \varepsilon \tag{2.2}$$

for all $x, y \in G$. Then there exists a unique quintic mapping $F : G \to X$ such that

$$\begin{aligned}
F(4x) &= 2^5F(2x),\\
EF(x,y) &= 0,\\
||f(x) - F(x)|| &\leq \frac{1}{3}\varepsilon
\end{aligned}$$

for all $x, y \in G$.

Proof. Letting $y = x$ in (2.2), we have

$$||f(4x) - 32f(2x)|| \leq \varepsilon.$$

To apply Lemma 2.1 for the mapping f, we may let $l = \frac{1}{4}$, $\delta = \varepsilon$, $K = 2$, $a_1 = a_2 = id_X$, $a_{k+i} = (-1)^i(5)id_X$ $(i = 1,2,3)$, $a_{3k} = -(21k)id_X$, $a_{4k-1} = a_{4k} = (-5k)id_X$, $a_{4k+1} = (4k+1)id_X$.

Then we have

$$\begin{aligned}
\delta = \varepsilon \leq \frac{1}{200}R &\leq \frac{3}{4}R = (1-l)R,\\
\sum_{i=1}^{K} \mathrm{lip}(a_i)\delta = 2\varepsilon &\leq \frac{1}{100}R \leq \frac{3}{4}R = (1-l)R,\\
\varepsilon + \sum_{i=K+1}^{N} \mathrm{lip}(a_i)\frac{l\delta}{1-l} &\leq \varepsilon + 90 \cdot \frac{\delta}{3} \leq \frac{1}{4}R = lR.
\end{aligned}$$

Thus we also obtain $\mathrm{lip}_R([n_X]^{-1}) \le l$ and so all conditions of Lemma 2.1 are satisfied. Hence we conclude that there exists a unique mapping $F : G \to X$ such that

$$F(4x) = 2^5 F(2x), \qquad EF(x,y) = 0$$

for all $x, y \in G$ and also we have

$$||f(x) - F(x)|| \le \frac{1}{3}\varepsilon$$

for all $x \in G$. □

3. Stability of the functional equation in a metric group

In this section, we prove the generalized Hyers-Ulam stability of the quintic functional equation $Ef(x,y) = 0$ in metric groups.

Let X be an Abelian metric group and let $R > 0$, $n \in \mathbb{N}$ be fixed. Suppose that the following condition holds:

For all $S \in [0, 2R]$ and all $y \in B(0,S)$, there exists a unique element $x \in B\left(0, \frac{S}{2}\right)$ in X such that $y = nx$. (3.0)

We use the operation $\left[\frac{1}{n}\right]$ to denote the unique element x from $B\left(0, \frac{S}{2}\right)$ in (3.0).

Lemma 3.1. *Let X be an Abelian metric group and let $R > 0$ be fixed. Suppose that the condition* (3.0) *holds. Then for every $k \in \mathbb{N}$ the mapping $[n_X^k]$ is locally $\left(\frac{R}{n^k}, R\right)$-invertible and $lip_R\left([n_X^k]^{-1}\right) \le n^k$.*

Proof. Since X is a metric group it is enough to check that $[n_X^k]$ is locally $\left(\frac{R}{n^k}, R\right)$-invertible at 0 and $\mathrm{lip}\left([n_X^k]^{-1}\right) \le \frac{1}{n^k}$.

Let $y \in B(0,R)$ be arbitrary. By the assumption, there exists uniquely $x_1 \in B\left(0, \frac{S}{n}\right)$ such that $y = nx$. Again there exists $x_2 \in B\left(0, \frac{S}{n^2}\right)$ such that $x_1 = nx_2$. Continuing this process we can get $x_k \in B\left(0, \frac{S}{n^k}\right)$ such that $y = n^k x_k$ We put $x_k = x = \left[\frac{1}{n}\right]^k y$. If we put $S = ||y||$, we can get $||x|| \le \frac{||y||}{n^k}$ and $n^k x = y$. Suppose that there exists $\tilde{x} \in B\left(0, \frac{R}{n^k}\right)$, $\tilde{x} \ne x$, such that $n^k \tilde{x} = y$. Then there exists $l \in \{1, \cdots, n\}$ such that $n^l x = n^l \tilde{x}$, $n^{l-1}x \ne n^{l-1}\tilde{x}$. This contradicts the uniqueness in (3.0).

For any $x, y \in B(0,S)$ and $S \in [0, 2R]$

$$n^k \left(\left[\frac{1}{n}\right]^k x - \left[\frac{1}{n}\right]^k y \right) = x - y.$$

By the uniqueness, we set

$$\left(\left[\frac{1}{n}\right]^k x - \left[\frac{1}{n}\right]^k y \right) = \left[\frac{1}{n}\right]^k (x - y).$$

Hence we have

$$\left\| \left[\frac{1}{n}\right]^k x - \left[\frac{1}{n}\right]^k y \right\| = \frac{1}{2^k} ||x - y||.$$

So $\left[\frac{1}{n}\right]^k$ is locally $\left(\frac{R}{n^k}, R\right)$-invertible at 0 and $\mathrm{lip}\left(([n]^k)_0^{-1}\right) \le \frac{1}{n^k}$. □

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Theorem 3.2. *Let $R > 0$. Let G be an Abelian group and X be a complete metric Abelian group satisfying the condition: for all $S \in [0, 2R]$ and all $y \in B(0, S)$ there exists uniquely $x \in B\left(0, \frac{S}{n}\right)$ such that $y = nx$. Let $\varepsilon \leq \frac{R}{20}$ be arbitrary and let $f : G \to X$ be a mapping such that*

$$\|Df(x, y)\| \leq \varepsilon \tag{3.1}$$

for all $x, y \in G$. Then there exists a unique quintic mapping $F : G \to X$ such that

$$DF(x, y) = 0, \qquad F(2x) = 2^5 F(x),$$

and

$$\|f(x) - F(x)\| \leq \frac{\varepsilon}{3}$$

for all $x, y \in G$.

Proof. Taking $n = 2$ in Lemma 3.1, we obtain that the mapping $[2^5_X]$ is locally $\left(\frac{R}{2^5}, R\right)$-invertible and $\mathrm{lip}_R\left([2^5_X]^{-1}\right) \leq \frac{1}{2^5}$. Putting $x = y = 0$ in (3.1), we obtain that $\|11f(0)\| \leq \varepsilon \leq R$. Set

$$\tilde{f}(x) := f(x) - \left(f(0) - \left[\frac{1}{11}\right](11f(0))\right).$$

Then we have

$$\left\|D\tilde{f}(x, y)\right\| = \left\|Df(x, y) - 11\left(f(0) - \left[\frac{1}{11}\right](11f(0))\right)\right\| = ||Df(x, y)|| \leq \varepsilon.$$

It follows from Lemma 3.1 that $||\tilde{f}(0)|| = \left\|\left[\frac{1}{11}\right](11f(0))\right\| \leq \frac{\varepsilon}{11}$. If we put $y = x$ in the above inequality $D\tilde{f}(x, y)$, then we get $\left\|\tilde{f}(2x) - 2^5\tilde{f}(x)\right\| \leq \varepsilon$. We can apply Lemma 2.1 for the mapping $\tilde{f}$. So let $l = \frac{1}{4}$, $\delta = \varepsilon$, $a_1 = a_2 = a_3 = (-1)^{i-1} id_X$, $a_{k+2i} = (-1)^{i+1}(k + (2i - 1)) id_X$ $(i = 1, 2)$, $a_{2k+2i-1} = (-1)(5i) id_X$ $(i = 1, 2)$, $a_{4k} = 3(4k + 1) id_X$.

Then

$$\delta \leq (1 - l)R, \quad \sum_{i=1}^{2} \mathrm{lip}(a_i)\delta \leq (1 - l)R, \quad \varepsilon + \sum_{i=3}^{9} \mathrm{lip}(a_i)\frac{l\delta}{1 - l} \leq lR.$$

Since the assumptions of Lemma 2.1 are satisfied, there exists a unique mapping $\tilde{F} : G \to X$ such that

$$\tilde{F}(2x) = 2^5\tilde{F}(x), \qquad D\tilde{F}(x, y) = 0, \qquad \left\|\tilde{f}(x) - \tilde{F}(x)\right\| \leq \frac{l\delta}{1 - l} = \frac{\varepsilon}{3} \tag{3.2}$$

for all $x, y \in G$. In order to get a function approximating to f we put $F(x) := \tilde{F}(x) + \left(f(0) - \left[\frac{1}{11}\right](11f(0))\right)$. Then $DF(x, y) = D\tilde{F}(x, y) + 11\left(f(0) - \left[\frac{1}{11}\right](11f(0))\right) = D\tilde{F}(x, y)$. Similarly, we have $||f(x) - F(x)|| = \left\|\tilde{f}(x) - \tilde{F}(x)\right\|$. Thus F satisfies

$$DF(x, y) = 0, \qquad ||f(x) - F(x)|| \leq \frac{\varepsilon}{3} \tag{3.3}$$

for all $x, y \in G$.

Finally, we prove the uniqueness part. Suppose that there exists another mapping F_1 satisfying (3.3). Let $\tilde{F_1}(x) := F_1(x) - \left(f(0) - \left[\frac{1}{11}\right](11f(0))\right)$. Then

$$D\tilde{F_1}(x, y) = 0, \qquad \left\|\tilde{f}(x) - \tilde{F_1}(x)\right\| \leq \frac{\varepsilon}{3} \tag{3.4}$$

for all $x, y \in G$. Since $||\tilde{f}(0)|| \le \frac{\varepsilon}{11}$, we obtain that $\left\|\tilde{F}_1(0)\right\| \le \varepsilon$. Putting $x = y = 0$ in $D\tilde{F}_1(x, y) = 0$ we get $11\tilde{F}_1(0) = 0$, By the uniqueness of the local division by 11, $\tilde{F}_1(0) = 0$. If we put $x = y$ in in (3.4), then we get $\tilde{F}_1(2x) = 32\tilde{F}_1(x)$. Thus $\tilde{F}_1$ satisfies (3.2), and by the uniqueness we obtain that $\tilde{F}_1 = \tilde{F}$. Thus $F_1 = \tilde{F}_1 + \left(f(0) - \left[\frac{1}{11}\right](11f(0))\right) = \tilde{F} + \left(f(0) - \left[\frac{1}{11}\right](11f(0))\right) = F$. □

Acknowledgement

The author was supported by the 2013 research fund of University of Ulsan had written this paper during visiting Research Institute of Mathematics Seoul National University.

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ULAM-GĂVRUTA-RASSIAS STABILITY OF AN ADDITIVE FUNCTIONAL INEQUALITY IN FUZZY NORMED MODULES

YEOL JE CHO, REZA SAADATI, AND YOUNG-OH YANG*

ABSTRACT. In this paper, we investigate the following additive functional inequality:

$$N(f(x) + f(y) + f(z) + f(w), t) \geq N(f(x) + f(y + z + w), t)$$

in fuzzy normed modules over a fuzzy C^*-algebra, which is applied to understand homomorphisms in fuzzy C^*-algebras.

1. Introduction

The stability problem of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms. Hyers [2] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [3] for additive mappings and by Th.M. Rassias [4] for linear mappings by considering an unbounded Cauchy difference. The paper of Th.M. Rassias [4] has provided a lot of influence in the development of what we call *generalized Hyers-Ulam stability* of functional equations. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach.

We recall a fundamental result in fixed point theory.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies the following conditions:

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.1. [6, 7] *Let (X, d) be a complete generalized metric space and $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for any element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all $n \geq 0$ or there exists a positive integer n_0 such that

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1991 *Mathematics Subject Classification.* Primary 39B72, 39B62, 47H10, 46L05.

Key words and phrases. Functional equation, fixed point, generalized Hyers-Ulam stability, functional inequality, linear mapping, fuzzy normed module, fuzzy C^*-algebra.

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(1) *$d(J^n x, J^{n+1}x) < \infty$ for all $n \geq n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;*
(4) *$d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

We use the definition of fuzzy normed spaces given in [9, 10, 11, 12, 13, 14, 15] to investigate a fuzzy version of the Hyers-Ulam stability for the Cauchy-Jensen functional equation in the setting of fuzzy normed algebras (see also [16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28]).

Definition 1.2. [9] Let X be a real vector space. A function $N : X \times \mathbb{R} \to [\nvdash, \nVdash]$ is called a *fuzzy norm* on X if, for all $x, y \in X$ and $s, t \in \mathbb{R}$,
(N1) $N(x, t) = 0$ for all $t \leq 0$;
(N2) $x = 0$ if and only if $N(x, t) = 1$ for all $t > 0$;
(N3) $N(cx, t) = N(x, \frac{t}{|c|})$ for all $c \in \mathbb{R}$ with $c \neq 0$;
(N4) $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$;
(N5) $N(x, \cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1$;
(N6) for any $x \in X$ with $x \neq 0$, $N(x, \cdot)$ is continuous on $\mathbb{R}$.

The pair (X, N) is called a *fuzzy normed vector space.*

Definition 1.3. [9] (1) Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to *be convergent* or *converge* if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by N-$\lim_{n\to\infty} x_n = x$.

(2) Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if, for any $\varepsilon > 0$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$ and $p > 0$, $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed vector space is a Cauchy sequence. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space.*

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is *continuous* at a point $x_0 \in X$ if, for each sequence $\{x_n\}$ converging to x_0 in X, the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X (see [9, 15]).

Definition 1.4. [17] A *fuzzy normed algebra* (X, N) is a fuzzy normed space (X, N) with algebraic structure such that
(N7) $N(xy, ts) \geq \min\{N(x, t), N(y, s)\}$ for all $x, y \in X$ and $t, s > 0$.

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Every normed algebra $(X, \|\cdot\|)$ defines a fuzzy normed algebra (X, N), where

$$N(x,t) = \frac{t}{t + \|x\|}$$

for all $t > 0$. This space is called the *induced fuzzy normed algebra*.

Definition 1.5. (1) Let (X, N) and (Y, N) be fuzzy normed algebras.

(1) An $\mathbb{R}$-linear mapping $f : X \to Y$ is called a *homomorphism* if

$$f(xy) = f(x)f(y)$$

for all $x, y \in X$.

(2) An $\mathbb{R}$-linear mapping $f : X \to X$ is called a *derivation* if

$$f(xy) = f(x)y + xf(y)$$

for all $x, y \in X$.

Definition 1.6. Let $(\mathcal{U}, \mathcal{N})$ be a fuzzy Banach algebra. Then an *involution* on $\mathcal{U}$ is a mapping $u \to u^*$ from $\mathcal{U}$ into $\mathcal{U}$ satisfying the following conditions:

(a) $u^{**} = u$ for all $u \in \mathcal{U}$;

(b) $(\alpha u + \beta v)^* = \overline{\alpha} u^* + \overline{\beta} v^*$ for all $u, v \in \mathcal{U}$;

(c) $(uv)^* = v^* u^*$ for all $u, v \in \mathcal{U}$.

If, in addition $N(u^*u, ts) = \min\{N(u,t), N(u,s)\}$ and $N(u^*, t) = N(u,t)$ for all $u \in \mathcal{U}$ and $t, s > 0$, then $\mathcal{U}$ is a fuzzy C^*-algebra. In this paper, we use $*$ for min.

2. Functional inequalities in fuzzy normed modules over a fuzzy C^*-algebra

Throughout this section, let A be a unital fuzzy C^*-algebra with the unitary group $U(A)$ and the unit e and let B be a fuzzy C^*-algebra. Assume that X is a fuzzy normed A-module with the norm N and Y is a fuzzy normed A-module with the norm N.

In this section, we investigate an A-linear mapping associated with the functional following inequality:

$$N(f(x) + f(y) + f(z) + f(w), t) \geq N(f(x) + f(y + z + w), t).$$

Theorem 2.1. *Let $f : X \to Y$ be a mapping such that*

$$N(f(x) + f(y) + f(z) + uf(w), t) \geq N(f(x) + f(y + z + uw), t) \tag{2.1}$$

for all $x, y, z, w \in X$, $u \in U(A)$ and $t > 0$. Then the mapping $f : X \to Y$ is A-linear.

Proof. Letting $x = y = z = w = 0$, $u = e \in U(A)$ and $t > 0$ in (2.1), we get

$$N(4f(0), t) \geq N(2f(0), t)$$

and so $f(0) = 0$. Letting $x = w = 0$ in (2.1), we get

$$N(f(y) + f(z), t) \geq Nf(y + z), t) \tag{2.2}$$

for all $y, z \in X$. Replacing y and z by x and $y + z + w$ in (2.2), respectively, we get

$$N(f(x) + f(y + z + w), t) \geq N(f(x + y + z + w), t)$$

for all $x, y, z, w \in X$ and $t > 0$ and so

$$N(f(x) + f(y) + f(z) + f(w), t) \geq N(f(x + y + z + w), t) \tag{2.3}$$

for all $x, y, z, w \in X$ and $t > 0$. Letting $z = w = 0$ and $y = -x$ in (2.3), we get

$$N(f(x) + f(-x), t) \geq N(f(0), t) = 1$$

for all $x \in X$ and $t > 0$ and so $f(-x) = -f(x)$ for all $x \in X$. Letting $z = -x - y$ and $w = 0$ in (2.3), we get

$$N(f(x) + f(y) - f(x + y), t) = N(f(x) + f(y) + f(-x - y), t) \geq N(f(0), t) = 1$$

for all $x, y \in X$ and $t > 0$. Thus we have

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in X$. Letting $z = -uw$ and $x = y = 0$ in (2.1), we get

$$N(-f(uw) + uf(w), t) = N(f(-uw) + uf(w), t) \geq N(2f(0), t) = 1$$

for all $w \in X$, $u \in U(A)$ and $t > 0$. Thus

$$f(uw) = uf(w) \tag{2.4}$$

for all $u \in U(A)$ and $w \in X$.

Now, let $a \in A$ with $a \neq 0$ and M be an integer greater than $4|a|$. Then we have

$$\left|\frac{a}{M}\right| < \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}.$$

By Theorem 1 of Kadison and Pedersen [29], there exist three elements $u_1, u_2, u_3 \in U(A)$ such that $3\frac{a}{M} = u_1 + u_2 + u_3$ and so, by (2.4),

$$\begin{aligned} f(ax) &= f\Big(\frac{M}{3} \cdot 3\frac{a}{M}x\Big) = M \cdot f\Big(\frac{1}{3} \cdot 3\frac{a}{M}x\Big) \\ &= \frac{M}{3} f(3\frac{a}{M}x) = \frac{M}{3} h(u_1x + u_2x + u_3x) = \frac{M}{3}(f(u_1x) + f(u_2x) + f(u_3x)) \\ &= \frac{M}{3}(u_1 + u_2 + u_3)f(x) = \frac{M}{3} \cdot 3\frac{a}{M} f(x) = af(x) \end{aligned}$$

for all $x \in X$. So $f : X \to Y$ is A-linear. This completes the proof. $\square$

Corollary 2.2. *Let $f : A \to B$ be a multiplicative mapping such that*

$$N(f(x) + f(y) + f(z) + \mu f(w), t) \geq N(f(x) + f(y + z + \mu w), t) \tag{2.5}$$

for all $x, y, z, w \in A$, $\mu \in \mathbb{T} := \{\lambda \in \mathbb{C} : |\lambda| = \nVdash\}$ and $t > 0$. Then the mapping $f : A \to B$ is a C^*-algebra homomorphism.

Proof. By Theorem 2.1, the multiplicative mapping $f : A \to B$ is $\mathbb{C}$-linear since C^*-algebras are fuzzy normed modules over $\mathbb{C}$. So, the multiplicative mapping $f : A \to B$ is a C^*-algebra homomorphism. □

3. Generalized Hyers-Ulam stability of functional inequalities

Throughout this section, assume that X is a real fuzzy normed linear space and t Y is a real fuzzy Banach space.

Theorem 3.1. *Let $f : X \to Y$ be an odd mapping for which there exists a function $\varphi : X^4 \times (0, \infty) \to [0, 1]$ such that there exists $L < 1$ such that*

$$\varphi(x, y, z, w, t) \geq \varphi\Big(2x, 2y, 2z, 2w, \frac{2t}{L}\Big)$$

for all $x, y, z, w \in X$ and $t > 0$ and

$$\begin{aligned} & N(f(x) + f(y) + f(z) + f(w), t) \\ \leq \ & N(f(x) + f(y + z + w), t) * \varphi(x, y, z, w, t) \end{aligned} \tag{3.1}$$

for all $x, y, z, w \in X$ and $t > 0$. Then there exists a unique Cauchy additive mapping $A : X \to Y$ satisfying

$$N(f(x) - A(x), t) \leq \varphi\left(0, x, x, -2x, \frac{(2 - 2L)t}{L}\right) \tag{3.2}$$

for all $x \in X$ and $t > 0$.

Proof. Consider the set

$$S := \{g : X \to Y\}$$

and introduce the *generalized metric* on S defined as follows:

$$d(g, h) = \inf\Big\{K \in \mathbb{R}_+ : N(g(x) - h(x), t) \geq \varphi\Big(0, x, x, -2x, \frac{t}{K}\Big), \ \forall x \in X\Big\}.$$

It is easy to show that (S, d) is complete (see the proof of Theorem 2.5 of Cădariu and Radu [30].)

Now, we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\Big(\frac{x}{2}\Big)$$

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for all $x \in X$. It follows from the proof of Theorem 3.1 of Cădariu and Radu [6] that

$$d(Jg, Jh) \le Ld(g, h)$$

for all $g, h \in S$. Since $f : X \to Y$ is odd, $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in X$. Letting $x = 0$ $z = y = w$ and $w = -2w$ in (3.1), we get

$$N(2f(w) - f(2w), t) = N(2f(w) + f(-2w), t) \ge \varphi(0, w, w, -2w, t) \tag{3.3}$$

for all $w \in X$ and $t > 0$. It follows from (3.3) that

$$N\Big(f(x) - 2f\Big(\frac{x}{2}\Big), t\Big) \ge \varphi\Big(0, \frac{x}{2}, \frac{x}{2}, -x, t\Big) \ge \varphi\Big(0, x, x, -2x, \frac{2t}{L}\Big)$$

for all $x \in X$ and $t > 0$. Hence $d(f, Jf) \le \frac{L}{2}$.

By Theorem 1.1, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, i.e.,

$$A\Big(\frac{x}{2}\Big) = \frac{1}{2}A(x) \tag{3.4}$$

for all $x \in X$. Then $A : X \to Y$ is an odd mapping. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that A is a unique mapping satisfying (3.4) such that there exists $K \in (0, \infty)$ satisfying

$$N(f(x) - A(x), t) \ge \varphi\Big(0, x, x, -2x, \frac{t}{K}\Big)$$

for all $x \in X$ and $t > 0$;

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 2^n f\Big(\frac{x}{2^n}\Big) = A(x) \tag{3.5}$$

for all $x \in X$;

(3) $d(f, A) \le \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, A) \le \frac{L}{2 - 2L}.$$

This implies that the inequality (3.2) holds.

It follows from (3.1) and (3.5) that

$$N(A(x) + A(y) + A(z) + A(w), t) \ge N(A(x) + A(y + z + w), t)$$

for all $x, y, z, w \in X$ and $t > 0$. Thus, by Theorem 2.1, the mapping $A : X \to Y$ is a Cauchy additive mapping. Therefore, there exists a unique Cauchy additive mapping $A : X \to Y$ satisfying (3.3). This completes the proof. □

Corollary 3.2. *Let $r > 1$, θ be nonnegative real numbers and let $f : X \to Y$ be an odd mapping such that*

$$\begin{aligned} & N(f(x)+f(y)+f(z)+f(w),t) \\ \geq\ & N(f(x)+f(y+z+w),t) \\ & * \frac{t}{t+\theta(||x||^r+||y||^r+||z||^r+||w||^r+||x||^{\frac{r}{4}}\cdot||y||^{\frac{r}{4}}\cdot||z||^{\frac{r}{4}}\cdot||w||^{\frac{r}{4}})} \end{aligned}$$

for all $x, y, z, w \in X$ and $t > 0$. Then there exists a unique Cauchy additive mapping $A : X \to Y$ such that

$$N(f(x)-A(x),t) \geq \frac{t}{t+\frac{2^r+2}{2^r-2}\theta||x||^r}$$

for all $x \in X$ and $t > 0$.

Proof. The proof follows from Theorem 3.1 by taking

$$\varphi(x,y,z,w,t) := \frac{t}{t+\theta(||x||^r+||y||^r+||z||^r+||w||^r+||x||^{\frac{r}{4}}\cdot||y||^{\frac{r}{4}}\cdot||z||^{\frac{r}{4}}\cdot||w||^{\frac{r}{4}})}$$

for all $x, y, z, w \in X$, which was introduced by Rassias et al. [32]. Then we can choose $L = 2^{1-r}$ and we get the desired result. This completes the proof. □

Acknowledgments

The second author was supported by the Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2013053358).

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Almost ideal statistical convergence and strongly almost ideal lacunary convergence of sequences of fuzzy numbers with respect to the Orlicz functions[†]

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Abstract: The purpose of this paper is to introduce the concepts of almost ideal lacunary statistical convergence (almost ideal statistical convergence) and strongly almost ideal lacunary convergence (strongly almost ideal convergence). we give some relations betweeen these concepts. At the same time, some connections between strongly almost ideal lacunary statistical convergence and almost ideal lacunary statistical convergence of sequences of fuzzy numbers are established. It also shows that if a sequence of fuzzy numbers is strongly almost ideal lacunary statistical convergence with respect to an Orlicz function then it is almost ideal lacunary statistical convergent.

Keywords: Fuzzy numbers; Statistical convergence of sequences of fuzzy numbers; Orlicz function

AMS subject classifications. 08A72, 26E50.

1 Introduction

The concepts of fuzzy sets and fuzzy set operations were first introduced by Zadeh[1]. Recently Matloka[2] introduced bounded and convergent sequences of fuzzy numbers, studied some of their properties, and showed that every convergent sequence of fuzzy numbers is bounded. In addition, sequences of fuzzy numbers have been discussed by Aytar and Pehlivan[3], Basarir and Mursaleen [4,5] and many others. The notion of statistical convergence was introduced by Fast [6] which is a very useful functional tool for studying the convergence problems of numerical sequences. Some applications of statistical convergence in number theory and mathematical analysis can be found in [7, 8]. The idea is based on the notion of natural density of subsets of N, and the natural density of s subset A of N is denoted by $\delta(A)$ and defined by

$$\delta(A) = \lim_{n\to\infty} \frac{1}{n}|\{k < n : k \in A\}|.$$

I-convergence is a generalization of statistical convergence, was introduced by Kastyrko, Salat and Wilczynski [9] by using the ideal I of subsets of the set of natural numbers N and further studied in [10]. Ideal convergence provided a general framework to study the properties of various type of convergence. Vijay Kumar and Kuldeep Kumar [11] introduced the concepts of I-convergence, I^*-convergence and I-Cauchy sequences for sequences of fuzzy numbers. In another direction, a new type of convergence, called lacunary statistical convergence, was introduced in [12] inspired by the investigations in [13-16]. A lacunary sequence is an increasing integer sequence $\theta = \{k_r\}_{r\in \mathbf{N}\cup\{0\}}$, such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$, let $I_r = (k_{r-1}, k_r]$ and $q_r = \frac{k_r}{k_{r-1}}$. In 2011, Pratulananda Das, Ekrem Savas and Sanjoy Kr. Ghosal [17] provided a new approach to two well-known summability methods by using ideals, and introduced new notions, namely I-statistical convergence and I-lacunary statistical convergence.

The study of Orlicz sequence spaces was initiated with a certain specific purpose in Banach space theory. Recall in [18] that an Orlicz function M is a continuous, convex, nondecreasing funcction defined for $x > 0$ such that $M(0) = 0$ and $M(x) > 0$. Lindenstrauss and Tzafriri [19] used Orlicz function to construct the sequence space

[†]This work is supported by National Natural Science Fund of China (11461062, 61262022) the Natural Science Foundation of Gansu Province (1208RJZA251).

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$$l_M=\{x\in w:\sum_{k=1}^{\infty}M(\tfrac{|x_k|}{\rho})<\infty \text{ for some } \rho>0\}.$$

It is well known that the space l_M is a Banach with the norm

$$\|x\|=\lim\{\rho>0:\sum_{k=1}^{\infty}M(\frac{|x_k|}{\rho})\leq 1\},$$

and this space is called an Orlicz sequence space. For $M(t)=t^p, 1\leq p<\infty$, the space l_M coincides with the classical sequence space l_p.

Almost convergent sequences was introduced by Lorentz [20] and Maddox [21,22]. x_n is said to be strongly almost convergent to a number l if

$$\lim_n \frac{1}{n}\sum_{k=1}^{n}|x_{k+m}-l|=0,$$

uniformly for m.

In this paper, we introduce the concepts of almost ideal statistical convergence and strongly almost ideal lacunary convergence of sequences of fuzzy numbers, and try to establish the relation between these two notions. In Section 2 we will give a brief overview about statistical convergence, fuzzy numbers, Orlicz function. Using the sequence $\theta=(k_r)$ and an Orlicz function, we will define the concepts of almost ideal statistical convergence and strongly almost ideal lacunary statistical convergence of sequences of fuzzy numbers with respect to the Orlicz function. In Section 3 we establish some relationships between almost ideal lacunary statistical convergence and almost ideal statistical convergence.

2 Definitions and preliminaries

A fuzzy sets of X is a nonempty subset $\{(x,u(x)):x\in X\}$ of $X\times[0,1]$ for some function $u:X\rightarrow[0,1]$. The function u itself is often used for the fuzzy set.

Let $C(R^n)$ denote the family of all nonempty, compact, convex subsets of R^n. If $\alpha,\beta\in R$ and $A,B\in C(R^n)$, then

$$\alpha(A+B)=\alpha A+\alpha B,\qquad (\alpha\beta)A=\alpha(\beta A),\qquad 1A=A,$$

and if $\alpha,\beta\geq 0$, then $(\alpha+\beta)A=\alpha A+\beta A$. The distance between A and B is defined by the Haussdorff metric

$$\delta_\infty(A,B)=\max\big\{\sup_{a\in A}\inf_{b\in B}\parallel a-b\parallel,\sup_{b\in B}\inf_{a\in A}\parallel a-b\parallel\big\},$$

where $\parallel . \parallel$ denotes the usual Euclidean norm in R^n. It is well known that $(C(R^n),\delta_\infty)$ is a complete mrtric space.

Definitions 2.1[15]. A fuzzy number is a function u from R^n to $[0,1]$, which satisfying the following conditions

(1) u is normal, that is, there exists an $x_0\in R^n$ such that $u(x_0)=1$;
(2) u is fuzzy convex, that is, for any $x,y\in R$ and $\lambda\in[0,1], u(\lambda x+(1-\lambda)y)\geq\min\{u(x),u(y)\}$;
(3) u is upper semi-continuous;
(4) the closure of $\{x\in R^n:u(x)>0\}$, denoted by $[u]^0$, is compact.

We write the fuzzy number space as $L(R^n)$.

For $0<\alpha\leq 1$, the α-level set $[u]^\alpha$ is defined by $[u]^\alpha=\{x\in R^n:u(x)\geq\alpha\}$. Then from $(1)-(4)$, it follows that $[u]^\alpha\in C(R^n)$. For the addition and scalar multiplication in $L(R^n)$, we have $[u+v]^\alpha=[u]^\alpha+[v]^\alpha, [ku]^\alpha=k[u]^\alpha$, where $u,v\in L(R^n), k\in R$. For each $1\leq q<\infty$,

$$d_q(u,v)=\big(\int_0^1[\delta_\infty([u]^\alpha,[v]^\alpha)]^q d\alpha\big)^{\frac{1}{q}}$$

and $d_\infty(u,v)=\sup_{0\leq\alpha\leq 1}\delta_\infty([u]^\alpha,[v]^\alpha)$, where δ_∞ is the Haussdorff mrtric. Throughout the paper, D will denote d_q with $1\leq q\leq\infty$.

Definitions 2.2.[15] A sequence $\{x_n\}$ of fuzzy numbers is said to be statistical convergent to a fuzzy number x_0 if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in N : D(x_n, x_0) \geq \varepsilon\}$ has natural density zero. The fuzzy number x_0 is called the statistical limit of the sequence $\{x_n\}$ and we write $st\text{-}\lim_{n\to\infty} x_n = x_0$.
Definitions 2.2.[15] If X is a non-empty set. A family of sets $I \subset 2^X$ is called an ideal in X if and only if

(1) $\varnothing \in I$;
(2) for each $A, B \in I$ we have $A \cup B \in I$;
(3) for each $A \in I$ and $B \subset A$ we have $B \in I$.

An ideal I is called non-trivial if $I \neq \varnothing$ and $X \notin I$. It immediately follows that $I \subset 2^X$ is a non-trivial ideal if and only if the class $F = F(I) = \{X - A : A \in I\}$ is a filter on X. The filter $F = F(I)$ is called the filter associated with the ideal I.
Definitions 2.2.[15] Let $I \subset 2^X$ be a non-trivial ideal in N. A sequence $\{x_n\}$ of fuzzy numbers is said to be I-convergent to a number x_0 if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in N : D(x_n, x_0) \geq \varepsilon\} \in I$. The fuzzy number x_0 is called the I-limit of the sequence $\{x_n\}$ and we write $I\text{-}\lim_{n\to\infty} x_n = x_0$.

3 Main results

Definition 3.1. Let $\theta = \{k_r\}$ be a lacunary sequence, M be an Orlicz function and $p = \{p_k\}$ be any sequence of strictly positive real numbers. A sequence $\{x_k\}$ of fuzzy numbers is said to be almost ideal lacunary statistical convergent to the fuzzy number $\tilde{l}$, with respect to the Orlicz function M, if for every $\varepsilon > 0$, $\{r \in N : \frac{1}{h_r}|\{k \in I_r : \big[M(\frac{D(t_{km}(x), \tilde{l})}{\rho})\big]^{p_k} \geq \varepsilon\}| \geq \delta\} \in I$ uniformly for m,
where

$$t_{km}(x) = \frac{x_m + x_{m+1} + ... + x_{m+k}}{k+1} = \frac{1}{k+1}\sum_{i=0}^{k} x_{m+i}.$$

As usual, we write $x_k \to \tilde{l}(\tilde{S}(M, p, \theta))$. The set of all almost ideal lacunary statistically convergent sequences will be denoted simply by $\tilde{S}(M, p, \theta)$. Especially, if $\theta = \{2^r\}$, or $M(x) = x$, $p_k = 1$ for all $k \in N$, we shall write $\tilde{S}(M, p)$ and $\tilde{S}(\theta)$ instead of $\tilde{S}(M, p, \theta)$, respectively.
Definition 3.2. Let $\theta = \{k_r\}$ be a lacunary sequence, M be an Orlicz function and $p = \{p_k\}$ be any sequence of strictly positive real numbers, $x = \{x_k\}$ be a sequence of fuzzy numbers. For some $\rho > 0$, we define the following sets

$$\tilde{W}(M, p, \theta) = \{x = \{x_k\} : \{r \in N : \frac{1}{h_r}\sum_{k\in I_r}\big[M(\frac{D(t_{km}(x), \tilde{l})}{\rho})\big]^{p_k} \geq \delta\} \in I\},$$

$$\tilde{W}_0(M, p, \theta) = \{x = \{x_k\} : \{r \in N : \frac{1}{h_r}\sum_{k\in I_r}\big[M(\frac{D(t_{km}(x), \tilde{0})}{\rho})\big]^{p_k} \geq \delta\} \in I\},$$

$\tilde{W}_\infty(M, p, \theta) = \{x = \{x_k\} : \exists K > 0, \text{ s.t } \{r \in N : \frac{1}{h_r}\sum_{k\in I_r}\big[M(\frac{D(t_{km}(x), \tilde{0})}{\rho})\big]^{p_k} \geq K\} \in I\}$,
where

$$\tilde{0}(t) = \begin{cases} 1, & t = (0, 0, 0..., 0), \\ 0, & \text{otherwise}. \end{cases}$$

If $x \in \tilde{W}(M, p, \theta)$, we say that x is strongly almost ideal lacunary convergent with respect to the Orlicz function M, and we write $x_k \to \tilde{l}(\tilde{W}(M, p, \theta))$. Especially, if $\theta = \{2^r\}$, we shall write above sets $\tilde{W}(M, p)$, $\tilde{W}_0(M, p)$, $\tilde{W}_\infty(M, p)$, respectively.
Theorem 3.1. Let the sequence $\{p_k\}$ be bounded. Then $\tilde{W}_0(M, P, \theta) \subset \tilde{W}(M, P, \theta) \subset \tilde{W}_\infty(M, P, \theta)$.

Proof. Let $x \in \tilde{W}(M,P,\theta)$. Note that

$$\begin{aligned}
&\frac{1}{h_r}\sum_{k\in I_r}[M(\frac{D(t_{km}(x),\tilde{0})}{2\rho})]^{p_k}\\
\leq\ &\frac{G}{h_r}\sum_{k\in I_r}\frac{1}{2^{p_k}}[M(\frac{D(t_{km}(x),\tilde{l})}{\rho})]^{p_k}+\frac{G}{h_r}\sum_{k\in I_r}\frac{1}{2^{p_k}}[M(\frac{D(\tilde{l},\tilde{0})}{\rho})]^{p_k}\\
\leq\ &\frac{G}{h_r}\sum_{k\in I_r}[M(\frac{D(t_{km}(x),\tilde{l})}{\rho})]^{p_k}+G\max\{1,\sup[M(\frac{D(\tilde{l},\tilde{0})}{\rho})]^{H}\},
\end{aligned}$$

where $H=\sup_k p_k$, $G=\max\{1,2^{H-1}\}$. There exists $K=\delta$, such that

$$\begin{aligned}
&\{r\in N:\frac{1}{h_r}\sum_{k\in I_r}[M(\frac{D(t_{km}(x),\tilde{0})}{\rho})]^{p_k}\geq K\}\\
\subset\ &\{r\in N:\frac{1}{h_r}\sum_{k\in I_r}[M(\frac{D(t_{km}(x),\tilde{l})}{\rho})]^{p_k}\geq \delta\}\in I.
\end{aligned}$$

Thus we get $x\in\tilde{W}_\infty(M,P,\theta)$. The proof of $\tilde{W}_0(M,P,\theta)\subset\tilde{W}(M,P,\theta)$ is obvious.

Theorem 3.2. Let M_1,M_2 be Orlicz functions. Then we have

(1) $\tilde{W}_0(M_1,P,\theta)\cap\tilde{W}_0(M_2,P,\theta)\subset\tilde{W}_0(M_1+M_2,P,\theta)$,

(2) $\tilde{W}(M_1,P,\theta)\cap\tilde{W}(M_2,P,\theta)\subset\tilde{W}(M_1+M_2,P,\theta)$,

(3) $\tilde{W}_\infty(M_1,P,\theta)\cap\tilde{W}_\infty(M_2,P,\theta)\subset\tilde{W}_\infty(M_1+M_2,P,\theta)$.

Proof. (1) Let $x\in\tilde{W}_0(M_1,P,\theta)\cap\tilde{W}_0(M_2,P,\theta)$. Then $x\in\tilde{W}_0(M_1,P,\theta)$ and $x\in\tilde{W}_0(M_2,P,\theta)$. Therefore for each $\delta>0$ and $\rho_1>0,\rho_2>0$, we have

$$\{r\in N:\frac{1}{h_r}\sum_{k\in I_r}\big[M_1(\frac{D(t_{km}(x),\tilde{0})}{\rho_1})\big]^{p_k}\geq\delta\}\in I,$$

$$\{r\in N:\frac{1}{h_r}\sum_{k\in I_r}\big[M_2(\frac{D(t_{km}(x),\tilde{0})}{\rho_2})\big]^{p_k}\geq\delta\}\in I.$$

Now let $\rho=\max\{\rho_1,\rho_2\}$. then

$$\begin{aligned}
&\frac{1}{h_r}\sum_{k\in I_r}\big[M_1(\frac{D(t_{km}(x),\tilde{0})}{\rho})+M_2(\frac{D(t_{km}(x),\tilde{0})}{\rho})\big]^{p_k}\\
\leq\ &\frac{G}{h_r}\sum_{k\in I_r}\{\big[M_1(\frac{D(t_{km}(x),\tilde{0})}{\rho_1})\big]^{p_k}+\big[M_2(\frac{D(t_{km}(x),\tilde{0})}{\rho_2})\big]^{p_k}\},
\end{aligned}$$

where $H=\sup_k p_k$, $G=\max\{1,2^{H-1}\}$. we have

$$\{r\in N:\frac{1}{h_r}\sum_{k\in I_r}\big[M_1(\frac{D(t_{km}(x),\tilde{0})}{\rho})+M_2(\frac{D(t_{km}(x),\tilde{0})}{\rho})\big]^{p_k}\geq\delta\}\in I.$$

thus $x\in\tilde{W}_0(M_1+M_2,P,\theta)$.

The proofs of (2),(3) are similar to (1).

Theorem 3.3. Let $\theta=\{k_r\}$ be a lacunary sequence, M be an Orlicz function and $0<h=\inf_k p_k\leq p_k\leq\sup_k p_k=H$. Then $\tilde{W}(M,P,\theta)\subset\tilde{S}(\theta)$.

Proof. Suppose that $\varepsilon_1 = \frac{\varepsilon}{\rho}$,. then we have

$$\begin{aligned}
&\frac{1}{h_r}\sum_{k\in I_r}[M(\frac{D(t_{km}(x),\tilde{l})}{\rho})]^{p_k}\\
\geq\ &\frac{1}{h_r}\sum_{k\in I_r, D(t_{km}(x),\tilde{l})\geq\varepsilon}[M(\frac{D(t_{km}(x),\tilde{l})}{\rho})]^{p_k}\\
\geq\ &\frac{1}{h_r}\sum_{k\in I_r, D(t_{km}(x),\tilde{l}))\geq\varepsilon}\min([M(\varepsilon_1)]^h,[M(\varepsilon_1)]^H)\\
=\ &\frac{1}{h_r}\mid\{k\in I_r : D(t_{km}(x),\tilde{l})\geq\varepsilon\}\mid .\min([M(\varepsilon_1)]^h,[M(\varepsilon_1)]^H)
\end{aligned}$$

Let $x\in\tilde{W}(M,P,\theta)$. Then $\{r\in N : \frac{1}{h_r}\sum\limits_{k\in I_r}[M(\frac{D(t_{km}(x),\tilde{l})}{\rho})]^{p_k}\geq\delta\}\in I$.

Thus we get

$$\begin{aligned}
&\{r\in N : \frac{1}{h_r}\mid\{k\in I_r : D(t_{km}(x),\tilde{l})\geq\varepsilon\}\mid\geq\delta\}\\
&\subset\{r\in N : \frac{1}{h_r}\sum_{k\in I_r}[M(\frac{D(t_{km}(x),l)}{\rho})]^{p_k}\geq K\delta\}\in I,
\end{aligned}$$

where $K=\min([M(\varepsilon_1)]^h,[M(\varepsilon_1)]^H)$. Hence $\tilde{W}(M,P,\theta)\subset\tilde{S}(\theta)$.

Theorem 3.4. Let $\theta=\{k_r\}$ be a lacunary sequence, M be an Orlicz function. Then we have

(1)If $\liminf_r q_r>1$, then $\tilde{W}(M,P)\subset\tilde{W}(M,P,\theta)$;

(2)If $\limsup_r q_r<\infty$, then $\tilde{W}(M,P,\theta)\subset\tilde{W}(M,P)$;

(3)If $1<\liminf_r q_r\leq\limsup_r q_r<\infty$, then $\tilde{W}(M,P)=\tilde{W}(M,P,\theta)$.

Proof. (1) Suppose $\liminf_r q_r>1$, then there exists $\alpha>0$ such that $q_r\geq 1+\alpha$ for all $r\geq 1$. Let $x\in\tilde{W}(M,P)$, we have

$$\begin{aligned}
&\frac{1}{h_r}\sum_{k\in I_r}[M(\frac{D(t_{km}(x),\tilde{l})}{\rho})]^{p_k}\\
=\ &\frac{1}{h_r}\sum_{k=1}^{k_r}[M(\frac{D(t_{km}(x),\tilde{l})}{\rho})]^{p_k}-\frac{1}{h_r}\sum_{k=1}^{k_{r-1}}[M(\frac{D(t_{km}(x),\tilde{l})}{\rho})]^{p_k}\\
=\ &\frac{k_r}{h_r}(\frac{1}{k_r}\sum_{k=1}^{k_r}[M(\frac{D(t_{km}(x),\tilde{l})}{\rho})]^{p_k})-\frac{k_{r-1}}{h_r}(\frac{1}{k_{r-1}}\sum_{k=1}^{k_{r-1}}[M(\frac{D(t_{km}(x),\tilde{l})}{\rho})]^{p_k})\\
\leq\ &\frac{1+\alpha}{\alpha}(\frac{1}{k_r}\sum_{k=1}^{k_r}[M(\frac{D(t_{km}(x),\tilde{l})}{\rho})]^{p_k})-\frac{1}{\alpha}(\frac{1}{k_{r-1}}\sum_{k=1}^{k_{r-1}}[M(\frac{D(t_{km}(x),\tilde{l})}{\rho})]^{p_k}).
\end{aligned}$$

Since

$$\{r\in N : \frac{1}{k_r}\sum_{k=1}^{k_r}[M(\frac{D(t_{km}(x),\tilde{l})}{\rho})]^{p_k}\geq\delta\}\in I,$$

$$\{r\in N : (\frac{1}{k_{r-1}}\sum_{k=1}^{k_{r-1}}[M(\frac{D(t_{km}(x),\tilde{l})}{\rho})]^{p_k}\geq\delta\}\in I,$$

we have

$$\{r\in N : \frac{1}{h_r}\sum_{k\in I_r}[M(\frac{D(t_{km}(x),\tilde{l})}{\rho})]^{p_k}\geq\delta\}\in I.$$

It follows that $\tilde{W}(M,P)\subset\tilde{W}(M,P,\theta)$.

(2) Suppose $\limsup_r q_r < \infty$. Then there exists $\beta > 0$ such that $q_r < \beta$ for all $r \geq 1$. Let $x \in \tilde{W}(M,P,\theta)$, and write
$A_r = \frac{1}{h_r}\sum_{k\in I_r}[M(\frac{D(t_{km}(x),\tilde{l})}{\rho})]^{p_k}$. Then $\{r \in N : A_r \geq \delta\} \in I$.
Now let n be any integer wtih $k_{r-1} < n \leq k_r$, then

$$\begin{aligned}
&\frac{1}{n}\sum_{k=1}^{n}[M(\frac{D(t_{km}(x),\tilde{l})}{\rho})]^{p_k} \leq \frac{1}{k_{r-1}}\sum_{k=1}^{k_r}[M(\frac{D(t_{km}(x),\tilde{l})}{\rho})]^{p_k}\\
= &\frac{1}{k_{r-1}}\{\sum_{k\in I_1}[M(\frac{D(t_{km}(x),\tilde{l})}{\rho})]^{p_k} + \sum_{k\in I_2}[M(\frac{D(t_{km}(x),\tilde{l})}{\rho})]^{p_k}\\
&+\dots+\sum_{k\in I_r}[M(\frac{D(t_{km}(x),\tilde{l})}{\rho})]^{p_k}\}\\
= &\frac{k_1}{k_{r-1}}A_1 + \frac{k_2-k_1}{k_{r-1}}A_2 + \dots + \frac{k_r-k_{r-1}}{k_{r-1}}A_r\\
\leq &(\sup_r A_r)\frac{k_r}{k_{r-1}}\\
= &(\sup_r A_r)q_r\\
< &(\sup_r A_r)\beta.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\{k \in N : \frac{1}{n}\sum_{k=1}^{n}[M(\frac{D(t_{km}(x),\tilde{l})}{\rho})]^{p_k} \geq \delta\}\\
\subset &\{r \in N : \frac{1}{h_r}\sum_{k\in I_r}[M(\frac{D(t_{km}(x),\tilde{l})}{\rho})]^{p_k} \geq \frac{1}{\beta}\delta\} \in I.
\end{aligned}$$

Hence $\tilde{W}(M,P,\theta) \subset \tilde{W}(M,P)$.

(3) From (1) and (2), (3) is obvious.

Definition 3.3. Let M be an Orlicz function and $p = \{p_k\}$ be any sequence of strictly positive real numbers. A sequence $\{x_k\}$ of fuzzy numbers is said to be almost ideal statistical convergent to the fuzzy number $\tilde{l}$ with respect to the Orlicz function M, if for every $\varepsilon > 0$,

$$\{n \in N : \frac{1}{n}|\{k \leq n : [M(\frac{D(t_{km}(x),\tilde{l})}{\rho})]^{p_k} \geq \varepsilon\}| \geq \delta\} \in I.$$

uniformly for m, where

$$t_{km}(x) = \frac{x_m + x_{m+1} + \dots + x_{m+k}}{k+1} = \frac{1}{k+1}\sum_{i=0}^{k} x_{m+i}.$$

As usual, we write $x_k \to \tilde{l}(S(M,P))$. The set of all almost ideal statistically convergent sequences will be denoted simply by $S(M,P)$. Especially, if $M(x) = x$, $p_k = 1$ for all $k \in N$, we shall write S instead of $S(M,P)$.

Definition 3.4. Let M be an Orlicz function and $p = \{p_k\}$ be any sequence of strictly positive real numbers, for some $\rho > 0$, we define the following sets

$$W(M,p) = \{x = (x_k) : \{n \in N : \frac{1}{n}\sum_{k=1}^{n}[M(\frac{D(t_{km}(x),\tilde{l})}{\rho})]^{p_k} \geq \delta\} \in I\},$$

$$W_0(M,p) = \{x = (x_k) : \{n \in N : \frac{1}{n}\sum_{k=1}^{n}[M(\frac{D(t_{km}(x),\tilde{0})}{\rho})]^{p_k} \geq \delta\} \in I\},$$

$W_\infty(M,P,\theta)=\{x=(x_k):\exists K>0$, s.t $\{n\in N:\frac{1}{n}\sum_{k=1}^{n}[M(\frac{D(t_{km}(x),\tilde{0})}{\rho})]^{p_k}\geq K\}\in I\}$,
where

$$0(t)=\begin{cases}1, & t=(0,0,0...,0),\\ 0, & \text{otherwise.}\end{cases}$$

If $x\in W(M,P)$, we say that x is strongly almost ideal convergent with respect to the Orlicz function M, and write $x_k\to\tilde{l}(W(M,P))$.

It similar to the proofs of Theorem 3.1, 3.2 and 3.3, for strongly almost ideal convergence of sequences of fuzzy numbers we have the following results.

Theorem 3.5. Let the sequence (p_k) be bounded. Then

$$W_0(M,P)\subset W(M,P)\subset W_\infty(M,P).$$

Theorem 3.6. Let M_1,M_2 be Orlicz functions. Then we have

(1) $W_0(M_1,P)\cap W_0(M_2,P)\subset W_0(M_1+M_2,P)$,

(2) $W(M_1,P)\cap W(M_2,P)\subset W(M_1+M_2,P)$,

(3) $W_\infty(M_1,P)\cap W_\infty(M_2,P)\subset W_\infty(M_1+M_2,P)$.

Theorem 3.7. Let M be an Orlicz function and $0<h=\inf_k p_k\leq p_k\leq\sup_k p_k=H$. Then $W(M,P)\subset S$.

In the following theorems, we shall discuss the relationship between the space $S(M,P)$ and $S(M,P,\theta)$. Let $q_r=\frac{k_r}{k_{r-1}}$.

Theorem 3.8. Let $\theta=(k_r)$ be a lacunary sequence, M be an Orlicz function. If $\liminf_r q_r>1$, we have $S(M,P)\subset\tilde{S}(M,P,\theta)$; If $\limsup q_r<\infty$, then $\tilde{S}(M,P,\theta)\subset S(M,P)$.

Proof. If $\liminf_r q_r>1$, then there exists a $\alpha>0$, such that $q_r\geq 1+\alpha$ for sufficiently large r. Since $h_r=k_r-k_{r-1}$ and $I_r=(k_{r-1},k_r]$, we have $\frac{h_r}{k_r}\geq\frac{\alpha}{\alpha+1}$. Let $x\in S(M,P)$, then for every $\varepsilon>0$ for all m and for sufficiently large r, we have

$$\begin{aligned}&\frac{1}{k_r}|\{k\leq k_r:[M(\frac{D(t_{km}(x),\tilde{l})}{\rho})]^{p_k}\geq\varepsilon\}|\\ \geq\ &\frac{1}{k_r}|\{k\in I_r:[M(\frac{D(t_{km}(x),\tilde{l})}{\rho})]^{p_k}\geq\varepsilon\}|\\ \geq\ &\frac{\alpha}{1+\alpha}.\frac{1}{h_r}|\{k\in I_r:[M(\frac{D(t_{km}(x),\tilde{l})}{\rho})]^{p_k}\geq\varepsilon\}|.\end{aligned}$$

Thus

$$\begin{aligned}&\{r\in N:\frac{1}{h_r}|\{k\in I_r:[M(\frac{D(t_{km}(x),\tilde{l})}{\rho})]^{p_k}\geq\varepsilon\}\ |\geq\delta\}\\ \subset\ &\{r\in N:\frac{1}{k_r}|\{k\leq k_r:[M(\frac{D(t_{km}(x),\tilde{l})}{\rho})]^{p_k}\geq\varepsilon\}\ |\geq\frac{\alpha}{1+\alpha}\delta\}\in I.\end{aligned}$$

Hence, $S(M,P)\subset\tilde{S}(M,P,\theta)$.

If $\limsup q_r<\infty$, then there exists a $\beta>0$, such that for every $r\in\mathbf{N}$, we have $q_r<\beta$. Let $x_k\to\tilde{l}(\tilde{S}(M,P,\theta))$ and $N_{rm}=\frac{1}{h_r}|\{k\in I_r:[M(\frac{D(t_{km}(x),\tilde{l})}{\rho})]^{p_k}\geq\epsilon\}|$, then $\{r\in N:N_{rm}\geq\delta\}\in I$, choose n such that $k_{r-1}\leq n\leq k_r$, we have

$$\begin{aligned}&\frac{1}{n}\mid\{k\leq n:[M(\frac{D(t_{km}(x),\tilde{l})}{\rho})]^{p_k}\geq\varepsilon\}\mid\leq\frac{1}{k_{r-1}}\mid\{k\leq k_r:[M(\frac{D(t_{km}(x),\tilde{l})}{\rho})]^{p_k}\geq\varepsilon\}\mid\\ =\ &\frac{k_1}{k_{r-1}}N_{1m}+\frac{k_2-k_1}{k_{r-1}}N_{2m}+...+\frac{k_r-k_{r-1}}{k_{r-1}}N_{rm}\leq(\sup_r N_{rm})\{\frac{k_1}{k_{r-1}}+\frac{k_2-k_1}{k_{r-1}}+...+\frac{k_r-k_{r-1}}{k_{r-1}}\}\\ \leq\ &(\sup_r N_{rm})\frac{k_r}{k_{r-1}}=(\sup_r N_{rm})q_r<(\sup_r N_{rm})\beta\end{aligned}$$

Which implies that

$$\begin{aligned} &\{r \in N : \frac{1}{n} \mid \{k \leq n : [M(\frac{D(t_{km}(x), \tilde{l})}{\rho})]^{p_k} \geq \varepsilon\} \mid \geq \delta\} \\ \subset \quad &\{r \in N : \frac{1}{h_r} \mid \{k \in I_r : [M(\frac{D(t_{km}(x), \tilde{l})}{\rho})]^{p_k} \geq \varepsilon\} \mid \geq \frac{1}{\beta}\delta\} \in I. \end{aligned}$$

Thus we get $\tilde{S}(M, P, \theta) \subset S(M, P)$.

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Some boundary value problems of fractional differential equations with fractional impulsive conditions

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This paper is concerned with the existence of solutions for nonlinear impulsive fractional differential equations with families of mixed and closed boundary conditions. In both cases, the fractional derivative of lower order is involved in the formulation of impulsive conditions. By means of the Banach fixed point theorem, Schaefer fixed point theorem and Nonlinear alternative of Leray-Schauder type, some existence results are obtained. Examples are given to illustrate the results.
Key words: *Fractional differential equations, Impulse, Mixed boundary conditions, Closed boundary conditions, Existence*

1 Introduction

The subject of fractional differential equations has recently evolved as an interesting and popular field of research. It is mainly due to the extensive applications of fractional calculus in the mathematical modeling of physical, engineering, and biological phenomena (see [1, 2, 3]). For some recent developments on the existence results of fractional differential equations, see for example [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18] and the references therein.

The theory and applications of impulsive initial and boundary value problems for ordinary differential equations is a well-developed area of analysis, which steadily receives attention of many authors [19, 20, 21, 22, 23]. As for impulsive fractional differential equations, we can refer to [24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35] and the references therein.

In this paper, we consider the existence and uniqueness of solutions for the impulsive fractional differential equations with fractional impulsive conditions

$$\begin{cases} {}^cD^\alpha x(t) = f(t, x(t)),\ t \in J := [0, T],\ t \neq t_k,\ k = 1, 2, \cdots, m, \\ \Delta x(t_k) = I_k(x(t_k^-)),\ \Delta({}^cD^\gamma x(t_k)) = I_k^*(x(t_k^-)),\ k = 1, 2, \cdots, m, \end{cases} \tag{1}$$

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subjected to two families of boundary conditions: (I) Mixed boundary conditions

$$Tx'(0) = -ax(0) - bx(T),\ Tx'(T) = cx(0) + dx(T), \tag{2}$$

and (II) Closed boundary conditions

$$x(T) = a_1x(0) + b_1Tx'(0),\ Tx'(T) = c_1x(0) + d_1Tx'(0), \tag{3}$$

where ${}^cD^\alpha$ is the Caputo fractional derivative of order $\alpha \in (1, 2)$ with the lower limit zero, $0 < \gamma < 1$, $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $I_k, I_k^* : \mathbb{R} \to \mathbb{R}$ are continuous, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ with $x(t_k^+) = \lim_{\epsilon\to 0^+} x(t_k + \epsilon)$, $x(t_k^-) = \lim_{\epsilon\to 0^-} x(t_k + \epsilon)$ representing the right and left limits of $x(t)$ at $t = t_k$, $\Delta({}^cD^\gamma x(t_k))$ has a similar meaning for ${}^cD^\gamma x(t_k)$ and $a, b, c, d,\ a_1, b_1, c_1, d_1$ are real constants such that

$$T(1 - d)(a + b) + T(1 + b)(c + d) \neq 0,$$

$$(a_1 - 1)(d_1 - 1)T + c_1(1 - b_1)T \neq 0.$$

Here we remark that the mixed boundary conditions (2) interpolate between Neumann ($a = b = c = d = 0$) and Dirichlet ($a, d \to \infty$ with finite values of b and c) boundary conditions. Notice that Zaremba boundary conditions ($x(0) = 0, x'(T) = 0$) can be considered as mixed boundary conditions with $a \to \infty$, $c = d = 0$ and the closed boundary conditions (3) include quasi-periodic boundary conditions ($b_1 = c_1 = 0$) and interpolate between periodic ($a_1 = d_1 = 1$, $b_1 = c_1 = 0$) and anti-periodic ($a_1 = d_1 = -1$, $b_1 = c_1 = 0$) boundary conditions.

We note that as pointed out in papers [32, 33, 34], the concept of piecewise continuous solutions used in some already published works to handle the impulsive fractional differential equations are not appropriate (see Lemma 3.1 in [32], Section 1 in [33] and Section 3 in [34]). The papers on this topic cited above except [32] all deal with the Caputo derivative and the impulsive conditions only involve integer order derivatives. Here we study the fractional differential equations with fractional impulsive conditions subjected to families of mixed and closed boundary conditions.

The rest of the paper is organized as follows. In Section 2 we introduce some preliminary results needed in the following sections. In Section 3 we present the existence results for the problems (1), (2) and (1), (3). Two examples are presented in the last section 4 to illustrate the results.

2 Preliminaries

Let us set $J_0 = [0, t_1]$, $J_1 = (t_1, t_2]$, $\cdots$, $J_{m-1} = (t_{m-1}, t_m]$, $J_m = (t_m, t_{m+1}]$, $J' := J\backslash\{t_1, t_2, \cdots, t_m\}$ and introduce the space $PC(J, \mathbb{R}) := \{u : J \to \mathbb{R} | u \in C(J_k, \mathbb{R}), k = 0, 1, 2, \cdots, m$, and there exist $u(t_k^+)$ and $u(t_k^-)$, $k = 1, 2, \cdots, m$, with $u(t_k^-) = u(t_k)\}$. It is clear that $PC(J, \mathbb{R})$ is a Banach space with the norm $\|u\| = \sup\{|u(t)| : t \in J\}$.

Definition 2.1 ([3]). *The Riemann-Liouville fractional integral of order q for a function f is defined as*

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)ds, \ \ q > 0,$$

provided the integral exists.

Definition 2.2 ([3]). *For a continuous function f, the Caputo derivative of order q is defined as*

$${}^cD^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} f^{(n)}(s)ds, \ n-1 < q < n, \ n = [q]+1,$$

where $[q]$ denotes the integer part of the real number q.

Lemma 2.1 ([3]). *Let $\alpha > 0$, then the differential equation*

$${}^cD^\alpha h(t) = 0$$

has solutions $h(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$ and

$$I^\alpha\, {}^cD^\alpha h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

here $c_i \in \mathbb{R}$, $i = 0, 1, 2, \cdots, n-1$, $n = [\alpha]+1$.

Definition 2.3. *A function $x \in PC(J, \mathbb{R})$ with its α-derivative exists on J' is said to be a solution of the problem (1), (2) (or the problem (1), (3)) if x satisfies the equation ${}^cD^\alpha x(t) = f(t, x(t))$ on J', the impulsive condition*

$$\Delta x(t_k) = I_k(x(t_k^-)), \ \Delta({}^cD^\gamma x(t_k)) = I_k^*(x(t_k^-)), \ k = 1, 2, \cdots, m,$$

and the boundary conditions (2) (or (3)).

Lemma 2.2. *Let $y \in C(J, \mathbb{R})$. A function x is a solution of the fractional integral equation*

$$x(t) = \begin{cases} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + e_0 + h_0 t, \quad t \in J_0; \\ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + e_0 + I_1(x(t_1^-)) - \Gamma(2-\gamma) t_1^\gamma I_1^*(x(t_1^-)) \\ \quad + h_0 t + \Gamma(2-\gamma) t \frac{I_1^*(x(t_1^-))}{t_1^{1-\gamma}}, \quad t \in J_1; \\ \cdots\cdots\cdots; \\ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + e_0 + \sum_{i=1}^k I_i(x(t_i^-)) \\ \quad -\Gamma(2-\gamma) \sum_{i=1}^k t_i^\gamma I_i^*(x(t_i^-)) + h_0 t \\ \quad + \Gamma(2-\gamma) t \sum_{i=1}^k \frac{I_i^*(x(t_i^-))}{t_i^{1-\gamma}}, \quad t \in J_k, k = 1, 2, \cdots, m, \end{cases} \tag{4}$$

where

$$e_0 = \frac{-(b+d)T \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + (1+b)T^2 \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds}{\Delta} + \frac{-(b+d)TA - (d-1)T^2 B}{\Delta},$$

3

$$h_0 = \frac{(ad-cb)\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds - (a+b)T\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}y(s)ds}{\Delta} + \frac{(ad-cb)A + (ad-a-b-cb)TB}{\Delta}$$

with

$$\Delta = T(1-d)(a+b) + T(1+b)(c+d) \neq 0,$$

$$A = \sum_{i=1}^{m} I_i(x(t_i^-)) - \Gamma(2-\gamma)\sum_{i=1}^{m} t_i^{\gamma} I_i^*(x(t_i^-)),\ B = \Gamma(2-\gamma)\sum_{i=1}^{m}\frac{I_i^*(x(t_i^-))}{t_i^{1-\gamma}}, \quad (5)$$

if and only if x is a solution of the fractional mixed boundary value problem

$$\begin{cases} {}^cD^{\alpha}x(t) = y(t),\ t \in J',\ 1 < \alpha < 2, \\ \Delta x(t_k) = I_k(x(t_k^-)),\ \Delta({}^cD^{\gamma}x(t_k)) = I_k^*(x(t_k^-)),\ k = 1, 2, \cdots, m, \\ Tx'(0) = -ax(0) - bx(T),\ Tx'(T) = cx(0) + dx(T). \end{cases} \quad (6)$$

Proof. From Lemma 2.1, we know that a general solution x of the equation ${}^cD^{\alpha}x(t) = y(t)$ on each interval J_k $(k = 0, 1, 2, \cdots, m)$ is given by

$$x(t) = I^{\alpha}y(t) + e_k + h_k t = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds + e_k + h_k t, \quad t \in J_k, \quad (7)$$

where $e_k, h_k \in \mathbb{R}$ are arbitrary constants. Then we have

$$x'(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}y(s)ds + h_k, \quad \text{for } t \in J_k.$$

Since ${}^cD^{\gamma}C = 0$ (C is a constant), ${}^cD^{\gamma}t = \frac{t^{1-\gamma}}{\Gamma(2-\gamma)}$, ${}^cD^{\gamma}I^{\alpha}y(t) = I^{\alpha-\gamma}y(t)$ (see [3]), then from (7), we get

$${}^cD^{\gamma}x(t) = \int_0^t \frac{(t-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}y(s)ds + \frac{h_k t^{1-\gamma}}{\Gamma(2-\gamma)}, \quad \text{for } t \in J_k.$$

Using the impulsive conditions in (6), we obtain that for $k = 1, 2, \cdots, m$

$$e_k - e_{k-1} + (h_k - h_{k-1})t_k = I_k(x(t_k^-)),$$

$$(h_k - h_{k-1})\frac{t_k^{1-\gamma}}{\Gamma(2-\gamma)} = I_k^*(x(t_k^-)).$$

That is to say we have, for $k = 1, 2, \cdots, m$,

$$e_k = e_0 + \sum_{i=1}^{k} I_i(x(t_i^-)) - \Gamma(2-\gamma)\sum_{i=1}^{k} t_i^{\gamma} I_i^*(x(t_i^-)), \quad (8)$$

$$h_k = h_0 + \Gamma(2-\gamma)\sum_{i=1}^{k}\frac{I_i^*(x(t_i^-))}{t_i^{1-\gamma}}. \quad (9)$$

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Applying the boundary conditions of (6), we find that (since $0 \in J_0, T \in J_m$)

$$Th_0 = -ae_0 - b\Big(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + e_m + h_m T\Big),$$

$$T\Big(\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds + h_m\Big) = ce_0 + d\Big(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + e_m + h_m T\Big).$$

These two equalities together with (8), (9) imply

$$\begin{aligned} e_0 = &\frac{-(b+d)T\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + (1+b)T^2 \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds}{\Delta} \\ &+ \frac{-(b+d)TA - (d-1)T^2 B}{\Delta}, \\ h_0 = &\frac{(ad-cb)\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds - (a+b)T\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds}{\Delta} \\ &+ \frac{(ad-cb)A + (ad-a-b-cb)TB}{\Delta}, \end{aligned}$$

where

$$\Delta = T(1-d)(a+b) + T(1+b)(c+d),$$

$$A = \sum_{i=1}^m I_i(x(t_i^-)) - \Gamma(2-\gamma)\sum_{i=1}^m t_i^\gamma I_i^*(x(t_i^-)),\ \ B = \Gamma(2-\gamma)\sum_{i=1}^m \frac{I_i^*(x(t_i^-))}{t_i^{1-\gamma}}.$$

Now from (8), (9), we obtain for $k = 0, 1, 2, \cdots, m$,

$$\begin{aligned} e_k + h_k t = &e_0 + \sum_{i=1}^k I_i(x(t_i^-)) - \Gamma(2-\gamma)\sum_{i=1}^k t_i^\gamma I_i^*(x(t_i^-)) \\ &+ h_0 t + \Gamma(2-\gamma)t\sum_{i=1}^k \frac{I_i^*(x(t_i^-))}{t_i^{1-\gamma}}. \end{aligned}$$

Substituting the values of e_0, h_0 in the above relation and by (7), we obtain the fractional integral equation (4).

Conversely, assume that x satisfies the fractional integral equation (4), i.e. for $t \in J_k$, $k = 0, 1, 2, \cdots, m$, we have

$$\begin{aligned} x(t) = &\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + e_0 + \sum_{i=1}^k I_i(x(t_i^-)) \\ &- \Gamma(2-\gamma)\sum_{i=1}^k t_i^\gamma I_i^*(x(t_i^-)) + h_0 t + \Gamma(2-\gamma)t\sum_{i=1}^k \frac{I_i^*(x(t_i^-))}{t_i^{1-\gamma}}. \end{aligned} \tag{10}$$

Since $1 < \alpha < 2$, we have ${}^cD^\alpha C = 0$ (C is a constant) and ${}^cD^\alpha t = 0$. Using the fact that ${}^cD^\alpha$ is the left inverse of I^α, we get

$${}^cD^\alpha x(t) = y(t), \quad t \in J',$$

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which means that x satisfies the first equation of the mixed boundary value problem (6). Next we shall verify that x satisfies the impulsive conditions. Taking fractional derivative ${}^cD^\gamma$ of the equation (10), we have, for $t \in J_k$,

$$ {}^cD^\gamma x(t) = \int_0^t \frac{(t-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} y(s)ds + \frac{h_0 t^{1-\gamma}}{\Gamma(2-\gamma)} + t^{1-\gamma} \sum_{i=1}^{k} \frac{I_i^*(x(t_i^-))}{t_i^{1-\gamma}}. \tag{11} $$

From (10), we obtain

$$ \begin{aligned} x(t_k^+) = & \int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + e_0 + \sum_{i=1}^{k} I_i(x(t_i^-)) \\ & - \Gamma(2-\gamma) \sum_{i=1}^{k} t_i^\gamma I_i^*(x(t_i^-)) + h_0 t_k + \Gamma(2-\gamma) t_k \sum_{i=1}^{k} \frac{I_i^*(x(t_i^-))}{t_i^{1-\gamma}}, \end{aligned} $$

$$ \begin{aligned} x(t_k^-) = & \int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + e_0 + \sum_{i=1}^{k-1} I_i(x(t_i^-)) \\ & - \Gamma(2-\gamma) \sum_{i=1}^{k-1} t_i^\gamma I_i^*(x(t_i^-)) + h_0 t_k + \Gamma(2-\gamma) t_k \sum_{i=1}^{k-1} \frac{I_i^*(x(t_i^-))}{t_i^{1-\gamma}}. \end{aligned} $$

Hence, we have, for $k = 1, 2, \cdots, m$,

$$ \Delta x(t_k) = I_k(x(t_k^-)) - \Gamma(2-\gamma) t_k^\gamma I_k^*(x(t_k^-)) + \Gamma(2-\gamma) t_k \frac{I_k^*(x(t_k^-))}{t_k^{1-\gamma}} = I_k(x(t_k^-)). $$

By (11), we have

$$ {}^cD^\gamma x(t_k^+) = \int_0^{t_k} \frac{(t_k-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} y(s)ds + \frac{h_0 t_k^{1-\gamma}}{\Gamma(2-\gamma)} + t_k^{1-\gamma} \sum_{i=1}^{k} \frac{I_i^*(x(t_i^-))}{t_i^{1-\gamma}}, $$

$$ {}^cD^\gamma x(t_k^-) = \int_0^{t_k} \frac{(t_k-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} y(s)ds + \frac{h_0 t_k^{1-\gamma}}{\Gamma(2-\gamma)} + t_k^{1-\gamma} \sum_{i=1}^{k-1} \frac{I_i^*(x(t_i^-))}{t_i^{1-\gamma}}. $$

Therefore, for $k = 1, 2, \cdots, m$,

$$ \Delta({}^cD^\gamma x(t_k)) = t_k^{1-\gamma} \frac{I_k^*(x(t_k^-))}{t_k^{1-\gamma}} = I_k^*(x(t_k^-)). $$

Finally, it follows from (10) that

$$ x'(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds + h_0 + \Gamma(2-\gamma) \sum_{i=1}^{k} \frac{I_i^*(x(t_i^-))}{t_i^{1-\gamma}}, \text{ for } t \in J_k. $$

This equation together with (10) implies that $x(0) = e_0$, $x'(0) = h_0$,

$$x'(T) = \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds + h_0 + \Gamma(2-\gamma)\sum_{i=1}^{m} \frac{I_i^*(x(t_i^-))}{t_i^{1-\gamma}},$$

$$\begin{aligned} x(T) = &\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + e_0 + \sum_{i=1}^{m} I_i(x(t_i^-)) \\ &- \Gamma(2-\gamma)\sum_{i=1}^{m} t_i^{\gamma} I_i^*(x(t_i^-)) + h_0 T + \Gamma(2-\gamma)T\sum_{i=1}^{m} \frac{I_i^*(x(t_i^-))}{t_i^{1-\gamma}}. \end{aligned}$$

Now, by a direct computation, it is clear that the mixed boundary conditions in (6) hold. Therefore x given by (4) satisfies the fractional mixed boundary value problem (6). This completes the proof. □

Lemma 2.3. *Let $y \in C(J, \mathbb{R})$. A function x is a solution of the fractional integral equation*

$$x(t) = \begin{cases} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + e_0 + h_0 t, \quad t \in J_0; \\ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + e_0 + I_1(x(t_1^-)) - \Gamma(2-\gamma)t_1^{\gamma} I_1^*(x(t_1^-)) \\ \quad + h_0 t + \Gamma(2-\gamma)t\frac{I_1^*(x(t_1^-))}{t_1^{1-\gamma}}, \quad t \in J_1; \\ \cdots\cdots\cdots; \\ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + e_0 + \sum_{i=1}^{k} I_i(x(t_i^-)) \\ \quad -\Gamma(2-\gamma)\sum_{i=1}^{k} t_i^{\gamma} I_i^*(x(t_i^-)) + h_0 t \\ \quad +\Gamma(2-\gamma)t\sum_{i=1}^{k} \frac{I_i^*(x(t_i^-))}{t_i^{1-\gamma}}, \quad t \in J_k, k = 1, 2, \cdots, m, \end{cases} \tag{12}$$

where

$$\begin{aligned} e_0 = &\frac{(d_1-1)T\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + (1-b_1)T^2\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds}{\Delta_1} \\ &+ \frac{(d_1-1)TA + (d_1-b_1)T^2 B}{\Delta_1}, \end{aligned}$$

$$\begin{aligned} h_0 = &\frac{-c_1\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + (a_1-1)T\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds}{\Delta_1} \\ &+ \frac{-c_1 A + (a_1-c_1-1)TB}{\Delta_1} \end{aligned}$$

with A, B defined in (5) and

$$\Delta_1 = (a_1-1)(d_1-1)T + c_1(1-b_1)T \neq 0,$$

if and only if x is a solution of the fractional closed boundary value problem

$$\begin{cases} {}^cD^{\alpha}x(t) = y(t), \ t \in J', \ 1 < \alpha < 2, \\ \Delta x(t_k) = I_k(x(t_k^-)), \ \Delta({}^cD^{\gamma}x(t_k)) = I_k^*(x(t_k^-)), \ k = 1, 2, \cdots, m, \\ x(T) = a_1 x(0) + b_1 T x'(0), \ Tx'(T) = c_1 x(0) + d_1 T x'(0). \end{cases} \tag{13}$$

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Proof. The proof is similar to the one of Lemma 2.2, however, for the sake of completeness, we provide the outline of the proof. Using the notations given in the proof of Lemma 2.2, we have

$$x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + e_k + h_k t, \quad t \in J_k,$$

$$x'(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds + h_k, \quad \text{for } t \in J_k,$$

$${}^cD^\gamma x(t) = \int_0^t \frac{(t-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} y(s)ds + \frac{h_k t^{1-\gamma}}{\Gamma(2-\gamma)}, \quad \text{for } t \in J_k.$$

Applying the boundary conditions of (13), we find that (since $0 \in J_0, T \in J_m$)

$$\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + e_m + h_m T = a_1 e_0 + b_1 T h_0,$$

$$T\Big(\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds + h_m\Big) = c_1 e_0 + d_1 T h_0.$$

Using the impulsive conditions in (13), we obtain that for $k = 1, 2, \cdots, m$

$$e_k - e_{k-1} + (h_k - h_{k-1})t_k = I_k(x(t_k^-)),$$

$$(h_k - h_{k-1})\frac{t_k^{1-\gamma}}{\Gamma(2-\gamma)} = I_k^*(x(t_k^-)).$$

Hence we obtain

$$e_0 = \frac{(d_1-1)T\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + (1-b_1)T^2\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds}{\Delta_1}$$
$$+\frac{(d_1-1)TA + (d_1-b_1)T^2 B}{\Delta_1},$$

$$h_0 = \frac{-c_1\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + (a_1-1)T\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds}{\Delta_1}$$
$$+\frac{-c_1 A + (a_1-c_1-1)TB}{\Delta_1},$$

with A, B defined in (5) and

$$\Delta_1 = (a_1-1)(d_1-1)T + c_1(1-b_1)T.$$

The remaining part of proof is similar to that of Lemma 2.2. □

Theorem 2.1 (Nonlinear alternative of Leray-Schauder type [36]). *Let X be a Banach space, C a nonempty convex subset of X, U a nonempty open subset of C with $0 \in U$. Suppose that $P : \overline{U} \to C$ is a continuous and compact map. Then either (a) P has a fixed point in $\overline{U}$, or (b) there exist a $x \in \partial U$ (the boundary of U) and $\lambda \in (0, 1)$ with $x = \lambda P(x)$.*

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Theorem 2.2 (Schaefer fixed point theorem [37]). *Let X be a normed space, P a continuous mapping of X into X which is compact on each bounded subset B of X. Then either (I) the equation $x = \lambda P x$ has a solution for $\lambda = 1$, or (II) the set of all such solutions x, for $0 < \lambda < 1$, is unbounded.*

3 Main results

This section deals with the existence and uniqueness of solutions to the problems (1), (2) and (1), (3).

Define an operator $F : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ as

$$\begin{aligned}(Fx)(t) = &\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s))ds + e_0 + \sum_{i=1}^{k} I_i(x(t_i^-)) \\ &- \Gamma(2-\gamma) \sum_{i=1}^{k} t_i^{\gamma} I_i^*(x(t_i^-)) + h_0 t \\ &+ \Gamma(2-\gamma) t \sum_{i=1}^{k} t_i^{\gamma-1} I_i^*(x(t_i^-)),\ t \in J_k,\ k = 0, 1, 2, \cdots, m, \end{aligned} \tag{14}$$

where

$$\begin{aligned}e_0 = &\frac{-(b+d)T \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s))ds + (1+b)T^2 \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, x(s))ds}{\Delta} \\ &+ \frac{-(b+d)TA - (d-1)T^2 B}{\Delta}, \end{aligned} \tag{15}$$

$$\begin{aligned}h_0 = &\frac{(ad-cb) \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s))ds - (a+b)T \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, x(s))ds}{\Delta} \\ &+ \frac{(ad-cb)A + (ad-a-b-cb)TB}{\Delta} \end{aligned} \tag{16}$$

with

$$\Delta = T(1-d)(a+b) + T(1+b)(c+d),$$

$$A = \sum_{i=1}^{m} I_i(x(t_i^-)) - \Gamma(2-\gamma) \sum_{i=1}^{m} t_i^{\gamma} I_i^*(x(t_i^-)),\ B = \Gamma(2-\gamma) \sum_{i=1}^{m} \frac{I_i^*(x(t_i^-))}{t_i^{1-\gamma}}.$$

Observe that the problem (1), (2) has solutions if and only if the operator equation $Fx = x$ has fixed points.

Lemma 3.1. *The operator $F : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ is completely continuous.*

Proof. Observe that F is continuous in view of the continuity of f, I_k and I_k^*. Let $\Omega \subseteq PC(J, \mathbb{R})$ be bounded, then there exist positive constants N_i, $i = 1, 2, 3$,

such that $|f(t,x(t))| \le N_1$, $|I_k(x(t_k^-))| \le N_2$ and $|I_k^*(x(t_k^-))| \le N_3$ for all $x \in \Omega$, $k = 1, 2, \cdots, m$. Thus, for $x \in \Omega$ and $t \in J$, we have

$$\begin{aligned} |(Fx)(t)| \le & \frac{N_1 T^\alpha}{\Gamma(\alpha+1)} + |e_0| + mN_2 + \Gamma(2-\gamma)N_3 \sum_{i=1}^m t_i^\gamma \\ & + |h_0|T + \Gamma(2-\gamma)TN_3 \sum_{i=1}^m t_i^{\gamma-1}, \end{aligned} \tag{17}$$

$$\begin{aligned} |e_0| \le & \frac{N_1|b+d|T^{\alpha+1}}{|\Delta|\Gamma(\alpha+1)} + \frac{|b+d|T\big(mN_2 + \Gamma(2-\gamma)N_3 \sum_{i=1}^m t_i^\gamma\big)}{|\Delta|} \\ & + \frac{N_1|1+b|T^{\alpha+1}}{|\Delta|\Gamma(\alpha)} + \frac{|d-1|T^2 N_3 \Gamma(2-\gamma) \sum_{i=1}^m t_i^{\gamma-1}}{|\Delta|}, \end{aligned} \tag{18}$$

$$\begin{aligned} |h_0| \le & \frac{N_1|ad-cb|T^\alpha}{|\Delta|\Gamma(\alpha+1)} + \frac{|ad-cb|\big(mN_2 + \Gamma(2-\gamma)N_3 \sum_{i=1}^m t_i^\gamma\big)}{|\Delta|} \\ & + \frac{N_1|a+b|T^\alpha}{|\Delta|\Gamma(\alpha)} + \frac{|ad-a-b-cb|TN_3\Gamma(2-\gamma) \sum_{i=1}^m t_i^{\gamma-1}}{|\Delta|}. \end{aligned} \tag{19}$$

Hence from (17)-(19), we obtain that, for all $x \in \Omega$ and $t \in J$,

$$\begin{aligned} |(Fx)(t)| \le & \frac{N_1 T^\alpha}{\Gamma(\alpha+1)} + \frac{N_1 T^{\alpha+1}}{|\Delta|}\Big(\frac{|b+d| + |ad-cb|}{\Gamma(\alpha+1)} + \frac{|1+b| + |a+b|}{\Gamma(\alpha)}\Big) \\ & + \frac{T\big(mN_2 + \Gamma(2-\gamma)N_3 \sum_{i=1}^m t_i^\gamma\big)}{|\Delta|}\big(|b+d| + |ad-cb|\big) \\ & + \frac{T^2 N_3 \Gamma(2-\gamma) \sum_{i=1}^m t_i^{\gamma-1}}{|\Delta|}\big(|d-1| + |ad-a-b-cb|\big) \\ & + mN_2 + \Gamma(2-\gamma)N_3\Big(\sum_{i=1}^m t_i^\gamma + T\sum_{i=1}^m t_i^{\gamma-1}\Big), \end{aligned}$$

which implies that the operator F is uniformly bounded on Ω.

On the other hand, let $x \in \Omega$ and for any $\tau_1.\tau_2 \in J_k$, $(k = 0, 1, 2, \cdots, m)$ with $\tau_1 < \tau_2$, we have

$$\begin{aligned} & |(Fx)(\tau_2) - (Fx)(\tau_1)| \\ \le & \Big|\int_0^{\tau_2} \frac{(\tau_2-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds - \int_0^{\tau_1} \frac{(\tau_1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds\Big| \\ & + |h_0|(\tau_2-\tau_1) + \Gamma(2-\gamma)\sum_{i=1}^k t_i^{\gamma-1}|I_i^*(x(t_i^-))|(\tau_2-\tau_1) \\ \le & \frac{N_1(\tau_2^\alpha - \tau_1^\alpha)}{\Gamma(\alpha+1)} + |h_0|(\tau_2-\tau_1) + \Gamma(2-\gamma)N_3\sum_{i=1}^k t_i^{\gamma-1}(\tau_2-\tau_1). \end{aligned}$$

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In virtue of (19), we deduce that the last term of the above inequality tends to zero as $\tau_2 \to \tau_1$. This implies that F is equicontinuous on the interval J_k. Hence by PC-type Arzela-Ascoli Theorem (see Theorem 2.1 [23]), the operator $F : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ is completely continuous. □

Theorem 3.1. *Assume that: (1) there exist $h \in L^\infty(J, \mathbb{R}^+)$ and $\varphi : [0, \infty) \to (0, \infty)$ continuous, nondecreasing such that $|f(t, x)| \leq h(t)\varphi(|x|)$ for $(t, x) \in J \times \mathbb{R}$; (2) there exist $\psi, \psi^* : [0, \infty) \to (0, \infty)$ continuous, nondecreasing such that $|I_k(x)| \leq \psi(|x|)$, $|I_k^*(x)| \leq \psi^*(|x|)$ for all $x \in \mathbb{R}$ and $k = 1, 2, \cdots, m$; (3) there exists a constant $M > 0$ such that*

$$\frac{M}{\varphi(M)\|h\|_{L^\infty} P + \psi(M)Q + \psi^*(M)\Gamma(2-\gamma)R} > 1, \tag{20}$$

where

$$P = \frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{T^{\alpha+1}}{|\Delta|}\Big(\frac{|b+d| + |ad-cb|}{\Gamma(\alpha+1)} + \frac{|1+b| + |a+b|}{\Gamma(\alpha)}\Big),$$

$$Q = m\Big(1 + \frac{T|b+d|}{|\Delta|} + \frac{T|ad-cb|}{|\Delta|}\Big),$$

and

$$R = \Big(1 + \frac{T|b+d|}{|\Delta|} + \frac{T|ad-cb|}{|\Delta|}\Big)\sum_{i=1}^{m} t_i^\gamma$$
$$+\Big(T + \frac{T^2|d-1|}{|\Delta|} + \frac{T^2|ad-a-b-cb|}{|\Delta|}\Big)\sum_{i=1}^{m} t_i^{\gamma-1}.$$

Then the impulsive fractional BVP (1), (2) has at least one solution on $[0, T]$.

Proof. Consider the operator $F : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ defined by (14). We will show that F satisfies the assumptions of the nonlinear alternative of Leray-Schauder type.

From Lemma 3.1, the operator $F : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ is continuous and completely continuous.

Let $x \in PC(J, \mathbb{R})$ be such that $x(t) = \lambda(Fx)(t)$ for some $\lambda \in (0, 1)$. Then using the computations in proving that F maps bounded sets into bounded sets in Lemma 3.1, we have

$$\begin{aligned}
|x(t)| \leq & \varphi(\|x\|)\|h\|_{L^\infty}\Big[\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{T^{\alpha+1}}{|\Delta|}\Big(\frac{|b+d| + |ad-cb|}{\Gamma(\alpha+1)} \\
& + \frac{|1+b| + |a+b|}{\Gamma(\alpha)}\Big)\Big] + \psi(\|x\|)\Big[\frac{Tm}{|\Delta|}\big(|b+d| + |ad-cb|\big) + m\Big] \\
& + \psi^*(\|x\|)\Gamma(2-\gamma)\Big[\frac{T\sum_{i=1}^m t_i^\gamma}{|\Delta|}\big(|b+d| + |ad-cb|\big) + \sum_{i=1}^{m} t_i^\gamma \\
& + T\sum_{i=1}^{m} t_i^{\gamma-1} + \frac{T^2\sum_{i=1}^m t_i^{\gamma-1}}{|\Delta|}\big(|d-1| + |ad-a-b-cb|\big)\Big] \\
\leq & \varphi(\|x\|)\|h\|_{L^\infty} P + \psi(\|x\|)Q + \psi^*(\|x\|)\Gamma(2-\gamma)R.
\end{aligned}$$

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Consequently, we have

$$\frac{\|x\|}{\varphi(\|x\|)\|h\|_{L^\infty}P+\psi(\|x\|)Q+\psi^*(\|x\|)\Gamma(2-\gamma)R}\leq 1.$$

Then by condition (20), there exists M such that $\|x\|\neq M$. Let

$$U=\{x\in PC(J,\mathbb{R}):\|x\|<M\}.$$

The operator $F:\overline{U}\to PC(J,\mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $x\in\partial U$ such that $x=\lambda Fx$ for some $\lambda\in(0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type (see Theorem 2.1), we deduce that F has a fixed point x in $\overline{U}$ which is a solution of the problem (1), (2). This completes the proof. □

Theorem 3.2. *Assume that there exist $h\in L^\infty(J,\mathbb{R}^+)$ and positive constants H_1,H_2 such that, for $t\in J$, $x\in\mathbb{R}$, $k=1,2,\cdots,m$,*

$$|f(t,x)|\leq h(t),\quad |I_k(x)|\leq H_1,\quad |I_k^*(x)|\leq H_2.$$

Then the impulsive fractional BVP (1), (2) has at least one solution on $[0,T]$.

Proof. From Lemma 3.1, we know that the operator $F:PC(J,\mathbb{R})\to PC(J,\mathbb{R})$ is continuous and compact on each bounded subset Ω of $PC(J,\mathbb{R})$.

Let us show that the set $V=\{u\in PC(J,\mathbb{R}):u=\lambda Fu, 0<\lambda<1\}$ is bounded. Let $x\in V$, then $x=\lambda Fx$ for some $0<\lambda<1$. For each $t\in J$, using similar estimations given in Lemma 3.1, we have

$$\begin{aligned}|x(t)|=&|\lambda(Fx)(t)|\leq\|h\|_{L^\infty}\Big[\frac{T^\alpha}{\Gamma(\alpha+1)}+\frac{T^{\alpha+1}}{|\Delta|}\Big(\frac{|b+d|+|ad-cb|}{\Gamma(\alpha+1)}\\&+\frac{|1+b|+|a+b|}{\Gamma(\alpha)}\Big)\Big]+H_1\Big[\frac{Tm}{|\Delta|}\big(|b+d|+|ad-cb|\big)+m\Big]\\&+H_2\Gamma(2-\gamma)\Big[\frac{T\sum_{i=1}^m t_i^\gamma}{|\Delta|}\big(|b+d|+|ad-cb|\big)+\sum_{i=1}^m t_i^\gamma\\&+T\sum_{i=1}^m t_i^{\gamma-1}+\frac{T^2\sum_{i=1}^m t_i^{\gamma-1}}{|\Delta|}\big(|d-1|+|ad-a-b-cb|\big)\Big].\end{aligned}$$

This implies that $\|x\|\leq M$ for all $x\in V$, i.e. V is bounded. Thus, by Theorem 2.2, the operator F has at least one fixed point. Hence the problem (1), (2) has at least one solution. The proof is completed. □

Theorem 3.3. *Assume that there exist $h\in L^\infty(J,\mathbb{R}^+)$ and positive constants L_1,L_2 such that, for $t\in J$, $x,y\in\mathbb{R}$, $k=1,2,\cdots,m$,*

$$|f(t,x)-f(t,y)|\leq h(t)|x-y|, \tag{21}$$

$$|I_k(x)-I_k(y)|\leq L_1|x-y|,\quad |I_k^*(x)-I_k^*(y)|\leq L_2|x-y|. \tag{22}$$

Moreover

$$\begin{aligned}\gamma=&\frac{\|h\|_{L^\infty}T^{\alpha+1}}{|\Delta|}\Big(\frac{|b+d|+|ad-cb|}{\Gamma(\alpha+1)}+\frac{|1+b|+|a+b|}{\Gamma(\alpha)}\Big)+\frac{\|h\|_{L^\infty}T^{\alpha}}{\Gamma(\alpha+1)}\\&+\Gamma(2-\gamma)L_2\Big(1+\frac{|b+d|T}{\Delta}+\frac{|ad-cb|T}{\Delta}\Big)\sum_{i=1}^m t_i^\gamma\\&+\Gamma(2-\gamma)L_2\Big(T+\frac{|d-1|T^2}{\Delta}+\frac{|ad-a-b-cb|T^2}{\Delta}\Big)\sum_{i=1}^m t_i^{\gamma-1}\\&+mL_1\Big(1+\frac{|b+d|T}{\Delta}+\frac{|ad-cb|T}{\Delta}\Big)<1.\end{aligned}\tag{23}$$

Then the impulsive fractional BVP (1), (2) has a unique solution on $[0,T]$.

Proof. For $x,y\in PC(J,\mathbb{R})$, we have

$$\begin{aligned}|(Fx)(t)-(Fy)(t)|\leq&\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,x(s))-f(s,y(s))|ds+|e_0^x-e_0^y|\\&+\sum_{i=1}^k|I_i(x(t_i^-))-I_i(y(t_i^-))|+\Gamma(2-\gamma)\sum_{i=1}^k t_i^\gamma|I_i^*(x(t_i^-))-I_i^*(y(t_i^-))|\\&+|h_0^x-h_0^y|T+\Gamma(2-\gamma)T\sum_{i=1}^k t_i^{\gamma-1}|I_i^*(x(t_i^-))-I_i^*(y(t_i^-))|\\\leq&\Big[\frac{\|h\|_{L^\infty}T^\alpha}{\Gamma(\alpha+1)}+mL_1+\Gamma(2-\gamma)L_2\sum_{i=1}^m(t_i^\gamma+Tt_i^{\gamma-1})\Big]\|x-y\|\\&+|e_0^x-e_0^y|+|h_0^x-h_0^y|T.\end{aligned}\tag{24}$$

Here e_0^x,h_0^x (or e_0^y,h_0^y) mean that e_0,h_0 defined by (15), (16) are related to x (or y). Since

$$\begin{aligned}|e_0^x-e_0^y|\leq&\Big[\frac{|b+d|\|h\|_{L^\infty}T^{\alpha+1}}{|\Delta|\Gamma(\alpha+1)}+\frac{|1+b|\|h\|_{L^\infty}T^{\alpha+1}}{|\Delta|\Gamma(\alpha)}\\&+\frac{|b+d|T}{\Delta}\big(mL_1+\Gamma(2-\gamma)L_2\sum_{i=1}^m t_i^\gamma\big)\\&+\frac{|d-1|T^2}{\Delta}\Gamma(2-\gamma)L_2\sum_{i=1}^m t_i^{\gamma-1}\Big]\|x-y\|,\end{aligned}\tag{25}$$

$$\begin{aligned}|h_0^x-h_0^y|T\leq&\Big[\frac{|ad-cb|\|h\|_{L^\infty}T^{\alpha+1}}{|\Delta|\Gamma(\alpha+1)}+\frac{|a+b|\|h\|_{L^\infty}T^{\alpha+1}}{|\Delta|\Gamma(\alpha)}\\&+\frac{|ad-cb|T}{\Delta}\big(mL_1+\Gamma(2-\gamma)L_2\sum_{i=1}^m t_i^\gamma\big)\\&+\frac{|ad-a-b-cb|T^2}{\Delta}\Gamma(2-\gamma)L_2\sum_{i=1}^m t_i^{\gamma-1}\Big]\|x-y\|,\end{aligned}\tag{26}$$

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then combining (25), (26) with (24), we obtain

$$
\begin{aligned}
|(Fx)(t)-(Fy)(t)| \le \Big[& \frac{\|h\|_{L^\infty}T^{\alpha+1}}{|\Delta|}\Big(\frac{|b+d|+|ad-cb|}{\Gamma(\alpha+1)}+\frac{|1+b|+|a+b|}{\Gamma(\alpha)}\Big) \\
& +\Gamma(2-\gamma)L_2\Big(1+\frac{|b+d|T}{\Delta}+\frac{|ad-cb|T}{\Delta}\Big)\sum_{i=1}^{m}t_i^{\gamma} \\
& +\Gamma(2-\gamma)L_2\Big(T+\frac{|d-1|T^2}{\Delta}+\frac{|ad-a-b-cb|T^2}{\Delta}\Big)\sum_{i=1}^{m}t_i^{\gamma-1} \\
& +\frac{\|h\|_{L^\infty}T^{\alpha}}{\Gamma(\alpha+1)}+mL_1\Big(1+\frac{|b+d|T}{\Delta}+\frac{|ad-cb|T}{\Delta}\Big)\Big]\|x-y\|.
\end{aligned}
$$

Which implies that

$$\|Fx-Fy\| \le \gamma\|x-y\|.$$

Now by condition (23), we know F is a contraction. Thus, the conclusion of this theorem follows from the contraction mapping principle. The proof is completed. □

In the rest of this section, we shall state some existence results for the problem (1), (3). We omit the proofs since these are similar to the ones obtained for the problem (1), (2) above.

Theorem 3.4. *Assume that: items (1) and (2) of assumptions in Theorem 3.1 hold. (3) there exists a constant $M_1 > 0$ such that*

$$\frac{M_1}{\varphi(M_1)\|h\|_{L^\infty}P_1+\psi(M_1)Q_1+\psi^*(M_1)\Gamma(2-\gamma)R_1} > 1, \tag{27}$$

where

$$P_1=\frac{T^\alpha}{\Gamma(\alpha+1)}+\frac{T^{\alpha+1}}{|\Delta_1|}\Big(\frac{|d_1-1|+|c_1|}{\Gamma(\alpha+1)}+\frac{|1-b_1|+|a_1-1|}{\Gamma(\alpha)}\Big),$$

$$Q_1=m\Big(1+\frac{|d_1-1|T}{|\Delta_1|}+\frac{|c_1|T}{|\Delta_1|}\Big)$$

and

$$
\begin{aligned}
R_1 = & \Big(1+\frac{|d_1-1|T}{|\Delta_1|}+\frac{|c_1|T}{|\Delta_1|}\Big)\sum_{i=1}^{m}t_i^{\gamma} \\
& +\Big(T+\frac{|d_1-b_1|T^2}{|\Delta_1|}+\frac{|a_1-c_1-1|T^2}{|\Delta_1|}\Big)\sum_{i=1}^{m}t_i^{\gamma-1}.
\end{aligned}
$$

Then the impulsive fractional BVP (1), (3) has at least one solution on $[0,T]$.

Theorem 3.5. *Let the assumptions of Theorem 3.2 hold, then the impulsive fractional BVP (1), (3) has at least one solution on* $[0,T]$.

Theorem 3.6. *Assume that all assumptions of Theorem 3.3 are satisfied but with the condition (23) replaced by*

$$\begin{aligned}\gamma = &\frac{\|h\|_{L^\infty}T^\alpha}{\Gamma(\alpha+1)} + \frac{\|h\|_{L^\infty}T^{\alpha+1}}{|\Delta_1|}\Big(\frac{|d_1-1|+|c_1|}{\Gamma(\alpha+1)} + \frac{|1-b_1|+|a_1-1|}{\Gamma(\alpha)}\Big)\\ &+\Gamma(2-\gamma)L_2\Big(1+\frac{|d_1-1|T}{|\Delta_1|}+\frac{|c_1|T}{|\Delta_1|}\Big)\sum_{i=1}^m t_i^\gamma\\ &+\Gamma(2-\gamma)L_2\Big(T+\frac{|d_1-b_1|T^2}{|\Delta_1|}+\frac{|a_1-c_1-1|T^2}{|\Delta_1|}\Big)\sum_{i=1}^m t_i^{\gamma-1}\\ &+mL_1\Big(1+\frac{|d_1-1|T}{|\Delta_1|}+\frac{|c_1|T}{|\Delta_1|}\Big)<1.\end{aligned}$$

Then the impulsive fractional BVP (1), (3) has a unique solution on $[0,T]$.

4 Examples

In this section we give two simple examples to show the applicability of our results.

Example 4.1. *Consider the following impulsive fractional mixed BVP*

$$\begin{cases} {}^cD^{\frac{7}{4}}x(t) = \frac{\cos t}{(t+6)^2}(x(t)+\arctan x(t)),\ t\in[0,1],\ t\neq\frac{1}{2},\\ \Delta x(\frac{1}{2}) = \frac{|x(\frac{1}{2}^-)|}{(17+|x(\frac{1}{2}^-)|)},\ \Delta({}^cD^{\frac{1}{4}}x(\frac{1}{2})) = \frac{|x(\frac{1}{2}^-)|}{(20+|x(\frac{1}{2}^-)|)},\\ x'(0) = -2x(0)-3x(1),\ x'(1) = 4x(0)+5x(1).\end{cases} \tag{28}$$

Here $\alpha=\frac{7}{4}$, $\gamma=\frac{1}{4}$, $T=1$, $m=1$ and $a=2$, $b=3$, $c=4$, $d=5$. Clearly, we can take $h(t)=\frac{2\cos t}{(t+6)^2}$, $L_1=\frac{1}{17}$ and $L_2=\frac{1}{20}$ such that (21) and (22) hold. As for the condition (23), since $\Delta=16$, $\|h\|_{L^\infty}=\frac{1}{18}$, $\Gamma(2-\gamma)=0.9191$, $\Gamma(\alpha+1)=1.6084$ and $\Gamma(\alpha)=0.9191$, we get

$$\gamma\approx 0.0956+0.1932+0.0345+0.0556=0.3789<1.$$

Thus all the assumptions of Theorem 3.3 are satisfied. Then by the conclusion of it, the impulsive fractional BVP (28) has a unique solution on $[0,1]$.

Example 4.2. *Consider the following impulsive fractional closed BVP*

$$\begin{cases} {}^cD^{\frac{3}{2}}x(t) = 5t^2+e^{-|x(t)|}+\sin x(t),\ t\in[0,1],\ t\neq\frac{1}{4},\\ \Delta x(\frac{1}{4}) = \frac{2|x(\frac{1}{4}^-)|}{(1+|x(\frac{1}{4}^-)|)},\ \Delta({}^cD^{\frac{1}{2}}x(\frac{1}{4})) = \cos x(\frac{1}{4}^-)+3,\\ x(1) = 4x(0)+x'(0),\ x'(1) = 2x(0)+3x'(0).\end{cases} \tag{29}$$

In the context of this problem, we have $\alpha=\frac{3}{2}$, $\gamma=\frac{1}{2}$, $T=1$, $m=1$ and $a_1=4$, $b_1=1$, $c_1=2$, $d_1=3$. And clearly $\Delta_1=6\neq0$,

$$|f(t,x)| = |5t^2+e^{-|x|}+\sin x|\le 7,\ t\in[0,1],\ x\in\mathbb{R},$$

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$$|I_k(x)| \leq 2,\ |I_k^*(x)| \leq 4,\ x \in \mathbb{R}.$$

Put $h(t) \equiv 7$, $H_1 = 2$ and $H_2 = 4$. Then from Theorem 3.5, the impulsive fractional BVP (29) has at least one solution on $[0, 1]$.

Acknowledgments The work is supported by the NNSFC (No. 11126170), Hunan Provincial Natural Science Foundation of China (Grant No. 11JJ4006) and Doctor Priming Fund Project of University of South China (Grant No.5-2011-XQD-008 and 2013XQD16).

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 19, NO.3, 2015, COPYRIGHT 2015 EUDOXUS PRESS, LLC

Volterra composition operators from $F(p,q,s)$ to logarithmic Bloch spaces *

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Abstract. This paper characterizes the boundedness and compactness of the Volterra composition operators from the general function space $F(p,q,s)$ to the logarithmic Bloch space B_L on the unit disk.

Keywords. Volterra composition operator; $F(p,q,s)$ space; logarithmic Bloch space

2010 Mathematics Subject Classification. 47B38, 47G10, 32A10, 32A18.

1. Introduction

Let D be the open unit disk in the complex plane $\mathbb{C}$, and $H(D)$ be the space of analytic functions on D. The space of analytic functions on D such that

$$\|f\|_L = \sup_{z\in D}(1-|z|^2)\log\frac{2}{1-|z|}|f'(z)| < \infty$$

is called logarithmic Bloch space B_L, which is a Banach space under the norm $\|f\|_{B_L} = |f(0)|+\|f\|_L$. Let B_L^0 denote the subspace of B_L consisting of those $f \in B_L$ for which $\lim\limits_{|z|\to 1}(1-|z|^2)\log\frac{2}{1-|z|}|f'(z)| = 0$. The space B_L^0 is called the little logarithmic Bloch space. Yoneda [8] studied the composition operators in the B_L space and the B_L^0 space.

Let $\alpha > 0$. The $\alpha-$Bloch space B_α is the space of all analytic functions on D such that $\sup\limits_{z\in D}(1-|z|^2)^\alpha|f'(z)| < \infty$. The little $\alpha-$Bloch space B_α^0 consists of all $f \in H(D)$ such that $\lim\limits_{|z|\to 1}(1-|z|^2)^\alpha|f'(z)| = 0$. It is clear that B_α is a Banach space with the norm $\|f\|_{B_\alpha} = |f(0)| + \sup\limits_{z\in D}(1-|z|^2)^\alpha|f'(z)|$, and for $0<\alpha<1$, $B_\alpha \subsetneq B_L \subsetneq B_1$. For more information about the B_α, see [14]. In [5] and [6] Ye studied the weighted composition operator and the extended Cesàro operator between B_α and B_L. And in [9] Yu studied the Li-Stević integral-type operators from B_L to B_α.

For $0 < p, s < \infty$ and $-2 < q < \infty$, f is said to belong to the general function space $F(p,q,s)$ if $f \in H(D)$ and

$$\|f\|_{F(p,q,s)} = |f(0)| + \{\sup_{a\in D}\int_D |f'(z)|^p(1-|z|^2)^q(1-|\varphi_a(z)|^2)^s dm(z)\}^{1/p} < \infty.$$

*The authors are supported by the Natural Science Foundation of China (N0. 11171285) and the Priority Academic Program Development of Jiangsu Higher Education Institutions.

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The little general function space $F_0(p,q,s)$ is the space of analytic functions such that $f \in F(p,q,s)$ and

$$\lim_{|a|\to 1}\int_D |f'(z)|^p(1-|z|^2)^q(1-|\varphi_a(z)|^2)^s dm(z) = 0.$$

Here dm is the Lebesgue area measure in D, and for $a \in D$, $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ is the Möbius transformation of D. We can get many function spaces if we take some specific parameters of p, q and s. For example, $F(p,q,s) = B_{\frac{q+2}{p}}$ and $F_0(p,q,s) = B^0_{\frac{q+2}{p}}$ for $s > 1$; $F(p,q,s) \subset B_{\frac{q+2}{p}}$ and $F_0(p,q,s) \subset B^0_{\frac{q+2}{p}}$ for $0 < s \le 1$. It also includes Q_s space, BMOA space, Bergman space and Besove space. In [12], Zhang studied the weighted composition operator from $F(p,q,s)$ to B_α. And Ye [7] investigated the weighted composition operator from $F(p,q,s)$ to B_L.

If $\varphi : D \to D$ is analytic, the composition operator induced by φ is

$$C_\varphi f = f \circ \varphi, \quad f \in H(D).$$

The composition operator has been studied by many researchers on various spaces. Let $g \in H(D)$, the Volterra type operator J_g is defined by the following:

$$J_g f(z) = \int_0^z f(\xi)g'(\xi)d\xi, \quad z \in D.$$

The companion operator I_g is defined as

$$I_g f(z) = \int_0^z f'(\xi)g(\xi)d\xi, \quad z \in D.$$

If $g(z) = z$ then J_g is the integration operator and if $g(z) = \log\frac{1}{1-z}$ then J_g is the Cesàro operator. Moreover $J_g f + I_g f = M_g f - f(0)g(0)$, where M_g is the multiplication operator $M_g f(z) = g(z)f(z)$, $f \in H(D)$.

Here, we consider the Volterra composition operators which defined as

$$J_{g,\varphi} f(z) = \int_0^z (f\circ\ \varphi)(\xi)(g\circ\varphi)'(\xi)d\xi$$

and

$$I_{g,\varphi} f(z) = \int_0^z (f\circ\ \varphi)'(\xi)(g\circ\varphi)(\xi)d\xi.$$

When $\varphi(z) = z$, then $J_{g,\varphi} = J_g, I_{g,\varphi} = I_g$. When $g = 1$, then $I_{g,\varphi} = C_\varphi$. Therefore we can regard the operators $J_{g,\varphi}$ and $I_{g,\varphi}$ as the generalization of C_φ, J_g and I_g. In addition, these operators are closely related with the product of composition operator and Volterra type operator. If we replace $g\circ\varphi$ by g we obtain the products $J_g C_\varphi$ and $I_g C_\varphi$. Here

$$J_g C_\varphi f(z) = \int_0^z (f\circ\ \varphi)(\xi)g'(\xi)d\xi$$

and

$$I_g C_\varphi f(z) = \int_0^z (f\circ\ \varphi)'(\xi)g(\xi)d\xi.$$

Therefore we can obtain the characterizations of boundedness and compactness of the operators $J_g C_\varphi$ and $I_g C_\varphi$ by modifying all results started for $J_{g,\varphi}$ and $I_{g,\varphi}$ respectively.

In [2], Li and Stević have characterized the boundedness and compactness of the operators $J_{g,\varphi}$ and $I_{g,\varphi}$ from $H^\infty(B)$ to Zygmund spaces and these oprators between Bloch-type spaces. The boundedness and compactness of the Volterra composition operators between weighted Bergman space and Bloch type spaces were investigated in [3].

In this paper, our main purpose is to characterize the conditions for $J_{g,\varphi}$ and $I_{g,\varphi}$ to be bounded operators or compact operators from $F(p,q,s)$ to B_L. Throughout this paper, C always denote positive constants and may be different at different occurrences.

2. Some lemmas

Lemma 2.1.[13] Suppose that $0<p,s<\infty$, $-2<q<\infty$, $q+s>-1$ and $f\in F(p,q,s)$. Then there is a positive constant C such that $\|f\|_{B_\alpha}\le C\|f\|_{F(p,q,s)}$, moreover, if $f\in F_0(p,q,s)$, then $f\in B_\alpha^0$, where $\alpha=\frac{q+2}{p}$.

Lemma 2.2.[4] Suppose $t>0$ and $f\in D$. Then $\sup\limits_{z\in D}(1-|z|)^t|f(z)|<\infty$ if and only if $\sup\limits_{z\in D}(1-|z|)^{t+1}|f'(z)|<\infty$.

The next lemma can be proved in a standard way (see, for example, Proposition 3.11 in [1]).

Lemma 2.3. Let $0<p,s<\infty$, $-2<q<\infty$, $q+s>-1$. The operator $J_{g,\varphi}$ $(I_{g,\varphi}):F(p,q,s)\to B_L$ is compact if and only if for any bounded sequence $\{f_n\}$ in $F(p,q,s)$ which converges to zero uniformly on compact subsets of D, we have $\|J_{g,\varphi}f_n\|_{B_L}\to 0$ $(\|I_{g,\varphi}f_n\|_{B_L}\to 0)$ as $n\to\infty$.

Lemma 2.4.[4] Let $\alpha>0$, and $f\in B_\alpha$. Then

(1) $|f(z)|\le C\|f\|_{B_\alpha}$, where $\alpha<1$;

(2) $|f(z)|\le C\log(\frac{2}{1-|z|^2})\|f\|_{B_\alpha}$, where $\alpha=1$;

(3) $|f(z)|\le \frac{C}{(\alpha-1)(1-|z|)^{\alpha-1}}\|f\|_{B_\alpha}$, where $\alpha>1$.

3. Main results and proofs

Theorem 3.1. Let $0<p,s<\infty$, $-2<q<\infty$, $q+s>-1$, $g\in H(D)$ and φ be an analytic self-map of D.

(1) If $2+q>p$, then $J_{g,\varphi}$ is a bounded operator from $F(p,q,s)$ to B_L if and only if

$$\sup_{z\in D}\frac{1-|z|^2}{(1-|\varphi(z)|^2)^{\alpha-1}}\log(\frac{2}{1-|z|})|g'(\varphi(z))||\varphi'(z)|<\infty,$$

where $\alpha=\frac{2+q}{p}$.

(2) If $2+q<p$, then $J_{g,\varphi}$ is a bounded operator from $F(p,q,s)$ to B_L if and only if

$$\sup_{z\in D}(1-|z|^2)\log(\frac{2}{1-|z|})|g'(\varphi(z))||\varphi'(z)|<\infty.$$

(3) If $2+q=p$, $s>1$, then $J_{g,\varphi}$ is a bounded operator from $F(p,q,s)$ to B_L if and only if

$$\sup_{z\in D}(1-|z|^2)\log(\frac{2}{1-|z|})\log(\frac{2}{1-|\varphi(z)|^2})|g'(\varphi(z))||\varphi'(z)|<\infty.$$

Proof. (1) Let $\alpha=\frac{2+q}{p}>1$.

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$\Leftarrow$ Assume that $f \in F(p,q,s)$ and g,φ satisfy the condition in (1). Then we have $\sup_{z\in D}(1-|z|^2)^{\alpha-1}|f(z)| \leq C\|f\|_{F(p,q,s)}$ by Lemma 2.1 and 2.2. It follows that

$$\begin{aligned}
&\sup_{z\in D}(1-|z|^2)\log(\frac{2}{1-|z|})|(J_{g,\varphi}f)'(z)| \\
=\ &\sup_{z\in D}(1-|z|^2)\log(\frac{2}{1-|z|}|f(\varphi(z))|)|g'(\varphi(z))||\varphi'(z)| \\
\leq\ &\sup_{z\in D}(1-|\varphi(z)|^2)^{\alpha-1}|f(\varphi(z))|\sup_{z\in D}\frac{1-|z|^2}{(1-|\varphi(z)|^2)^{\alpha-1}}\log(\frac{2}{1-|z|})|g'(\varphi(z))||\varphi'(z)| \\
\leq\ &C\|f\|_{F(p,q,s)}.
\end{aligned}$$

In addition, $(J_{g,\varphi}f)(0) = 0$. Hence $J_{g,\varphi}$ is bounded from $F(p,q,s)$ to B_L.

$\Rightarrow$ Suppose that $J_{g,\varphi}$ is bounded from $F(p,q,s)$ to B_L. For $w \in D$, set $f_w(z) = \frac{1-|\varphi(w)|^2}{(1-z\overline{\varphi(w)})^\alpha}$, then $\|f_w\|_{F(p,q,s)} \leq C$ by [12]. Thus

$$\begin{aligned}
&(1-|w|^2)\log(\frac{2}{1-|w|})|(J_{g,\varphi}f_w)'(w)| \\
=\ &(1-|w|^2)\log(\frac{2}{1-|w|})|f_w(\varphi(w))||g'(\varphi(w))||\varphi'(w)| \\
=\ &(1-|w|^2)\log(\frac{2}{1-|w|})\frac{1}{(1-|\varphi(w)|^2)^{\alpha-1}}|g'(\varphi(w))||\varphi'(w)| \\
\leq\ &\|J_{g,\varphi}f_w\|_{B_L} \\
\leq\ &\|J_{g,\varphi}\|\cdot\|f_w\|_{F(p,q,s)} \\
\leq\ &C,
\end{aligned}$$

which showing that $\sup_{z\in D}\frac{1-|z|^2}{(1-|\varphi(z)|^2)^{\alpha-1}}\log(\frac{2}{1-|z|})|g'(\varphi(z))||\varphi'(z)| < \infty$.

(2) Let $\alpha = \frac{2+q}{p} < 1$.

$\Leftarrow$ Assume that $f \in F(p,q,s)$ and g,φ satisfy the condition in (2). Then we have $|f(z)| \leq C\|f\|_{F(p,q,s)}$ by Lemma 2.1 and 2.4. It follows that

$$\begin{aligned}
&\sup_{z\in D}(1-|z|^2)\log(\frac{2}{1-|z|})|(J_{g,\varphi}f)'(z)| \\
=\ &\sup_{z\in D}(1-|z|^2)\log(\frac{2}{1-|z|})|f(\varphi(z))||g'(\varphi(z))||\varphi'(z)| \\
\leq\ &\sup_{z\in D}|f(z)|\sup_{z\in D}(1-|z|^2)\log(\frac{2}{1-|z|})|g'(\varphi(z))||\varphi'(z)| \\
\leq\ &C\|f\|_{F(p,q,s)}.
\end{aligned}$$

In addition, $(J_{g,\varphi}f)(0) = 0$. Hence $J_{g,\varphi}$ is bounded from $F(p,q,s)$ to B_L.

$\Rightarrow$ Suppose that $J_{g,\varphi}$ is bounded from $F(p,q,s)$ to B_L. We can obtain

$$\sup_{z\in D}(1-|z|^2)\log(\frac{2}{1-|z|})|g'(\varphi(z))||\varphi'(z)| < \infty$$

by taking $f(z) = 1$ respectively.

(3) Let $\alpha = \frac{2+q}{p} = 1$ and $s > 1$.

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$\Leftarrow$ Assume that $f \in F(p,q,s)$ and g, φ satisfy the condition in (3). Then $f \in F(p,q,s) = B_1$ and $|f(z)| \le C \log(\frac{2}{1-|z|^2})\|f\|_{F(p,q,s)}$ by Lemma 2.4. It follows that

$$\begin{aligned}
&\sup_{z\in D}(1-|z|^2)\log(\frac{2}{1-|z|})|(J_{g,\varphi}f)'(z)| \\
= &\sup_{z\in D}(1-|z|^2)\log(\frac{2}{1-|z|})|f(\varphi(z))||g'(\varphi(z))||\varphi'(z)| \\
\le &C\|f\|_{F(p,q,s)}\sup_{z\in D}(1-|z|^2)\log(\frac{2}{1-|z|})\log(\frac{2}{1-|\varphi(z)|^2})|g'(\varphi(z))||\varphi'(z)| \\
< &\infty.
\end{aligned}$$

In addition, $(J_{g,\varphi}f)(0) = 0$. Hence $J_{g,\varphi}$ is bounded from $F(p,q,s)$ to B_L.

$\Rightarrow$ Suppose that $J_{g,\varphi}$ is bounded from $F(p,q,s)$ to B_L. For $w \in D$, set $f_w(z) = \log\frac{2}{1-\overline{\varphi(w)}z}$, then $\|f_w\|_{F(p,q,s)} \le C$ by [12]. Thus

$$\begin{aligned}
&(1-|w|^2)\log(\frac{2}{1-|w|})|(J_{g,\varphi}f_w)'(w)| \\
= &(1-|w|^2)\log(\frac{2}{1-|w|})|f_w(\varphi(w))||g'(\varphi(w))||\varphi'(w)| \\
= &(1-|w|^2)\log(\frac{2}{1-|w|})\log(\frac{2}{1-|\varphi(w)|^2})|g'(\varphi(w))||\varphi'(w)| \\
\le &\|J_{g,\varphi}f_w\|_{B_L} \\
\le &\|J_{g,\varphi}\|\cdot\|f_w\|_{F(p,q,s)} \\
\le &C,
\end{aligned}$$

which showing that $\sup_{z\in D}(1-|z|^2)\log(\frac{2}{1-|z|})\log(\frac{2}{1-|\varphi(z)|^2})|g'(\varphi(z))||\varphi'(z)| < \infty$.

Theorem 3.2. Let $0 < p, s < \infty$, $-2 < q < \infty$, $q + s > -1$, $g \in H(D)$ and φ be an analytic self-map of D.

(1) If $2 + q > p$, then $J_{g,\varphi}$ is a compact operator from $F(p,q,s)$ to B_L if and only if

$$\sup_{z\in D}(1-|z|^2)\log(\frac{2}{1-|z|})|g'(\varphi(z))||\varphi'(z)| < \infty \qquad (3.1)$$

and

$$\lim_{|\varphi(z)|\to 1}\frac{1-|z|^2}{(1-|\varphi(z)|^2)^{\alpha-1}}\log(\frac{2}{1-|z|})|g'(\varphi(z))||\varphi'(z)| = 0, \qquad (3.2)$$

where $\alpha = \frac{2+q}{p}$.

(2) If $2 + q < p$, then $J_{g,\varphi}$ is a compact operator from $F(p,q,s)$ to B_L if and only if (3.1) holds.

(3) If $2 + q = p$, $s > 1$, then $J_{g,\varphi}$ is a compact operator from $F(p,q,s)$ to B_L if and only if (3.1) holds and

$$\lim_{|\varphi(z)|\to 1}(1-|z|^2)\log(\frac{2}{1-|z|})\log(\frac{2}{1-|\varphi(z)|^2})|g'(\varphi(z))||\varphi'(z)| = 0. \qquad (3.3)$$

Proof. (1) Let $\alpha = \frac{2+q}{p} > 1$.

$\Leftarrow$ For any sequence $\{f_n\}$ in $F(p,q,s)$ such that $\|f_n\|_{F(p,q,s)} \leq 1$ and $f_n \to 0$ uniformly on compact subsets of D, it is required to show that by Lemma 2.3, $\|J_{g,\varphi}f_n\|_{B_L} \to 0$ as $n \to \infty$. From (3.2), we have that for every $\varepsilon > 0$, there exists a $r \in (0,1)$, such that $r < |\varphi(z)| < 1$ implies

$$\frac{1-|z|^2}{(1-|\varphi(z)|^2)^{\alpha-1}}\log(\frac{2}{1-|z|})|g'(\varphi(z))||\varphi'(z)| < \varepsilon.$$

Since $f_n \to 0$ uniformly on compact subsets of D, then there exists an $N > 0$, such that for all $n \geq N$, $\sup\limits_{|w|\leq r}|f_n(w)| < \varepsilon$. Using Lemma 2.1, 2.2 and (3.1), for all $n \geq N$,

$$\begin{aligned}
& \sup_{z\in D}(1-|z|^2)\log(\frac{2}{1-|z|})|(J_{g,\varphi}f_n)'(z)| \\
\leq \;& \sup_{|\varphi(z)|\leq r}(1-|z|^2)\log(\frac{2}{1-|z|})|f_n(\varphi(z))||g'(\varphi(z))||\varphi'(z)| \\
& + \sup_{|\varphi(z)|> r}(1-|z|^2)\log(\frac{2}{1-|z|})|f_n(\varphi(z))||g'(\varphi(z))||\varphi'(z)| \\
\leq \;& \sup_{|\varphi(z)|> r}(1-|\varphi(z)|^2)^{\alpha-1}|f_n(\varphi(z))| \sup_{|\varphi(z)|> r}\frac{1-|z|^2}{(1-|\varphi(z)|^2)^{\alpha-1}}\log(\frac{2}{1-|z|})|g'(\varphi(z))||\varphi'(z)| \\
& + \sup_{|\varphi(z)|\leq r}|f_n(\varphi(z))| \sup_{|\varphi(z)|\leq r}(1-|z|^2)\log(\frac{2}{1-|z|})|g'(\varphi(z))||\varphi'(z)| \\
\leq \;& C\varepsilon.
\end{aligned}$$

Note that $(J_{g,\varphi}f_n)(0) = 0$, we obtain $\|J_{g,\varphi}f_n\|_{B_L} \to 0$ as $n \to \infty$.

$\Rightarrow$ Suppose that $J_{g,\varphi}$ is compact from $F(p,q,s)$ to B_L, then $J_{g,\varphi} : F(p,q,s) \to B_L$ is bounded. Taking $f(z) = 1$, we see that $\sup\limits_{z\in D}(1-|z|^2)\log(\frac{2}{1-|z|})|g'(\varphi(z))||\varphi'(z)| < \infty$. Let $\{z_n\}$ be a sequence in D such that $\lim\limits_{n\to\infty}|\varphi(z_n)| = 1$. We choose the test function f_n defined by

$$f_n(z) = \frac{1-|\varphi(z_n)|^2}{(1-\overline{\varphi(z_n)}z)^{\alpha}},$$

then $\|f_n\|_{F(p,q,s)} \leq C$ and $f_n \to 0$ uniformly on compact subsets of D by computation. In view of Lemma 2.3, it follows that $\|J_{g,\varphi}f_n\|_{B_L} \to 0$ as $n \to \infty$. Since

$$\begin{aligned}
\|J_{g,\varphi}f_n\|_{B_L} &\geq (1-|z_n|^2)\log(\frac{2}{1-|z_n|})|(J_{g,\varphi}f_n)'(z_n)| \\
&= (1-|z_n|^2)\log(\frac{2}{1-|z_n|})|f_n(\varphi(z_n))||g'(\varphi(z_n))||\varphi'(z_n)| \\
&= \frac{1-|z_n|^2}{(1-|\varphi(z_n)|^2)^{\alpha-1}}\log(\frac{2}{1-|z_n|})|g'(\varphi(z_n))||\varphi'(z_n)|,
\end{aligned}$$

and $\lim\limits_{n\to\infty}|\varphi(z_n)| = 1$, we obtain

$$\lim_{n\to\infty}\frac{1-|z_n|^2}{(1-|\varphi(z_n)|^2)^{\alpha-1}}\log(\frac{2}{1-|z_n|})|g'(\varphi(z_n))||\varphi'(z_n)| = 0.$$

Hence

$$\lim_{|\varphi(z)|\to 1}\frac{1-|z|^2}{(1-|\varphi(z)|^2)^{\alpha-1}}\log(\frac{2}{1-|z|})|g'(\varphi(z))||\varphi'(z)| = 0.$$

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(2) Let $\alpha = \frac{2+q}{p} < 1$.

$\Leftarrow$ For any sequence $\{f_n\}$ in $F(p,q,s)$ such that $\|f_n\|_{F(p,q,s)} \le 1$ and $f_n \to 0$ uniformly on compact subsets of D, it is required to show that by Lemma 2.3, $\|J_{g,\varphi} f_n\|_{B_L} \to 0$ as $n \to \infty$. Using Lemma 2.1 and [11, Lemma 3.2] we have $\sup_{z\in D} |f_n(z)| \to 0$ as $n \to \infty$. Then by (3.1)

$$\begin{aligned}
& \sup_{z\in D}(1-|z|^2)\log(\frac{2}{1-|z|})|(J_{g,\varphi}f_n)'(z)| \\
= \; & \sup_{z\in D}(1-|z|^2)\log(\frac{2}{1-|z|})|f_n(\varphi(z))||g'(\varphi(z))||\varphi'(z)| \\
\le \; & \sup_{z\in D}|f_n(z)| \sup_{z\in D}(1-|z|^2)\log(\frac{2}{1-|z|})|g'(\varphi(z))||\varphi'(z)| \\
\to \; & 0,
\end{aligned}$$

as $n \to \infty$. Note that $(J_{g,\varphi}f_n)(0) = 0$, we get $\|J_{g,\varphi}f_n\|_{B_L} \to 0$ as $n \to \infty$.

$\Rightarrow$ It has been proved in case $\alpha = \frac{2+q}{p} > 1$.

(3) Let $\alpha = \frac{2+q}{p} = 1$ and $s > 1$.

$\Leftarrow$ For any sequence $\{f_n\}$ in $F(p,q,s)$ such that $\|f_n\|_{F(p,q,s)} \le 1$ and $f_n \to 0$ uniformly on compact subsets of D, it is required to show that by Lemma 2.3, $\|J_{g,\varphi} f_n\|_{B_L} \to 0$ as $n \to \infty$. By Lemma 2.1 and 2.4, we get $|f_n(z)| \le C\log(\frac{2}{1-|z|^2})$. From (3.3), we have that for every $\varepsilon > 0$, there exists a $r \in (0,1)$, such that $r < |\varphi(z)| < 1$ implies

$$(1-|z|^2)\log(\frac{2}{1-|z|})\log(\frac{2}{1-|\varphi(z)|^2})|g'(\varphi(z))||\varphi'(z)| < \varepsilon.$$

Additionally, for above $\varepsilon > 0$, there exists an $N > 0$, such that for all $n \ge N$, $\sup_{|w|\le r}|f_n(w)| < \varepsilon$. Therefore, by (3.1), for all $n \ge N$,

$$\begin{aligned}
& \sup_{z\in D}(1-|z|^2)\log(\frac{2}{1-|z|})|(J_{g,\varphi}f_n)'(z)| \\
\le \; & \sup_{|\varphi(z)|\le r}(1-|z|^2)\log(\frac{2}{1-|z|})|f_n(\varphi(z))||g'(\varphi(z))||\varphi'(z)| \\
& + \sup_{|\varphi(z)|> r}(1-|z|^2)\log(\frac{2}{1-|z|})|f_n(\varphi(z))||g'(\varphi(z))||\varphi'(z)| \\
\le \; & \varepsilon \sup_{|\varphi(z)|\le r}(1-|z|^2)\log(\frac{2}{1-|z|})|g'(\varphi(z))||\varphi'(z)| \\
& + C\sup_{|\varphi(z)|> r}(1-|z|^2)\log(\frac{2}{1-|z|})\log(\frac{2}{1-|\varphi(z)|^2})|g'(\varphi(z))||\varphi'(z)| \\
\le \; & C\varepsilon.
\end{aligned}$$

Note that $(J_{g,\varphi}f_n)(0) = 0$, we get $\|J_{g,\varphi}f_n\|_{B_L} \to 0$ as $n \to \infty$.

$\Rightarrow$ Suppose that $J_{g,\varphi}$ is compact from $F(p,q,s)$ to B_L. The condition (3.1) has been proved in case $\alpha = \frac{2+q}{p} > 1$. Let $a_n = \log\frac{2}{1-|\varphi(z_n)|^2}$, we take $f_n(z) = \frac{3}{a_n}(\log\frac{2}{1-\overline{\varphi(z_n)}z})^2 - \frac{2}{a_n^2}(\log\frac{2}{1-\overline{\varphi(z_n)}z})^3$ such that $\lim_{n\to\infty}|\varphi(z_n)| = 1$. Then $\sup_n \|f_n\|_{F(p,q,s)} = \sup_n \|f_n\|_{B_1} \le C$ and $f_n \to 0$ uniformly on

compact subsets of D by computation. In view of Lemma 2.3, it follows that $\|J_{g,\varphi}f_n\|_{B_L} \to 0$ as $n \to \infty$. Thus

$$\begin{aligned}
& (1-|z_n|^2)\log(\frac{2}{1-|z_n|})|(J_{g,\varphi}f_n)'(z_n)| \\
= & (1-|z_n|^2)\log(\frac{2}{1-|z_n|})|f_n(\varphi(z_n))||g'(\varphi(z_n))||\varphi'(z_n)| \\
= & (1-|z_n|^2)\log(\frac{2}{1-|z_n|})\log(\frac{2}{1-|\varphi(z_n)|^2})|g'(\varphi(z_n))||\varphi'(z_n)| \\
\to & \ 0,
\end{aligned}$$

as $n \to \infty$, which implies

$$\lim_{|\varphi(z)|\to 1}(1-|z|^2)\log(\frac{2}{1-|z|})\log(\frac{2}{1-|\varphi(z)|^2})|g'(\varphi(z))||\varphi'(z)| = 0.$$

Theorem 3.3. Let $0 < p, s < \infty$, $\ -2 < q < \infty$, $\ q + s > -1$, $\ g \in H(D)$ and φ be an analytic self-map of D. Then $I_{g,\varphi}$ is a bounded operator from $F(p,q,s)$ to B_L if and only if

$$\sup_{z\in D}\frac{1-|z|^2}{(1-|\varphi(z)|^2)^\alpha}\log(\frac{2}{1-|z|})|g(\varphi(z))||\varphi'(z)| < \infty, \qquad (3.4)$$

where $\alpha = \frac{2+q}{p}$.

Proof. Let $\alpha = \frac{2+q}{p}$.

$\Leftarrow$ Assume that $f \in F(p,q,s)$ and g, φ satisfy the condition in (3.4). By Lemma 2.1, we have

$$\begin{aligned}
& \sup_{z\in D}(1-|z|^2)\log(\frac{2}{1-|z|})|(I_{g,\varphi}f)'(z)| \\
= & \sup_{z\in D}(1-|z|^2)\log(\frac{2}{1-|z|})|f'(\varphi(z))||g(\varphi(z))||\varphi'(z)| \\
\le & \sup_{z\in D}(1-|\varphi(z)|^2)^\alpha|f'(\varphi(z))|\sup_{z\in D}\frac{1-|z|^2}{(1-|\varphi(z)|^2)^\alpha}\log(\frac{2}{1-|z|})|g(\varphi(z))||\varphi'(z)| \\
\le & C\|f\|_{F(p,q,s)}.
\end{aligned}$$

In addition, $(I_{g,\varphi}f)(0) = 0$. Hence $I_{g,\varphi}$ is bounded from $F(p,q,s)$ to B_L.

$\Rightarrow$ Suppose that $I_{g,\varphi}$ is bounded from $F(p,q,s)$ to B_L. For $w \in D$, set $f_w(z) = \frac{(z-\varphi(w))(1-|\varphi(w)|^2)}{(1-z\overline{\varphi(w)})^{\alpha+1}}$, then $\|f_w\|_{F(p,q,s)} \le C$ by [12]. Thus

$$\begin{aligned}
& (1-|w|^2)\log(\frac{2}{1-|w|})|(I_{g,\varphi}f_w)'(w)| \\
= & (1-|w|^2)\log(\frac{2}{1-|w|})|f_w'(\varphi(w))||g(\varphi(w))||\varphi'(w)| \\
= & (1-|w|^2)\log(\frac{2}{1-|w|})\frac{1}{(1-|\varphi(w)|^2)^\alpha}|g(\varphi(w))||\varphi'(w)| \\
\le & \|I_{g,\varphi}f_w\|_{B_L} \\
\le & \|I_{g,\varphi}\|\cdot\|f_w\|_{F(p,q,s)} \\
\le & C,
\end{aligned}$$

which showing that $\sup_{z\in D} \frac{1-|z|^2}{(1-|\varphi(z)|^2)^\alpha} \log \frac{2}{1-|z|} |g'(\varphi(z))||\varphi'(z)| < \infty$.

Theorem 3.4. Let $0 < p, s < \infty$, $-2 < q < \infty$, $q + s > -1$, $g \in H(D)$ and φ be an analytic self-map of D. Then $I_{g,\varphi}$ is a compact operator from $F(p, q, s)$ to B_L if and only if

$$\sup_{z\in D}(1-|z|^2)\log(\frac{2}{1-|z|})|g'(\varphi(z))||\varphi'(z)| < \infty \qquad (3.5)$$

and

$$\lim_{|\varphi(z)|\to 1} \frac{1-|z|^2}{(1-|\varphi(z)|^2)^\alpha} \log(\frac{2}{1-|z|})|g(\varphi(z))||\varphi'(z)| = 0, \qquad (3.6)$$

where $\alpha = \frac{2+q}{p}$.

Proof. $\Leftarrow$ For any sequence $\{f_n\}$ in $F(p, q, s)$ such that $\|f_n\|_{F(p,q,s)} \le 1$ and $f_n \to 0$ uniformly on compact subset of D, it is required to show that by Lemma 2.3, $\|I_{g,\varphi} f_n\|_{B_L} \to 0$ as $n \to \infty$. From (3.6), we have that for every $\varepsilon > 0$, there exists a $r \in (0, 1)$, such that $r < |\varphi(z)| < 1$ implies

$$\frac{1-|z|^2}{(1-|\varphi(z)|^2)^\alpha} \log(\frac{2}{1-|z|})|g(\varphi(z))||\varphi'(z)| < \varepsilon.$$

Additionally, for above $\varepsilon > 0$, there exists an $N > 0$, such that for all $n \ge N$, $\sup_{|w|\le r} |f_n'(w)| < \varepsilon$. Therefore, by (3.5), for all $n \ge N$,

$$\begin{aligned}
& \sup_{z\in D}(1-|z|^2)\log(\frac{2}{1-|z|})|(I_{g,\varphi} f_n)'(z)| \\
\le\ & \sup_{|\varphi(z)|\le r}(1-|z|^2)\log(\frac{2}{1-|z|})|f_n'(\varphi(z))||g(\varphi(z))||\varphi'(z)| \\
& + \sup_{|\varphi(z)|>r}(1-|z|^2)\log(\frac{2}{1-|z|})|f_n'(\varphi(z))||g(\varphi(z))||\varphi'(z)| \\
\le\ & \varepsilon \sup_{|\varphi(z)|\le r}(1-|z|^2)\log(\frac{2}{1-|z|})|g(\varphi(z))||\varphi'(z)| \\
& + \sup_n \|f_n\|_{F(p,q,s)} \sup_{|\varphi(z)|>r} \frac{1-|z|^2}{(1-|\varphi(z)|^2)^\alpha}\log(\frac{2}{1-|z|})|g(\varphi(z))||\varphi'(z)| \\
\le\ & C\varepsilon.
\end{aligned}$$

Note that $(I_{g,\varphi} f_n)(0) = 0$, we get $\|I_{g,\varphi} f_n\|_{B_L} \to 0$ as $n \to \infty$.

$\Rightarrow$ Suppose that $I_{g,\varphi}$ is compact from $F(p, q, s)$ to B_L, then $I_{g,\varphi} : F(p, q, s) \to B_L$ is bounded. Taking $f(z) = 1$, we see that $\sup_{z\in D}(1-|z|^2)\log(\frac{2}{1-|z|})|g(\varphi(z))||\varphi'(z)| < \infty$. Let $\{z_n\}$ be a sequence in D such that $\lim_{n\to\infty} |\varphi(z_n)| = 1$. We choose the test function f_n defined by

$$f_n(z) = \frac{(z-\varphi(z_n))(1-|\varphi(z_n)|^2)}{(1-\overline{\varphi(z_n)}z)^{\alpha+1}},$$

then $\|f_n\|_{F(p,q,s)} \le C$ and $f_n \to 0$ uniformly on compact subset of D as $n \to \infty$. In view of Lemma 2.3, it follows that $\|I_{g,\varphi} f_n\|_{B_L} \to 0$ as $n \to \infty$. Since

$$\|I_{g,\varphi} f_n\|_{B_L} \ \ge\ (1-|z_n|^2)\log(\frac{2}{1-|z_n|})|(I_{g,\varphi} f_n)'(z_n)|$$

$$= (1-|z_n|^2)\log(\frac{2}{1-|z_n|})|f_n'(\varphi(z_n))||g(\varphi(z_n))||\varphi'(z_n)|$$
$$= \frac{1-|z_n|^2}{(1-|\varphi(z_n)|^2)^\alpha}\log(\frac{2}{1-|z_n|})|g(\varphi(z_n))||\varphi'(z_n)|,$$

and $\lim_{n\to\infty}|\varphi(z_n)|=1$, we obtain

$$\lim_{n\to\infty}\frac{1-|z_n|^2}{(1-|\varphi(z_n)|^2)^\alpha}\log(\frac{2}{1-|z_n|})|g(\varphi(z_n))||\varphi'(z_n)|=0.$$

Hence

$$\lim_{|\varphi(z)|\to 1}\frac{1-|z|^2}{(1-|\varphi(z)|^2)^\alpha}\log(\frac{2}{1-|z|})|g(\varphi(z))||\varphi'(z)|=0.$$

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THE (p,q)-ABSOLUTELY SUMMING OPERATORS ON THE KÖTHE BOCHNER FUNCTION SPACES.

MONA KHANDAQJI, AHMED AL-RAWASHDEH

ABSTRACT. This paper is devoted to study some properties of the class of positive (p,q)-absolutely summing operators from a Banach lattice into a Banach space. We shall obtain a new characterization of such operators and generalize some inequalities on the Köthe Bochner function spaces .

1. Introduction

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If Y is a normed space and $y_1, y_2, ..., y_n \in Y$, $n \geqslant 1$, let Y^*represents the dual of Y, then we write $\alpha_p\left(y_i;(1\leq i\leq n)\right)=\left(\sum_{i=1}^{n}\|y_i\|^p\right)^{\frac{1}{p}}$, and $\varepsilon_p\left(y_i;(1\leq i\leq n)\right)=\sup_{\|y^*\|\leq 1}\left(\sum_{i=1}^{n}|y^*(y_i)|^p\right)^{\frac{1}{p}}$, $y^*\in Y^*$. Let $L(X,Y)$ denote the class of all linear, bounded operators from a Banach space X into a Banach space Y. Throughout this paper , we will take X is a real Banach lattice with $X^+ = \{x \in X : x \geqslant 0\}$. An operator $T \in L(X,Y)$ is called a positive (p,q)-absolutely summing operator, $1 \leq p \leq q \leq \infty$, if there is a constant $C > 0$ such that for every finite set of elements $\{x_i\}_{1\leq i\leq n}$ in X^+ the inequality

$$\alpha_p\left(T(x_i):(1\leq i\leq n)\right)\leq C\varepsilon_q\left(x_i:(1\leq i\leq n)\right), \tag{1.1}$$

holds.

The positive (p,q)-absolutely summing norm of T is $\|T\|^+_{(p,q)} = \inf C$. The space of positive (p,q)-absolutely summing operators will be denoted by $\Pi^+_{(p,q)}(X,Y)$, $1 \leq p,q \leq \infty$. This space becomes a Banach space with the norm $\|\cdot\|^+_{(p,q)}$ given by the infimum of the constants verifying (1.1). Observe that

$$\|T\|^+_{(p,q)} = \sup\left\{\begin{array}{c}\alpha_p\left(T(x_i):(1\leq i\leq n)\right):\{x_i\}_{1\leq i\ \leq n}\ in\ X^+,\\ with\quad \varepsilon_q\left(x_i:(1\leq i\leq n)\right)\leq 1\end{array}\right\} \tag{1.2}$$

and $\alpha_p\left(T(x_i):(1\leq i\leq n)\right) \leq \|T\|^+_{(p,q)}\varepsilon_q\left(x_i:(1\leq i\leq n)\right)$. As usual, if $p=\infty$ (resp. $q=\infty$), the left (resp. right) hand side of (1.1) is replaced by $\sup_i\|T(x_i)\|$ (resp. $\sup_{\|y^*\|\leq 1}\sup_i|y^*(y_i)|$). The theory of $(p,q)-$absolutely summing operators is a unified theory of various important classes of operators in connection with the classes of nuclear and Hilbert-Schmidt operators. If $p=q=1$, then the class is called positive absolutely summing operator and denoted by $\Pi^+(X,Y)$ and the norm defined on it by $\|\cdot\|^+$.

2000 *Mathematics Subject Classification.* 47B10, 46E40.

Key words and phrases. Absolutely Summing operators;Köthe Bochner function spaces.

1

In the sequel we use the following useful relation which is a simple implication of duality $(\ell^p)^* = \ell^q$:

$$\varepsilon_q \left(x_i : (1 \leq i \leq n)\right) = \sup_{\alpha_p(\lambda_i:(1\leq i\leq n))\leq 1} \left\| \sum_{i=1}^{n} \lambda_i x_i \right\|_X , \tag{1.3}$$

with $\lambda_i > 0$, $(1 \leq p, q < \infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$ [5].

Let $(T, \sum, \mu)$ be a finite complete measure space and let $L^0 = L^0(T)$ denote the space of all (equivalence classes) of Σ-measurable real-valued functions.

For $f, g \in L^0$, $f \leq g$ means that $f(t) \leq g(t)$ $\mu-$almost every where, $t \in T$. A Banach space $(E, \|\cdot\|_E)$ is said to be a Köthe space if :

(1) For $f, g \in L^0$, $|f| \leq |g|$ and $g \in E$ imply $f \in E$ and $\|f\|_E \leq \|g\|_E$.

(2) For each $A \in \Sigma$, if $\mu(A)$ is finite, then $\chi_A \in E$. See [4].

So Köthe spaces are Banach lattices in the obvious order ($f \geqslant 0$ if $f(t) \geqslant 0$ $\mu-$almost every where $t \in T$). A Köthe space E has absolutely continuous norm if for each $f \in E$ and each decreasing sequence (A_n) converges to 0, then $\| \chi_{A_n} f\|_E \to 0$. Let E be a Köthe space on the measure space $(T, \sum, \mu)$ and $(X, \|\cdot\|_X)$ be a real Banach space. Then $E(X)$ is the space of all equivalence classes of strongly measurable functions $f : T \to X$, such that $\|f(\cdot)\|_X \in E$ equipped with the norm

$$|||f|||_{E(X)} = \|\|f(\cdot)\|_X\|_E .$$

The space $(E(X), |||\cdot|||_{E(X)})$ is a Banach space called the Köthe Bochner function space [3].

The most important class of Köthe Bochner function spaces are the Lebesgue-Bochner spaces $L^p(X)$, $(1 \leq p < \infty)$ and their generalization the Orlicz-Bochner spaces $L^\Phi(X)$. If X is a Banach lattice, the Köthe Bochner function spaces $E(X)$ are Banach lattices, we put $f \geqslant 0$, when $f(t) \geqslant 0$ $\mu-$almost every where $t \in T$.

The characterization of the positive absolutely summing operator was introduced on Lebesgue-Bochner spaces $L^P(X)$, $(1 \leq p < \infty)$ and Köthe Bochner function spaces in [1], [2], [6], [7]. In this paper, we shall investigate the space of positive (p, q)-absolutely summing operators for the Köthe Bochner function spaces with a real Banach lattice X, which is analogous with that given by D. Popa in [7], where he showed that $\Pi^+(E(X), Y) = \Pi^+(E, \Pi^+(X, Y))$.

For $T \in L(E(X), Y)$, define a function $\widetilde{T} : E \to L(X, Y)$ by

$$\left(\widetilde{T} f\right)(x) = T(fx), f \in E,\ x \in X. \tag{1.4}$$

Then $\widetilde{T}$ is clearly linear and bounded operator.

2. Main results

We now introduce the following lemmas which play important role in the proof of our main theorem

Lemma 2.1. *Let X be a Banach lattice, Y a Banach space and let E be a Köthe space. If $T \in \Pi^+_{(p,q)}(E(X), Y)$, for $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, then $\widetilde{T} f \in \Pi^+_{(p,q)}(X, Y)$, for all $f \in E$.*

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Proof. Let $T \in \Pi^{+}_{(p,q)}(E(X), Y)$ and $\widetilde{T}$ be defined as in (1.4). First we show that $\widetilde{T}f \in \Pi^{+}_{(p,q)}(X,Y)$, for all $f \in E$, $f \geqslant 0$. If $f \in E^{+}$, then for every finite set of elements $\{x_i\}_{1\leq i\leq n}$ in X^{+}, we have

$$\begin{aligned}\alpha_p\left(\left(\widetilde{T}f\right)(x_i) : (1 \leq k \leq n)\right) &= \alpha_p\left(T(fx_i) : (1 \leq i \leq n)\right)\\ &\leq \|T\|^{+}_{(p,q)}\,\varepsilon_q\left(fx_i : (1 \leq i \leq n)\right)\end{aligned}$$

Now, using (1.3)

$$\begin{aligned}\varepsilon_q\left(fx_i : (1 \leq i \leq n)\right) &= \sup_{\alpha_p(\lambda_i:(1\leq i\leq n))\ \leq 1}\left\|\left\|\sum_{i=1}^{n}\lambda_i f x_i\right\|\right\|_{E(X)}, with\ \ \lambda_i > 0\\ &= \sup_{\alpha_p(\lambda_i:(1\leq i\leq n))\ \leq 1}\left\|\left\|\sum_{i=1}^{n}\lambda_i f(\cdot)\,x_i\right\|_X\right\|_E\\ &= \sup_{\alpha_p(\lambda_i:(1\leq i\leq n))\ \leq 1}\left\|f(\cdot)\left\|\sum_{i=1}^{n}\lambda_i x_i\right\|_X\right\|_E\\ &= \|f\|_E \sup_{\alpha_p(\lambda_i:(1\leq i\leq n))\leq 1}\left\|\sum_{i=1}^{n}\lambda_i x_i\right\|_X\\ &= \|f\|_E\ \varepsilon_q\left(x_i : (1 \leq i \leq n)\right).\end{aligned}$$

Hence,

$$\alpha_p\left(\left(\widetilde{T}\ f\right)(x_i) : (1 \leq i \leq n)\right) \leq \|T\|^{+}_{(p,q)}\|f\|_E\ \ \varepsilon_q\left(x_i : (1 \leq i \leq n)\right),$$

this implies that $\widetilde{T}f \in \Pi^{+}_{(p,q)}(X,Y)$, for all$f \in E, f \geqslant 0$. In general, since $f = f^{+} - f^{-}$, where f^{+}, $f^{-} \in E^{+}$ then $\widetilde{T}f \in \Pi^{+}_{(p,q)}(X,Y)$, for all $f \in E$. ■

Lemma 2.2. *Let X be a Banach lattice, Y a Banach space and let E be a Köthe space. Consider a set of simple functions $\{f_i\}_{1\leq i\leq n}$ in $E(X)$, which is defined by $f_i = \sum_{j=1}^{k} x_{ij}\ \chi_{A_j}$, where χ_{A_j} are the characteristic functions of A_j's, $x_{ij} \in X^{+}$, for $i = 1, 2, \ldots, n$, $j = 1, 2, ...k$. Then for each $\lambda_i > 0$ with $\alpha_p\left(\lambda_i : (1 \leq i \leq n)\right) \leq 1$, we have*

$$\varepsilon_q\left(f_i : (1 \leq i \leq n)\right) > \varepsilon_1\left(\chi_{A_j}\left\|\sum_{i=1}^{n}\lambda_i x_{ij}\right\|_X : (1 \leq j \leq k)\right).$$

Proof. For $t \in T$, $f_i(t) = \sum_{j=1}^{k} x_{ij}\ \chi_{A_j}(t), i = 1, 2, \ldots, n$. Without loss of generality we may assume that the A_j's are pairwise disjoint measurable sets of T with $\cup_{j=1}^{k} A_j = T$. Now for each $\lambda_i > 0$ with $\alpha_p\left(\lambda_i : (1 \leq i \leq n)\right) \leq 1$, we have

form (1.3) we have the following

$$
\begin{aligned}
\varepsilon_q\left(f_i : (1 \leq i \leq n)\right) &> \left\|\left\|\sum_{i=1}^{n} \lambda_i f_i\right\|\right\|_{E(X)} \\
&= \left\|\left\|\sum_{i=1}^{n} \lambda_i f_i\left(\cdot\right)\right\|_X\right\|_E \\
&= \left\|\left\|\sum_{i=1}^{n} \lambda_i \left(\sum_{j=1}^{k} x_{ij}\ \chi_{A_j}\left(\cdot\right)\right)\right\|_X\right\|_E \\
&= \left\|\sum_{j=1}^{k} \chi_{A_j}\left(\cdot\right)\left\|\sum_{i=1}^{n} \lambda_i x_{ij}\right\|_X\right\|_E \\
&= \varepsilon_1\left(\chi_{A_j}\left\|\sum_{i=1}^{n} \lambda_i x_{ij}\right\|_X : (1 \leq j \leq k)\right).
\end{aligned}
$$

■

Lemma 2.3. *Let X be a Banach lattice, Y a Banach space and let E be a Köthe space with absolutely continuous norm. For $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, if $T \in L\left(E\left(X\right), Y\right)$ and $\widetilde{T} \in \Pi^+\left(E, \Pi^+_{(p,q)}\left(X, Y\right)\right)$, then $T \in \Pi^+_{(p,q)}\left(E\left(X\right), Y\right)$ with $\|T\|^+_{(p,q)} \leq \left\|\widetilde{T}\right\|^+$.*

Proof.

Lemma 1. **Proof.** Let $T \in L\left(E\left(X\right), Y\right)$ and $\widetilde{T} \in \Pi^+\left(E, \Pi^+_{(p,q)}\left(X, Y\right)\right)$. It is known that if the Köthe space E has absolutely continuous norm, then the subspace of all simple functions is dense in Köthe Bochner function spaces $E\left(X\right)$, [3]. So it is sufficient to show that $T \in \Pi^+_{(p,q)}\left(E\left(X\right), Y\right)$ on the restriction of T to the subspace of simple functions from $E\left(X\right)$. Choose a set of simple functions $\{f_i : f_i \geqslant 0\}_{1 \leq i \leq n}$ in $E\left(X\right)$, then as in Lemma (2.2) assume that for each $i = 1, 2, \ldots, n,\ f_i(t) = \sum_{j=1}^{k} x_{ij}\ \chi_{A_j}(t),\ t \in T$ with A_j's are pairwise disjoint measurable sets of T with $\cup_{j=1}^{k} A_j = T$, and χ_{A_j}'s are characteristic functions of A_j's, $x_{ij} \in X^+$, $j = 1, 2, ...k$. Since f_i 's $(i = 1, \ldots, n)$ represent classes of functions, we may assume that $\mu\left(A_j\right) > 0$, for each $j = 1, 2, ..., k$. Now given $\varepsilon > 0$ there exist $\lambda_i > 0$ with $\alpha_p\left(\lambda_i : (1 \leq i \leq n)\right) \leq 1$ that satisfy ■

$$
\varepsilon_q\left(x_{ij} : (1 \leq i \leq n)\right) < \left\|\sum_{i=1}^{n} \lambda_i x_{ij}\right\|_X + \varepsilon. \tag{2.1}
$$

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Thus,

$$\begin{aligned} \alpha_p\left(\left(T\left(f_i\right)\right):(1 \leq i \leq n)\right) &= \left(\sum_{i=1}^{n}\left\|T\left(f_i\right)\right\|^p\right)^{\frac{1}{p}} \\ &= \left(\sum_{i=1}^{n}\left[\left\|\sum_{j=1}^{k} T\left(\chi_{A_j} x_{ij}\right)\right\|\right]^p\right)^{\frac{1}{p}} \\ &\leq \sum_{j=1}^{k}\left(\sum_{i=1}^{n}\left\|\widetilde{T}\left(\chi_{A_j}\right)\left(x_{ij}\right)\right\|^p\right)^{\frac{1}{p}}. \end{aligned}$$

Using inequality (2.1) and that $\widetilde{T}\left(\chi_{A_j}\right) \in \Pi^{+}_{(p,q)}(X,Y)$ since $\widetilde{T} \in \Pi^{+}\left(E, \Pi^{+}_{(p,q)}(X,Y)\right)$:

$$\begin{aligned} \alpha_p\left(\left(T\left(f_i\right)\right):(1 \leq i \leq n)\right) &\leq \sum_{j=1}^{k}\left(\sum_{i=1}^{n}\left\|\widetilde{T}\left(\chi_{A_j}\right)\left(x_{ij}\right)\right\|^p\right)^{\frac{1}{p}} \qquad (2.2) \\ &\leq \sum_{j=1}^{k}\left\|\widetilde{T}\left(\chi_{A_j}\right)\right\|^{+}_{(p,q)} \varepsilon_q\left(x_{ij}:(1 \leq i \leq n)\right) \\ &< \sum_{j=1}^{k}\left\|\widetilde{T}\left(\chi_{A_j}\ \left(\left\|\sum_{i=1}^{n}\lambda_{ij}\ x_{ij}\right\|_X + \varepsilon\right)\right)\right\|^{+}_{(p,q)} \\ &\leq \left\|\widetilde{T}\right\|^{+} \varepsilon_1\left(\chi_{A_j}\left(\left\|\sum_{i=1}^{n}\lambda_i x_{ij}\right\|_X + \varepsilon\right):(1 \leq j \leq n)\right) \end{aligned}$$

for some $\lambda_i > 0$ with $\alpha_p\left(\lambda_i:(1 \leq i \leq n)\right) \leq 1$.

Using Lemma 2.2, then for each λ_i with $\alpha_p\left(\lambda_i:(1 \leq i \leq n)\right) \leq 1$, we have

$$\varepsilon_q\left(f_i:(1 \leq i \leq n)\right) > \varepsilon_1\left(\chi_{A_j}\left\|\sum_{i=1}^{n}\lambda_i x_{ij}\right\|_X:(1 \leq j \leq k)\right) \qquad (2.3)$$

.

Then from inqualities (2.2), (2.3) ,we have

$$\alpha_p\left(\left(T\left(f_i\right)\right):(1 \leq i \leq n)\right) < \left\|\widetilde{T}\right\|^{+} \varepsilon_q\left(f_i:(1 \leq i \leq n)\right) + \varepsilon\ \left\|\chi_T\right\|_E.$$

As $\left\|\chi_T\right\|_E$ is finite and ε is an arbitrary, then

$$\alpha_p\left(\left(T\left(f_i\right)\right):(1 \leq i \leq n)\right) \leq \left\|\widetilde{T}\right\|^{+} \varepsilon_q\left(f_i:(1 \leq i \leq n)\right).$$

This gives us the result that $T \in \Pi^{+}_{(p,q)}\left(E\left(X\right), Y\right)$ with $\left\|T\right\|^{+}_{(p,q)} \leq \left\|\widetilde{T}\right\|^{+}$. ■

The following theorem explains the relation between the spaces of positive (p,q)-absolutely summing operators over the Köthe spaces

Theorem 2.1. *Let X be a Banach lattice, Y a Banach space and E be a Köthe space. Then for $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$,*

$$\Pi^{+}_{(p,q)}\left(E\left(X\right), Y\right) \subseteq \Pi^{+}_{(p,q)}\left(E, \Pi^{+}\left(X,Y\right)\right).$$

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Proof. Let $T \in \Pi^{+}_{(p,q)}(E(X), Y)$. Using Lemma 2.1, $\widetilde{T}f \in \Pi^{+}_{(p,q)}(X, Y)$, for all $f \in E$. Hence, from equation(1.2) given $\varepsilon > 0$, there exist a finite sequence $\{f_i : f_i \geq 0\}_{1 \leq i \leq n}$ in E^{+} and $(x_{ij}, j \in \sigma_i)$ in X^{+},where σ_i is a finite set with $\sigma_i \subseteq N$ such that for each $i = 1, 2, \ldots, n$, and $\varepsilon_q(x_{ij} : (j \in \sigma_i)) \leq 1$

$$\left[\left\|\widetilde{T}(f_i)\right\|^{+}_{(p,q)}\right]^{p} - \frac{\varepsilon}{n} < \alpha^{p}_{p}\left[\widetilde{T}(f_i)(x_{ij}), (j \in \sigma_i)\right],$$

Therefore,

$$\begin{aligned}
\sum_{i=1}^{n}\left[\left\|\widetilde{T}(f_i)\right\|^{+}_{(p,q)}\right]^{p} - \varepsilon &< \sum_{i=1}^{n}\alpha^{p}_{p}\left[\widetilde{T}(f_i)(x_{ij}),\ (j \in \sigma_i)\right] \qquad (2.4)\\
&= \alpha^{p}_{p}\left[\widetilde{T}(f_i)(x_{ij}), (j \in \sigma_i, 1 \leq i \leq n)\right]\\
&\leq \alpha^{p}_{p}\left[T(f_i x_{ij}), (j \in \sigma_i, 1 \leq i \leq n)\right]\\
&\leq \left[\|T\|^{+}_{(p,q)}\right]^{p}\varepsilon^{p}_{q}(f_i x_{ij} : (j \in \sigma_i, 1 \leq i \leq n)).
\end{aligned}$$

Next we want to show that for each $i = 1, 2, \ldots, n$, if $\varepsilon_q(x_{ij} : (j \in \sigma_i)) \leq 1$, then

$$\varepsilon_q(f_i x_{ij} : (j \in \sigma_i, 1 \leq i \leq n)) \leq \varepsilon_q(f_i : (1 \leq i \leq n)).$$

But, from (1.3)

$$\begin{aligned}
&\varepsilon_q(f_i x_{ij} : (j \in \sigma_i, 1 \leq i \leq n))\\
&= \sup_{\alpha_p(\lambda_{ij} : (j \in \sigma_i, 1 \leq i \leq n)) \ \leq 1}\left\|\left\|\sum_{i=1}^{n}\sum_{j \in \sigma_i}\lambda_{ij} f_i x_{ij}\right\|\right\|_{E(X)}, with\ \lambda_{ij} > 0.
\end{aligned}$$

As $\lambda_{ij} > 0$ with $\varepsilon_q(x_{ij} : (j \in \sigma_i)) \leq 1$ and $\alpha_p(\lambda_{ij} : (j \in \sigma_i, 1 \leq i \leq n)) \leq 1$, we get that $\alpha_p(\lambda_{ij} : (j \in \sigma_i)) \leq 1$ for each $i = 1, \ldots, n$. Therefore, for each $t \in T$,

$$\begin{aligned}
\left\|\sum_{i=1}^{n}\sum_{j \in \sigma_i}\lambda_{ij} f_i(t) x_{ij}\right\|_{X} &= \sup_{\|x^*\| \leq 1}\sum_{i=1}^{n}\sum_{j \in \sigma_i}\lambda_{ij} f_i(t) x^*(x_{ij}), x^* \in X^*\\
&\leq \sup_{\|x^*\| \leq 1}\sum_{i=1}^{n} f_i(t)\left(\sum_{j \in \sigma_i}(\lambda_{ij})^{p}\right)^{\frac{1}{p}}\left(\sum_{j \in \sigma_i} x^*(x_{ij})^{q}\right)^{\frac{1}{q}}\\
&\leq \sum_{i=1}^{n} f_i(t).
\end{aligned}$$

Thus

$$\begin{aligned}
\varepsilon_q(f_i x_{ij} : (j \in \sigma_i, (1 \leq i \leq n))) &= \sup_{\alpha_p(\lambda_{ij} : (j \in \sigma_i)\ , 1 \leq i \leq n)}\left\|\left\|\sum_{i=1}^{n}\sum_{j \in \sigma_i}\lambda_{ij} f_i x_{ij}\right\|\right\|_{E(X)} \qquad (2.5)\\
&\leq \left\|\sum_{i=1}^{n} f_i\right\|_{E}\\
&= \varepsilon_1(f_i : (1 \leq i \leq n))\\
&\leq \varepsilon_q(f_i : (1 \leq i \leq n)).
\end{aligned}$$

Inequalities (2.4) and (2.5) induce,

$$\sum_{i=1}^{n}\left[\left\|\widetilde{T}\left(f_i\right)\right\|_{(p,q)}^{+}\right]^{p}-\varepsilon<\left[\|T\|_{(p,q)}^{+}\right]^{p}\varepsilon_q^p\left(f_i:(1\leq i\leq n)\right).$$

As ε is an arbitrary, hence

$$\alpha_p\left(\left(\widetilde{T}\left(f_i\right)\right):(1\leq i\leq n)\right)\leq\|T\|_{(p,q)}^{+}\varepsilon_q\left(f_i:(1\leq i\leq n)\right).$$

and $\left\|\widetilde{T}\right\|^{+}\leq\|T\|_{(p,q)}^{+}$. ■

Now let us state another result, which discuss the equality between the spaces of positive (p,q)-absolutely summing operators over the Köthe spaces. It easily follows from Lemma 2.3 and Theorem 2.1.

Theorem 2.2. *Let X be a Banach lattice, Y a Banach space and let E be a Köthe space with absolutely continuous norm . Then for $1\leq p,q\leq\infty$ with $\frac{1}{p}+\frac{1}{q}=1$, we have*

$$\Pi_{(p,q)}^{+}\left(E\left(X\right),Y\right)=\Pi^{+}\left(E,\Pi_{(p,q)}^{+}\left(X,Y\right)\right).$$

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An Estimate for the Four-Dimensional Discrete Derivative Green's Function and Its Applications in FE Superconvergence

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In this article we first introduce definitions of the regularized derivative Green's function, the discrete derivative Green's function, the discrete derivative δ function, and the L^2-projection operator in four dimensions. Then the $W^{2,1}$-seminorm estimates for the regularized derivative Green's function and the discrete derivative Green's function are derived. Finally, we show the applications of the $W^{2,1}$-seminorm estimate for the discrete derivative Green's function in finite element (FE) superconvergence.

1 Introduction

It is well known that estimates for the Green's function play very important roles in the study of the superconvergence (especially, pointwise superconvergence) of the finite element method (see [1–8]). For one- and two-dimensional elliptic problems, one have obtained many optimal estimates for the Green's function (see [1]). Recently, for three-dimensional elliptic problems, the $W^{2,1}$-seminorm optimal estimate with order $O(|\ln h|^{\frac{2}{3}})$ for the discrete Green's function and the $W^{1,1}$-seminorm optimal estimate with order $O(|\ln h|^{\frac{4}{3}})$ for the discrete derivative Green's function were derived (see [5, 9]). For the four-dimensional setting, we also obtained the corresponding $W^{1,1}$-seminorm estimate with order $O(|\ln h|^{\frac{5}{4}})$ for the discrete derivative Green's function (see [10]).

In this article, we will discuss the $W^{2,1}$-seminorm estimate for the four-dimensional discrete derivative Green's function. Although this estimate can be derived by the inverse property from the $W^{1,1}$-seminorm estimate, we will give a better result by the other way.

we shall use the symbol C to denote a generic constant, which is independent from the discretization parameter h and which may not be the same in each occurrence and also use the standard notations for the Sobolev spaces and their norms.

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We consider the following variable coefficient elliptic problem:

$$\mathcal{L}u \equiv -\sum_{i,j=1}^{4} \partial_j(a_{ij}\partial_i u) + a_0 u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where $\Omega \subset \mathcal{R}^4$ is a bounded polytopic domain and $\partial_i u = \partial_{x_i} u(x_1, x_2, x_3, x_4), i = 1, 2, 3, 4$ stand for usual partial derivatives. We assume a_{ij}, a_0, and f are sufficiently smooth given functions, and $a_{ij} = a_{ji}$. Let $\{\mathcal{T}^h\}$ be a regular family of partitions of $\bar{\Omega}$. Denote a space of continuous piecewise m-degree $(m \geq 1)$ polynomials with respect to this kind of partitions by $S^h(\Omega)$ and let $S_0^h = S^h(\Omega) \cap H_0^1(\Omega)$. Discretizing the above elliptic equation using S_0^h as approximating space means finding $u_h \in S_0^h$ such that $a(u_h\,,\, v) = (f\,,\, v)$ for all $v \in S_0^h$, where

$$a(u_h\,,\, v) \equiv \int_\Omega (\sum_{i,j=1}^{4} a_{ij}\partial_i u_h \partial_j v + a_0 u_h v)\, dx_1 dx_2 dx_3 dx_4,$$

and

$$(f\,,\, v) \equiv \int_\Omega f v\, dx_1 dx_2 dx_3 dx_4.$$

This gives the Galerkin orthogonality relation

$$a(u - u_h\,,\, v) = 0 \;\; \forall\, v \in S_0^h. \tag{1.2}$$

For every $Z \in \Omega$, we define the discrete derivative δ function $\partial_{Z,\ell}\delta_Z^h \in S_0^h$, the regularized derivative Green's function $\partial_{Z,\ell}G_Z^* \in H^2(\Omega) \cap H_0^1(\Omega)$, the discrete derivative Green's function $\partial_{Z,\ell}G_Z^h \in S_0^h$ and the L^2-projection $P_h u \in S_0^h$ such that (see [1])

$$(v,\, \partial_{Z,\ell}\delta_Z^h) = \partial_\ell v(Z) \;\; \forall\, v \in S_0^h, \tag{1.3}$$

$$a(\partial_{Z,\ell}G_Z^*,\, v) = (\partial_{Z,\ell}\delta_Z^h,\, v) \;\; \forall\, v \in H_0^1(\Omega), \tag{1.4}$$

$$a(\partial_{Z,\ell}G_Z^* - \partial_{Z,\ell}G_Z^h,\, v) = 0 \;\; \forall\, v \in S_0^h, \tag{1.5}$$

$$(u - P_h u,\, v) = 0 \;\; \forall\, v \in S_0^h, \tag{1.6}$$

where $\ell \in \mathcal{R}^4$ and $|\ell| = 1$. $\partial_\ell v(Z)$ stands for the onesided directional derivative

$$\partial_\ell v(Z) = \lim_{|\Delta Z| \to 0} \frac{v(Z + \Delta Z) - v(Z)}{|\Delta Z|}, \;\; \Delta Z = |\Delta Z|\ell.$$

In the following sections, we will bound the terms $|\partial_{Z,\ell}G_Z^*|_{2,\,1}$ and $\left|\partial_{Z,\ell}G_Z^h\right|_{2,\,1}$ and give an application of $\left|\partial_{Z,\ell}G_Z^h\right|_{2,\,1}$ in FE superconvergence.

2 The $W^{2,1}$-Seminorm Estimate for the Discrete Derivative Green's Function

To derive the estimate for the discrete derivative Green's function, we introduce the weight function defined by

$$\phi \equiv \phi(X) = \left(|X-\bar{X}|^2+\theta^2\right)^{-2} \ \ \forall\, X \in \bar{\Omega}, \tag{2.1}$$

where $\bar{X} \in \bar{\Omega}$ is a fixed point, $\theta = \gamma h$, and $\gamma \in [4, +\infty)$ is a suitable real number.

For every $\alpha \in \mathcal{R}$, we give the following notations:

$$|\nabla^n v|^2 = \sum_{|\beta|=n} \left|D^\beta v\right|^2, |\nabla^n v|_{\phi^\alpha,\,\Omega} = \left(\int_\Omega \phi^\alpha \left|\nabla^n v\right|^2 dX\right)^{\frac{1}{2}}, \|v\|^2_{m,\,\phi^\alpha,\,\Omega} = \sum_{n=0}^{m} |\nabla^n v|^2_{\phi^\alpha,\,\Omega},$$

where $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)$, $|\beta| = \beta_1+\beta_2+\beta_3+\beta_4$, and $\beta_i \geq 0,\ i = 1,2,3,4$ are integers. In particular, for the case of $m = 0$, we write

$$\|v\|_{\phi^\alpha,\,\Omega} = \left(\int_\Omega \phi^\alpha |v|^2\, dX\right)^{\frac{1}{2}}.$$

We assume that there exists an q_0 $(1 < q_0 \leq \infty)$ such that

$$\mathcal{L}:\ W^{2,\,q}(\Omega) \cap W_0^{1,\,q}(\Omega) \longrightarrow L^q(\Omega)\ (1 < q < q_0)$$

is a homeomorphism (see [1]). It means that for all $v \in W^{2,\,q}(\Omega) \cap W_0^{1,\,q}(\Omega)$, we have the so-called a priori estimate:

$$\|v\|_{2,\,q,\,\Omega} \leq C(q)\|\mathcal{L}v\|_{0,\,q,\,\Omega}, \tag{2.2}$$

where $C(q)$ denotes a positive constant only depending on q. Next, we give some lemmas used in the proofs of our main results.

For the the weight function ϕ, we have the following lemma.

Lemma 2.1. *For ϕ the weight function defined by (2.1), we have the following estimates*:

$$|\nabla^n \phi^\alpha| \leq C(\alpha, n)\phi^{\alpha+\frac{n}{4}},\ \alpha \in \mathcal{R},\ n = 1, 2. \tag{2.3}$$

$$\int_\Omega \phi^\alpha\, dX \leq C(\alpha-1)^{-1}\theta^{-4(\alpha-1)}\ \forall\, \alpha > 1, \tag{2.4}$$

$$\int_\Omega \phi\, dX \leq C(k)|\ln\theta|,\ \ \theta \leq k < 1. \tag{2.5}$$

$$\int_\Omega \phi^\alpha\, dX \leq C(1-\alpha)^{-1}\ \forall\ 0 < \alpha < 1, \tag{2.6}$$

Remark 1. The results (2.3)-(2.5) have been proved in [10] and the result (2.6) is also easily obtained.

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For the L^2-projection operator P_h and the discrete derivative δ function $\partial_{Z,\ell}\delta_Z^h$, we have the following lemmas 2.2 and 2.3 (see [10])

Lemma 2.2. *For $P_h w$ the L^2-projection of $w \in W^{1,q}(\Omega) \cap C(\bar{\Omega})$, and $1 \le q \le \infty$, we have the following stability estimate*:

$$\|P_h w\|_{1,q} \le C\|w\|_{1,q}. \tag{2.7}$$

Lemma 2.3. *For $\partial_{Z,\ell}\delta_Z^h$ the discrete derivative δ function defined by (1.3), we have the following estimate*:

$$|\partial_{Z,\ell}\delta_Z^h(X)| \le Ch^{-5}\mathrm{e}^{-Ch^{-1}|X-Z|}, \tag{2.8}$$

where $X, Z \in \bar{\Omega}$, and C is independent of h, X, and Z.

Further, we have the following weighted-norm estimate for $\partial_{Z,\ell}\delta_Z^h$.

Lemma 2.4. *For $\partial_{Z,\ell}\delta_Z^h$ the discrete derivative δ function defined by (1.3) and each real number $\alpha > 0$, we have the following estimate*:

$$\left\|\partial_{Z,\ell}\delta_Z^h\right\|_{\phi^{-\alpha}} \le Ch^{-3+2\alpha}. \tag{2.9}$$

Proof. From (2.1) and (2.8),

$$\begin{aligned} \left\|\partial_{Z,\ell}\delta_Z^h\right\|_{\phi^{-\alpha}}^2 &\le C\int_\Omega \left(|X-Z|^2+\theta^2\right)^{2\alpha} h^{-10}e^{-Ch^{-1}|X-Z|}dX \\ &\le C\int_0^\infty \left(r^2+\theta^2\right)^{2\alpha} h^{-10}e^{-Ch^{-1}r}r^3dr. \end{aligned}$$

Set $h^{-1}r = t$, then

$$\begin{aligned} \left\|\partial_{Z,\ell}\delta_Z^h\right\|_{\phi^{-\alpha}}^2 &\le Ch^{-6+4\alpha}\int_0^\infty \left(t^2+\gamma^2\right)^{2\alpha} e^{-Ct}t^3dt \\ &\le Ch^{-6+4\alpha}. \end{aligned}$$

The desired result (2.9) is immediately obtained.

Lemma 2.5. *For $\partial_{Z,\ell}G_Z^*$ the regularized derivative Green's function and $0 < \alpha < 1$, we have the following weighted-norm estimate*:

$$\left\|\nabla^2\partial_{Z,\ell}G_Z^*\right\|_{\phi^{-\alpha}} \le C\left(1+\alpha^{\frac{\alpha-1}{2(\alpha+1)}}\right)h^{-3+2\alpha}. \tag{2.10}$$

Proof.

$$\begin{aligned} \left\|\nabla^2\partial_{Z,\ell}G_Z^*\right\|_{\phi^{-\alpha}}^2 &= \int_\Omega \phi^{-\alpha}\left|\nabla^2\partial_{Z,\ell}G_Z^*\right|^2 dX = \int_\Omega \left(\phi^{-\frac{\alpha}{2}}\left|\nabla^2\partial_{Z,\ell}G_Z^*\right|\right)^2 dX \\ &\le C\left(\int_\Omega \left|\nabla^2\left(\phi^{-\frac{\alpha}{2}}\partial_{Z,\ell}G_Z^*\right)\right|^2 dX + \int_\Omega \left|\nabla^2\phi^{-\frac{\alpha}{2}}\partial_{Z,\ell}G_Z^*\right|^2 dX \right. \\ &\quad \left. + \int_\Omega \left|\nabla\phi^{-\frac{\alpha}{2}}\right|^2\left|\nabla\partial_{Z,\ell}G_Z^*\right|^2 dx\right) \\ &\le C\left(\left\|\nabla^2\left(\phi^{-\frac{\alpha}{2}}\partial_{Z,\ell}G_Z^*\right)\right\|_0^2 + \left\|\partial_{Z,\ell}G_Z^*\right\|_{\phi^{1-\alpha}}^2 + \left|\partial_{Z,\ell}G_Z^*\right|_{1,\phi^{\frac{1}{2}-\alpha}}^2\right) \end{aligned}$$

$$
\begin{aligned}
&\le C\left(\left\|\mathcal{L}\left(\phi^{-\frac{\alpha}{2}}\partial_{Z,\ell}G_Z^*\right)\right\|_0^2+\|\partial_{Z,\ell}G_Z^*\|^2_{\phi^{1-\alpha}}+|\partial_{Z,\ell}G_Z^*|^2_{1,\,\phi^{\frac{1}{2}-\alpha}}\right)\\
&\le C\left(\int_\Omega \phi^{-\alpha}\left|\mathcal{L}\partial_{Z,\ell}G_Z^*\right|^2dX+\int_\Omega\left|\nabla\phi^{-\frac{\alpha}{2}}\right|^2\left|\nabla\partial_{Z,\ell}G_Z^*\right|^2dX\right.\\
&\quad+\int_\Omega\left|\nabla\phi^{-\frac{\alpha}{2}}\right|^2\left|\partial_{Z,\ell}G_Z^*\right|^2dX+\int_\Omega\left|\nabla^2\phi^{-\frac{\alpha}{2}}\right|^2\left|\partial_{Z,\ell}G_Z^*\right|^2dX\\
&\quad\left.+\|\partial_{Z,\ell}G_Z^*\|^2_{\phi^{1-\alpha}}+|\partial_{Z,\ell}G_Z^*|^2_{1,\,\phi^{\frac{1}{2}-\alpha}}\right)\\
&\le C\left(\|\mathcal{L}\partial_{Z,\ell}G_Z^*\|^2_{\phi^{-\alpha}}+|\partial_{Z,\ell}G_Z^*|^2_{1,\,\phi^{\frac{1}{2}-\alpha}}+\|\partial_{Z,\ell}G_Z^*\|^2_{\phi^{1-\alpha}}\right)\\
&\le C\left\|\partial_{Z,\ell}\delta_Z^h\right\|^2_{\phi^{-\alpha}}+C\left|a\left(\partial_{Z,\ell}G_Z^*,\phi^{\frac{1}{2}-\alpha}\partial_{Z,\ell}G_Z^*\right)\right|+C\left\|\partial_{Z,\ell}G_Z^*\right\|^2_{\phi^{1-\alpha}}\\
&\le C\left\|\partial_{Z,\ell}\delta_Z^h\right\|^2_{\phi^{-\alpha}}+C\left|\left(\partial_{Z,\ell}\delta_Z^h,\phi^{\frac{1}{2}-\alpha}\partial_{Z,\ell}G_Z^*\right)\right|+C\left\|\partial_{Z,\ell}G_Z^*\right\|^2_{\phi^{1-\alpha}}\\
&\le C\left\|\partial_{Z,\ell}\delta_Z^h\right\|^2_{\phi^{-\alpha}}+C\left\|\partial_{Z,\ell}G_Z^*\right\|^2_{\phi^{1-\alpha}}.
\end{aligned}
$$

Namely,

$$\left\|\nabla^2\partial_{Z,\ell}G_Z^*\right\|^2_{\phi^{-\alpha}}\le C\left\|\partial_{Z,\ell}\delta_Z^h\right\|^2_{\phi^{-\alpha}}+C\left\|\partial_{Z,\ell}G_Z^*\right\|^2_{\phi^{1-\alpha}}. \tag{2.11}$$

Now we need to bound the term $\|\partial_{Z,\ell}G_Z^*\|^2_{\phi^{1-\alpha}}$. Obviously, when $0<\alpha<1$,

$$\|\partial_{Z,\ell}G_Z^*\|^2_{\phi^{1-\alpha}}\le\|\phi\|^{1-\alpha}_{0,\,1+\alpha}\|\partial_{Z,\ell}G_Z^*\|^2_{0,\,\frac{1+\alpha}{\alpha}}\le C\left(\alpha^{-1}\theta^{-4\alpha}\right)^{\frac{1-\alpha}{1+\alpha}}\|\partial_{Z,\ell}G_Z^*\|^2_{0,\,\frac{1+\alpha}{\alpha}}, \tag{2.12}$$

where we used (2.4) in the above arguments. From (1.3), (1.4), (1.6), (2.7), the inverse property, and the Poincaré inequality, we have

$$
\begin{aligned}
\|\partial_{Z,\ell}G_Z^*\|^{\frac{1+\alpha}{\alpha}}_{0,\,\frac{1+\alpha}{\alpha}} &= \left(\partial_{Z,\ell}G_Z^*,|\partial_{Z,\ell}G_Z^*|^{\frac{1}{\alpha}}\,\mathrm{sgn}\partial_{Z,\ell}G_Z^*\right)\\
&= a(\partial_{Z,\ell}G_Z^*,w)=(\partial_{Z,\ell}\delta_Z^h,w)=\partial_\ell P_hw(Z)\\
&\le |P_hw|_{1,\,\infty}\le Ch^{-\frac{4}{q}}\,|P_hw|_{1,\,q}\\
&\le Ch^{-\frac{4}{q}}\,|w|_{1,\,q},
\end{aligned} \tag{2.13}
$$

where $q\ge1$, and $w\in W^{2,\,1+\alpha}(\Omega)\bigcap W_0^{1,\,1+\alpha}(\Omega)$ satisfies

$$a(v,w)=\left(v,|\partial_{Z,\ell}G_Z^*|^{\frac{1}{\alpha}}\,\mathrm{sgn}\partial_{Z,\ell}G_Z^*\right)\ \forall\,v\in H_0^1(\Omega).$$

Set $q=\frac{4(1+\alpha)}{3-\alpha}>1$, thus $\frac{1}{q}=\frac{1}{1+\alpha}-\frac{1}{4}$. By the Sobolev Embedding Theorem [11] and the a priori estimate (2.2), we get

$$|w|_{1,\,q}\le C\,\|w\|_{2,\,1+\alpha}\le C\,\|\partial_{Z,\ell}G_Z^*\|^{\frac{1}{\alpha}}_{0,\,\frac{1+\alpha}{\alpha}}. \tag{2.14}$$

Combining (2.13) and (2.14) yields

$$\|\partial_{Z,\ell}G_Z^*\|^2_{0,\,\frac{1+\alpha}{\alpha}}\le Ch^{-\frac{8}{q}}=Ch^{\frac{2(\alpha-3)}{\alpha+1}}. \tag{2.15}$$

From (2.12) and (2.15),

$$\|\partial_{Z,\ell} G_Z^*\|^2_{\phi^{1-\alpha}} \le C(\alpha) h^{-6+4\alpha}, \tag{2.16}$$

where $C(\alpha) = C\alpha^{\frac{\alpha-1}{\alpha+1}}$. Combining (2.9), (2.11), and (2.16) yields the result (2.10).

Lemma 2.6. *For $\partial_{Z,\ell} G_Z^*$ the regularized derivative Green's function, we have the following $W^{2,1}$-seminorm estimate*:

$$|\partial_{Z,\ell} G_Z^*|_{2,1} \le Ch^{-1} |\ln h|^{\frac{1}{2}}. \tag{2.17}$$

Proof. Obviously,

$$|\partial_{Z,\ell} G_Z^*|^2_{2,1} \le \int_\Omega \phi^\alpha \, dX \cdot \left\|\nabla^2 \partial_{Z,\ell} G_Z^*\right\|^2_{\phi^{-\alpha}}. \tag{2.18}$$

Choosing $0 < \alpha < 1$, and using (2.6) and (2.10), we have

$$\begin{aligned} |\partial_{Z,\ell} G_Z^*|^2_{2,1} &\le C(1-\alpha)^{-1}\left(1+\alpha^{\frac{\alpha-1}{\alpha+1}}\right) h^{-6+4\alpha} \\ &= C(1-\alpha)^{-1} h^{-6+4\alpha} + C(1-\alpha)^{-1} h^{-6+4\alpha} \alpha^{\frac{\alpha-1}{\alpha+1}} \\ &\equiv \varphi_1(\alpha) + \varphi_2(\alpha). \end{aligned} \tag{2.19}$$

We easily obtain

$$\varphi_1(\alpha_0) = \min_{0<\alpha<1} \varphi_1(\alpha) = Ch^{-2}|\ln h|, \tag{2.20}$$

where $\alpha_0 = 1 + \frac{1}{4\ln h}$. Further,

$$\varphi_2(\alpha_0) \le Ch^{-2}|\ln h|. \tag{2.21}$$

From (2.19)–(2.21), we immediately obtain the result (2.17). In order to derive the estimate of the discrete derivative Green's function $\partial_{Z,\ell} G_Z^h$, we need the following estimate (see [10]).

Lemma 2.7. *For $\partial_{Z,\ell} G_Z^*$ and $\partial_{Z,\ell} G_Z^h$, the regularized derivative Green's function and the discrete derivative Green's function, respectively, we have the following estimate*:

$$\left|\partial_{Z,\ell} G_Z^* - \partial_{Z,\ell} G_Z^h\right|_{1,1} \le C|\ln h|^{\frac{1}{2}}. \tag{2.22}$$

We now give the main result in this article.

Theorem 2.1. *For $\partial_{Z,\ell} G_Z^h$ the discrete derivative Green's function, we have the following estimate*:

$$\left|\partial_{Z,\ell} G_Z^h\right|^h_{2,1} \le Ch^{-1}|\ln h|^{\frac{1}{2}}, \tag{2.23}$$

where $\left|\partial_{Z,\ell} G_Z^h\right|^h_{2,1} = \sum_{e\in\mathcal{T}^h} \left|\partial_{Z,\ell} G_Z^h\right|_{2,1,e}$.

Proof. We denote by $\Pi\partial_{Z,\ell}G_Z^*$ the interpolant to $\partial_{Z,\ell}G_Z^*$. Thus, by the triangle inequality, the interpolation error estimate, and the inverse property, we have

$$
\begin{aligned}
& \left|\partial_{Z,\ell}G_Z^h\right|_{2,1}^h \leq \left|\partial_{Z,\ell}G_Z^* - \partial_{Z,\ell}G_Z^h\right|_{2,1}^h + \left|\partial_{Z,\ell}G_Z^*\right|_{2,1} \\
\leq\ & \left|\partial_{Z,\ell}G_Z^*\right|_{2,1} + \left|\partial_{Z,\ell}G_Z^* - \Pi\partial_{Z,\ell}G_Z^*\right|_{2,1}^h + \left|\Pi\partial_{Z,\ell}G_Z^* - \partial_{Z,\ell}G_Z^h\right|_{2,1}^h \\
\leq\ & C\left|\partial_{Z,\ell}G_Z^*\right|_{2,1} + Ch^{-1}\left|\Pi\partial_{Z,\ell}G_Z^* - \partial_{Z,\ell}G_Z^h\right|_{1,1} \\
\leq\ & C\left|\partial_{Z,\ell}G_Z^*\right|_{2,1} + Ch^{-1}\left|\partial_{Z,\ell}G_Z^* - \Pi\partial_{Z,\ell}G_Z^*\right|_{1,1} \\
& +Ch^{-1}\left|\partial_{Z,\ell}G_Z^* - \partial_{Z,\ell}G_Z^h\right|_{1,1} \\
\leq\ & C\left|\partial_{Z,\ell}G_Z^*\right|_{2,1} + Ch^{-1}\left|\partial_{Z,\ell}G_Z^* - \partial_{Z,\ell}G_Z^h\right|_{1,1}.
\end{aligned}
\tag{2.24}
$$

Combining (2.17), (2.22), and (2.24) yields the result (2.23).

3 An Application in FE Superconvergence

In this section, we give an application of the estimate for the discrete derivative Green's function in FE superconvergence.

Let Πu be the interpolant to u, the solution of (1.1), and u_h the finite element approximation to u. We first assume the following lemma holds.

Lemma 3.1. *Suppose $v \in S_0^h(\Omega)$ and $u \in W^{m+2,\infty}(\Omega)\bigcap H_0^1(\Omega)$. Then we have the following weak estimate*:

$$|a(u - \Pi u, v)| \leq Ch^{m+2}\|u\|_{m+2,\infty}|v|_{2,1}^h, \ \ m \geq 2, \tag{3.1}$$

where $|v|_{2,1}^h = \sum_{e\in\mathcal{T}^h}|v|_{2,1,e}$.

Finally, we give the following superconvergent estimate.

Theorem 3.1. *Let $\{\mathcal{T}^h\}$ be a regular family of partitions of $\bar\Omega$ and $u \in W^{m+2,\infty}(\Omega)\bigcap H_0^1(\Omega)$. For u_h and Πu, the finite element approximation of degree m and the corresponding interpolant to u, respectively. Then we have the following superconvergent estimates*:

$$|u_h - \Pi u|_{1,\infty,\Omega} \leq Ch^{m+1}|\ln h|^{\frac{1}{2}}\|u\|_{m+2,\infty}, \ \ m \geq 2. \tag{3.2}$$

Proof. For every $Z \in \Omega$, applying the definition of $\partial_{Z,\ell}G_Z^h$ and the Galerkin orthogonality relation (1.2), we derive

$$\partial_{Z,\ell}(u_h - \Pi u)(Z) = a(u_h - \Pi u\,,\, \partial_{Z,\ell}G_Z^h) = a(u - \Pi u\,,\, \partial_{Z,\ell}G_Z^h). \tag{3.3}$$

From (2.23), (3.1), and (3.3), we immediately obtain the result (3.2).

Acknowledgments This work was supported by the National Natural Science Foundation of China Grant 11161039, the Zhejiang Provincial Natural Science Foundation of China Grant LY13A010007 and the Natural Science Foundation of Ningbo City Grant 2013A610104.

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APOLLONIUS TYPE ADDITIVE FUNCTIONAL EQUATIONS IN C^*-TERNARY ALGEBRAS AND JB^*-TRIPLES

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ABSTRACT. In this paper, we consider the following Apollonius type additive functional equation:

$$f(z-x)+f(z-y)=-\frac{1}{2}f(x+y)+2f\Big(z-\frac{x+y}{4}\Big)$$

and investigate homomorphisms between C^*-ternary algebras and derivations on C^*-ternary algebras and homomorphisms between JB^*-triples and derivations on JB^*-triples under general control functions. Our paper extend the results obtained by Park and Rassias [Choonkil Park and Themistocles M. Rassias, Homomorphisms in C^*-ternary algebras and JB^*-triples, J. Math. Anal. Appl. 337(2008), 13–20].

1. Introduction and preliminaries

A C^**-ternary algebra* is a complex Banach space A equipped with a ternary product $(x,y,z)\mapsto[x,y,z]$ of A^3 into A, which is $\mathbb{C}$-linear in the outer variables, conjugate $\mathbb{C}$-linear in the middle variable and associative in the sense that

$$[x,y,[z,w,v]]=[x,[w,z,y],v]=[[x,y,z],w,v]$$

and satisfies

$$\|[x,y,z]\|\le\|x\|\cdot\|y\|\cdot\|z\|,\quad \|[x,x,x]\|=\|x\|^3$$

(see [26]).

If a C^*-ternary algebra $(A,[\cdot,\cdot,\cdot])$ has the identity, i.e., an element $e\in A$ such that $x=[x,e,e]=[e,e,x]$ for all $x\in A$, then it is routine to verify that A endowed with $x\circ y:=[x,e,y]$ and $x^*:=[e,x,e]$ is a unital C^*-algebra. Conversely, if $(A,\circ)$ is a unital C^*-algebra, then $[x,y,z]:=x\circ y^*\circ z$ makes A into a C^*-ternary algebra.

A $\mathbb{C}$-linear mapping $H:A\to B$ is called a C^**-ternary algebra homomorphism* if

$$H([x,y,z])=[H(x),H(y),H(z)]$$

1991 *Mathematics Subject Classification.* 39B52, 17A40, 46B03, 47Jxx, 46K70, 47B48.

Key words and phrases. Apollonius type additive functional equation, C^*-ternary algebra homomorphism, generalized Hyers-Ulam stability, C^*-ternary derivation, JB^*-triple homomorphism, JB^*-triple derivation.

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The first author was supported by the Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2013053358).

for all $x, y, z \in A$. A $\mathbb{C}$-linear mapping $\delta : A \to A$ is called a *C^*-ternary derivation* if

$$\delta([x, y, z]) = [\delta(x), y, z] + [x, \delta(y), z] + [x, y, \delta(z)]$$

for all $x, y, z \in A$ (see [14]).

Ternary structures and their generalization, the so-called n-ary structures, raise certain hopes in view of their applications in physics (see [11]).

Suppose that $\mathcal{J}$ is a complex vector space endowed with a real trilinear composition $\mathcal{J} \times \mathcal{J} \times \mathcal{J} \ni (\S, \dagger, \ddagger) \mapsto \{\S\dagger^*\ddagger\} \in \mathcal{J}$ which is complex bilinear in (x, z) and conjugate linear in y. Then $\mathcal{J}$ is called a *Jordan triple system* if $\{xy^*z\} = \{zy^*x\}$ and

$$\{\{xy^*z\}u^*v\} + \{\{xy^*v\}u^*z\} - \{xy^*\{zu^*v\}\} = \{z\{yx^*u\}^*v\}.$$

We are interested in Jordan triple systems having a Banach space structure. A complex Jordan triple system $\mathcal{J}$ with a Banach space norm $\|\cdot\|$ is called a *J^*-triple* if, for all $x \in \mathcal{J}$, the operator $x\Box x^*$ is hermitian in the sense of Banach algebra theory. Here the operator $x\Box x^*$ on $\mathcal{J}$ is defined by $(x\Box x^*)y := \{xx^*y\}$. This implies that $x\Box x^*$ has the real spectrum $\sigma(x\Box x^*) \subset \mathbb{R}$. A J^*-triple $\mathcal{J}$ is called a *JB^*-triple* if every $x \in \mathcal{J}$ satisfies $\sigma(x\Box x^*) \geq 0$ and $\|x\Box x^*\| = \|x\|^2$.

A $\mathbb{C}$-linear mapping $H : \mathcal{J} \to \mathcal{L}$ is called a *JB^*-triple homomorphism* if

$$H(\{xyz\}) = \{H(x)H(y)H(z)\}$$

for all $x, y, z \in \mathcal{J}$. A $\mathbb{C}$-linear mapping $\delta : \mathcal{J} \to \mathcal{J}$ is called a *JB^*-triple derivation* if

$$\delta(\{xyz\}) = \{\delta(x)yz\} + \{x\delta(y)z\} + \{xy\delta(z)\}$$

for all $x, y, z \in \mathcal{J}$ (see [12]).

Ulam [25] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms:

We are given a group G and a metric group G' with metric $\rho(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G$?

Hyers [5] considered the case of approximately additive mappings $f : E \to E'$, where E and E' are Banach spaces and f satisfies the *Hyers inequality*:

$$\|f(x + y) - f(x) - f(y)\| \leq \epsilon$$

for all $x, y \in E$. It was shown that the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping satisfying

$$\|f(x) - L(x)\| \leq \epsilon.$$

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Th.M. Rassias [17] provided a generalization of Hyers' Theorem which allows the *Cauchy difference to be unbounded.*

Theorem 1.1. ([17]) *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality:*

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \le \frac{2\epsilon}{2 - 2^p}\|x\|^p \tag{1.2}$$

for all $x \in E$. If $p < 0$, then the inequality (1.1) holds for all $x, y \in E$ with $x, y \ne 0$ and the inequality (1.2) holds for all $x \in E$ with $x \ne 0$. Also, if, for all $x \in E$, the function $f(tx)$ is continuous in $t \in \mathbb{R}$, then L is linear.

Th.M. Rassias [18] during the 27^{th} International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \ge 1$. Gajda [2] who followed the same approach as in Th.M. Rassias [17] gave an affirmative solution to this question for $p > 1$. For further research developments in stability of functional equations, the readers referred to the works of Cho et al. [3], Găvruta [4], Jung [10], Park [15], Th.M. Rassias [19]–[22], Th.M. Rassias and Šemrl [23], F. Skof [24] and the references cited therein.

On the other hand, in an inner product space, the following equality:

$$\|z - x\|^2 + \|z - y\|^2 = \frac{1}{2}\|x - y\|^2 + 2\left\|z - \frac{x+y}{2}\right\|^2$$

holds, which is called the *Apollonius' identity*. The following functional equation, which was motivated by the following equation:

$$Q(z-x) + Q(z-y) = \frac{1}{2}Q(x-y) + 2Q\Big(z - \frac{x+y}{2}\Big), \tag{1.3}$$

is quadratic. For this reason, the function equation (1.3) is called a *quadratic functional equation of Apollonius type* and each solution of the functional equation (1.3) is called a *quadratic mapping of Apollonius type*. Jun and Kim [9] investigated the quadratic functional equation of Apollonius type.

In this paper, employing the above equality (1.3), we introduce a new functional equation, which is called the *Apollonius type additive functional equation* and whose solution of the functional equation is called the *Apollonius type additive mapping*:

$$L(z-x) + L(z-y) = -\frac{1}{2}L(x+y) + 2L\Big(z - \frac{x+y}{4}\Big).$$

Also, we investigate homomorphisms and derivations in C^*-ternary algebras and homomorphisms and derivations in JB^*-triples.

2. Homomorphisms between C^*-ternary algebras

Throughout this section, assume that A is a C^*-ternary algebra with the norm $\|\cdot\|_A$ and B is a C^*-ternary algebra with the norm $\|\cdot\|_B$.

In this section, we investigate homomorphisms between C^*-ternary algebras.

Lemma 2.1. ([16]) *Let $f : A \to B$ be a mapping such that*

$$\left\| f(z-x) + f(z-y) + \frac{1}{2} f(x+y) \right\|_B \le \left\| 2f\left(z - \frac{x+y}{4}\right) \right\|_B \tag{2.1}$$

for all $x, y, z \in A$. Then f is additive.

Theorem 2.2. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} 8^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) = 0$$

and let $f : A \to B$ be a mapping such that

$$\|f(z-\mu x) + \mu f(z-y) + \frac{1}{2} f(x+y)\|_B \le \left\| 2f\left(z - \frac{x+y}{4}\right) \right\|_B \tag{2.2}$$

and

$$\|f([x,y,z]) - [f(x), f(y), f(z)]\|_B \le \varphi(x,y,z) \tag{2.3}$$

for all $\mu \in \mathbb{T}^{1} := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and $x, y, z \in A$. Then the mapping $f : A \to B$ is a C^*-ternary algebra homomorphism.

Proof. Let $\mu = 1$ in (2.3). By Lemma 2.1, the mapping $f : A \to B$ is additive. Letting $y = -x$ and $z = 0$, we get

$$\|f(-\mu x) + \mu f(x)\|_B \le \|2f(0)\|_B = 0$$

for all $x \in A$ and $\mu \in \mathbb{T}^{1}$. So we have

$$-f(\mu x) + \mu f(x) = f(-\mu x) + \mu f(x) = 0$$

for all $x \in A$ and all $\mu \in \mathbb{T}^{1}$. Hence $f(\mu x) = \mu f(x)$ for all $x \in A$ and $\mu \in \mathbb{T}^{1}$. By the same reasoning as in the proof of Theorem 2.1 of [13], the mapping $f : A \to B$ is

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$\mathbb{C}$-linear. It follows from (2.4) that

$$\begin{aligned}
&\|f([x,y,z]) - [f(x), f(y), f(z)]\|_B \\
= &\lim_{n\to\infty} 8^n \left\| f\left(\frac{[x,y,z]}{2^n \cdot 2^n \cdot 2^n}\right) - \left[f\left(\frac{x}{2^n}\right), f\left(\frac{y}{2^n}\right), f\left(\frac{z}{2^n}\right)\right]\right\|_B \\
\leq &\lim_{n\to\infty} 8^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) \\
= &\ 0
\end{aligned}$$

for all $x, y, z \in A$. Thus we have

$$f([x,y,z]) = [f(x), f(y), f(z)]$$

for all $x, y, z \in A$. Hence the mapping $f : A \to B$ is a C^*-ternary algebra homomorphism. This completes the proof. □

Corollary 2.3. *Let $r \neq 1$, θ be nonnegative real numbers and let $f : A \to B$ be a mapping such that*

$$\|f(z - \mu x) + \mu f(z - y) + \frac{1}{2} f(x + y)\|_B \leq \|2f(z - \frac{x+y}{4})\|_B$$

and

$$\|f([x,y,z]) - [f(x), f(y), f(z)]\|_B \leq \theta(\|x\|_A^{3r} + \|y\|_A^{3r} + \|z\|_A^{3r})$$

for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and $x, y, z \in A$. Then the mapping $f : A \to B$ is a C^*-ternary algebra homomorphism.

Proof. From Theorem 2.2 and $\varphi(x, y, z) = \theta(\|x\|_A^{3r} + \|y\|_A^{3r} + \|z\|_A^{3r})$, we can complete the proof. □

3. Derivations on C^*-ternary algebras

Throughout this section, assume that A is a C^*-ternary algebra with the norm $\|\cdot\|_A$.

In this section, we investigate derivations on C^*-ternary algebras.

Theorem 3.1. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} 8^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) = 0$$

and let $f : A \to A$ be a mapping satisfying (2.3) such that

$$\|f([x,y,z]) - [f(x), y, z] - [x, f(y), z] - [x, y, f(z)]\|_A \leq \varphi(x, y, z) \tag{3.1}$$

for all $x, y, z \in A$. Then the mapping $f : A \to A$ is a C^*-ternary derivation.

Proof. By the same reasoning as in the proof of Theorem 2.2, the mapping $f : A \to A$ is $\mathbb{C}$-linear. It follows from (3.1) that

$$\begin{aligned} & \|f([x,y,z]) - [f(x),y,z] - [x,f(y),z] - [x,y,f(z)]\|_A \\ = & \lim_{n\to\infty} 8^n \left\| f\left(\frac{[x,y,z]}{8^n}\right) - \left[f\left(\frac{x}{2^n}\right), \frac{y}{2^n}, \frac{z}{2^n}\right] - \left[\frac{x}{2^n}, f\left(\frac{y}{2^n}\right), \frac{z}{2^n}\right] - \left[\frac{x}{2^n}, \frac{y}{2^n}, f\left(\frac{z}{2^n}\right)\right]\right\|_A \\ \le & \lim_{n\to\infty} 8^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) \\ = & \ 0 \end{aligned}$$

for all $x, y, z \in A$. So we have

$$f([x,y,z]) = [f(x),y,z] + [x,f(y),z] + [x,y,f(z)]$$

for all $x, y, z \in A$. Thus the mapping $f : A \to A$ is a C^*-ternary derivation. This completes the proof. □

Corollary 3.2. *Let $r \neq 1$, θ be nonnegative real numbers and let $f : A \to A$ be a mapping satisfying* (2.3) *such that*

$$\begin{aligned} & \|f([x,y,z]) - [f(x),y,z] - [x,f(y),z] - [x,y,f(z)]\|_A \\ \le & \ \theta(\|x\|_A^{3r} + \|y\|_A^{3r} + \|z\|_A^{3r}) \end{aligned}$$

for all $x, y, z \in A$. Then the mapping $f : A \to A$ is a C^*-ternary derivation.

Proof. From Theorem 3.1 and $\varphi(x,y,z) = \theta(\|x\|_A^{3r} + \|y\|_A^{3r} + \|z\|_A^{3r})$, we can completes the proof. □

4. Homomorphisms between JB^*-triples

Throughout this paper, assume that $\mathcal{J}$ is a JB^*-triple with the norm $\|\cdot\|_{\mathcal{J}}$ and $\mathcal{L}$ is a JB^*-triple with the norm $\|\cdot\|_{\mathcal{L}}$.

In this section, we investigate homomorphisms between JB^*-triples.

Theorem 4.1. *Let $\varphi : \mathcal{J}^3 \to [0,\infty)$ be a function such that*

$$\lim_{n\to\infty} 8^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) = 0$$

and let $f : \mathcal{J} \to \mathcal{L}$ be a mapping such that

$$\left\| f(z - \mu x) + \mu f(z - y) + \frac{1}{2} f(x+y) \right\|_{\mathcal{L}} \le \|2f(z - \frac{x+y}{4})\|_{\mathcal{L}} \tag{4.1}$$

and

$$\|f(\{xyz\}) - \{f(x)f(y)f(z)\}\|_{\mathcal{L}} \le \varphi(x,y,z) \tag{4.2}$$

for all $\mu \in \mathbb{T}^{1}$ and $x, y, z \in \mathcal{J}$. Then the mapping $f : \mathcal{J} \to \mathcal{L}$ is a JB^*-triple homomorphism.

Proof. By the same reasoning as in the proof of Theorem 2.2, the mapping $f : \mathcal{J} \to \mathcal{L}$ is $\mathbb{C}$-linear. It follows from (4.2) that

$$\begin{aligned}
&\|f(\{xy\}]) - \{f(x)f(y)f(z)\}\|_{\mathcal{L}} \\
= &\lim_{n\to\infty} 8^n \Big\| f\Big(\frac{\{xyz\}}{2^n \cdot 2^n \cdot 2^n}\Big) - \Big\{ f\Big(\frac{x}{2^n}\Big) f\Big(\frac{y}{2^n}\Big) f\Big(\frac{z}{2^n}\Big)\Big\}\Big\|_{\mathcal{L}} \\
\le &\lim_{n\to\infty} 8^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) \\
= &\ 0
\end{aligned}$$

for all $x, y, z \in \mathcal{J}$. Thus we have

$$f(\{xyz\}) = \{f(x)f(y)f(z)\}$$

for all $x, y, z \in \mathcal{J}$. Hence the mapping $f : \mathcal{J} \to \mathcal{L}$ is a JB^*-triple homomorphism. This completes the proof. □

Corollary 4.2. *Let $r \neq 1$, θ be nonnegative real numbers and let $f : \mathcal{J} \to \mathcal{L}$ be a mapping such that*

$$\Big\| f(z - \mu x) + \mu f(z - y) + \frac{1}{2} f(x + y) \Big\|_{\mathcal{L}} \le \Big\| 2f(z - \frac{x + y}{4}) \Big\|_{\mathcal{L}}$$

and

$$\|f(\{xyz\}) - \{f(x)f(y)f(z)\}\|_{\mathcal{L}} \le \theta(\|x\|_{\mathcal{J}}^{3r} + \|y\|_{\mathcal{J}}^{3r} + \|z\|_{\mathcal{J}}^{3r})$$

for all $\mu \in \mathbb{T}^{1/\!\!\!\!/}$ and $x, y, z \in \mathcal{J}$. Then the mapping $f : \mathcal{J} \to \mathcal{L}$ is a JB^*-triple homomorphism.

Proof. From Theorem 4.1 and $\varphi(x, y, z) = \theta(\|x\|_A^{3r} + \|y\|_A^{3r} + \|z\|_A^{3r})$, we can completes the proof. □

5. Derivations on JB^*-triples

Throughout this paper, assume that $\mathcal{J}$ is a JB^*-triple with the norm $\|\cdot\|_{\mathcal{J}}$.

In this section, we investigate derivations on JB^*-triples.

Theorem 5.1. *Let $\varphi : \mathcal{J}^3 \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} 8^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) = 0$$

and let $f : \mathcal{J} \to \mathcal{J}$ be a mapping satisfying (4.1) *such that*

$$\|f(\{xyz\}) - \{f(x)yz\} - \{xf(y)z\} - \{xyf(z)\}\|_{\mathcal{J}} \le \varphi(x, y, z) \tag{5.1}$$

for all $x, y, z \in \mathcal{J}$. Then the mapping $f : \mathcal{J} \to \mathcal{J}$ is a JB^*-triple derivation.

Proof. By the same reasoning as in the proof of Theorem 2.2, the mapping $f : \mathcal{J} \to \mathcal{J}$ is $\mathbb{C}$-linear. It follows from (5.1) that

$$\begin{aligned}
& \|f(\{xyz\}) - \{f(x)yz\} - \{xf(y)z\} - \{xyf(z)\}\|_{\mathcal{J}} \\
= & \lim_{n\to\infty} 8^n \Big\| f\Big(\frac{\{xyz\}}{8^n}\Big) - \Big\{ f\Big(\frac{x}{2^n}\Big)\frac{y}{2^n}\frac{z}{2^n}\Big\} - \Big\{\frac{x}{2^n} f\Big(\frac{y}{2^n}\Big)\frac{z}{2^n}\Big\} - \Big\{\frac{x}{2^n}\frac{y}{2^n} f\Big(\frac{z}{2^n}\Big)\Big\}\Big\|_{\mathcal{J}} \\
\le & \lim_{n\to\infty} 8^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) \\
= & \ 0
\end{aligned}$$

for all $x, y, z \in \mathcal{J}$. So we have

$$f(\{xyz\}) = \{f(x)yz\} + \{xf(y)z\} + \{xyf(z)\}$$

for all $x, y, z \in \mathcal{J}$. Thus the mapping $f : \mathcal{J} \to \mathcal{J}$ is a JB^*-triple derivation. This completes the proof. □

Corollary 5.2. *Let $r \neq 1$, θ be nonnegative real numbers and let $f : \mathcal{J} \to \mathcal{J}$ be a mapping satisfying* (4.1) *such that*

$$\begin{aligned}
& \|f(\{xyz\}) - \{f(x)yz\} - \{xf(y)z\} - \{xyf(z)\}\|_{\mathcal{J}} \\
\le & \ \theta(\|x\|_{\mathcal{J}}^{3r} + \|y\|_{\mathcal{J}}^{3r} + \|z\|_{\mathcal{J}}^{3r})
\end{aligned}$$

for all $x, y, z \in \mathcal{J}$. Then the mapping $f : \mathcal{J} \to \mathcal{J}$ is a JB^*-triple derivation.

Proof. From Theorem 5.1 and $\varphi(x, y, z) = \theta(\|x\|_A^{3r} + \|y\|_A^{3r} + \|z\|_A^{3r})$, we can completes the proof. □

Acknowledgments

The first author was supported by the Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2013053358).

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On Grüss Type Integral Inequality Involving the Saigo's Fractional Integral Operators

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Abstract

Using Saigo's fractional integral operators, we establish a generalized version of the Grüss type integral inequality related to the bounded integrable functions, whose bounds are integrable functions. Some special cases of our results are also considered.

Key words: Grüss integral inequalities, Gauss hypergeometric function, fractional integral operators.

1. Introduction

Fractional differential equations appear more and more frequently for modeling of relevant systems in several fields of applied sciences. These equations play important roles, not only in mathematics, but also in physics, dynamical systems, control systems and engineering, to create the mathematical model of many physical phenomena [1]-[6]. To define upper and lower bounds to solutions of these fractional differential equations, one has to adopt the study of fractional integral inequalities. Moreover, fractional integral inequalities have many applications in numerical quadrature, transform theory, probability, and in statistical problems, but the most useful ones are in establishing uniqueness of solutions in fractional boundary value problems. For detailed applications on the subject, one may refer [7]-[21] and the references cited therein.

In [22], Grüss proved an inequality, which establishes a connection between the integral of the product of two functions and the product of the integrals of individual functions. This well known Grüss inequality [22] (see also [23, p. 296]) is defined as follows:
Let f and g are two continuous functions defined on $[a, b]$, such that $m \leq f(t) \leq M$ and $p \leq g(t) \leq P$, for each $t \in [a, b]$, where m, p, M, P are given real constants, then

$$\left| \frac{1}{(b-a)} \int_a^b f(t)\, g(t)\, dt - \frac{1}{(b-a)^2} \int_a^b f(t)\, dt \int_a^b g(t)\, dt \right| \leq \frac{1}{4}(M-m)(P-p), \tag{1}$$

and the constant $1/4$ is the best possible.
In the theory of approximations, the Chebyshev and Grüss inequalities [23]-[25], are useful to give a

lower bound or an upper bound for the functionals. Therefore, in the literature several generalizations of the Grüss type integral inequality have been addressed extensively by several researchers (see [26]-[43]). Recently, Dahmani *et al.* [44] using Riemann-Liouville fractional integral operators, established a generalization of Grüss integral inequality as under:
Let f and g are two integrable functions with constant bounds defined on $[0,\infty)$, such that

$$m \leq f(t) \leq M, \quad p \leq g(t) \leq P, \quad m,p,M,P \in \mathbb{R}, \quad t \in [0,\infty),$$

then for $\alpha > 0$

$$\left|\frac{t^{\alpha}}{\Gamma(\alpha+1)} I^{\alpha} f\left(t\right) g\left(t\right) - I^{\alpha} f\left(t\right) I^{\alpha} g\left(t\right)\right| \leq \left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right)^{2} (M-m)(P-p), \tag{2}$$

where $I^{\alpha} f(t)$ denote the familiar Riemann-Liouville fractional integral operator of a function $f(t)$. By replacing the constants appeared as bounds of the functions f and g, by four integrable functions, Tariboon *et al.* [45] investigates more general forms of the inequality (2). Moreover, Secer *et al.* [46] provides q-extension of the result due to Tariboon *et al.* [45] and derive certain interesting consequent results and special cases.
Recently, Kalla and Rao [47], Purohit and Raina [48], [49], Purohit *et al.* [50], Baleanu *et al.* [51] and Baleanu and Purohit [52] added one more dimension to this study by introducing certain new integral inequalities for synchronous functions, involving the fractional hypergeometric integral operators.

In this paper, our aim is to establish a new generalization of the Grüss integral inequality related to the integrable functions whose bounds are four integrable functions, involving Saigo's fractional integral operators [53]. We also develop some consequent results and special cases of the main results.

Before stating the fractional integral inequalities, we mention below the basic definitions and notations of some well-known operators of fractional calculus, which are used further in this paper.

Let $\alpha > 0$, $\beta, \eta \in \mathbb{R}$, then the Saigo fractional integral $I_{0,t}^{\alpha,\beta,\eta}$ of order α for a real-valued continuous function $f(t)$ is defined by ([53], see also [54, p. 19], [55]):

$$I_{0,t}^{\alpha,\beta,\eta} f(t) = \frac{t^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \, {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{\tau}{t}\right) f(\tau) d\tau, \tag{3}$$

where, the function ${}_2F_1(-)$ appearing as a kernel for the operator (3) is the Gaussian hypergeometric function defined by

$${}_2F_1\left(a,b;c;t\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!}, \tag{4}$$

and $(a)_n$ is the Pochhammer symbol

$$(a)_n = a(a+1)\cdots(a+n-1), \quad (a)_0 = 1.$$

The Saigo fractional integral operator $I_{0,t}^{\alpha,\beta,\eta}$ includes both the Riemann-Liouville and the Erdélyi-Kober fractional integral operators given by the following relationships:

$$I^{\alpha} f(t) = I_{0,t}^{\alpha,-\alpha,\eta} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau \quad (\alpha > 0) \tag{5}$$

and

$$I^{\alpha,\eta} f(t) = I_{0,t}^{\alpha,0,\eta} f(t) = \frac{t^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \, \tau^{\eta} f(\tau) d\tau \quad (\alpha > 0, \eta \in \mathbb{R}). \tag{6}$$

For $f(t) = t^{\mu}$ in (3), we get the known formula [53]:

$$I_{0,t}^{\alpha,\beta,\eta} t^{\mu} = \frac{\Gamma(\mu+1)\Gamma(\mu+1-\beta+\eta)}{\Gamma(\mu+1-\beta)\Gamma(\mu+1+\alpha+\eta)} t^{\mu-\beta}, \tag{7}$$

$$(\alpha > 0, min(\mu, \mu-\beta+\eta) > -1, t > 0)$$

which shall be used in the sequel.

2. A Generalized Grüss Integral Inequality

Our results in this section are based on the following lemma, giving functional relation for Saigo's fractional integral operators, with an integrable functions:

Lemma 1. *Let f, φ_1 and φ_2 are integrable functions defined on $[0,\infty)$, such that*

$$\varphi_1(t) \leq f(t) \leq \varphi_2(t), \quad for\ all\ t \in [0,\infty). \tag{8}$$

Then, for $t > 0, \alpha > \max\{0, -\beta\}$, $\beta < 1$ and $\beta - 1 < \eta < 0$, we have

$$\begin{aligned}
&\frac{\Gamma(1-\beta+\eta)t^{-\beta}}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} I_{0,t}^{\alpha,\beta,\eta} f^2(t) - \left(I_{0,t}^{\alpha,\beta,\eta} f(t)\right)^2 \\
&= (I_{0,t}^{\alpha,\beta,\eta}\varphi_2(t) - I_{0,t}^{\alpha,\beta,\eta} f(t))(I_{0,t}^{\alpha,\beta,\eta} f(t) - I_{0,t}^{\alpha,\beta,\eta}\varphi_1(t)) \\
&\quad - \frac{\Gamma(1-\beta+\eta)t^{-\beta}}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} I_{0,t}^{\alpha,\beta,\eta}(\varphi_2(t) - f(t))(f(t) - \varphi_1(t)) \\
&\quad + \frac{\Gamma(1-\beta+\eta)t^{-\beta}}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} I_{0,t}^{\alpha,\beta,\eta}\varphi_1(t)f(t) - I_{0,t}^{\alpha,\beta,\eta}\varphi_1(t) I_{0,t}^{\alpha,\beta,\eta} f(t) \\
&\quad + \frac{\Gamma(1-\beta+\eta)t^{-\beta}}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} I_{0,t}^{\alpha,\beta,\eta}\varphi_2(t)f(t) - I_{0,t}^{\alpha,\beta,\eta}\varphi_2(t) I_{0,t}^{\alpha,\beta,\eta} f(t) \\
&\quad + I_{0,t}^{\alpha,\beta,\eta}\varphi_1(t) I_{0,t}^{\alpha,\beta,\eta}\varphi_2(t) - \frac{\Gamma(1-\beta+\eta)t^{-\beta}}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} I_{0,t}^{\alpha,\beta,\eta}\varphi_1(t)\varphi_2(t).
\end{aligned} \tag{9}$$

Proof. By the hypothesis of inequality (8), for any $\tau, \rho > 0$, it follows that

$$\begin{aligned}
&(\varphi_2(\rho) - f(\rho))(f(\tau) - \varphi_1(\tau)) + (\varphi_2(\tau) - f(\tau))(f(\rho) - \varphi_1(\rho)) \\
&\quad - (\varphi_2(\tau) - f(\tau))(f(\tau) - \varphi_1(\tau)) - (\varphi_2(\rho) - f(\rho))(f(\rho) - \varphi_1(\rho)) \\
&= f^2(\tau) + f^2(\rho) - 2f(\tau)f(\rho) + \varphi_2(\rho)f(\tau) + \varphi_1(\tau)f(\rho) - \varphi_1(\tau)\varphi_2(\rho) \\
&\quad + \varphi_2(\tau)f(\rho) + \varphi_1(\rho)f(\tau) - \varphi_1(\rho)\varphi_2(\tau) - \varphi_2(\tau)f(\tau) + \varphi_1(\tau)\varphi_2(\tau) \\
&\quad - \varphi_1(\tau)f(\tau) - \varphi_2(\rho)f(\rho) + \varphi_1(\rho)\varphi_2(\rho) - \varphi_1(\rho)f(\rho).
\end{aligned} \tag{10}$$

Consider

$$\begin{aligned}
F(t,\tau) &= \frac{t^{-\alpha-\beta}(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\ {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1 - \frac{\tau}{t}\right) \qquad (\tau \in (0,t)\ ;\ t > 0) \\
&= \frac{1}{\Gamma(\alpha)}\ \frac{(t-\tau)^{\alpha-1}}{t^{\alpha+\beta}} + \frac{(\alpha+\beta)(-\eta)}{\Gamma(\alpha+1)}\ \frac{(t-\tau)^{\alpha}}{t^{\alpha+\beta+1}} +
\end{aligned} \tag{11}$$

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$$\frac{(\alpha+\beta)(\alpha+\beta+1)(-\eta)(-\eta+1)}{\Gamma(\alpha+2)}\ \frac{(t-\tau)^{\alpha+1}}{t^{\alpha+\beta+2}}+\cdots .$$

where $\tau\in(0,t)\quad(t>0)$. Multiplying both sides of (10) by $F(t,\tau)$ (where $F(t,\tau)$ is given by (11)) and integrating the resulting identity with respect to τ from 0 to t, and using (3), we get

$$\begin{aligned}
&(\varphi_2(\rho)-f(\rho))(I_{0,t}^{\alpha,\beta,\eta}f(t)-I_{0,t}^{\alpha,\beta,\eta}\varphi_1(t))+(I_{0,t}^{\alpha,\beta,\eta}\varphi_2(t)-I_{0,t}^{\alpha,\beta,\eta}f(t))(f(\rho)-\varphi_1(\rho))\\
&\quad-I_{0,t}^{\alpha,\beta,\eta}(\varphi_2(t)-f(t))(f(t)-\varphi_1(t))-(\varphi_2(\rho)-f(\rho))(f(\rho)-\varphi_1(\rho))I_{0,t}^{\alpha,\beta,\eta}\{1\}\\
&=I_{0,t}^{\alpha,\beta,\eta}f^2(t)+f^2(\rho)I_{0,t}^{\alpha,\beta,\eta}\{1\}-2f(\rho)I_{0,t}^{\alpha,\beta,\eta}f(t)+\varphi_2(\rho)I_{0,t}^{\alpha,\beta,\eta}f(t)+f(\rho)I_{0,t}^{\alpha,\beta,\eta}\varphi_1(t)\\
&\quad-\varphi_2(\rho)I_{0,t}^{\alpha,\beta,\eta}\varphi_1(t)+f(\rho)I_{0,t}^{\alpha,\beta,\eta}\varphi_2(t)+\varphi_1(\rho)I_{0,t}^{\alpha,\beta,\eta}f(t)-\varphi_1(\rho)I_{0,t}^{\alpha,\beta,\eta}\varphi_2(t)\\
&\quad-I_{0,t}^{\alpha,\beta,\eta}\varphi_2(t)f(t)+I_{0,t}^{\alpha,\beta,\eta}\varphi_1(t)\varphi_2(t)-I_{0,t}^{\alpha,\beta,\eta}\varphi_1(t)f(t)-\varphi_2(\rho)f(\rho)I_{0,t}^{\alpha,\beta,\eta}\{1\}\\
&\quad+\varphi_1(\rho)\varphi_2(\rho)I_{0,t}^{\alpha,\beta,\eta}\{1\}-\varphi_1(\rho)f(\rho)I_{0,t}^{\alpha,\beta,\eta}\{1\}. \qquad (12)
\end{aligned}$$

Next, multiplying both sides of (12) by $F(t,\rho)$ $(\rho\in(0,t),\ t>0)$, where $F(t,\rho)$ is given by (11), and integrating with respect to ρ from 0 to t, we obtain

$$\begin{aligned}
&2(I_{0,t}^{\alpha,\beta,\eta}\varphi_2(t)-I_{0,t}^{\alpha,\beta,\eta}f(t))(I_{0,t}^{\alpha,\beta,\eta}f(t)-I_{0,t}^{\alpha,\beta,\eta}\varphi_1(t))-2I_{0,t}^{\alpha,\beta,\eta}(\varphi_2(t)-f(t))(f(t)-\varphi_1(t))I_{0,t}^{\alpha,\beta,\eta}\{1\}\\
&=2I_{0,t}^{\alpha,\beta,\eta}\{1\}\,I_{0,t}^{\alpha,\beta,\eta}f^2(t)-2\left(I_{0,t}^{\alpha,\beta,\eta}f(t)\right)^2+2I_{0,t}^{\alpha,\beta,\eta}\varphi_1(t)I_{0,t}^{\alpha,\beta,\eta}f(t)-2I_{0,t}^{\alpha,\beta,\eta}\{1\}\,I_{0,t}^{\alpha,\beta,\eta}\varphi_1(t)f(t)\\
&\quad+2I_{0,t}^{\alpha,\beta,\eta}\varphi_2(t)I_{0,t}^{\alpha,\beta,\eta}f(t)-2I_{0,t}^{\alpha,\beta,\eta}\{1\}\,I_{0,t}^{\alpha,\beta,\eta}\varphi_2(t)f(t)\\
&\quad-2I_{0,t}^{\alpha,\beta,\eta}\varphi_1(t)I_{0,t}^{\alpha,\beta,\eta}\varphi_2(t)+2I_{0,t}^{\alpha,\beta,\eta}\{1\}\,I_{0,t}^{\alpha,\beta,\eta}\varphi_1(t)\varphi_2(t), \qquad (13)
\end{aligned}$$

which upon using the formula (7), leads to the desired result (9).

Now, we obtain a general integral inequality, which gives an estimation for the fractional integral of a product in terms of the product of the individual function fractional integrals, involving Saigo's fractional hypergeometric operators. We give our results related to the integrable functions f and g, whose bounds are integrable functions and satisfying the Cauchy-Schwarz inequality.

Theorem 1. *Let f and g be two integrable functions on $[0,\infty)$ and $\varphi_1,\varphi_2,\psi_1$ and ψ_2 are four integrable functions on $[0,\infty)$, such that*

$$\varphi_1(t)\le f(t)\le\varphi_2(t),\quad \psi_1(t)\le g(t)\le\psi_2(t),\qquad for\ all\ t\in[0,\infty). \qquad (14)$$

Then, for $t>0,\alpha>\max\{0,-\beta\},\ \beta<1$ and $\beta-1<\eta<0$, we have

$$\left|\frac{\Gamma(1-\beta+\eta)t^{-\beta}}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)}I_{0,t}^{\alpha,\beta,\eta}f(t)g(t)-I_{0,t}^{\alpha,\beta,\eta}f(t)I_{0,t}^{\alpha,\beta,\eta}g(t)\right|\le\sqrt{\mathcal{T}(f,\varphi_1,\varphi_2)\ \mathcal{T}(g,\psi_1,\psi_2)}. \qquad (15)$$

where

$$\begin{aligned}
\mathcal{T}(u,v,w)&=(I_{0,t}^{\alpha,\beta,\eta}w(t)-I_{0,t}^{\alpha,\beta,\eta}f(t))(I_{0,t}^{\alpha,\beta,\eta}u(t)-I_{0,t}^{\alpha,\beta,\eta}v(t))\\
&\quad+\frac{\Gamma(1-\beta+\eta)t^{-\beta}}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)}I_{0,t}^{\alpha,\beta,\eta}v(t)u(t)-I_{0,t}^{\alpha,\beta,\eta}v(t)I_{0,t}^{\alpha,\beta,\eta}u(t)\\
&\quad+\frac{\Gamma(1-\beta+\eta)t^{-\beta}}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)}I_{0,t}^{\alpha,\beta,\eta}w(t)u(t)-I_{0,t}^{\alpha,\beta,\eta}w(t)I_{0,t}^{\alpha,\beta,\eta}u(t)
\end{aligned}$$

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$$+ I_{0,t}^{\alpha,\beta,\eta} v(t) I_{0,t}^{\alpha,\beta,\eta} w(t) - \frac{\Gamma(1-\beta+\eta)t^{-\beta}}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} I_{0,t}^{\alpha,\beta,\eta} v(t)w(t). \tag{16}$$

Proof. Let f and g are two integrable functions on $[0,\infty)$ and satisfying inequality (14), then for any $\tau, \rho > 0$, let us define a function

$$\mathcal{H}(\tau,\rho) = (f(\tau) - f(\rho))(g(\tau) - g(\rho)), \quad \tau, \rho \in (0,t), t > 0. \tag{17}$$

On multiplying both sides of (17) by $F(t,\tau)F(t,\rho)$, where $F(t,\tau)$ and $F(t,\rho)$ are given by (11), and integrating with respect to τ and ρ, respectively, from 0 to t, we can write

$$\frac{t^{-2\alpha-2\beta}}{2\Gamma^2(\alpha)} \int_0^t \int_0^t (t-\tau)^{\alpha-1} (t-\rho)^{\alpha-1} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1 - \frac{\tau}{t}\right) {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1 - \frac{\rho}{t}\right) \mathcal{H}(\tau,\rho) d\tau d\rho$$
$$= \frac{\Gamma(1-\beta+\eta)t^{-\beta}}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} I_{0,t}^{\alpha,\beta,\eta} f(t)g(t) - I_{0,t}^{\alpha,\beta,\eta} f(t) I_{0,t}^{\alpha,\beta,\eta} g(t). \tag{18}$$

Now, by applying the Cauchy-Schwarz inequality, we have

$$\left(\frac{\Gamma(1-\beta+\eta)t^{-\beta}}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} I_{0,t}^{\alpha,\beta,\eta} f(t)g(t) - I_{0,t}^{\alpha,\beta,\eta} f(t) I_{0,t}^{\alpha,\beta,\eta} g(t)\right)^2$$
$$\leq \left(\frac{\Gamma(1-\beta+\eta)t^{-\beta}}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} I_{0,t}^{\alpha,\beta,\eta} f^2(t) - \left(I_{0,t}^{\alpha,\beta,\eta} f(t)\right)^2\right)$$
$$\times \left(\frac{\Gamma(1-\beta+\eta)t^{-\beta}}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} I_{0,t}^{\alpha,\beta,\eta} g^2(t) - \left(I_{0,t}^{\alpha,\beta,\eta} g(t)\right)^2\right). \tag{19}$$

On the other hand, we observe that each term of the series in (11) is positive, and hence, the function $F(t,\tau)$ remains positive, for all $\tau \in (0, t)$ $(t > 0)$. Therefore, under the hypothesis of Lemma 1, it is obvious to see that either if a function f is integrable and nonnegative on $[0, \infty)$, then $I_{0,t}^{\alpha,\beta,\eta} f(t) \geq 0$; or if a function f is integrable and nonpositive on $[0, \infty)$, then $I_{0,t}^{\alpha,\beta,\eta} f(t) \leq 0$.

Now, by noting the relation that

$$(\varphi_2(t) - f(t))(f(t) - \varphi_1(t)) \geq 0$$

and

$$(\psi_2(t) - g(t))(g(t) - \psi_1(t)) \geq 0,$$

we have

$$\frac{\Gamma(1-\beta+\eta)t^{-\beta}}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} I_{0,t}^{\alpha,\beta,\eta} (\varphi_2(t) - f(t))(f(t) - \varphi_1(t)) \geq 0,$$

and

$$\frac{\Gamma(1-\beta+\eta)t^{-\beta}}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} I_{0,t}^{\alpha,\beta,\eta} (\psi_2(t) - g(t))(g(t) - \psi_1(t)) \geq 0.$$

Thus, upon using Lemma 1, we get

$$\frac{\Gamma(1-\beta+\eta)t^{-\beta}}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} I_{0,t}^{\alpha,\beta,\eta} f^2(t) - \left(I_{0,t}^{\alpha,\beta,\eta} f(t)\right)^2$$
$$\leq (I_{0,t}^{\alpha,\beta,\eta} \varphi_2(t) - I_{0,t}^{\alpha,\beta,\eta} f(t))(I_{0,t}^{\alpha,\beta,\eta} f(t) - I_{0,t}^{\alpha,\beta,\eta} \varphi_1(t))$$
$$+ \frac{\Gamma(1-\beta+\eta)t^{-\beta}}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} I_{0,t}^{\alpha,\beta,\eta} \varphi_1(t) f(t) - I_{0,t}^{\alpha,\beta,\eta} \varphi_1(t) I_{0,t}^{\alpha,\beta,\eta} f(t)$$

$$
\begin{aligned}
&+ \frac{\Gamma(1-\beta+\eta)t^{-\beta}}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} I_{0,t}^{\alpha,\beta,\eta}\varphi_2(t)f(t) - I_{0,t}^{\alpha,\beta,\eta}\varphi_2(t) I_{0,t}^{\alpha,\beta,\eta} f(t) \\
&+ I_{0,t}^{\alpha,\beta,\eta}\varphi_1(t) I_{0,t}^{\alpha,\beta,\eta}\varphi_2(t) - \frac{\Gamma(1-\beta+\eta)t^{-\beta}}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} I_{0,t}^{\alpha,\beta,\eta}\varphi_1(t)\varphi_2(t) \\
&= \mathcal{T}(f,\varphi_1,\varphi_2). \qquad (20)
\end{aligned}
$$

Similarly, we can write

$$
\frac{\Gamma(1-\beta+\eta)t^{-\beta}}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} I_{0,t}^{\alpha,\beta,\eta} g^2(t) - \left(I_{0,t}^{\alpha,\beta,\eta} g(t)\right)^2 \leq \mathcal{T}(g,\psi_1,\psi_2). \qquad (21)
$$

Evidently, on making use of the relations (19), (20) and (21), one can easily arrive at the desired inequality (15), which proves Theorem 1.

3. Consequent Results and Special Cases

We now briefly consider some consequences of the result derived in the previous section. Following [53], by suitably specializing the values of parameters α, β and η the result presented in this paper would find further Grüss type integral inequalities involving the Erdélyi-Kober and the Riemann-Liouville type fractional integral operator, on taking relations (6) and (5) into account. To this end, let us set $\beta = 0$ and make use of the relation (6), then Theorem 1 yield the following integral inequality involving the Erdélyi-Kober type fractional integral operators:

Corollary 1. *Let $f, g, \varphi_1, \varphi_2, \psi_1$ and ψ_2 are integrable functions on $[0,\infty)$, satisfying inequalities of (14), then, for $t > 0, \alpha > 0$ and $-1 < \eta < 0$, we have*

$$
\left| \frac{\Gamma(1+\eta)}{\Gamma(1+\alpha+\eta)} I^{\alpha,\eta} f(t)g(t) - I^{\alpha,\eta} f(t) I^{\alpha,\eta} f(t) \right| \leq \sqrt{\mathcal{T}'(f,\varphi_1,\varphi_2)\ \mathcal{T}'(g,\psi_1,\psi_2)}. \qquad (22)
$$

where

$$
\begin{aligned}
\mathcal{T}'(u,v,w) = {} & (I^{\alpha,\eta} w(t) - I^{\alpha,\eta} f(t))(I^{\alpha,\eta} u(t) - I^{\alpha,\eta} v(t)) \\
& + \frac{\Gamma(1+\eta)}{\Gamma(1+\alpha+\eta)} I^{\alpha,\eta} v(t)u(t) - I^{\alpha,\eta} v(t) I^{\alpha,\eta} u(t) \\
& + \frac{\Gamma(1+\eta)}{\Gamma(1+\alpha+\eta)} I^{\alpha,\eta} w(t)u(t) - I^{\alpha,\eta} w(t) I^{\alpha,\eta} u(t) \\
& + I^{\alpha,\eta} v(t) I^{\alpha,\eta} w(t) - \frac{\Gamma(1+\eta)}{\Gamma(1+\alpha+\eta)} I^{\alpha,\eta} v(t)w(t). \qquad (23)
\end{aligned}
$$

Next, if we replace β by $-\alpha$, and make use of the relation (5), then Theorems 1 corresponds to the known results due to Tariboon *et al.* [45, p. 5, Theorem 9].

Further, if we put $\varphi_1(t) = m, \varphi_2(t) = M, \psi_1(t) = p$ and $\psi_2(t) = P$, where $m, M, p, P \in \mathbb{R}, \forall\ t \in [0,\infty)$, then the Theorem 1 yields the following result obtained by Kalla and Rao [47] in a slightly different form:

Corollary 2. *Let f and g be two integrable functions on $[0,\infty)$, satisfying the following inequalities*

$$
m \leq f(t) \leq M, \quad p \leq g(t) \leq P, \quad m, M, p, P \in \mathbb{R} \quad \text{for all } t \in [0,\infty). \qquad (24)
$$

Then, for $t > 0, \alpha > \max\{0, -\beta\}$, $\beta < 1$ and $\beta - 1 < \eta < 0$, we have

$$\left| \frac{\Gamma(1-\beta+\eta)t^{-\beta}}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} I_{0,t}^{\alpha,\beta,\eta} f(t)g(t) - I_{0,t}^{\alpha,\beta,\eta} f(t) I_{0,t}^{\alpha,\beta,\eta} f(t) \right|$$
$$\leq \left(\frac{\Gamma(1-\beta+\eta)t^{-\beta}}{2\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} \right)^2 (M-m)(P-p). \tag{25}$$

Now, again if we replace β by $-\alpha$, and make use of the relation (5), then the above result of Corollary 2 corresponds to the known results due to Dahmani *et al.* [44].

Lastly, we conclude this paper by remarking that, we have introduced a new general extensions of Grüss type integral inequality, which gives an estimation for the fractional integral of a product in terms of the product of the individual function fractional integrals involving Saigo's fractional integral operators. Our main result related to the integrable functions f and g, whose bounds are integrable functions. Therefore, by suitably specializing the arbitrary function $\varphi_1(t), \varphi_2(t), \psi_1(t)$ and $\psi_2(t)$, one can further easily obtain additional integral inequalities involving the Riemann-Liouville, Erdélyi-Kober and Saigo type fractional integral operators from our main result.

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The value distribution of solutions of some types of q-shift difference Riccati equations *

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Abstract

The purpose of this paper is to investigated the value distribution of solutions of some types of q-shift difference Riccati equations and obtained some results about the zeros, fixed points and deficient value of solutions of such equations.

Key words: system; q-shift; difference equation; fixed-point.

Mathematical Subject Classification (2010): 39A50, 30D 35.

1 Introduction and Main Results

Throughout this paper, the term "meromorphic" will always mean meromorphic in the complex plane $\mathbb{C}$. Considering meromorphic function f, we shall assume that readers are familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions such as $m(r, f)$, $N(r, f)$, $T(r, f)$, etc. of Nevalinna theory, (see Hayman [12], Yang [19] and Yi and Yang [20]). We use $\rho(f), \lambda(f)$ and $\lambda(\frac{1}{f})$ to denote the order, the exponent of convergence of zeros and the exponent of convergence of poles of $f(z)$ respectively, and we also use the notion $\tau(f)$ to denote the exponent of convergence of fixed points of f that is defined as

$$\tau(f) = \limsup_{r\to\infty} \frac{N(r, \frac{1}{f(z)-z})}{T(r, f)},$$

*This work was supported by the NSFC(11301233, 61202313), the Natural Science Foundation of Jiang-Xi Province in China(20132BAB211001), the Foundation of Education Department of Jiangxi (GJJ14644,GJJ14774) of China.

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and $S(r,f)$ denotes any quantity satisfying $S(r,f) = o(T(r,f))$ for all r on a set F of logarithmic density 1, the logarithmic density of a set F is defined by

$$\limsup_{r\to\infty} \frac{1}{\log r} \int_{[1,r]\cap F} \frac{1}{t} dt.$$

Throughout this paper, the set F of logarithmic density will be not necessarily the same at each occurrence.

In 2007, Bergweiler and Langley [2] investigated the existence of zeros of $f(z+c)-f(z)$ and $\frac{f(z+c)-f(z)}{f(z)}$, and obtained several profound and significant result.

Theorem 1.1 *[2]. There exists $\delta_0 \in (0, 1/2)$ with the following property. Let f be a transcendental entire function with $\rho(f) \le \rho < \frac{1}{2} + \delta_0 < 1$. Then $\frac{f(z+1)-f(z)}{f(z)}$ has infinitely many zeros.*

In 2008, Chen and Shon [4] considered zeros and fixed points of differences and divided differences of entire functions with $\rho(f) = 1$ and obtained the following theorem.

Theorem 1.2 *[4]. Let $c \in \mathbb{C} \setminus \{0\}$ and let f be a transcendental entire function with $\rho(f) = \rho = 1$ that has infinitely many zeros with $\lambda(f) = \lambda < 1$. Then $f(z+c) - f(z)$ has infinitely many zeros and infinitely many fixed points.*

For meromorphic function f, Chen and Shon [5] further investigated the zeros and fixed points of $f(z+c) - f(z)$ and obtained

Theorem 1.3 *[5]. Let $c \in \mathbb{C} \setminus \{0\}$ and let f be a meromorphic function of order of growth $\rho(f) = \rho \le 1$. Suppose that f satisfies $\lambda(1/f) < \lambda(f) < 1$ or has infinitely many zeros (with $\lambda(f) = 0$) and finitely many poles. Then $f(z+c) - f(z)$ has infinitely many zeros and satisfies $\lambda(f(z+c) - f(z)) = \lambda(f)$.*

In 2009, Fletcher et al [8] studied the zeros of $f(qz)-f(z)$ and $\frac{f(qz)-f(z)}{f(z)}$, and obtained the following results

Theorem 1.4 *[8]. Let $q \in \mathbb{C}$ with $|q| > 1$. Let f be a transcendental meromorphic function in the plane with*

$$L(f) = \liminf_{r\to\infty} \frac{T(r,f)}{(\log r)^2} = 0.$$

Then at least one of $f(qz) - f(z)$ and $\frac{f(qz)-f(z)}{f(z)}$ has infinitely many zeros.

As we all know, the equation

$$f(z+c) = \frac{a(z) + b(z)f(z)}{c(z) + d(z)f(z)}$$

is called as the difference Riccati equation, where $a(z), b(z), c(z), d(z)$ are meromorphic, $a(z)d(z) - b(z)c(z) \not\equiv 0$, and $c \in \mathbb{C} \setminus \{0\}$. Similarly, we can call the q-shift difference equation

$$f(qz + \eta) = \frac{a(z) + b(z)f(z)}{c(z) + d(z)f(z)} \tag{1}$$

the q-shift difference Riccati equation, where $q \in \mathbb{C} \setminus \{0\}, \eta \in \mathbb{C}$.

In 2000, Gundersen et al [9] studied the growth order of solutions of the Schröder equation $f(qz) = R(z, f(z))$ and prove that $\rho(f) = \frac{\log deg_f R}{\log |q|}$ for $|q| > 1$ and some other conditions. In 2011, Liu and Qi [14] further investigated the growth of solutions of some q-shift difference equation $f(qz + \eta) = R(z, f(z))$ and extended some results given by Gundersen [9]. Recently, there exist many papers about the value distribution and growth of solutions of complex difference, q-difference and q-shift difference equations (including [3, 6, 11, 13, 16, 17, 18]).

Example 1.1 *Let $q = 2, \eta = 1$, we have that $f(z) = z$ satisfies equation*

$$f(2z + 1) = \frac{1 + 4(z + 1)f(z)}{1 + 2f(z)}.$$

Then $f(z) - z = z - z \equiv 0$. Thus, this shows that solution of q-shift difference Riccati equation has infinitely many fixed points.

Example 1.2 *Let $q = 2, \eta = 1$, we have that $f(z) = 1 + z$ satisfies equation*

$$f(2z + 1) = \frac{6 + \frac{2z^2+4}{(z+1)} f(z)}{1 + f(z)}.$$

Then $f(z) - z = 1 + z - z = 1 \neq 0$. Thus, this shows that solution of q-shift difference Riccati equation has no any fixed point.

Example 1.3 *Let $q = 2, \eta = 1$, we have that $f(z) = z^2$ satisfies equation*

$$f(2z + 1) = \frac{1 + 4z + (4z^2 + 5z + 4)f(z)}{1 + f(z)}.$$

Then $f(z) - z = z^2 - z = 0$. Thus, this shows that solution of q-shift difference Riccati equation has finite many fixed points.

From Examples 1.1-1.3, it is a natural question to ask: *What conditions will guarantee that the solutions of q-shift difference Riccati equations has infinitely many fixed points?* Inspired by the ideas of Refs. [4, 5, 8, 14], we will investigate the zeros and fixed points of solutions of some types of complex q-shift Riccati equations and obtain the following results.

Theorem 1.5 *For equation (1), let $c(z) \equiv b(z) \equiv 1$ and $a(z), d(z)$ be nonconstant rational functions such that $a(z)d(z) \not\equiv 0$, and let $q \in \mathbb{C} \setminus \{0\}, \eta \in \mathbb{C}$ and $|q| \neq 1$. If f is a transcendental meromorphic solution of (1). Thus, we have*

(i) if $\frac{a(z)}{d(z)}$ is not a constant, then $f(z)$ has no deficient value;

(ii) if $[d(z)p_m(qz+\eta)-1]p_m(z)-a(z) \not\equiv 0$, then $f(z)-p_m(z)$ has infinitely many zeros, where $p_m(z)$ is a polynomial of degree m. In particularly, if $[d(z)(qz+\eta)-1]z-a(z) \not\equiv 0$, then $f(z)$ has infinitely many fixed points.

Theorem 1.6 *For equation (1), let $d(z) \equiv 1$ and $a(z), b(z), c(z)$ be rational functions such that $a(z) \not\equiv 0$, and let $q \in \mathbb{C} \setminus \{0\}, \eta \in \mathbb{C}$ and $|q| \neq 1$. If f is a transcendental meromorphic solution of (1). Thus, we have*

(i) if $a(z)$ is not a constant, then $0, \infty$ are not deficient values of $f(z)$. Furthermore, if only one of $a(z), b(z), c(z)$ has the maximum degree, then $f(z)$ has no deficient value;

(ii) if $deg_z\{c(z)(qz+\eta) - b(z)z - a(z)\} \neq 2$, then $f(z)$ has infinitely many fixed points and $\tau(f) = \rho(f)$;

(iii) if $b(z) \equiv c(z)$, $zb(z) - a(z) = \frac{mkz^2}{(m+k)^2}$ and $b(z) \neq \left\{-\frac{z(qz+\eta)}{qz+\eta-z}, \frac{mz}{m+k}, \frac{kz}{m+k}\right\}$, where m, k are two distinct constants and $m + k \neq 0$, then $f(qz+\eta) - f(z)$ has infinitely many fixed points and $\tau(f(qz+\eta) - f(z)) = \rho(f)$;

(iv) if $b(z) \equiv c(z)$, $z^2b(z)^2 + 4(1+z)a(z) \equiv h(z)^2$, $h(z)$ is a nonconstant rational function, and satisfy $b(z)[h(qz+\eta) - (qz+\eta+1)h(z)] - (qz+\eta)b(qz+\eta)h(z) \not\equiv 0$, $(2+z)b(z) - h(z) \not\equiv 0$ and $(2+3z)b(z) + h(z) \not\equiv 0$, then $\frac{f(qz+\eta)-f(z)}{f(z)}$ has infinitely many fixed points and $\tau(\frac{f(qz+\eta)-f(z)}{f(z)}) = \rho(f)$.

2 Some Lemmas

Similar to the proof of [1, Theorem 3.1] or [10, Theorem 2.1], we can also get the following theorem analogous to the Clunie Lemma [7].

Lemma 2.1 *[14, Theorem 2.4] Let f be a nonconstant zero-order meromorphic solution of*

$$f^n P(z, f) = Q(z, f),$$

where $P(z, f)$ and $Q(z, f)$ are q-shift difference polynomials in f. If the degree of $Q(z, f)$ as a polynomial in $f(z)$ and its q-shifts is at most n, then

$$m(r, P(z, f)) = S(r, f),$$

on a set of logarithmic density 1.

The following result is a difference counterpart to the standard result due to Mohon'ko and Mohon'ko [15] and the proof is similar to that of [10, Theorem 3.2].

Lemma 2.2 *[14, Theorem 2.5] Let f be a nonconstant zero-order meromorphic solution of $P(z, f) = 0$, where $P(z, f)$ is a q-shift difference polynomial in $f(z)$. If $P(z, a) \not\equiv 0$ for a small function $a(z)$ of $f(z)$, then*

$$m(r, \frac{1}{f-a}) = S(r, f),$$

on a set of logarithmic density 1.

Lemma 2.3 *[14, Theorem 2.1] Let $f(z)$ be a nonconstant zero-order meromorphic function and $q \in \mathbb{C} \setminus \{0\}$. Then*

$$m\left(r, \frac{f(qz+\eta)}{f(z)}\right) = S(r, f),$$

on a set of logarithmic density 1.

3 The proof of Theorem 1.5

Suppose that $f(z)$ is a transcendental solution of (1). Since $a(z), b(z), c(z), d(z)$ are rational functions, by using the same argument as in [9], we can get that $f(z)$ is of zero order.

(i) We will firstly prove that any constant $\mu \in \mathbb{C}$ is not a deficient value of $f(z)$. For (1), Since $c(z) \equiv b(z) \equiv 1$, from (1) we have

$$P_1(z,f) := d(z)f(qz+\eta)f(z) + f(qz+\eta) - f(z) - a(z) \equiv 0.$$

Then $P_1(z,\mu) = d(z)\mu^2 - a(z) \equiv 0$. If $\mu = 0$, a contradiction with $a(z)d(z) \not\equiv 0$. If $\mu \neq 0$, since $\frac{a(z)}{d(z)}$ is not a constant, we can get a contradiction too. Thus, $P_1(z,\mu) \not\equiv 0$. From Lemma 2.2, we have

$$m(r, \frac{1}{f-\mu}) = S(r,f)$$

on a set F of logarithmic density 1. Thus, it follows that

$$N(r, \frac{1}{f-\mu}) = T(r,f) + S(r,f)$$

on a set F of logarithmic density 1. Therefore, for any constant $\mu \in \mathbb{C}$, we have

$$\delta(\mu, f) = 1 - \limsup_{r\to\infty} \frac{N(r, \frac{1}{f-\mu})}{T(r,f)} \leq 1 - \limsup_{r\to\infty, r\in F} \frac{N(r, \frac{1}{f-\mu})}{T(r,f)} = 0.$$

Thus, $\delta(\mu, f) = 0$, that is, μ is not a deficient value of $f(z)$.

Next, we will prove that ∞ is not a deficient value of $f(z)$. Let $y(z) = \frac{1}{f(z)}$. Then $T(r,y) = T(r,f) + S(r,f)$. Since $f(z)$ is of zero order, then $y(z)$ is of zero order. Substituting $f(z) = \frac{1}{y(z)}$ into (1), we have

$$P_2(z,f) := a(z)y(qz+\eta)y(z) + y(qz+\eta) - y(z) - d(z) \equiv 0.$$

Since $d(z) \not\equiv 0$, we have $P_2(z,0) \not\equiv 0$. By Lemma 2.2, we have

$$m(r, \frac{1}{y}) = S(r,y),$$

on a set F of logarithmic density 1. Thus,

$$N(r,f) = N(r, \frac{1}{y}) = T(r,y) + S(r,y) = T(r,f) + S(r,f),$$

on a set F of logarithmic density 1. So, we can get that ∞ is not a deficient value of $f(z)$.

(ii) Let $g(z) = f(z) - p_m(z)$. Since $f(z)$ is a transcendental meromorphic function with zero order and $p_m(z)$ is a nonconstant polynomial of degree m, then $g(z)$ is transcendental with zero order and $T(r,g) = T(r,f) + O(\log r)$. Substituting $f(z) = g(z) + p_m(z)$ into (1), we have

$$P_3(z,g) := d(z)g(z)g(qz+\eta) + d(z)[g(qz+\eta) + p_m(z)]p_m(z) - a(z) - g(z) - p_m(z) \equiv 0.$$

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Since $[d(z)p_m(qz+\eta)-1]p_m(z)-a(z)\not\equiv 0$, we have $P_3(z,0)\not\equiv 0$. By Lemma 2.2, we have

$$N(r,\frac{1}{f(z)-p_m(z)})=N(r,\frac{1}{g(z)})=T(r,g)+S(r,g)=T(r,f)+S(r,f),$$

on a set F of logarithmic density 1. Thus, we can prove that $f(z)-p_m(z)$ has infinitely many zeros.

Thus, this completes the proof of Theorem 1.5.

4 The proof of Theorem 1.6

Suppose that $f(z)$ is a transcendental solution of (1). Since $a(z), b(z), c(z), d(z)$ are rational functions, by using the same argument as in [9], we can get that $f(z)$ is of zero order.

(i) We firstly prove that ∞ is not a deficient value of $f(z)$. Since $d(z)\equiv 1$ and from (1), we have

$$f(qz+\eta)f(z)=-c(z)f(qz+\eta)+b(z)f(z)+a(z). \tag{2}$$

Since $f(z)$ is of zero order, from Lemma 2.1 and (2), we have

$$m(r,f(qz+\eta))=S(r,f), \tag{3}$$

on a set F of logarithmic density 1. Since $m(r,f)\leq m(r,\frac{f(z)}{f(qz+\eta)})+m(r,f(qz+\eta))$, by applying Lemma 2.3, from (3) we have

$$m(r,f)=S(r,f),$$

on a set F of logarithmic density 1. Thus, we have $N(r,f)=T(r,f)+S(r,f)$ on a set F of logarithmic density 1. Similar to the method as in Theorem 1.5, we can get that ∞ is not a deficient value of $f(z)$.

Now, we prove that 0 is not a deficient value of $f(z)$. Set

$$P_4(z,f):=f(qz+\eta)f(z)+c(z)f(qz+\eta)-b(z)f(z)-a(z)=0. \tag{4}$$

Since $a(z)\not\equiv 0$, then we have $P_4(z,0)=-a(z)\not\equiv 0$. Thus, it follows by Lemma 2.2 that $m(r,\frac{1}{f})=S(r,f)$ on a set F of logarithmic density 1. So, $N(r,\frac{1}{f})=T(r,f)+S(r,f)$ on a set F of logarithmic density 1. By using the same argument as in Theorem 1.5, we can get that 0 is not a deficient value of $f(z)$.

Furthermore, we will prove that any nonzero finite constant $\mu\in\mathbb{C}$ is not a deficient value of $f(z)$. For any nonzero finite constant μ, it follows from (4) that

$$P_4(z,\mu)=\mu^2+\mu(c(z)-b(z))-a(z).$$

Since only one of $a(z), b(z), c(z)$ has the maximum degree, then we have $P_4(z,\mu)\not\equiv 0$. It follows from Lemma 2.2 that $m(r,\frac{1}{f-\mu})=S(r,f)$ on a set F of logarithmic density 1. So, $N(r,\frac{1}{f-\mu})=T(r,f)+S(r,f)$ on a set F of logarithmic density 1. By using the same argument as in Theorem 1.5, we can get that μ is not a deficient value of $f(z)$.

(ii) Let $g(z) = f(z) - z$. Thus, $T(r, g) = T(r, f) + O(\log r)$. Since $f(z)$ is transcendental with zero order, then g is transcendental with zero order. Substituting $f(z) = g(z) + z$ into (1), and set

$$\begin{aligned} P_5(z, g(z)) :=& g(z)g(qz+\eta) + zg(qz+\eta) + (qz+\eta)g(z) + z(qz+\eta) + c(z)g(qz+\eta) \\ &+ (qz+\eta)c(z) - b(z)g(z) - zb(z) - a(z) = 0. \end{aligned}$$

It follows that $P_5(z, 0) = (qz+\eta)z + (qz+\eta)c(z) - zb(z) - a(z)$. Since $deg_z\{(qz+\eta)c(z) - zb(z) - a(z)\} \neq 2$, then $P_5(z, 0) \not\equiv 0$. By Lemma 2.2, we have

$$m(r, \frac{1}{g}) = S(r, g),$$

on a set F of logarithmic density 1. Thus,

$$N(r, \frac{1}{f(z) - z}) = N(r, \frac{1}{g}) = T(r, g) + S(r, g) = T(r, f) + S(r, f),$$

on a set F of logarithmic density 1. So, we can prove that $f(z)$ has infinitely many fixed points and $\tau(f) = \rho(f)$.

(iii) Since $b(z) \equiv c(z)$, set $\Delta_q f(z) := f(qz+\eta) - f(z)$, it follows from (1) that

$$f(qz+\eta) - f(z) - z = \Delta_q f(z) - z = -\frac{f^2 + zf(z) + zb(z) - a(z)}{f(z) + b(z)}. \tag{5}$$

Since $zb(z) - a(z) = \frac{mkz^2}{(m+k)^2}$, we deduce from (5) that

$$\Delta_q f(z) - z = -\frac{(f(z) + \frac{mz}{m+k})(f(z) + \frac{kz}{m+k})}{f(z) + b(z)}. \tag{6}$$

From (4), we have

$$P_4(z, -\frac{mz}{m+k}) = m^2(qz+\eta)z + (m+k)mb(z)(qz+\eta-z) - (m+k)^2 a(z) = 0, \tag{7}$$

$$P_4(z, -\frac{kz}{m+k}) = k^2(qz+\eta)z + (m+k)kb(z)(qz+\eta-z) - (m+k)^2 a(z) = 0. \tag{8}$$

We will prove that $P_4(z, -\frac{mz}{m+k}) \equiv 0$ and $P_4(z, -\frac{kz}{m+k}) \equiv 0$ can not hold at the same time. If $P_4(z, -\frac{mz}{m+k}) \equiv 0$ and $P_4(z, -\frac{kz}{m+k}) \equiv 0$, it follows from (7) and (8) that

$$(qz+\eta)z + b(z)[(q-1)z + \eta] \equiv 0,$$

which is a contradiction with the assumptions of Theorem 1.6(iii). Thus, we get that $P_4(z, -\frac{mz}{m+k}) \not\equiv 0$ or $P_4(z, -\frac{kz}{m+k}) \not\equiv 0$. Suppose that $P_4(z, -\frac{mz}{m+k}) \not\equiv 0$, it follows by Lemma 2.2 that

$$m(r, \frac{1}{f(z) + \frac{mz}{m+k}}) = S(r, f),$$

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on a set F of logarithmic density 1. Thus,

$$N(r, \frac{1}{f(z)+\frac{mz}{m+k}}) = T(r,f) + S(r,f), \tag{9}$$

on a set F of logarithmic density 1.

From (6), we will claim that the zeros of $f(z)+\frac{mz}{m+k}$ are all the zeros of $\Delta_q f(z)-z$ except at most finite many zeros. If z_0 is a common zero of $f(z)+\frac{mz}{m+k}$ and $f(z)+b(z)$, it follows that $b(z_0)-\frac{mz_0}{m+k}=0$. Since $b(z)\not\equiv\frac{mz}{m+k}$ and $b(z)$ are rational functions, it follows that the zeros of $b(z)-\frac{mz}{m+k}$ is finite. Thus, $f(z)+\frac{mz}{m+k}$ and $f(z)+b(z)$ only have finite many common zeros. And since $f(z)+\frac{mz}{m+k}$ and $f(z)+\frac{kz}{m+k}$ have all common poles. Thus, it follows that

$$N\left(r, \frac{f(z)+b(z)}{(f(z)+\frac{mz}{m+k})(f(z)+\frac{kz}{m+k})}\right) = N\left(r, \frac{1}{f(z)+\frac{mz}{m+k}}\right) + O(\log r). \tag{10}$$

Thus, it follows from (9) and (10) that

$$N\left(r, \frac{1}{\Delta_q f(z)-z}\right) = N\left(r, \frac{1}{f(z)+\frac{mz}{m+k}}\right) + O(\log r) = T(r,f)+S(r,f),$$

on a set F of logarithmic density 1.

If $P_4(z,-\frac{kz}{m+k})\not\equiv 0$, similar to the above discussion, we can get that

$$N\left(r, \frac{1}{\Delta_q f(z)-z}\right) = N\left(r, \frac{1}{f(z)+\frac{kz}{m+k}}\right) + O(\log r) = T(r,f)+S(r,f),$$

on a set F of logarithmic density 1.

Therefore, $f(qz+\eta)-f(z)$ has infinitely many fixed points and $\tau(f(qz+\eta)-f(z)) = \rho(f)$.

(iv) Since $b(z)\equiv c(z)$, it follows from (1) that

$$\frac{\Delta_q f(z)}{f(z)} - z = -\frac{(z+1)f(z)^2 + zb(z)f(z) - a(z)}{f(z)(f(z)+b(z))}. \tag{11}$$

Since $z^2b(z)^2 + 4(1+z)a(z) \equiv h(z)^2$, set $\alpha_1(z) = \frac{h(z)-zb(z)}{2(1+z)}$ and $\alpha_2(z) = -\frac{h(z)+zb(z)}{2(1+z)}$, it follows from (11) that

$$\frac{\Delta_q f(z)}{f(z)} - z = -\frac{(z+1)(f(z)-\alpha_1(z))(f(z)-\alpha_2(z))}{f(z)(f(z)+b(z))}. \tag{12}$$

For (4), we have

$$P_4(z,\alpha_1(z)) = \alpha_1(z)\alpha_1(qz+\eta) + b(z)[\alpha_1(qz+\eta)-\alpha_1(z)] - a(z) = 0, \tag{13}$$
$$P_4(z,\alpha_2(z)) = \alpha_2(z)\alpha_2(qz+\eta) + b(z)[\alpha_2(qz+\eta)-\alpha_2(z)] - a(z) = 0. \tag{14}$$

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We will prove that $P_4(z,\alpha_1(z)) \equiv 0$ and $P_4(z,\alpha_2(z)) \equiv 0$ can not hold at the same time. If $P_4(z,\alpha_1(z)) \equiv 0$ and $P_4(z,\alpha_2(z)) \equiv 0$, by a series of calculation, we can get that $b(z)[h(qz+\eta)-(qz+\eta+1)h(z)]-(qz+\eta)b(qz+\eta)h(z) \equiv 0$, which is a contradiction with the assumption of Theorem 1.6(iv).

If $P_4(z,\alpha_1(z)) \not\equiv 0$, similar to the above discussion as in Theorem 1.6(iii), we can get

$$N(r,\frac{1}{f(z)-\alpha_1(z)}) = T(r,f)+S(r,f), \tag{15}$$

on a set F of logarithmic density 1. Since $a(z), b(z), h(z)$ are nonconstant rational functions and $(2+z)b(z)-h(z) \not\equiv 0$, by using the same argument as in Theorem 1.6(iii), we can get

$$N\left(r,\frac{1}{\frac{\Delta_q f(z)}{f(z)}-z}\right) = N\left(r,\frac{1}{f(z)-\alpha_1(z)}\right)+O(\log r) = T(r,f)+S(r,f),$$

on a set F of logarithmic density 1.

If $P_4(z,\alpha_2(z)) \not\equiv 0$, similar to the above discussion, we can get that

$$N\left(r,\frac{1}{\frac{\Delta_q f(z)}{f(z)}-z}\right) = N\left(r,\frac{1}{f(z)-\alpha_2(z)}\right)+O(\log r) = T(r,f)+S(r,f),$$

on a set F of logarithmic density 1.

Therefore, $\frac{f(qz+\eta)-f(z)}{f(z)}$ has infinitely many fixed points and $\tau(\frac{f(qz+\eta)-f(z)}{f(z)}) = \rho(f)$.

Thus, we complete the proof of Theorem 1.6.

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ENERGY DECAY RATES FOR VISCOELASTIC WAVE EQUATION WITH DYNAMIC BOUNDARY CONDITIONS

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Abstract. In this paper we consider the energy decay rates for viscoelastic wave equation with dynamic boundary conditions. Motivated by results of Gerbi and Said-Houari [7], Cavalcanti and Oquendo[11], Li and Zhao [12] and we intend to study the energy decay for problem (1.1). By using the perturbed energy method, we proved the general energy decay rate.

1. INTRODUCTION

In this paper, we consider the following viscoelastic wave equation with dynamic boundary condition:

$$\begin{aligned}
&u^{''} - \triangle u - \alpha\Delta u^{'} + \int_0^t h(t-\tau)div[a(x)\nabla u(\tau)]d\tau = 0 \text{ in } \Omega\times(0,\infty)\ ,\\
&u = 0 \text{ on } \Gamma_0\times(0,\infty)\ ,\\
&u^{''} + \frac{\partial u}{\partial \nu} + \alpha\frac{\partial u^{'}}{\partial\nu} - \int_0^t h(t-\tau)[a(x)\nabla u(\tau)]\cdot\nu d\tau + |u^{'}|^{m-2}u^{'} = |u|^{p-2}u\\
&\qquad \text{on } \Gamma_1\times(0,\infty)\ ,\\
&u^{'}(x,0) = u_1(x), u(x,0) = u_0(x) \text{ in } \Omega.
\end{aligned} \tag{1.1}$$

Here $m \geq 2, p \geq 2$ and Ω is a bounded open subset of $R^n (n \geq 1)$ with a smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1, \overline{\Gamma_0}\cap\overline{\Gamma_1} = \emptyset$, where Γ_0 and Γ_1 are measurable over $\partial\Omega$, endowed with the $(n-1)$-dimensional Lebesgue measure $\lambda_{n-1}(\Gamma_i), i = 0, 1$, ν is the unit outward normal to $\partial\Omega$, α is a positive constant, h is a positive uniformly decay C^1 function, a is a bounded, nonnegative C^1 function, and u_0, u_1 are given functions.

This problem has its origin in the mathematical description of viscoelastic materials. It is well known that viscoelastic materials exhibit natural damping which is due to the special property of theses materials to retain a memory of their past history. From the mathematical point of view, these damping effects are modeled by integro-differential operators. Thus problem (1.1) arises in the mathematical description of viscoelastic materials with dynamic boundary conditions. Therefore, the viscoelastic materials with dynamic boundary problems are important and interest as they have wide applications in natural science. The term $\Delta u'$, indicates that the stress is proportional not only to

1991 *Mathematics Subject Classification.* 35L05, 35L75, 74D10, 74J05.

Key words and phrases. Energy decay rates, Viscoelastic wave equation, Dynamic boundary conditions.

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the strain rate(see [7]). Here, we are does not neglect acceleration terms on the boundary. Such type of boundary conditions are usually called dynamic boundary conditions. They are not only important from the theoretical point of view but also arise in several physical applications. Many authors have investigated energy decay rates for nonlinear viscoelastic wave equation with boundary dissipation(see [3],[4],[5],[6],[9],[17],[18],[20],[21],[22],[24],[25],[26]).

Cavalcanti et al.[2], proved the exponential decay for the solution of semilinear viscoelastic wave equations with localized damping.

$$\begin{aligned}
&u^{''} - \triangle u - \int_0^t g(t-\tau)\Delta u(\tau)d\tau + a(x)u^{'} + |u|^r u = 0 \text{ in } \Omega\times(0,\infty)\ ,\\
&u = 0 \text{ on } \partial\Omega, t \geq 0\ ,\\
&u^{'}(x,0) = u_1(x), u(x,0) = u_0(x) \text{ in } \Omega,
\end{aligned}$$

where Ω is a bounded domain of $R^n(n \geq 1)$ with smooth boundary $\partial\Omega$, $r > 0$, g is positive nonincreasing function defined on R_+ and $a : \Omega \longrightarrow R_+$ is a bounded function, which may vanish on a part of the domain.

Under the conditions

$$\xi_1 g(t) \leq g^{'}(t) \leq \xi_2 g(t), t \geq 0,$$

for some constants $\xi_1, \xi_2 > 0$ and $a(x) \geq a_0 > 0$ in a subdomain $\omega \subset \Omega$, with measure(ω)> 0 and satisfying some geometry restrictions, the authors established an exponential rate of decay.

In [11], Cavalcanti and Oquendo consider

$$u^{''} - \kappa_0\triangle u + \int_0^t div[a(x)g(t-s)\nabla u(\tau)]d\tau + b(x)h(u^{'}) + f(u) = 0,$$

under similar conditions on the relaxation function g and $a(x) + b(x) \geq \rho > 0$ for all $x \in \Omega$. They improved the result of [2] by establishing exponential stability for g decaying exponentially and h nonlinear. And Messaoudi [15] investigated a general decay rate for a viscoelastic wave equation with Dirichlet boundary conditions under a more general condition on g:

$$g^{'}(t) \leq -\xi(t)g(t),\ |\frac{\xi^{'}(t)}{\xi(t)}| \leq k,\ \xi(t) > 0,\ \xi^{'}(t) \leq 0,\ \forall t \geq 0.$$

Recently Li and Zhao [12] consider the following problem and proved the uniform energy decay for nonlinear viscoelastic wave equation with boundary damping.

$$\begin{aligned}
&u^{''} - \kappa_0\triangle u + \int_0^t g(t-s)div[a(x)\nabla u(s)]ds + b(x)h(u^{'}) = 0,\\
&\qquad\qquad (x,t) \in \Omega\times(0,\infty),\\
&-\frac{\partial u}{\partial\nu} + \int_0^t g(t-s)[a(x)\nabla u(s)]\cdot\nu ds = f(u), \quad (x,t) \in \Gamma_0\times(0,\infty),\\
&u(x,t) = 0, \quad (x,t) \in \Sigma_1 = \Gamma_1\times(0,\infty),\\
&u^{'}(x,0) = u_1(x), u(x,0) = u_0(x), \quad x \in \Omega.
\end{aligned}$$

Moreover Park et al.[23], Wu and Chen[26] consider the uniform decay estimates of solutions for nonlinear viscoelastic wave equations with boundary dissipations. Dynamic boundary problems for wave equation have been considered by Gerbi and Said-Houari [7]:

$$\begin{aligned}
&u^{''} - \triangle u - \alpha \triangle u^{'} = |u|^{p-2}u, \quad x \in \Omega, t > 0,\\
&u(x,t) = 0, \quad x \in \Gamma_0, t > 0,\\
&u^{''}(x,t) = -a[\frac{\partial u}{\partial \nu}(x,t) + \frac{\partial u^{'}}{\partial \nu}(x,t) + \gamma(|u^{'}|^{m-2}u^{'})(x,t)], \quad x \in \Gamma_1, t > 0,\\
&u^{'}(x,0) = u_1(x), u(x,0) = u_0(x), \quad x \in \Omega.
\end{aligned}$$

Motivated by theses results, in this work we prove the energy decay for problem (1.1). Our idea comes from [7],[11],[12], which we are concerned with the decay rates for viscoelastic wave equation with dynamic boundary problem (1.1) and we also give a decay result of global solutions under a weaker assumptions on the relaxation function $h(t)$ (see [11],(H1) below). Therefore, our result allows a larger class of relaxation functions and improves earlier results in [13] and [12]. The main point for showing desired decay rates in constructing a Lyapunov functional $\mathcal{L}$, which is equivalent to the energy of the problem (1.1), satisfying

$$\mathcal{L}^{'}(t) \le -c\xi(t)\mathcal{L}(t), \quad \text{for some } c > 0 (\text{see } (3.28)).$$

The present work is organized as follows: In section 2, we present some assumptions, notations, some known results for our work and state our main results. In section 3, we proved the general energy decay rate by the perturbed energy method.

2. Preliminaries and main results

We begin with some notations and known results for problem (1.1). First we recall the following assumptions made in [10] and [19].

(H1) $h : R_+ \to R_+$ is a bounded C^1 function satisfying $h(0) > 0$, and there exists a positive differentiable function $\xi(t)$ such that

$$h^{'}(t) \le -\xi(t)h(t), t \ge 0,$$

and $\xi(t)$ and satisfies

$$|\frac{\xi^{'}(t)}{\xi(t)}| \le k, \xi^{'}(t) \le 0, \forall t > 0, \int_0^\infty \xi(t)dt = \infty.$$

(H2) $a : \Omega \to R_+$ is a nonnegative bounded function and $a(x) > a_0 \ge 0$ on Ω with

$$1 - ||a||_{L^\infty} \int_0^\infty h(s)ds = l > 0.$$

(H3) We assume $m > \frac{r}{r+1-p}, 2 < p < r$ if $n \ge 3$ and $p > 2$ if $n = 1, 2$, where $r = \frac{2(n-1)}{n-2}$.

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Throughout this paper, we define

$$H^1_{\Gamma_0}(\Omega) = \{u \in H^1(\Omega) \mid u = 0 \quad on \quad \Gamma_0\},$$

and the following scalar products

$$(u, v) = \int_\Omega u(x)v(x)dx, \quad (u, v)_{\Gamma_1} = \int_{\Gamma_1} u(x)v(x)d\Gamma,$$

and the following norms

$$||u||_{L^p(\Omega)} = (\int_\Omega |u(x)|^p dx)^{1/p}, \quad ||u||_{L^p(\Gamma_1)} = (\int_{\Gamma_1} |u(x)|^p d\Gamma)^{1/p}.$$

To simplify the notations, we denote $||u||_{L^2(\Omega)}, ||u||_{L^2(\Gamma_1)}$ by $||u||, ||u||_{\Gamma_1}$ respectively. We define the functions $I, J : H^1_{\Gamma_0} \to R$ by

$$\begin{aligned} I(t) = I(u(t)) &= ||\nabla u(t)||^2 - ||\sqrt{a(x)}\nabla u(t)||^2 \int_0^t h(s)ds \\ &+(h \circ \nabla u)(t) - ||u(t)||^p_{p,\Gamma_1} \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} J(t) = J(u(t)) &= \frac{1}{2}||\nabla u(t)||^2 - \frac{1}{2}||\sqrt{a(x)}\nabla u(t)||^2 \int_0^t h(s)ds \\ &+\frac{1}{2}(h \circ \nabla u)(t) - \frac{1}{p}||u(t)||^p_{p,\Gamma_1}, \end{aligned} \tag{2.2}$$

where $(h \circ \nabla u)(t) = \int_0^t h(t-s)||\sqrt{a(x)}(\nabla u(t) - \nabla u(s))||^2 ds, \forall u \in H^1_{\Gamma_0}$. Then we define the energy of a solution u of (1.1) as follows:

$$E(t) = J(u(t)) + \frac{1}{2}||u^{'}(t)||^2 + \frac{1}{2}||u^{'}(t)||^2_{\Gamma_1}. \tag{2.3}$$

We recall the trace Sobolev embedding

$$H^1_{\Gamma_0}(\Omega) \hookrightarrow L^2(\Gamma_1) \text{ for } 2 \le q \le r = 2(n-1)/(n-2) \tag{2.4}$$

and the embedding inequality

$$||u||_{q,\Gamma_1} \le B_q||\nabla u||, \tag{2.5}$$

where B_q is the optimal constant. We will also by using the following Sobolev-poincaré embedding

$$H^1_0(\Omega) \hookrightarrow L^q(\Omega),$$

so

$$\begin{aligned} &||v||_q \le B_{*q}||\nabla v||, \quad \forall v \in H^1_0(\Omega) \\ &\text{for } 2 \le q \le 2n/(n-2) \text{ if } n \ge 3 \text{ and } q \ge 2, \text{ if } n = 1, 2 \ , \end{aligned} \tag{2.6}$$

where B_{*q} is the optimal constant. For $t \ge 0$, we define

$$d(t) = \inf_{u \in H^1_{\Gamma_0}(\Omega)} \sup_{u|_{\Gamma_1} \ne 0, \lambda \ge 0} J(\lambda u). \tag{2.7}$$

Then similar as in [10],[19], we can prove the following lemma.

Lemma 2.1. *For $t \geq 0$, we have*

$$0 < d_1 \leq d_2(u) = \sup_{\lambda \geq 0} J(\lambda u),$$

where

$$(2.8)\quad d_1 = \frac{p-2}{2p}\left(\frac{l}{B_p^2}\right)^{p/(p-2)},$$

$$d_2(u(t)) = \frac{p-2}{2p}\left(\frac{(1-\int_0^t h(s)ds)||\sqrt{a(x)}\nabla u(t)||^2 + (h \circ \nabla u)(t)}{||u(t)||^2_{p,\Gamma_1}}\right)^{p/(p-2)}.$$

We now state a local existence theorem of the problem (1.1)(see [8], [10],[19]).

Theorem 2.1. *Suppose that $(H2),(H3)$ hold and assume that $(u_0,u_1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$, then problem (1.1) has a unique local solution which satisfies*

$$(2.9)\quad u \in C([0,T]; H^1_{\Gamma_0}(\Omega)) \cap C^1([0,T]; L^2(\Omega)), u^{'} \in L^m([0,T] \times \Gamma_1),$$

for some $T > 0$ and the energy identity

$$(2.10)\qquad E(t) + \int_0^t ||u^{'}(s)||^m_{m,\Gamma_1} ds - \frac{1}{2}\int_0^t (h^{'} \circ \nabla u)(s) ds$$
$$+\alpha \int_0^t ||\nabla u^{'}(s)||^2 ds + \frac{1}{2}\int_0^t h(s)||\sqrt{a(x)}\nabla u(s)||^2 ds = E(0)$$

holds for $0 \leq t \leq T$.

Remark 2.1. From (2.10), we see that

$$(2.11)\qquad \frac{d}{dt}E(t) = -||u^{'}(t)||^m_{m,\Gamma_1} - \alpha||\nabla u^{'}(t))||^2$$
$$+\frac{1}{2}(h^{'} \circ \nabla u)(t) - \frac{1}{2}h(t)||\sqrt{a(x)}\nabla u(t)||^2 \leq 0.$$

Theorem 2.2. *([10], [19]). Suppose that $(H2)$ and $(H3)$ hold. Assume that $(u_0,u_1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$ and satisfy*

$$(2.12)\qquad I(0) > 0, \qquad E(0) < d.$$

Then the solution of (1.1) is global and bounded in time, and satisfies

$$I(t) > 0,$$
$$(2.13)\qquad l||\nabla u(t)||^2 + ||u^{'}(t)||^2 \leq \frac{2p}{p-2}E(t) \leq \frac{2p}{p-2}E(0)$$

for all $0 \leq t < \infty$.

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Remark 2.2. From (2.5) and (2.13), we deduce that

$$\begin{aligned} ||u(t)||_{p,\Gamma_1}^p &\le B_p^p ||\nabla u(t)||^p \\ &\le B_p^p \left(\frac{2p}{(p-2)l} \right)^{p/2} E^{\frac{p-2}{2}}(0) E(t) \\ &= \frac{2p}{p-2} \left(\frac{2p}{p-2} E(0) \right)^{\frac{p-2}{2}} \left(\frac{B_p^2}{l} \right)^{p/2} E(t) \\ &= \frac{2p}{p-2} \left(\frac{E(0)}{d_1} \right)^{\frac{p-2}{2}} E(t). \end{aligned} \tag{2.14}$$

We can now state the asymptotic behavior of the solution of problem (1.1).

Theorem 2.3. *Suppose that (H1)-(H3) hold. $m \ge 2$ and $m < r = 2(n-1)/(n-2)$ if $m \ge 3$. Assume that $(u_0, u_1) \in H_{\Gamma_0}^1(\Omega) \times L^2(\Omega)$ and satisfy (2.12). Then for each $t_0 > 0$, there exist two positive constants c and r such that the solution of (1.1) satisfies*

$$E(t) \le c e^{-r \int_{t_0}^t \xi(s) ds}, \quad t \ge t_0.$$

3. Decay rate of global solution

For positive constants ϵ_1, ϵ_2, let us define the perturbed modified energy by

$$F(t) = E(t) + \epsilon_1 \Phi(t) + \epsilon_2 \Psi(t), \tag{3.1}$$

where

$$\Phi(t) = \int_\Omega u(t) u'(t) dx + \frac{\alpha}{2} \int_\Omega |\nabla u(t)|^2 dx + \int_{\Gamma_1} u(t) u'(t) d\Gamma, \tag{3.2}$$

and

$$\begin{aligned} \Psi(t) = &- \int_\Omega a(x) u'(t) \int_0^t h(t-\tau)(u(t) - u(\tau)) d\tau dx \\ &- \int_{\Gamma_1} a(x) u'(t) \int_0^t h(t-\tau)(u(t) - u(\tau)) d\tau d\Gamma. \end{aligned} \tag{3.3}$$

Now, we give a lemma which indicates that the generalized energy $E(t)$ and the functional $F(t)$ are equivalent in the following sense.

Lemma 3.1. *Let u be the solution obtained in Theorem 2.1, then there exist two positive constants β_1 and β_2 such that*

$$\beta_1 E(t) \le F(t) \le \beta_2 E(t). \tag{3.4}$$

Proof. By Young's inequality and poincaré's inequality, it yields

$$\begin{aligned} |\Phi(t)| &\le \frac{1}{2} ||u'(t)||^2 + \frac{B_{*2}^2}{2} ||\nabla u(t)||^2 \\ &+ \frac{\alpha}{2} ||\nabla u(t)||^2 + \frac{B_2^2}{2} ||\nabla u(t)||^2 + \frac{1}{2} ||u'(t)||_{2,\Gamma_1}^2 \\ &\le \frac{1}{2} ||u'(t)||^2 + \frac{1}{2}(\alpha + B_2^2 + B_{*2}^2) ||\nabla u(t)||^2 + \frac{1}{2} ||u'(t)||_{2,\Gamma_1}^2. \end{aligned} \tag{3.5}$$

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Using the definition of Ψ, Young's inequality and poincaré's inequality, we have

$$
\begin{aligned}
|\Psi(t)| &\leq \frac{1}{2}||u^{'}(t)||^2 + \frac{B_{*2}^2(1-l)}{2}(h\circ\nabla u)(t) \\
&+\frac{1}{2}||u^{'}(t)||^2_{2,\Gamma_1} + \frac{B_2^2(1-l)}{2}(h\circ\nabla u)(t) \\
&= \frac{1}{2}||u^{'}(t)||^2 + \frac{1}{2}||u^{'}(t)||^2_{2,\Gamma_1} + \frac{(1-l)}{2}(B_{*2}^2 + B_2^2)(h\circ\nabla u)(t). \qquad (3.6)
\end{aligned}
$$

Using (3.1),(3.5) and (3.6), we only to choose ϵ_1, ϵ_2 small, then there exist two positive constants β_1, β_2 such that (3.4) hold.

Lemma 3.2. *([10], [19]). Under the conditions of Theorem 2.3, there exists a constant c depending on $B_m, \lambda_{n-1}(\Gamma_1), E(0)$ and m only, such that the solution of (1.1) satisfies*

$$||u(t)||^m_{m,\Gamma_1} \leq c(||\nabla u(t)||^2 + ||u(t)||^p_{p,\Gamma_1}).$$

Lemma 3.3. *([14], [19]). Let $u \in L^\infty(0,T;H^1_0(\Omega))$ be the solution of (1.1), then*

$$
\begin{aligned}
&\left(\int_\Omega \int_0^t h(t-\tau)(u(t)-u(\tau))d\tau dx\right)^{\rho+2} \\
&\leq B_{*2}^{\rho+2}(1-l)\left(\frac{8\rho E(0)}{(\rho-2)l}\right)^{\rho/2}(h\circ\nabla u)(t). \qquad (3.7)
\end{aligned}
$$

Lemma 3.4. *([10], [19]). Under the conditions of Theorem 2.3, the functional $\Phi(t)$ defined by (3.2) satisfies*

$$
\begin{aligned}
\Phi^{'}(t) &\leq ||u^{'}(t)||^2 + ||u^{'}(t)||^2_{2,\Gamma_1} \qquad (3.8) \\
&+\left(1+\alpha\delta+\frac{1+(1+\delta)(1-l)^2}{2}+\frac{c\delta}{m}\right)||\nabla u(t)||^2 + \frac{\alpha}{4\delta}||\nabla u^{'}(t)||^2 \\
&+\frac{(4\delta+1)(1-l)}{8\delta}(h\circ\nabla u)(t) + (1+\frac{c\delta}{m})||u(t)||^p_{p,\Gamma_1} \\
&+\frac{(m-1)}{m\delta}||u^{'}(t)||^m_{m,\Gamma_1}
\end{aligned}
$$

for some $\delta > 0$.

Proof. By using (1.1), we have

$$\begin{aligned}
\Phi'(t) &= \int_\Omega u(t)u''(t)dx + \int_{\Gamma_1} u(t)u''(t)d\Gamma \\
&+ \int_\Omega |u'(t)|^2 dx + \int_{\Gamma_1} |u'(t)|^2 d\Gamma + \alpha \int_\Omega \nabla u(t)\cdot\nabla u'(t)dx \\
&= \int_\Omega u(t)\left(\Delta u(t) + \alpha\Delta u'(t) - \int_0^t h(t-\tau)div(a(x)\nabla u(\tau))d\tau\right)dx \\
&+ \int_{\Gamma_1} u(t)u''(t)d\Gamma + \int_\Omega |u'(t)|^2 dx + \int_{\Gamma_1} |u'(t)|^2 d\Gamma + \alpha\int_\Omega \nabla u(t)\cdot\nabla u'(t)dx \\
&= \int_\Omega |u'(t)|^2 dx + \int_{\Gamma_1} |u'(t)| d\Gamma + \alpha\int_\Omega \nabla u(t)\cdot\nabla u'(t)dx - \int_\Omega |\nabla u(t)|^2 dx \\
&\quad + \int_\Omega \nabla u(t)\cdot(\int_0^t h(t-\tau)a(x)\nabla u(\tau)d\tau)dx \\
&\quad + \int_{\Gamma_1} u(t)(|u(t)|^{p-2}u(t) - |u'(t)|^{m-2}u'(t))d\Gamma \\
&= \int_\Omega |u'(t)|^2 dx + \int_{\Gamma_1} |u'(t)|^2 d\Gamma - \int_\Omega |\nabla u(t)|^2 dx + I_1 + I_2 + I_3. \qquad (3.9)
\end{aligned}$$

In what follows, we will estimate $I_i, i = 1,2,3$ in (3.9). Using Young's inequality, Hölder's inequality, poincaré's inequality, and Lemma 3.2, we obtain

$$\begin{aligned}
I_1 &= \alpha\int_\Omega \nabla u(t)\cdot\nabla u'(t)dx \qquad (3.10)\\
&\le \alpha(\delta\int_\Omega |\nabla u(t)|^2 dx + \frac{1}{4\delta}\int_\Omega |\nabla u'(t)|^2 dx) \\
&= \alpha(\delta||\nabla u(t)||^2 + \frac{1}{4\delta}||\nabla u'(t)||^2),
\end{aligned}$$

$$\begin{aligned}
I_2 &= \int_\Omega \nabla u(t)\cdot(\int_0^t h(t-\tau)a(x)\nabla u(\tau)d\tau)dx \\
&\le \frac{1}{2}\int_\Omega |\nabla u(t)|^2 dx + \frac{1}{2}\int_\Omega |\int_0^t h(t-\tau)[a(x)(\nabla u(\tau) - \nabla u(t)) \\
&\qquad\qquad + a(x)\nabla u(t)]d\tau)|^2 dx \\
&\le \frac{1}{2}\int_\Omega |\nabla u(t)|^2 dx + (\frac{1}{2} + \frac{\delta}{2})\int_\Omega |\int_0^t h(t-\tau)a(x)\nabla u(t)d\tau|^2 dx \\
&+ (\frac{1}{2} + \frac{1}{8\delta})\int_\Omega |\int_0^t h(t-\tau)a(x)(\nabla u(\tau) - \nabla u(t))d\tau|^2 dx \\
&\le \frac{1}{2}||\nabla u(t)||^2 + (\frac{1}{2} + \frac{\delta}{2})(||a||_{L^\infty}\int_0^t h(s)ds)^2||\nabla u(t)||^2 \\
&+ (\frac{1}{2} + \frac{1}{8\delta})(||a||_{L^\infty}\int_0^t h(s)ds)\int_0^t h(t-\tau)||\sqrt{a(x)}(\nabla u(\tau) - \nabla u(t))||^2 d\tau \\
&\le \frac{1 + (1+\delta)(1-l)^2}{2}||\nabla u(t)||^2 + \frac{(4\delta+1)(1-l)}{8\delta}(h\circ\nabla u)(t), \qquad (3.11)
\end{aligned}$$

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and

$$I_3 = \int_{\Gamma_1} u(t)(|u(t)|^{p-2}u(t) - |u'(t)|^{m-2}u'(t))d\Gamma$$
$$\leq ||u(t)||^p_{p,\Gamma_1} + \frac{\delta}{m}||u(t)||^m_{m,\Gamma_1} + \frac{m-1}{m\delta}||u'(t)||^m_{m,\Gamma_1}$$
$$\leq ||u(t)||^p_{p,\Gamma_1} + \frac{\delta}{m}c(||\nabla u(t)||^2 + ||u(t)||^p_{p,\Gamma_1}) + \frac{m-1}{m\delta}||u'(t)||^m_{m,\Gamma_1}, \tag{3.12}$$

where c is constant in Lemma 3.2. Combing theses estimates (3.9)-(3.12), we deduce that

$$\begin{aligned}\Phi'(t) \leq\ & ||u'(t)||^2 + ||u'(t)||^2_{\Gamma_1} + (1 + \alpha\delta + \frac{1+(1+\delta)(1-l)^2}{2} + \frac{c\delta}{m})||\nabla u(t)||^2 \\ & + \frac{\alpha}{4\delta}||\nabla u'(t)||^2 + \frac{(4\delta+1)(1-l)}{8\delta}(h \circ \nabla u)(t) \\ & + (1 + \frac{c\delta}{m})||u(t)||^p_{p,\Gamma_1} + \frac{m-1}{m\delta}||u'(t)||^m_{m,\Gamma_1} \quad \text{for any } \delta > 0.\end{aligned}$$

Lemma 3.5. *Under the conditions of Theorem 2.3, the functional $\Psi(t)$, defined by (3.3) satisfies*

$$\begin{aligned}\Psi'(t) \leq\ & (\delta - a_0 h_0)||u'(t)||^2 + \{\delta + 2\delta(1-l)^2 \\ & + \delta B^{2p-2}_{2p-2}\left(\frac{2p}{(p-2)l}E(0)\right)^{p-2}\}||\nabla u(t)||^2 + \frac{\alpha}{2}||\nabla u'(t)||^2 \\ & + \{\frac{1-l}{4\delta} + \frac{\alpha(1-l)}{2} + (2\delta + \frac{1}{4\delta})(1-l) \\ & \quad + \frac{B^m_m c}{4\delta m}\left(\frac{8pE(0)}{(p-2)l}\right)^{\frac{m-2}{2}} + \frac{B^2_2(1-l)}{4\delta}\}(h \circ \nabla u)(t) \\ & + \left(\frac{c}{4\delta}B^2_{*2} + \frac{c}{4\delta}B^2_2\right)(h' \circ \nabla u)(t) \\ & + (\delta + (1-l))||u'(t)||^2_{\Gamma_1} + \frac{\delta(m-1)}{m}(1-l)||u'(t)||^m_{m,\Gamma_1}\end{aligned} \tag{3.13}$$

for some $\delta > 0$.

Proof. By using (1.1) and the green formula we have

$$
\begin{aligned}
\Psi'(t) =& \int_\Omega a(x)u''(t)\int_0^t h(t-\tau)(u(t)-u(\tau))d\tau dx \\
&- \int_\Omega a(x)u'(t)\int_0^t h'(t-\tau)(u(t)-u(\tau))d\tau dx \\
&-(\int_0^t h(s)ds)\int_\Omega |\sqrt{a(x)}u'(t)|^2 dx \\
&- \int_{\Gamma_1} a(x)u''(t)\int_0^t h(t-\tau)(u(t)-u(\tau))d\tau d\Gamma \\
&-(\int_0^t h(s)ds)\int_{\Gamma_1} |\sqrt{a(x)}u'(t)|^2 d\Gamma \\
&- \int_\Omega a(x)[\Delta u(t)+\alpha\Delta u'(t) - \int_0^t h(t-\tau)div(a(x)\nabla u(\tau))d\tau] \\
&\qquad \times \int_0^t h(t-\tau)(u(t)-u(\tau))d\tau dx \\
&- \int_\Omega a(x)u'(t)\int_0^t h'(t-\tau)(u(t)-u(\tau))d\tau dx \\
&-(\int_0^t h(s)ds)\int_\Omega |\sqrt{a(x)}u'(t)|^2 dx \\
&- \int_{\Gamma_1} a(x)u'(t)\int_0^t h'(t-\tau)(u(t)-u(\tau))d\tau d\Gamma \\
&-(\int_0^t h(s)ds)\int_{\Gamma_1} |\sqrt{a(x)}u'(t)|^2 d\Gamma \\
=& \int_\Omega [\nabla u(t)+\alpha\nabla u'(t) - \int_0^t h(t-\tau)a(x)\nabla u(\tau)d\tau] \\
&\qquad \times \int_0^t h(t-\tau)a(x)(\nabla u(t)-\nabla u(\tau))d\tau dx \\
&- \int_{\Gamma_1} (|u'(t)|^{m-2}u'(t) - |u(t)|^{p-2}u(t)) \\
&\qquad \times \int_0^t h(t-\tau)a(x)(u(t)-u(\tau))d\tau d\Gamma \\
&- \int_\Omega a(x)u'(t)\int_0^t h'(t-\tau)(u(t)-u(\tau))d\tau dx \\
&-(\int_0^t h(s)ds)\int_\Omega |\sqrt{a(x)}u'(t)|^2 dx \\
&- \int_{\Gamma_1} a(x)u'(t)\int_0^t h'(t-\tau)(u(t)-u(\tau))d\tau d\Gamma \\
&-(\int_0^t h(s)ds)\int_{\Gamma_1} |\sqrt{a(x)}u'(t)|^2 d\Gamma.
\end{aligned}
$$

$$
\begin{aligned}
&= \int_\Omega \nabla u(t) \int_0^t h(t-\tau)a(x)(\nabla u(t) - \nabla u(\tau))d\tau dx \\
&\quad + \alpha \int_\Omega \nabla u'(t) \int_0^t h(t-\tau)a(x)(\nabla u(t) - \nabla u(\tau))d\tau dx \\
&\quad - \int_\Omega \int_0^t h(t-\tau)a(x)\nabla u(\tau)d\tau \int_0^t h(t-\tau)a(x)(\nabla u(t) - \nabla u(\tau))d\tau dx \\
&\quad - \int_{\Gamma_1} |u'(t)|^{m-2}u'(t) \int_0^t h(t-\tau)a(x)(u(t) - u(\tau))d\tau d\Gamma \\
&\quad + \int_{\Gamma_1} |u(t)|^{p-2}u(t) \int_0^t h(t-\tau)a(x)(u(t) - u(\tau))d\tau d\Gamma \\
&\quad - \int_\Omega a(x)u'(t) \int_0^t h'(t-\tau)(u(t) - u(\tau))d\tau dx \\
&\quad - (\int_0^t h(s)ds) \int_\Omega |\sqrt{a(x)}u'(t)|^2 dx \\
&\quad - \int_{\Gamma_1} a(x)u'(t) \int_0^t h'(t-\tau)(u(t) - u(\tau))d\tau d\Gamma \\
&\quad - (\int_0^t h(s)ds) \int_{\Gamma_1} |\sqrt{a(x)}u'(t)|^2 d\Gamma. \\
&= \sum_{i=1}^{9} I_i. \qquad (3.14)
\end{aligned}
$$

In the sequel, we will estimate $I_i, i = 1, ..., 9$ in the right hand side of (3.14). Exploiting Young's inequality, Hölder's inequality, Cauchy-Schwarz's inequality, and Lemma 3.3, we get

$$
\begin{aligned}
I_1 &= \int_\Omega \nabla u(t) \int_0^t h(t-\tau)a(x)(\nabla u(t) - \nabla u(\tau))d\tau dx \\
&\le \delta ||\nabla u(t)||^2 + \frac{1-l}{4\delta}(h \circ \nabla u)(t), \qquad (3.15)
\end{aligned}
$$

$$
\begin{aligned}
I_2 &= \alpha \int_\Omega \nabla u'(t) \int_0^t h(t-\tau)a(x)(\nabla u(t) - \nabla u(\tau))d\tau dx \\
&\le \frac{\alpha}{2}||\nabla u'(t)||^2 + \frac{\alpha(1-l)}{2}(h \circ \nabla u)(t), \qquad (3.16)
\end{aligned}
$$

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$$
\begin{aligned}
I_3 &= -\int_\Omega \int_0^t h(t-\tau)a(x)\nabla u(\tau)d\tau \int_0^t h(t-\tau)a(x)(\nabla u(t)-\nabla u(\tau))d\tau dx \\
&\le \delta|| \int_0^t h(t-\tau)a(x)\nabla u(\tau)d\tau||^2 \\
&\quad + \frac{1}{4\delta}|| \int_0^t h(t-\tau)a(x)(\nabla u(t)-\nabla u(\tau))d\tau dx||^2 \\
&\le \delta|| \int_0^t h(t-\tau)a(x)(\nabla u(\tau)-\nabla u(t)+\nabla u(t))d\tau||^2 \\
&\quad + \frac{1}{4\delta}||a||_{L^\infty}(\int_0^t h(s)ds)(h\circ\nabla u)(t) \\
&\le (2\delta+\frac{1}{4\delta})(1-l)(\int_0^t h(s)ds)(h\circ\nabla u)(t) \qquad (3.17) \\
&\quad +2\delta(1-l)^2||\nabla u(t))d\tau||^2 \quad \text{for any } \delta>0,
\end{aligned}
$$

$$
\begin{aligned}
I_4 &= -\int_{\Gamma_1} |u'(t)|^{m-2}u'(t)\int_0^t h(t-\tau)a(x)(u(t)-u(\tau))d\tau d\Gamma \\
&\le \int_0^t h(t-\tau)\int_{\Gamma_1} a(x)^{\frac{m-1}{m}}|u'(t)|^{m-1}a(x)^{\frac{1}{m}}|u(t)-u(\tau)|d\Gamma d\tau \\
&\le \int_0^t h(t-\tau)\left(\int_{\Gamma_1}\left(a(x)^{\frac{m-1}{m}}|u'(t)|^{m-1}\right)^{\frac{m}{m-1}}d\Gamma\right)^{\frac{m-1}{m}} \\
&\qquad \times\left(\int_{\Gamma_1}\left(a(x)^{\frac{1}{m}}|u(t)-u(\tau)|\right)^m d\Gamma\right)^{\frac{1}{m}}d\tau \\
&\le \int_0^t h(t-\tau)\left(\int_{\Gamma_1} a(x)|u'(t)|^m d\Gamma\right)^{\frac{m-1}{m}} \\
&\qquad \times\left(\int_{\Gamma_1} a(x)|u(t)-u(\tau)|^m d\Gamma\right)^{\frac{1}{m}}d\tau \\
&\le \frac{\delta(m-1)}{m}\int_0^t h(t-\tau)d\tau\int_{\Gamma_1} a(x)|u'(t)|^m d\Gamma \\
&\quad +\frac{1}{4\delta m}\int_0^t h(t-\tau)\int_{\Gamma_1} a(x)|u(t)-u(\tau)|^m d\Gamma d\tau \\
&\le \frac{\delta(m-1)}{m}||a||_{L^\infty}\int_0^t h(\tau)d\tau\int_{\Gamma_1}|u'(t)|^m d\Gamma \\
&\quad +\frac{B_m^m}{4\delta m}\int_0^t h(t-\tau)\int_\Omega\sqrt{a(x)}|\nabla u(t)-\nabla u(\tau)|^m dx d\tau \\
&\le \frac{\delta(m-1)}{m}(1-l)||u'(t)||_{m,\Gamma_1}^m \qquad (3.18) \\
&\quad +\frac{B_m^m c}{4\delta m}\left(\frac{8pE(0)}{(p-2)l}\right)^{\frac{m-2}{2}}(h\circ\nabla u)(t) \text{ for any } \delta>0,
\end{aligned}
$$

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where we used Lemma 3.3,

$$\begin{aligned} I_5 &= \int_{\Gamma_1} |u(t)|^{p-2}u(t)\int_0^t h(t-\tau)a(x)(u(t)-u(\tau))d\tau d\Gamma \\ &\le \delta\int_{\Gamma_1}|u(t)|^{2p-2}d\Gamma + \frac{B_2^2(1-l)}{4\delta}(h\circ\nabla u)(t) \\ &\le \delta B_{2p-2}^{2p-2}\left(\frac{2pE(0)}{(p-2)l}\right)^{p-2}||\nabla u(t)||^2 \\ &\quad + \frac{B_2^2(1-l)}{4\delta}(h\circ\nabla u)(t) \text{ for any } \delta>0, \end{aligned} \tag{3.19}$$

where we used $H^1(\Omega)\hookrightarrow L^{2(p+1)}(\Omega)$,

$$\begin{aligned} I_6 &= -\int_\Omega a(x)u^{'}(t)\int_0^t h^{'}(t-\tau)(u(t)-u(\tau))d\tau dx \\ &\le |\int_\Omega a(x)u^{'}(t)\int_0^t h^{'}(t-\tau)(u(t)-u(\tau))d\tau dx| \\ &\le \delta||u^{'}(t)||^2 + \frac{cB_{*2}^2}{4\delta}(h^{'}\circ\nabla u)(t) \text{ for any } \delta>0, \end{aligned} \tag{3.20}$$

$$\begin{aligned} I_7 &= -(\int_0^t h(s)ds)\int_\Omega |\sqrt{a(x)}u^{'}(t)|^2dx \\ &\le -(\int_0^{t_0} h(s)ds)\int_\Omega |\sqrt{a(x)}u^{'}(t)|^2dx \\ &\le -a_0h_0||u^{'}(t)||^2, \text{ for all } t\ge t_0, \end{aligned} \tag{3.21}$$

where $h_0=\int_0^{t_0}h(s)ds$,

$$\begin{aligned} I_8 &= -\int_{\Gamma_1} a(x)u^{'}(t)\int_0^t h^{'}(t-\tau)(u(t)-u(\tau))d\tau dx \\ &\le \delta||u^{'}(t)||_{\Gamma_1}^2 + \frac{cB_2^2}{4\delta}(h^{'}\circ\nabla u)(t), \text{ for any } \delta>0, \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} I_9 &= -(\int_0^t h(s)ds)\int_{\Gamma_1} |\sqrt{a(x)}u^{'}(t)|^2d\Gamma \\ &\le (1-l)||u^{'}(t)||_{\Gamma_1}^2. \end{aligned} \tag{3.23}$$

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Combining (3.14)-(3.23), we get

$$\begin{aligned}
\Psi^{'}(t) &\le (\delta - a_0h_0)||\nabla u^{'}(t)||^2 + \frac{\alpha}{2}||\nabla u^{'}(t)||^2 \\
&+ \left(\delta + 2\delta(1-l)^2 + \delta B_{2p-2}^{2p-2}\left(\frac{2pE(0)}{(p-2)l}\right)^{p-2}\right)||\nabla u(t))d\tau||^2 \\
&+[\frac{1-l}{4\delta} + \frac{\alpha(1-l)}{2} + (2\delta + \frac{1}{4\delta})(1-l)^2 + \frac{B_m^m c}{4\delta m}\left(\frac{8pE(0)}{(p-2)l}\right)^{\frac{m-2}{2}} \\
&\qquad + \frac{B_2^2(1-l)}{4\delta}](h\circ\nabla u)(t) \\
&+ \left(\frac{cB_{*2}^2}{4\delta} + \frac{cB_2^2}{4\delta}\right)(h^{'}\circ\nabla u)(t) + [\delta + (1-l)]||u^{'}(t)||_{\Gamma_1}^2 \\
&+ \frac{\delta(m-1)}{m}(1-l)||u^{'}(t)||_{m,\Gamma_1}^m \text{ for any } \delta > 0 \text{ and for all } t \ge t_0.
\end{aligned}$$

Now,we are ready to prove the general result.

Proof of Theorem 3.1. Since h is positive, we see that, for any $t_0 > 0$,

$$\int_0^t h(s)ds \ge \int_0^{t_0} h(s)ds := h_0, \quad t \ge t_0.$$

Combining (2.11), (3.8), (3.13), then from (3.1), we have

$$\begin{aligned}
F^{'}(t) &= E^{'}(t) + \epsilon_1\Phi^{'}(t) + \epsilon_2\Psi^{'}(t) \\
&\le (\epsilon_1 + \epsilon_2\delta - \epsilon_2 a_0 h_0)||u^{'}(t)||^2 \\
&+[-\frac{1}{2}h(t)a_0 + \epsilon_1\left(1 + \alpha\delta + \frac{1+(1+\delta)(1-l)^2}{2} + \frac{c\delta}{m}\right) \\
&\qquad + \epsilon_2\left(\delta + 2\delta(1-l)^2 + \delta B_{2p-2}^{2p-2}\left(\frac{2pE(0)}{(p-2)l}\right)^{p-2}\right)]||\nabla u(t))||^2 \\
&+(-\alpha + \epsilon_1\frac{\alpha}{4\delta} + \epsilon_2\frac{\alpha}{2})||\nabla u^{'}(t)||^2 \\
&+[\epsilon_1\frac{(4\delta+1)(1-l)}{8\delta} + \epsilon_2(\frac{1-l}{4\delta} + \frac{\alpha(1-l)}{2} + (2\delta + \frac{1}{4\delta})(1-l)^2 \\
&\quad + \frac{B_m^m c}{4\delta m}\left(\frac{8pE(0)}{(p-2)l}\right)^{\frac{m-2}{2}} + \frac{B_2^2(1-l)}{4\delta})](h\circ\nabla u)(t) \\
&+[\epsilon_1 + \epsilon_2(\delta + (1-l))]||u^{'}(t)||_{\Gamma_1}^2 + \epsilon_1(1 + \frac{c\delta}{m})||u(t)||_{p,\Gamma_1}^p \\
&+[-1 + \epsilon_1\frac{(m-1)}{m\delta} + \epsilon_2\frac{\delta(m-1)}{m}(1-l)]||u^{'}(t)||_{m,\Gamma_1}^m \\
&+[\frac{1}{2} + \epsilon_2(\frac{cB_{*2}^2}{4\delta} + \frac{cB_2^2}{4\delta})](h^{'}\circ\nabla u)(t)
\end{aligned}$$

$$
\begin{aligned}
(3.24)\qquad &\leq -[\epsilon_2(a_0h_0-\delta)-\epsilon_1]||u^{'}(t)||^2 \\
&-[\frac{1}{2}h(t)a_0-\epsilon_1(1+\alpha\delta+\frac{1+(1+\delta)(1-l)^2}{2}+\frac{c\delta}{m})-\epsilon_2K_\delta]||\nabla u(t))||^2 \\
&-[\alpha-\epsilon_1(\frac{\alpha}{4\delta}B_2^2)-\epsilon_2\frac{\alpha}{2}]||\nabla u^{'}(t)||^2 \\
&+[\epsilon_1\frac{(4\delta+1)(1-l)}{8\delta}+\epsilon_2K](h\circ\nabla u)(t) \\
&+\epsilon_1(1+\frac{c\delta}{m})||u(t)||_{p,\Gamma_1}^p \\
&-[1-\epsilon_1\frac{(m-1)}{m\delta}-\epsilon_2\frac{\delta(m-1)}{m}(1-l)]||u^{'}(t)||_{m,\Gamma_1}^m \\
&+[\frac{1}{2}+\epsilon_2(\frac{cB_{*2}^2}{4\delta}+\frac{cB_2^2}{4\delta})](h^{'}\circ\nabla u)(t),
\end{aligned}
$$

where

$$
\begin{aligned}
K_\delta &= \delta+2\delta(1-l)^2+\delta B_{2p-2}^{2p-2}\left(\frac{2p}{(p-2)l}E(0)\right)^{p-2} > 0, \\
K &= \frac{1-l}{4\delta}+\frac{\alpha(1-l)}{2}+(2\delta+\frac{1}{4\delta})(1-l)^2 \\
&\quad +\frac{B_m^m c}{4\delta m}\left(\frac{8pE(0)}{(p-2)l}\right)^{\frac{m-2}{2}}+\frac{B_2^2(1-l)}{4\delta} > 0.
\end{aligned}
$$

At this point, first we choose $\delta > 0$ satisfying

$$0 < \delta < a_0h_0 - \frac{\epsilon_1}{\epsilon_2},$$

and be such that

$$b_1 = \epsilon_2(a_0h_0-\delta)-\epsilon_1 > 0.$$

Now, we choose ϵ_1 and ϵ_2 sufficiently small so that satisfy Lemma 3.1 and following:

$$
\begin{aligned}
b_2 &= \frac{1}{2}h(t)a_0-\epsilon_1(1+\alpha\delta+\frac{1+(1+\delta)(1-l)^2}{2}+\frac{c\delta}{m})-\epsilon_2K_\delta > 0, \\
b_3 &= \alpha-\epsilon_1(\frac{\alpha}{4\delta}+B_2^2)-\epsilon_2\frac{\alpha}{2} > 0, \\
b_4 &= 1-\epsilon_1\frac{m-1}{m\delta}-\epsilon_2\frac{\delta(m-1)}{m}(1+l) > 0, \\
b_5 &= \frac{1}{2}+\epsilon_2(\frac{c}{4\delta}B_{*2}^2+\frac{c}{4\delta}) > 0.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
(3.25)\qquad F^{'}(t) &\leq -b_1||u^{'}(t)||^2-b_2||\nabla u(t)||^2-b_3||\nabla u^{'}(t)||^2 \\
&+(\epsilon_1\frac{(4\delta+1)(1-l)}{8\delta}+\epsilon_2K)(h\circ\nabla u)(t) \\
&+\epsilon_1(1+\frac{c\delta}{m})||u(t)||_{p,\Gamma_1}^p-b_4||u^{'}(t)||_{m,\Gamma_1}^m+b_5(h^{'}\circ\nabla u)(t).
\end{aligned}
$$

We choose $\epsilon_1 > 0$ sufficiently small, it follows that (3.25), we arrive at

$$F^{'}(t) \leq -cE(t) + c(h \circ \nabla u)(t), \quad \forall t \geq t_0. \tag{3.26}$$

From (H1), (2.11) and (3.26), we obtain

$$\begin{aligned} \xi(t)F^{'}(t) &\leq -c\xi(t)E(t) + c\xi(t)(h \circ \nabla u)(t) \\ &\leq -c\xi(t)E(t) - c(h^{'} \circ \nabla u)(t) \\ &\leq -c\xi(t)E(t) - cE^{'}(t), \quad \forall t \geq t_0. \end{aligned} \tag{3.27}$$

That is

$$L^{'}(t) \leq -c\xi(t)E(t) \leq -r\xi(t)L(t), \quad t \geq t_0, \tag{3.28}$$

where $L(t) = \xi F(t) + cE(t)$ is clearly equivalent to $E(t)$ and r is a positive constant. A simple integration of (3.28) leads to

$$L(t) \leq L(t_0)e^{-r\int_{t_0}^{t}\xi(s)ds}, \quad \forall t \geq t_0. \tag{3.29}$$

Thus, Lemma 3.1. and (3.29) yield

$$E(t) \leq ce^{-r\int_{t_0}^{t}\xi(s)ds}, \quad \forall t \geq t_0.$$

This completes the proof of Theorem 2.3.

Acknowledgments

Y.H. Kang was supported by research grants from the Catholic University of Daegu in 2014 year(20141080).

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Lower order Fractional Monotone Approximation

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Abstract

Here is presented the theory of lower order fractional simultaneous monotone uniform polynomial approximation with rates using mixed lower order fractional linear differential operators.

To obtain that, we use first ordinary simultaneous polynomial approximation with respect to the highest lower order right and left fractional derivatives of the function under approximation using their moduli of continuity. Then we use the total right and left fractional simultaneous polynomial approximation with rates, as well their convex combination. Based on the last and elegant analytical techniques, we derive preservation of monotonicity by mixed lower order fractional linear differential operators.

2010 AMS Mathematics Subject Classification : 26A33, 41A10, 41A17, 41A25, 41A29.
Keywords and Phrases: Monotone Approximation, Caputo fractional derivative, fractional linear differential operator, modulus of continuity.

1 Introduction

The topic of monotone approximation started in [6] has become a major trend in approximation theory. A typical problem in this subject is: given a positive integer k, approximate a given function whose kth derivative is ≥ 0 by polynomials having this property.

In [3] the authors replaced the kth derivative with a linear differential operator of order k. We mention this motivating result.

Theorem 1 *Let h, k, p be integers, $0 \leq h \leq k \leq p$ and let f be a real function, $f^{(p)}$ continuous in $[-1,1]$ with first modulus of continuity $\omega_1 \left(f^{(p)}, x\right)$ there. Let $a_j(x)$, $j = h, h+1, ..., k$ be real functions, defined and bounded on $[-1,1]$ and assume $a_h(x)$ is either $\geq$ some number $\alpha > 0$ or $\leq$ some number $\beta < 0$ throughout $[-1,1]$. Consider the operator*

$$L = \sum_{j=h}^{k} a_j(x) \left[\frac{d^j}{dx^j}\right] \tag{1}$$

and suppose, throughout $[-1,1]$,

$$L(f) \geq 0. \tag{2}$$

Then, for every integer $n \geq 1$, there is a real polynomial $Q_n(x)$ of degree $\leq n$ such that

$$L(Q_n) \geq 0 \text{ throughout } [-1,1] \tag{3}$$

and

$$\max_{-1 \leq x \leq 1} |f(x) - Q_n(x)| \leq C n^{k-p} \omega_1 \left(f^{(p)}, \frac{1}{n}\right), \tag{4}$$

where C is independent of n or f.

The purpose of this article is to extend completely Theorem 1 to the lower order fractional level. All involved ordinary derivatives will become now fractional derivatives of lower order and even more we will have fractional simultaneous approximation.

We need

Definition 2 *([4], p. 50) Let $\alpha > 0$ and $\lceil \alpha \rceil = m$, ($\lceil \cdot \rceil$ ceiling of the number). Consider $f \in AC^m([0,1])$ (space of functions f with $f^{(m-1)} \in AC([0,1])$, absolutely continuous functions), $z \in [0,1]$. We define the left Caputo fractional derivative of f of order α as follows:*

$$\left(D_{*z}^{\alpha} f\right)(x) = \frac{1}{\Gamma(m-\alpha)} \int_z^x (x-t)^{m-\alpha-1} f^{(m)}(t)\, dt, \tag{5}$$

for any $x \in [z,1]$, where Γ is the gamma function.

We set

$$D_{*z}^{0} f(x) = f(x),$$
$$D_{*z}^{m} f(x) = f^{(m)}(x), \ \ \forall\, x \in [z,1]. \tag{6}$$

Definition 3 *([5]) Let $\alpha > 0$ and $\lceil \alpha \rceil = m$. Consider $f \in AC^m([0,1])$, $z \in [0,1]$. We define the right Caputo fractional derivative of f of order α as follows:*

$$\left(D_{z-}^{\alpha} f\right)(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^z (t-x)^{m-\alpha-1} f^{(m)}(t)\, dt, \tag{7}$$

for any $x \in [0, z]$.

We set

$$D_{z-}^{0} f(x) = f(x),$$
$$D_{z-}^{m} f(x) = (-1)^{m} f^{(m)}(x), \ \ \forall\, x \in [0, z]. \tag{8}$$

In particular we give

Definition 4 *Let* $0 < \alpha < 1$ *and* $f \in AC([0, 1])$ *(absolutely continuous functions),* $z \in [0, 1]$. *We define the left Caputo fractional derivative of* f *of order* α *as follows:*

$$(D_{*z}^{\alpha} f)(x) = \frac{1}{\Gamma(1-\alpha)} \int_{z}^{x} (x-t)^{-\alpha} f'(t)\, dt, \tag{9}$$

for any $x \in [z, 1]$.

We set

$$D_{*z}^{0} f(x) = f(x),$$
$$D_{*z}^{1} f(x) = f'(x), \ \ \forall\, x \in [z, 1]. \tag{10}$$

Definition 5 *Let* $0 < \alpha < 1$ *and* $f \in AC([0, 1])$, $z \in [0, 1]$. *We define the right Caputo fractional derivative of* f *of order* α *as follows:*

$$\left(D_{z-}^{\alpha} f\right)(x) = \frac{-1}{\Gamma(1-\alpha)} \int_{x}^{z} (t-x)^{-\alpha} f'(t)\, dt, \tag{11}$$

for any $x \in [0, z]$.

We set

$$D_{z-}^{0} f(x) = f(x),$$
$$D_{z-}^{1} f(x) = -f'(x), \ \ \forall\, x \in [0, z]. \tag{12}$$

Definition 6 *Let* $f \in C([0, 1])$, *we define the Bernstein polynomials*

$$(B_N f)(t) = \sum_{k=0}^{N} f\left(\frac{k}{N}\right) \binom{N}{k} t^{k} (1-t)^{N-k}, \tag{13}$$

$\forall\, t \in [0, 1]$, $N \in \mathbb{N}$, *of degree* N.

Our article [1] was the base to develop the general article [2]. We rely alot on [2].

We need the following special result from [2], here is the $n = 1$ case.

Theorem 7 *Let* $1 < \beta < 2$ *and* $0 < \alpha < 1$ *such that* $\beta = 1 + \alpha$. *Let* $f \in AC^{2}([0, 1])$, *and* $f'' \in L_{\infty}([0, 1])$, $N \in \mathbb{N}$. *Set*

$$P_{N+1}(f)(x) := f(0) + \int_{0}^{x} B_N(f')(t)\, dt, \tag{14}$$

3

for all $0 \leq x \leq 1$, *a polynomial of degree* $(N+1)$. *Set also*

$$T_N^{\beta,\alpha}(f) := \frac{2^{1-\alpha}}{\Gamma(\alpha+1) N^{\frac{\alpha}{2}}} \left[\sup_{x \in [0,1]} \omega_1 \left(D_{x-}^{\beta} f, \frac{1}{2(\alpha+1) N^{\frac{1}{2}}} \right)_{[0,x]} + \right.$$

$$\left. \sup_{x \in [0,1]} \omega_1 \left(D_{*x}^{\beta} f, \frac{1}{2(\alpha+1) N^{\frac{1}{2}}} \right)_{[x,1]} \right] < \infty. \tag{15}$$

for every $N \in \mathbb{N}$.

Then

1) the quantity within the bracket of (15) is finite,

2) $P'_{N+1}(f) = B_N(f')$,

3)

$$\left\| P_{N+1}^{(i)}(f) - f^{(i)} \right\|_{\infty,[0,1]} \leq T_N^{\beta,\alpha}(f), \quad i = 0, 1. \tag{16}$$

As $N \to \infty$ *we derive with rates* $P_{N+1}^{(i)}(f) \xrightarrow{u} f^{(i)}$ *(uniformly),* $i = 0, 1$.

We completely left fractionalize Theorem 7, to have

Theorem 8 *Here all terms and assumptions as in Theorem 7 and* $\alpha_j \in [0,1]$, $j \in \mathbb{Z}_+$. *Then*

$$\left\| D_{*0}^{\alpha_j}(f) - D_{*}^{\alpha_j}(P_{N+1}(f)) \right\|_{\infty,[0,1]} \leq \frac{T_N^{\beta,\alpha}(f)}{\Gamma(j^* - \alpha_j + 1)}, \tag{17}$$

where $j^* = \lceil \alpha_j \rceil = 0$ *or* 1.

Observe that (17) generalizes (16).

Proof. By [2], see there Theorem 9. ■

We completely right fractionalize Theorem 7, to have

Theorem 9 *Here all terms and assumptions as in Theorem 7, and* $\alpha_j \in [0,1]$, $j \in \mathbb{Z}_+$. *Then*

$$\left\| D_{1-}^{\alpha_j}(f) - D_{1-}^{\alpha_j}(P_{N+1}(f)) \right\|_{\infty,[0,1]} \leq \frac{T_N^{\beta,\alpha}(f)}{\Gamma(j^* - \alpha_j + 1)}, \tag{18}$$

where $j^* = \lceil \alpha_j \rceil = 0$ *or* 1.

Observe that (18) generalizes (16).

Proof. By [2], see there Theorem 10. ■

It follows the important

Corollary 10 *Here all as in Theorem 7, $\lambda \in [0,1]$. Then*

$$\left\| \left(\lambda D_{*0}^{\alpha_j} (f) + (1-\lambda) D_{1-}^{\alpha_j} (f) \right) - \right.$$

$$\left. \left(\lambda D_{*0}^{\alpha_j} (P_{N+1}(f)) + (1-\lambda) D_{1-}^{\alpha_j} (P_{N+1}(f)) \right) \right\|_{\infty,[0,1]} \le \frac{T_N^{\beta,\alpha}(f)}{\Gamma(j^* - \alpha_j + 1)}, \quad (19)$$

where $\alpha_j \in [0,1]$, $j \in \mathbb{Z}_+$; $j^* = \lceil \alpha_j \rceil = 0$ or 1.

Proof. By [2], see there Corollary 11. ■

2 Main Results

Next comes our main result: the totally lower order fractional simultaneous monotone uniform approximation, using mixed fractional differential operators.

Theorem 11 *Let $1 < \beta < 2$ and $0 < \alpha < 1 : \beta = 1 + \alpha$. Let $f \in AC^2([0,1])$ with $f'' \in L_\infty([0,1])$, $N \in \mathbb{N}$. Here let $k, \rho \in \mathbb{N} : 0 < k \le \rho$, and $\alpha_0 = 0 < \alpha_1 < \alpha_2 < ... < \alpha_k < ... < \alpha_\rho \le 1$. Let $\lambda \in [0,1]$, and $\alpha_j(x)$, $j = 0, 1, ..., k$ be real functions, defined and bounded on $[0,1]$, and suppose $\alpha_0(x)$ is either $\ge \overline{\alpha} > 0$ or $\le \overline{\beta} < 0$ on $[0,1]$. We set*

$$l_\tau :\equiv \sup_{x \in [0,1]} \left| \alpha_0^{-1}(x) \alpha_\tau(x) \right|, \quad \tau = 0, 1, ..., k. \quad (20)$$

Here $T_N^{\beta,\alpha}(f)$, $N \in \mathbb{N}$, as in (15), $(N \to \infty)$. Consider the mixed fractional linear differential operator

$$L_\lambda^* := \sum_{j=0}^{k} \alpha_j(x) \left[\lambda D_{*0}^{\alpha_j} + (1-\lambda) D_{1-}^{\alpha_j} \right]. \quad (21)$$

Then, for any $N \in \mathbb{N}$, there exists a real polynomial Q_{N+1} of degree $(N+1)$ such that

1)

$$\left\| \left(\lambda D_{*0}^{\alpha_j} + (1-\lambda) D_{1-}^{\alpha_j} \right)(f) - \right.$$

$$\left. \left(\lambda D_{*0}^{\alpha_j} + (1-\lambda) D_{1-}^{\alpha_j} \right)(Q_{N+1}(f)) \right\|_{\infty,[0,1]} \le \frac{T_N^{\beta,\alpha}(f)}{\Gamma(2-\alpha_j)}, \ j = 1, ..., \rho, \quad (22)$$

and

2)

$$\|f - Q_{N+1}(f)\|_{\infty,[0,1]} \le T_N^{\beta,\alpha}(f) \left[\sum_{\tau=1}^{k} \frac{l_\tau}{\Gamma(2-\alpha_\tau)} + 2 \right]. \quad (23)$$

Assuming $L_\lambda^* f(x) \ge 0$, for all $x \in [0,1]$ we get $(L_\lambda^*(Q_{N+1}(f)))(x) \ge 0$ for all $x \in [0,1]$.

Proof. Here let $k, \rho \in \mathbb{N} : 0 < k \leq \rho$, and $\alpha_0 = 0 < \alpha_1 < \alpha_2 < ... < \alpha_k < ... < \alpha_\rho \leq 1$, that is $\lceil \alpha_0 \rceil = 0$; $\lceil \alpha_j \rceil = 1$, $j = 1, ..., \rho$. We set

$$l_\tau := \sup_{x \in [0,1]} \left| \alpha_0^{-1}(x) \alpha_\tau(x) \right| < \infty, \quad 0 \leq \tau \leq k, \tag{24}$$

with $l_0 = 1$, and

$$\rho_N := T_N^{\beta,\alpha}(f) \left(\sum_{\tau=1}^{k} \frac{l_\tau}{\Gamma(2 - \alpha_\tau)} + 1 \right). \tag{25}$$

I. Suppose, throughout $[0,1]$, $\alpha_0(x) \geq \overline{\alpha} > 0$. Call

$$Q_{N+1}(f)(x) := P_{N+1}(f)(x) + \rho_N, \tag{26}$$

where $P_{N+1}(f)(x)$ as in (14).

Then by (19) we obtain

$$\left\| \left(\lambda D_{*0}^{\alpha_j} + (1 - \lambda) D_{1-}^{\alpha_j} \right) (f(x) + \rho_N) - \right.$$

$$\left. \left(\lambda D_{*0}^{\alpha_j} + (1 - \lambda) D_{1-}^{\alpha_j} \right) (Q_{N+1}(f)(x)) \right\|_{\infty,[0,1]} \leq \frac{T_N^{\beta,\alpha}(f)}{\Gamma(2 - \alpha_j)}, \quad j = 1, ..., \rho. \tag{27}$$

And of course it holds

$$\|(f(x) + \rho_N) - Q_{N+1}(f)(x)\|_{\infty,[0,1]} \overset{(16)}{\leq} T_N^{\beta,\alpha}(f). \tag{28}$$

So that we find

$$\|f - Q_{N+1}(f)\|_{\infty,[0,1]} \overset{(28)}{\leq} \rho_N + T_N^{\beta,\alpha}(f) = \tag{29}$$

$$T_N^{\beta,\alpha}(f) \left(\sum_{\tau=1}^{k} \frac{l_\tau}{\Gamma(2 - \alpha_j)} + 1 \right) + T_N^{\beta,\alpha}(f) = T_N^{\beta,\alpha}(f) \left[\sum_{\tau=1}^{k} \frac{l_\tau}{\Gamma(2 - \alpha_j)} + 2 \right],$$

proving (23).

From (27) and (9), (11), we get

$$\left\| \left(\lambda D_{*0}^{\alpha_j} + (1 - \lambda) D_{1-}^{\alpha_j} \right) (f(x)) - \right.$$

$$\left. \left(\lambda D_{*0}^{\alpha_j} + (1 - \lambda) D_{1-}^{\alpha_j} \right) (Q_{N+1}(f))(x) \right\|_{\infty,[0,1]} \leq \frac{T_N^{\beta,\alpha}(f)}{\Gamma(2 - \alpha_j)}, \quad j = 1, ..., \rho, \tag{30}$$

proving (22).

Next we use the assumption $L_\lambda^* f(x) \geq 0$, all $x \in [0,1]$, to get

$$\alpha_0^{-1}(x) L_\lambda^* (Q_{N+1}(f))(x) = \alpha_0^{-1}(x) L_\lambda^* f(x) + \rho_N +$$

$$\sum_{j=0}^{k} \alpha_0^{-1}(x)\,\alpha_j(x)\left\{\left(\lambda D_{*0}^{\alpha_j} + (1-\lambda)\,D_{1-}^{\alpha_j}\right)[Q_{N+1}(f)(x) - f(x) - \rho_N]\right\} \overset{((27),(28))}{\geq}$$

$$\rho_N - \left(\sum_{\tau=1}^{k} \frac{l_\tau}{\Gamma(2-\alpha_j)} + 1\right) T_N^{\beta,\alpha}(f) = \rho_N - \rho_N = 0. \tag{31}$$

Hence $L_\lambda^*(Q_{N+1}(f))(x) \geq 0$, all $x \in [0,1]$.

II. Suppose, throughout $[0,1]$, $\alpha_0(x) \leq \overline{\beta} < 0$. Call

$$Q_{N+1}(f)(x) := P_{N+1}(f)(x) - \rho_N, \tag{32}$$

where $P_{N+1}(f)(x)$ as in (14).

Then by (19) we obtain

$$\left\|\left(\lambda D_{*0}^{\alpha_j} + (1-\lambda)\,D_{1-}^{\alpha_j}\right)(f(x) - \rho_N) - \right.$$

$$\left.\left(\lambda D_{*0}^{\alpha_j} + (1-\lambda)\,D_{1-}^{\alpha_j}\right)(Q_{N+1}(f)(x))\right\|_{\infty,[0,1]} \leq \frac{T_N^{\beta,\alpha}(f)}{\Gamma(2-\alpha_j)}, \quad j = 1, ..., \rho. \tag{33}$$

And of course it holds

$$\|(f(x) - \rho_N) - Q_{N+1}(f)(x)\|_{\infty,[0,1]} \overset{(16)}{\leq} T_N^{\beta,\alpha}(f). \tag{34}$$

Similarly we obtain again (22) and (23).

Next we use the assumption $L_\lambda^* f(x) \geq 0$, all $x \in [0,1]$, to get

$$\alpha_0^{-1}(x)\,L_\lambda^*(Q_{N+1}(f))(x) = \alpha_0^{-1}(x)\,L_\lambda^* f(x) - \rho_N +$$

$$\sum_{j=0}^{k} \alpha_0^{-1}(x)\,\alpha_j(x)\left\{\left(\lambda D_{*0}^{\alpha_j} + (1-\lambda)\,D_{1-}^{\alpha_j}\right)[Q_{N+1}(f)(x) - f(x) + \rho_N]\right\} \overset{((33),(34))}{\leq}$$

$$-\rho_N + \left(\sum_{\tau=1}^{k} \frac{l_\tau}{\Gamma(2-\alpha_j)} + 1\right) T_N^{\beta,\alpha}(f) = -\rho_N + \rho_N = 0. \tag{35}$$

Hence $L_\lambda^*(Q_{N+1}(f))(x) \geq 0$, for any $x \in [0,1]$. ■

Corollary 12 *(to Theorem 11, $\lambda = 1$ case) Let* $L_1^* := \sum_{j=0}^{k} \alpha_j(x)\,D_{*0}^{\alpha_j}$.

Then, for any $N \in \mathbb{N}$, there exists a real polynomial Q_{N+1} of degree $(N+1)$ such that

1)

$$\left\|D_{*0}^{\alpha_j}(f) - D_{*0}^{\alpha_j}(Q_{N+1}(f))\right\|_{\infty,[0,1]} \leq \frac{T_N^{\beta,\alpha}(f)}{\Gamma(2-\alpha_j)}, \ j = 1, ..., \rho, \tag{36}$$

2) inequality (23) is again valid.

Assuming $L_1^* f(x) \geq 0$, for all $x \in [0,1]$, we get $(L_1^*(Q_{N+1}(f)))(x) \geq 0$ for all $x \in [0,1]$.

7

Corollary 13 *(to Theorem 11, $\lambda = 0$ case) Let* $L_0^* := \sum_{j=0}^{k} \alpha_j(x) D_{1-}^{\alpha_j}$.

Then, for any $N \in \mathbb{N}$, there exists a real polynomial Q_{N+1} of degree $(N+1)$ such that

1)

$$\left\| D_{1-}^{\alpha_j} f - D_{1-}^{\alpha_j} \left(Q_{N+1}(f) \right) \right\|_{\infty,[0,1]} \leq \frac{T_N^{\beta,\alpha}(f)}{\Gamma(2-\alpha_j)}, \ j = 1, ..., \rho, \tag{37}$$

2) inequality (23) is again valid.

Assuming $L_0^* f(x) \geq 0$, we get $L_0^* (Q_{N+1}(f))(x) \geq 0$ for any $x \in [0,1]$.

Finally we give

Corollary 14 *(to Theorem 11, $\lambda = \frac{1}{2}$ case) Let* $L_{\frac{1}{2}}^* := \sum_{j=0}^{k} \alpha_j(x) \left[\frac{D_{*0}^{\alpha_j} + D_{1-}^{\alpha_j}}{2} \right]$.

Then, for any $N \in \mathbb{N}$, there exists a real polynomial Q_{N+1} of degree $(N+1)$ such that

1)

$$\left\| \left(D_{*0}^{\alpha_j} + D_{1-}^{\alpha_j} \right)(f) - \left(D_{*0}^{\alpha_j} + D_{1-}^{\alpha_j} \right) \left(Q_{N+1}(f) \right) \right\|_{\infty,[0,1]} \leq \frac{2T_N^{\beta,\alpha}(f)}{\Gamma(2-\alpha_j)}, \ j = 1, ..., \rho, \tag{38}$$

2) inequality (23) is again valid.

Assuming $L_{\frac{1}{2}}^* f(x) \geq 0$, we get $L_{\frac{1}{2}}^* (Q_{N+1}(f))(x) \geq 0$ for any $x \in [0,1]$.

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Modified quadrature method with Sidi transformation for three dimensional axisymmetric potential problems with Dirichlet conditions ☆

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Abstract

This paper is the application of the modified quadrature method for the numerical evaluation of boundary integral equations of three dimensional (3D) axisymmetric Laplace equations with Dirichlet conditions on curved polygonal boundaries. The Sidi transformation is used to remove the logarithmic singularities in the integral kernels and then the modified quadrature method is presented to approximate the weakly integrals. Numerical examples show that the error of numerical solution for the boundary integral equation of the 3D Laplace equation can converge with the order $O\ (h^3)$ by use of a Sidi transformation, and the optimal condition number for the according discrete system is $O\ (h^{-1})$, where h is the uniform mesh step size.

Keywords: Laplace equation, axisymmetric, modified quadrature method, convergence, stability.

2010 Mathematics Subject Classification: 65R20, 65N38.

1. INTRODUCTION

Consider the Laplace problem with a Dirichlet condition

$$\begin{cases} \Delta\phi(\mathbf{x}) = 0, & \text{in} \quad V \\ \phi(\mathbf{x}) = g(\mathbf{x}), & \text{on} \quad \partial V \end{cases} \tag{1.1}$$

where V is an axisymmetric domain, which is formed by rotating a two dimensional bounded simply connected region Ω around $z-$axis, Δ is a three dimensional Laplace operator, and $g(\mathbf{x})$ is a known function on ∂V.

For practical problems, high precise and validated solutions of (1.1) are required. Some well known methods, such as finite element methods (FEMs) , collocation methods (CMs) and finite difference methods (FDMs) can be used to solve the mentioned equation above [5, 6, 9–11, 13, 15–19].

By ring potential theory, Eq. (1.1) is converted into the first kind boundary integral equation (BIE) and its solution can be represented via the following single layer potential

$$\phi(\mathbf{y}) = \int_S \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\sigma(\mathbf{x})dS, \quad \mathbf{x} = (x_1, x_2, x_3) \in S, \quad \mathbf{y} = (y_1, y_2, y_3) \in V, \tag{1.2}$$

☆The work is supported by the National Natural Science Foundation of China (11371079).
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September 6, 2014

where the density function $\sigma(\mathbf{x})$ is sought on S, and $S = \partial V$ is the surface of the body V. Here $|\mathbf{x} - \mathbf{y}|$ denotes the Euclidean distance between the point $\mathbf{x}$ and $\mathbf{y}$. Using the cylindrical coordinate (ρ, z, θ), Eq. (1.2) can be written as

$$\phi(\mathbf{y}) = \int_\Gamma \rho\phi^\star(\mathbf{x}, \mathbf{y})\sigma(\mathbf{x})d\Gamma, \quad \mathbf{x} = (\rho, z) \in \Gamma, \ \ \mathbf{y} = (\rho_0, z_0) \in \Omega^{'}, \tag{1.3}$$

where Γ is the polygonal boundary of the transformed cylindrical coordinate region $\Omega^{'}$ of region Ω, and

$$\phi^\star(\mathbf{x}, \mathbf{y}) = \phi^\star(\rho, z; \rho_0, z_0) = \int_0^{2\pi} \frac{1}{4\pi|\mathbf{x} - \mathbf{y}|} d\theta. \tag{1.4}$$

By use of the Hankel transformation [21], the fundamental solution of three dimensional axisymmetric Laplace problem can also be represented by

$$\phi^\star(\rho, z; \rho_0, z_0) = \frac{Q_{-1/2}(q)}{2\pi\sqrt{\rho\rho_0}}, \tag{1.5}$$

where

$$q = \frac{2\rho\rho_0 + (\rho - \rho_0)^2 + (z - z_0)^2}{2\rho\rho_0}$$

and $Q_{-1/2}(q)$ is the second kind of Legendre function, which has the following asymptotic expression

$$Q_{-1/2}(q) = -\frac{1}{2}\ln(\frac{q-1}{32}) + O((q-1)\ln(q-1)), \quad q \to 1.$$

Since the single potential function is continuous on the boundary Γ, we have

$$g(\mathbf{y}) = \int_\Gamma \rho\phi^\star(\mathbf{x}, \mathbf{y})\sigma(\mathbf{x})d\Gamma, \quad \mathbf{y} \in \Gamma. \tag{1.6}$$

Eq. (1.6) is the weakly singular BIE system of the first kind , whose solution exists and is unique as long as $C_\Gamma \neq 1$ [7], where C_Γ is the logarithmic capacity (i.e., the transfinite diameter). Once the solution $\sigma(\mathbf{x})$ is solved from (1.6), we can obtain the solution in $\Omega^{'}$ by (1.3).

In this paper, before discretization the boundary integral equation is converted to an equivalent equation over $[0, 1]$ using Sidi transformation[22] at the corners which vary more slowly than arc-length near each corner. This has the effect of producing a transformed equation with a solution which is smooth on $[0, 1]$. Then the modified quadrature method is applied to deal with the weakly logarithmically singular integrals in (1.6). The modified quadrature method consists of regularization of the kernel together with trapezoidal approximation of the integral, and this method is superior in efficiency to both Galerkin method and the collocation method since each component in the discrete matrix is directly evaluated by the quadrature [3, 4]. The modified quadrature method was first prosed to solve the first kind BIEs of two dimensional Laplace problems in [23] and the experimental O (h^3) order of convergence was reported in smooth domains. Later, in the case of a circle, Abou El-Seoud was able to prove the existence of solution and an O (h^2) order of convergence in [24]. Saranen proved the order of convergence O (h^3) if the solution is smooth enough in [4]. In this paper, our numerical results show the high convergent rate of modified quadrature method for (1.1) by using Sidi transformation is O (h^3). In addition, the experimental O (h^{-1}) order of the optimal condition number for

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the according discrete system of (1.6) is reported, which shows the modified quadrature method for weakly singular integral problems possesses an excellent stability.

The remainder of this paper is organized as follows: in Section 2, the singularity analysis for the integral kernel in (1.6) is given in detail. In Section 3, modified quadrature method is applied for the BIEs of axisymmetric Laplace equation. In Section 4, some numerical examples are provided to show the efficiency of the proposed methods.

2. SINGULARITY ANALYSIS OF THE INTEGRAL KERNEL

Assume that the non-smooth points on the boundary can be seen as the corners on the boundary $\Gamma = (\partial\Omega/\{(0,z)\}|z \in R) \cup (S' \cap \{(0,z)|z \in R\})$, where $S = \partial V$ is the surface of the body V, and the corners divide the boundary Γ into d segments, i.e., $\Gamma = \cup_{j=1}^{d}\Gamma_j$ $(d > 1)$ with $C_\Gamma \neq 1$, and $\Gamma_j \in C^{2l+1}$ $(j = 1, ..., d,\ l \in N)$. Also assume that Γ_j $(j = 1, \cdots, d)$ have the following parametric representations

$$\mathbf{x}_j(t) = (x_{1j}, x_{2j}) = (\rho_j(t), z_j(t)) : [0,1] \to \Gamma_j, \quad j = 1, \cdots, d, \tag{2.1}$$

where $|\mathbf{x}'_j(t)| = [|\rho'_j(t)|^2 + |z'_j(t)|^2]^{1/2} \geq c > 0$, $x_j(0)$ and $x_j(1)$ denote two endpoints of Γ_j. Under the parameter mapping (2.1), Eq. (1.6) can be converted into the first kind integral equations including definite integrals over [0, 1]

$$g(\mathbf{x}_i(t)) = \sum_{j=1}^{d} \int_0^1 \rho_j(\tau)\phi^*(\mathbf{x}_j(\tau), \mathbf{x}_i(t))|\mathbf{x}'_j(\tau)|\sigma(\mathbf{x}_j(\tau))\mathrm{d}\tau, \quad i = 1, \cdots, d. \tag{2.2}$$

In order to degrade the singularities at corners [8, 14], we apply the Sidi transformation [22] to the variable mapping, which is defined by

$$\varphi_\mu(t) = \frac{\Theta_\mu(t)}{\Theta_\mu(1)}; \qquad \Theta_\mu(t) = \int_0^t (\sin\pi u)^\mu \mathrm{d}u, \quad \mu \in N. \tag{2.3}$$

From the equality

$$\Theta_\mu(t) = \frac{\mu-1}{\mu}\Theta_{\mu-2}(t) - \frac{1}{\pi\mu}(\sin\pi t)^{\mu-1}\cos\pi t,$$

which can be obtained by integration by parts, we have the recursion relation

$$\varphi_\mu(t) = \varphi_{\mu-2}(t) - \frac{\Gamma(\mu)/2}{2\sqrt{\pi}\Gamma((\mu+1)/2)}(\sin\pi t)^{\mu-1}\cos\pi t.$$

where $\Gamma(z)$ is the Gamma function. It is easy to see that $\Theta'_\mu(t)$ has μ order derivatives at the point $t = 0$ and $t = 1$. Hence, $\varphi'_\mu(t)$ has also the same order derivatives. For convenience of discussion, we rewrite Eq. (2.2) as the following operator equation

$$\mathbf{KW} = \mathbf{F}, \tag{2.4}$$

where

$$\mathbf{K} = [\mathbf{K_{ij}}]_{i,j=1}^{d}, \quad \mathbf{F} = (f_1, f_2, \cdots, f_d)^T, \quad f_j = g_j(t) = g_j(\varphi_\mu(t)),$$

$$\mathbf{W} = (w_1(\tau), w_2(\tau), \cdots, w_d(\tau))^T, \quad w_j(\tau) = |\mathbf{x}'_j(\tau)|\varphi'_\mu(\tau)\sigma_j(\varphi_\mu(\tau)).$$

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In (2.4), the kernel $k_{ij}(t,\tau)$ of $\mathbf{K_{ij}}$ can be written as

$$\begin{aligned} k_{ij}(t,\tau) &= \rho_j(\tau)T^*(x_j(\tau),x_i(t)) \\ &= \frac{\alpha_{ij}(t,\tau)}{2\pi}\left[-\frac{1}{2}\ln(\frac{q_{ij}(t,\tau)-1}{32}) + O((q_{ij}(t,\tau)-1)\ln(q_{ij}(t,\tau)-1))\right] \\ &= -\frac{\alpha_{ij}(t,\tau)}{2\pi}\Big(\ln r_{\rho z}(t,\tau) - \ln(8\beta_{ij}(t,\tau))\Big) + \tilde{E}_{ij}(t,\tau),\ \rho_i(\tau)\neq 0, \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} \rho_j(\tau) &= \rho_j(\varphi_\mu(\tau)),\quad z_j(\tau) = z_j(\varphi_\mu(\tau)), \\ \alpha_{ij}(t,\tau) &= [\tfrac{\rho_j(\tau)}{\rho_i(t)}]^{1/2},\quad \beta_{ij}(t,\tau) = [\rho_j(\tau)\rho_i(t)]^{1/2}, \\ r_{\rho,z}(t,\tau) &= [(\rho_j(\tau)-\rho_i(t))^2 + (z_j(\tau)-z_i(t))^2]^{1/2}, \\ \tilde{E}_{ij}(t,\tau) &= O(\rho_j(\tau)(q_{ij}(t,\tau)-1)\ln(q_{ij}(t,\tau)-1), \\ q_{ij}(t,\tau) &= 1 + [(\rho_j(\tau)-\rho_i(t))^2 + (z_j(\tau)-z_i(t))^2](2\rho_j(\tau)\rho_i(t))^{-1}. \end{aligned}$$

Now we split $\mathbf{K_{ii}}$ into a singularity part and a compact perturbation part as follows

$$\mathbf{K_{ii}} = \mathbf{A_{ii}} + \mathbf{B_{ii}}.$$

Let a_{ii} be the kernel of singularity operators $\mathbf{A}_{ii}$ $(i = 1,2,\cdots,d)$,

$$a_{ii}(t,\tau) = -\frac{\alpha_{ii}(t,\tau)}{2\pi}\ln\left|2e^{-1/2}\sin(\pi(t-\tau))\right|,$$

and let $\tilde{a}_{ii}$ be the kernel of $\tilde{\mathbf{A}}_{\mathbf{ii}}$

$$\tilde{a}_{ii}(t,\tau) = -\frac{1}{2\pi}\ln\left|2e^{-1/2}\sin(\pi(t-\tau))\right|.$$

Thus, we have $a_{ii}(t,\tau) = \alpha_{ii}(t,\tau)\tilde{a}_{ii}(t,\tau)$.

Denoted by $\mathbf{A} = \mathrm{diag}(\mathbf{A_{11}},\mathbf{A_{22}},\cdots,\mathbf{A_{dd}})$ and $\mathbf{B} = [\mathbf{B_{ij}}]_{i,j=1}^d = \mathbf{K}-\mathbf{A}$. Then the integral operator equation (2.4) can be splitted into

$$(\mathbf{A}+\mathbf{B})\mathbf{W} = \mathbf{F}, \tag{2.6}$$

where the kernel of $\mathbf{B_{ij}}$ is

$$b_{ij}(t,\tau) = \begin{cases} k_{ii}(t,\tau) - a_{ii}(t,\tau), & i = j, \\ k_{ij}(t,\tau), & i\neq j. \end{cases} \tag{2.7}$$

Clearly, the operator $\tilde{\mathbf{A}}_{\mathbf{ii}}$ is defined on the circular contour with radius $e^{-1/2}$. From [2, 7, 8], we can see that $\tilde{\mathbf{A}}_{\mathbf{ii}}$ has the special property

$$\tilde{\mathbf{A}}_{\mathbf{ii}}\sigma(t) = \frac{\sqrt{2}}{\pi}(\sum_{m\in Z^*}|m|^{-1}\hat{\sigma}(m)e^{imt} + \hat{\sigma}(0)),\quad t\in[0,2\pi],$$

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where $Z^* = Z\backslash\{0\}$, Z is the set of all integers, and $\hat{\sigma}(t)$ is the Fourier transform of $\sigma(t) \in C^\infty$ (i.e., the p order derivative of σ is continuous and 2π-periodic for all $p \geq 0$). As is easily shown, $|\tilde{\mathbf{A}}_{\mathbf{ii}}\sigma|_{p+1} = |\sigma|_p$, for all $\sigma \in H^p$, where $|\sigma|_p$ on C^∞ is the norm of the usual space H^p. So $\tilde{\mathbf{A}}_{\mathbf{ii}}$ is an isometry from H^p to H^{p+1} [12]. As above, the operators $\mathbf{A}_{\mathbf{ii}}(i = 1, \cdots, d)$ are isometry operators from $H^p[0,1]$ to $H^{p+1}[0,1]$ for any real number m. Hence, $\mathbf{A}$ is also an isometry operator from $(H^p[0,1])^d$ to $(H^{p+1}[0,1])^d$. Furthermore, Eq. (2.6) becomes

$$(\mathbf{I} + \mathbf{A}^{-1}\mathbf{B})\mathbf{W} = \mathbf{A}^{-1}\mathbf{F} = \widetilde{\mathbf{F}}, \tag{2.8}$$

Let P_j $(j = 1, 2, \cdots, d)$ be the corner points of the boundary Γ. Define a function $\chi_j \in [-1, 1]$

$$\chi_j = \begin{cases} -1 & \text{as } j = 1 \text{ or } j = d, \\ \dfrac{\pi - \theta_j}{\pi} & \text{as } 1 < j < d, \end{cases}$$

where θ_j is the interior angle of the middle corner P_j.

Since the boundary Γ is a closed polygon, the solution singularity for (1.6) occurs at the corner points P_1, ..., P_d of the boundary Γ [7, 11, 18]. The solution $w_j(\mathbf{x}) = \frac{\partial\phi_j(\mathbf{x})}{\partial\nu^-} - \frac{\partial\phi_j(\mathbf{x})}{\partial\nu^+}$ has a singularity $O(|s - s_j|^{\beta_j})$ when $s = s_j$, where $\beta_j = -\frac{|\chi_j|}{(1+|\chi_j|)}$, $\beta_j \in [-\frac{1}{2}, 0)$, and ν and s_j denote the unit normal and the arc parameter at P_j respectively.

Lemma 2.1. [20] (1) Let the function $\sigma_j(s) = s^{\beta_j}u_j(s)$ $(0 > \beta_j \geq -1/2)$, where $u_j(s)$ is differentiable enough on $[0, 1]$ with $u_j(0) \neq 0$. Then the function $w_j(t)$ can be expressed by

$$w_j(t) = c_1 u_j(0) t^{(\mu+1)\beta_j+\mu}(1 + O(t^2)) \text{ as } t \to 0^+, \tag{2.9}$$

where c_1 is a constant independent of t.

(2) Let the function $\sigma_j(s) = (1 - s)^{\beta_j}\tilde{u}_j(s)$ $(0 > \beta_j \geq -1/2)$, where $\tilde{u}_j(s)$ is differentiable enough on $[0, 1]$ with $\tilde{u}_j(1) \neq 0$. Then the function $w_j(t)$ can be expressed by

$$w_j(t) = c_2\tilde{u}_j(1)(1 - t)^{(\mu+1)\beta_j+\mu}(1 + O((1 - t)^2))) \text{ as } t \to 1^-, \tag{2.10}$$

where c_2 is a constant independent of t.

By Lemma 2.1 and $\beta_j \geq -\frac{1}{2}$, we can obtain $(\mu + 1)\beta_j + \mu \geq 0$ for $\mu \geq 1$. Furthermore, we have the following remark.

Remark 1 Although $\sigma_j(s)$ has singularities at endpoints $s = 0$ and $s = 1$, $w_j(t)$ has no singularity by Sidi transformation at $t = 0$ and $t = 1$.

Bellow we study singularity for the kernels $b_{ij}(t, \tau)$ $(i, j = 1, \cdots, d)$. Obviously, if $\Gamma_i \cap \Gamma_j = \emptyset$, $b_{ij}(t, \tau)$ are continuous on $[0, 1]^2$, and if $\Gamma_i \cap \Gamma_j \neq \emptyset$, $b_{ij}(t, \tau)$ have singularities at the points $(t, \tau) = (0, 1)$ and $(t, \tau) = (1, 0)$, where $\emptyset$ denotes the empty set. Here, we only discuss the case in that $(t, \tau) = (1, 0)$, the singularity analysis for the case in that $(t, \tau) = (0, 1)$ can be similarly obtained. Defining the following function

$$\tilde{b}_{ij}(t, \tau) = b_{ij}(t, \tau)\sin^\mu(\pi t), \quad \mu \geq 1, \quad \Gamma_i \cap \Gamma_j \neq \emptyset. \tag{2.11}$$

Theorem 2.2. Let $\tilde{b}_{ij}(t,\tau)$ be defined by (2.11), then $\tilde{b}_{ij}(t,\tau)$ and $\frac{\partial^k \tilde{b}_{ij}(t,\tau)}{\partial \tau^k}$ $(k = 1, 2)$ are smooth on $[0, 1]^2$.

Proof. By using the continuity of $\tilde{b}_{ii}(t,\tau)$ and the boundness of $\sin^{\mu}(\pi t)$, we can immediately complete the proof for the case $i = j$. Let $\Gamma_i \cap \Gamma_j = P_i = (0, 0)$ ($|i - j| = 1$ or $d - 1$) and $\theta_i \in (0, 2\pi)$ is the corresponding interior angle. Then we have $r_{\rho,z}(t,\tau) = 0$, which shows the kernel $b_{i,j}(t,\tau)$ has a logarithmically singularity at $(t,\tau) = (1, 0)$. Suppose that $a_i(t) = |x_i(t)|$ and $a_j(\tau) = |x_j(\tau)|$. Then we can obtain

$$\tilde{b}_{i,j}(t,\tau) = I_1(t,\tau) + I_2(t,\tau). \tag{2.12}$$

where

$$I_1(t,\tau) = -\frac{\alpha_{ij}(t,\tau)}{2\pi}\frac{\sin^{\mu}(\pi\tau)}{\Theta_{\mu}(1)}\ln\left|a_i^2(t) + a_j^2(\tau) - 2a_i(t)a_j(\tau)\cos\theta_i\right| - \frac{\alpha_{ij}(t,\tau)}{2\pi}\frac{\sin^{\mu}(\pi\tau)}{\Theta_{\mu}(1)}\ln\left|8\beta_{ij}(t,\tau)\right|,$$

and

$$I_2(t,\tau) = O((q_{ij}(t,\tau) - 1)\ln(q_{ij}(t,\tau) - 1))\frac{\sin^{\mu}(\pi\tau)}{\Theta_{\mu}(1)}.$$

Since $\frac{\sin^{\mu}(\pi\tau)}{\Theta_{\mu}(1)}$ has order μ zero at $\tau = 0$ and $\tau = 1$, the function $I_2(t,\tau)$ is continuous on $[0, 1]^2$. Let

$$I_1(t,\tau) = I_{11}(t,\tau) + I_{12}(t,\tau), \tag{2.13}$$

where

$$I_{11}(t,\tau) = -\frac{\alpha_{ij}(t,\tau)}{2\pi}\frac{\sin^{\mu}(\pi\tau)}{\Theta_{\mu}(1)}\ln\left|a_i^2(t) + a_j^2(\tau) - 2a_i(t)a_j(\tau)\cos\theta_i\right|$$

and

$$I_{12}(t,\tau) = -\frac{\alpha_{ij}(t,\tau)}{2\pi}\frac{\sin^{\mu}(\pi\tau)}{\Theta_{\mu}(1)}\ln\left|8\beta_{ij}(t,\tau)\right|.$$

Obviously, $I_{12}(t,\tau)$ is a smooth function on $[0, 1]^2$. Now we will prove $I_{11}(t,\tau)$ is also smooth on the same interval. We write

$$I_{11}(t,\tau) = I_{111}(t,\tau) + I_{112}(t,\tau), \tag{2.14}$$

where

$$I_{111}(t,\tau) = -\frac{\alpha_{ij}(t,\tau)}{2\pi}\frac{\sin^{\mu}(\pi\tau)}{\Theta_{\mu}(1)}\ln\left|1 - 2a_i(t)a_j(\tau)/(a_i^2(t) + a_j^2(\tau))\right|,$$

and

$$I_{112}(t,\tau) = -\frac{\alpha_{ij}(t,\tau)}{2\pi}\frac{\sin^{\mu}(\pi\tau)}{\Theta_{\mu}(1)}\ln[a_i^2(t) + a_j^2(\tau)].$$

By

$$\left|2a_i(t)a_j(\tau)/(a_i^2(t) + a_j^2(\tau))\right| \leq \left|\cos\theta_i\right| < 1,$$

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we know that $I_{111}(t,\tau)$ is smooth on $[0,1]^2$. Let $0 < t,\ \tau < \varepsilon$, then we have

$$a_i(t) = O(\varepsilon), \quad a_j(\tau) = O(\varepsilon), \quad \left|\sin^{\mu}(\pi\tau)\right| = O(\varepsilon^{\mu})$$

and

$$\left|I_{112}(t,\tau)\right| = O(\varepsilon^{\mu}\ln\varepsilon) \to 0, \quad \varepsilon \to 0.$$

Hence, $I_{112}(t,\tau)$ is continuous on $[0,1]^2$. Furthermore,

$$\begin{aligned}
\Big|\frac{\partial I_{111}(t,\tau)}{\partial\tau}\Big| &\le \Big|\frac{\alpha_{ij}(t,\tau)}{2\pi}\frac{\sin^{\mu-1}(\pi\tau)\cos(\pi\tau)}{\Theta_{\mu}(1)}\ln\left|a_i^2(t)+a_j^2(\tau)\right|\Big| \\
&+ \Big|\frac{\alpha_{ij}(t,\tau)}{2\pi}\frac{\sin^{\mu}(\pi\tau)}{\Theta_{\mu}(1)}\frac{(x_{1j}(\varphi_\mu(t))x'_{1j}(\varphi_\mu(t)) + (x_{2j}(\varphi_\mu(t))x'_{2j}(\varphi_\mu(t)))\varphi'_\mu(t)}{a_i^2(t)+a_j^2(\tau)}\Big| \\
&+ \Big|\frac{1}{2\pi}\frac{\partial\alpha_{ij}(t,\tau)}{\partial\tau}\frac{\sin^{\mu}(\pi\tau)}{\Theta_{\mu}(1)}\Big| \\
&= O(\varepsilon^{\mu-1}\ln\varepsilon) + O(\varepsilon^{\mu})O(\varepsilon^{2\mu})/O(\varepsilon^{2\mu+1}) + O(\varepsilon^{\mu}\ln\varepsilon) \\
&= O(\varepsilon^{\mu-1}\ln\varepsilon) + O(\varepsilon^{\mu-1}) \to 0, \quad \varepsilon\to 0.
\end{aligned} \tag{2.15}$$

Similarly, we have

$$\Big|\frac{\partial^2 I_{111}(t,\tau)}{\partial\tau^2}\Big| \le O(\varepsilon^{\mu-2}\ln\varepsilon) + O(\varepsilon^{\mu-2}) \to 0, \quad \text{as} \quad \varepsilon \to 0,$$

which shows that $\frac{\partial^2 I_{111}(t,\tau)}{\partial\tau^2}$ and $\frac{\partial I_{111}(t,\tau)}{\partial\tau}$ are continuous on $[0,1]^2$. Similar to the proof mentioned above, we can prove $\frac{\partial^2 I_{111}(t,\tau)}{\partial t^2}$ and $\frac{\partial I_{111}(t,\tau)}{\partial t}$ are continuous on $[0,1]^2$. So, we can obtain $I_{111}(t,\tau) \in C^2[0,1]^2$ for $\mu \ge 3$ and $I(t,\tau) \in C^2[0,1]^2$ for $\mu \ge 3$.

For $\theta_i = \pi$, we have

$$\begin{aligned}
\left|\tilde{b}_{ij}(t,\tau)\right| &\le \frac{\alpha_{ij}(t,\tau)}{\pi}\frac{\sin^{\mu}(\pi\tau)}{\Theta_{\mu}(1)}(\ln\left|a_i(t)+a_j(\tau)\right| \\
&+ \ln 8\beta_{ij}(t,\tau))\Big| + \Big|O((q_{ij}(t,\tau)-1)\ln(q_{ij}(t,\tau)-1)\frac{\sin^{\mu}(\pi\tau)}{\Theta_{\mu}(1)}\Big| \\
&= O(\varepsilon^{\mu}\ln\varepsilon) + O(\varepsilon^{\mu+2}\ln\varepsilon).
\end{aligned} \tag{2.16}$$

Similar to the case $\theta_i \in (0,2\pi)$, we also can prove $\tilde{b}_{ij}(t,\tau) \in C^2[0,1]^2$.

3. MODIFIED QUADRATURE FORMULA AND DISCRETIZATION FOR INTEGRAL OPERATORS

A reformulated rule based on the modified quadrature formula [4] to achieve the quadrature values for integrals with the logarithmically singular kernels can be described as follows

$$\int_0^1 \ln|t-\tau|\sigma(\tau)\mathrm{d}\tau = h\sum_{j=1,t\neq\tau_j}^{N}\ln|t-\tau_j|\sigma(\tau_j) + h\ln\Big(\frac{|x^{'}(t)|}{2\pi}\Big)\sigma(t) - h\sum_{j=1}^{N-1}\ln(2|\sin\pi jh|)\sigma(t), \tag{3.1}$$

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where $\tau_j = jh$ and the step size is $h = 1/N$.

Now we apply (3.1) to solve the boundary integral equation (1.6). Assume that $h_j = 1/n_j$ ($n_j \in N, j = 1, \cdots, d$) is the mesh size of Γ_j, $t_{jm} = \tau_{jm} = (m - 1/2)h_j$ ($m = 1, \cdots, n_j$) are the nodes on Γ_j.

(1) When $i \neq j$, we can use trapezoidal rule [1] to construct the Nystöm approxiamation $\mathbf{K^h_{ij}}$ of $\mathbf{K_{ij}}$

$$(\mathbf{K^h_{ij}} w_j)(t) = h_j \sum_{s=1}^{n_j} k_{ij}(t, \tau_{js}) w_j(\tau_{js}), \quad i, j = 1, \cdots, d, \tag{3.2}$$

which has the error order at least $O\,(h^3)$ by using the Sidi transformation[22]. The trapezoidal rule approximation $\mathbf{B^{h_j}_{ij}}$ for the boundary integral operator $\mathbf{B_{ij}}$ is

$$(\mathbf{B^{h_j}_{ij}} w_j)(t) = h_j \sum_{m=1}^{n_j} b_{ij}(t, \tau_{jm}) w_j(\tau_{jm}), \quad t \in [0, 1], \quad i, j = 1, ..., d, \tag{3.3}$$

which has the error bounds [10, 22]

$$\left|(\mathbf{B_{ij}} w_j)(t) - (\mathbf{B^{h_j}_{ij}} w_j)(t)\right|_\infty \leq \begin{cases} O\,(h_j^{2\ell+1}), & \text{for } \Gamma_j = \Gamma_j \text{ or } \Gamma_j \cap \Gamma_j = \emptyset, \quad \ell \in N, \\ O\,(h_j^{\omega_j(\gamma+1)+1}), & \text{for } \Gamma_i \cap \Gamma_j \in \{P_m\}, \end{cases}$$

where

$$\omega_j = \begin{cases} \min\{\beta_j + 1, 2\}, & \gamma \text{ even}, \\ \\ \min\{\beta_j + 1, 1\}, & \gamma \text{ odd}. \end{cases}$$

(2) When $i = j$, integral operator $\mathbf{K_{ii}} = \mathbf{A_{ii}} + \mathbf{B_{ii}}$ has a logarithmically singularity, we use the reformulated formula (3.1) to construct the approximation $\mathbf{A^h_{ii}}$ and $\mathbf{B^h_{ii}}$ of $\mathbf{A_{ii}}$ and $\mathbf{B_{ii}}$, respectively,

$$\begin{aligned}(\mathbf{A^h_{ii}} w_i)(t) = &-\frac{h_i \alpha_{ii}(t, \tau_{is})}{2\pi}\Big(\sum_{t=1, t\neq\tau_{is}}^{n_i} \ln\left|2e^{-1/2}\sin(\pi(t - \tau_{is}))\right| w_i(\tau_{is}) - \gamma_i w_i(t) \\ &- \ln|\frac{e^{1/2} x_i'(t)\varphi_\mu'(t)}{2\pi}| w_i(t)\Big), \quad i = 1, \cdots, d.\end{aligned} \tag{3.4}$$

where $\gamma_i = \sum_{i=1}^{n_i-1} \ln(2|\sin(\pi i h_i)|)$ and $h_i = 1/n_i$ ($i = 1, \cdots, d$). The error of (3.4) is

$$\left|(\mathbf{A^h_{ii}} w_i)(t) - (\mathbf{A_{ii}} w_i)(t)\right|_\infty \leq c_i h_i^3, \tag{3.5}$$

where c_i is a constant number. Accordingly, we have

$$\begin{aligned}(\mathbf{B^h_{ii}} w_i)(t) = &\, h_i \sum_{s=1, t\neq\tau_{is}} b_{ij}(t, \tau_{is}) w_i(\tau_{is}) \\ &+ \frac{h_i \alpha_{ii}(t, \tau_{is})}{2\pi}\left[\ln|8\rho_i(t)| - \ln|\frac{[(\rho_i'(t))^2 + (z_i'(t))^2]^{1/2} x_i'(t)\varphi_\mu'(t)}{2\pi e^{-1/2}}|\right] w_i(t), \\ &i = 1, \cdots, d.\end{aligned} \tag{3.6}$$

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Hence, the approxiamation $\mathbf{K^h_{ii}}$ of $\mathbf{K_{ii}}$ can be constructed by

$$\begin{aligned}(\mathbf{K^h_{ii}}w_i)(t) = {} & h_i \sum_{s=1,s\neq\tau_{is}}^{n_i} k_{ii}(t,\tau_{is})w_i(\tau_{is}) + \frac{h_i\alpha_{ii}(t,\tau_{is})}{2\pi}c_iw_i(t) \\ & + \frac{h_i\alpha_{ii}(t,\tau_{is})}{2\pi}\ln|\frac{e^{1/2}x_i'(t)\varphi_\mu'(t)}{2\pi}|w_i(t) \\ & - \frac{\alpha_{ii}(t,\tau_{is})h_i}{2\pi}\ln\left|\frac{[(\rho_i'(t))^2+(z_i'(t))^2]^{1/2}x_i'(t)\varphi_\mu'(t)}{16\pi e^{-1/2}r_i(t)}\right|w_i(t), \quad i=1,\cdots,d. \end{aligned} \tag{3.7}$$

Thus the approximate equation of (2.4) is

$$\mathbf{K^hW^h} = \mathbf{F^h}, \tag{3.8}$$

where

$$\mathbf{K^h} = [\mathbf{K^h_{ij}}]^d_{i,j=1}, \quad \mathbf{F^h} = (f_1^h, f_2^h, \cdots, f_d^h)^t,$$
$$\mathbf{f^h_j} = (g(\mathbf{x}_j(t_{j1})), g(\mathbf{x}_j(t_{j2})), \cdots, g(\mathbf{x}_j(t_{jd})))^t, \quad j=1,\cdots,d,$$
$$\mathbf{W^h} = (w_1^h, w_2^h, \cdots, w_d^h)^t, \quad w_i^h = (w_1^h(\tau_{i1}), w_2^h(\tau_{i2}), \cdots, w_d^h((\tau_{in_i})))^t, \quad i=1,\cdots,d.$$

In fact, E.q. (3.8) can be rewritten as

$$(\mathbf{I^h} + (\mathbf{A^h})^{-1}\mathbf{B^h})\mathbf{W^h} = \tilde{\mathbf{F}}^\mathbf{h}, \tag{3.9}$$

where $\mathbf{A^h} = \mathrm{diag}(\mathbf{A^h_{11}}, \mathbf{A^h_{22}}, \cdots, \mathbf{A^h_{dd}})$, $\mathbf{B^h} = [\mathbf{B^h_{ij}}]^d_{i,j=1}$, and $\tilde{\mathbf{F}}^\mathbf{h} = (\mathbf{A^h})^{-1}\mathbf{F^h}$.

Obviously, (3.9) has $n\ (=\sum_{j=1}^d n_j)$ unknowns. Once $\mathbf{W^h}$ is solved from (3.6), $\phi(\mathbf{y})$ $(\mathbf{y}\in\Omega')$ can be calculated by

$$\phi^h(\mathbf{y}) = \sum_{j=1}^d \sum_{i=1}^{n_j} \rho_j(\varphi_\mu(t_i))\phi^*(\mathbf{x}_j(\varphi_\mu(t_i),\mathbf{y})w_{ij}. \tag{3.10}$$

4. NUMERICAL EXAMPLES

In this section, two numerical examples about the axisymmetric Laplace equation are computed by modified quadrature method in this paper. Let $e_n(\mathbf{x}) = \left|\phi_{\mathrm{exact}}(\mathbf{x}) - \phi_n(\mathbf{x})\right|$ be the errors at the polar coordinate point $\mathbf{x}=(\rho,z)$ by modified quadrature method using $n\ (=\sum_{j=1}^d n_j)$ boundary nodes, and let $r_n = e_n/e_{n/2}$ be the error ratio by modified quadrature rule, where $n = 2\pi/h$.

Example 4.1 Consider a finite solid cylinder with bottom radius $R=2$, and the height $L=3$. Temperature on the upper base is $g=2$, and the temperature on the other sides is 0. The analytical solution is

$$\phi_{\mathrm{exact}} = \sum_{k=1}^{\infty} \frac{2g}{x_k^0}\frac{\sinh x_k^0 z/R}{\sinh x_k^0 L/R}\frac{J_0(x_k^0 r/R)}{J_1(x_k^0)}.$$

where J_0 and J_1 are 0 order and 1 order Bessel functions respectively.

Assume that each boundary Γ_j $(j=1,\cdots,3)$ be divided into 2^k $(k=3,\cdots,6)$ segments. We use φ_3 transformation

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to compute the points in the subregion $\Omega_1 = \{(r,z)|r = 0.1R : \frac{1}{N_r}R : 0.9R,\ z = 0.1L : \frac{1}{N_z}L : 0.9L,$ where $N_r = N_z = 50\}$. By (2.3), we obtain $\varphi_3(t) = \frac{1}{4}\cos^3\pi t - \frac{3}{4}\cos\pi t + \frac{1}{2}$, where $t \in [0,1]$.

The total node number of the whole boundary are n $(= 3\times 2^k,\ k = 3,\cdots,6)$. The errors and error ratio of the polar coordinate points $P_1 = (0.5,1)$, $P_2 = (0.5,1.5)$ and $P_3 = (0.5,2)$ by modified quadrature method are listed in Table 2. From the numerical results, we can see that $\log_2 r_n \approx 3$, which shows the experimental convergence order of ϕ is $O\ (h^3)$. The condition numbers for the matrix $\mathbf{K^h} = \mathbf{A^h} + \mathbf{B^h}$ are listed in Table 1, where $|\lambda_{\min}|$ and $|\lambda_{\max}|$ are minimum and maximum eigenvalues respectively. Let Cond. be the traditional 2-norm condition number, from Table 1, we can see $\Xi_k = \text{Cond.}|_{(2^k,2^k,2^k)}/\text{Cond.}|_{(2^{k-1},2^{k-1},2^{k-1})} \approx 2\ (k = 4,5,6)$, which show the modified quadrature method has excellent stability.

Fig. 1 shows that isolines of temperature distribution in the subregion Ω_1 and the errors for points along the line $0.04 \le r \le 1.96;\ z = 0.06$ by modified quadrature rule. Fig. 2 and Fig. 3 show the isolines of errors in the subregion Ω_1 using modified quadrature rule.

Table 1: The condition number for Example 1

n	$(2^3,2^3,2^3)$	$(2^4,2^4,2^4)$	$(2^5,2^5,2^5)$	$(2^6,2^6,2^6)$
$\|\lambda_{\min}\|$	2.912E-3	1.409E-3	6.982E-4	3.164E-4
$\|\lambda_{\max}\|$	1.016	1.017	1.017	1.017
Cond	3.490E+2	7.219E+2	1.457E+3	3.215E+3
Ξ_k	–	2.068	2.018	2.207

Table 2: The errors for Example 1

n	$(2^3,2^3,2^3)$	$(2^4,2^4,2^4)$	$(2^5,2^5,2^5)$	$(2^6,2^6,2^6)$
$e_n(P_1)$	5.840E-4	8.299E-5	1.051E-5	1.302E-6
$r_n(P_1)$	–	$2^{2.815}$	$2^{2.981}$	$2^{3.012}$
$e_n(P_2)$	6.961E-4	1.016E-4	1.296E-5	1.629E-6
$r_n(P_2)$	–	$2^{2.777}$	$2^{2.970}$	$2^{2.992}$
$e_n(P_3)$	3.800E-4	6.858E-5	9.002E-6	1.184E-6
$r_n(P_3)$	–	$2^{2.470}$	$2^{2.929}$	$2^{2.927}$

Example 4.2 Consider a finite solid cylinder with bottom radius $R = 5$, and the height $L = 7$. Temperature on the upper base is $g = r^2$, and the temperature on the other sides is 0. The analytical solution is

$$\phi_{\text{exact}} = 2R^2 \sum_{k=1}^{\infty} \frac{(x_k^0)^2 - 4}{(x_k^0)^3} \frac{\sinh x_k^0 z/R}{\sinh x_k^0 L/R} \frac{J_0(x_k^0 r/R)}{J_1(x_k^0)}.$$

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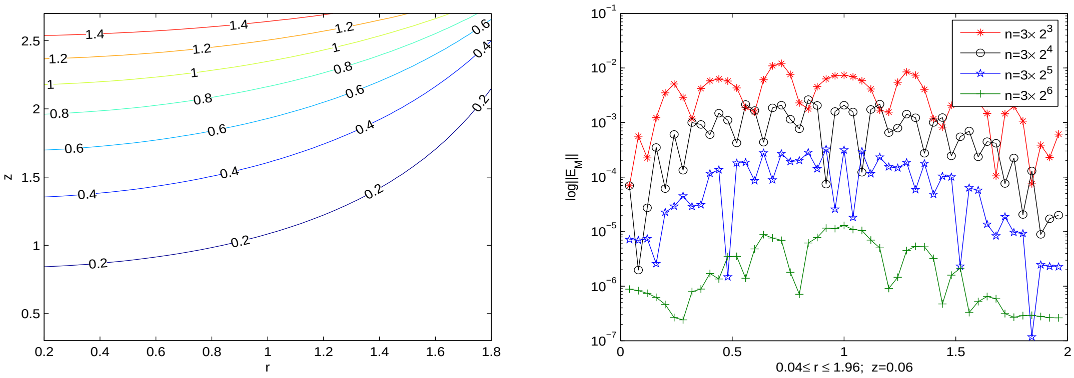

Figure 1: The distributions of temperature when $n = 3\times2^6$ (left) and errors for points along the line $0.04 \le r \le 1.96$; $z = 0.06$ by φ_3 transformation (right)

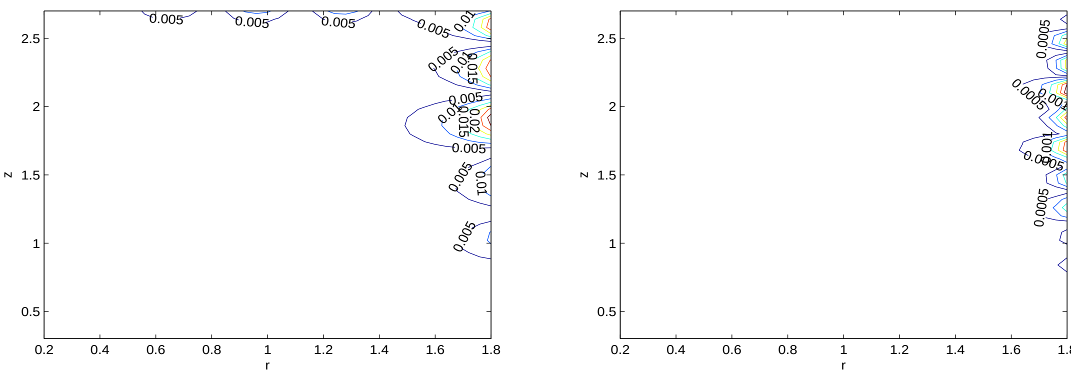

Figure 2: Isolines of errors of temperature by 3×2^3 boundary nodes (left) and Isolines of errors of temperature by 3×2^4 boundary nodes (right)

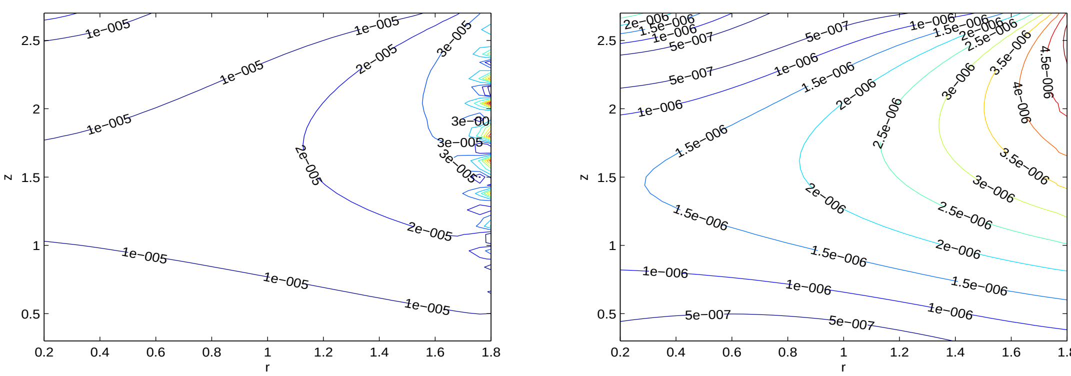

Figure 3: Isolines of errors of temperature by 3×2^5 boundary nodes (left) and Isolines of errors of temperature by 3×2^6 boundary nodes (right)

Let each boundary Γ_j $(j = 1, \cdots, 3)$ be divided into 2^k $(k = 3, \cdots, 6)$ segments. We also use φ_3 transformation to compute the polar coordinate points in the subregion $\Omega_2 = \{(r,z)|r = 0.1R : \frac{1}{N_r}R : 0.9R,\ z = 0.1L : \frac{1}{N_z}L :$

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$0.9L$, where $N_r = 50,\ N_z = 50\}$.

The total node number of all piece-wised boundaries are $n\ (= 3 \times 2^k,\ k = 3, \cdots, 6)$. The errors and error ratio of $P_1 = (0.5, 1), P_2 = (0.5, 3)$ and $P_3 = (0.5, 5)$ by modified quadrature method are listed in Table 4. From the numerical results, we can see that $\log_2 r_n \approx 3$, which shows the same experimental convergence order of ϕ is $O\ (h^3)$. The condition numbers for the matrix $\mathbf{K^h} = \mathbf{A^h} + \mathbf{B^h}$ are listed in Table 3, where $|\lambda_{\min}|$ and $|\lambda_{\max}|$ are minimum and maximum eigenvalues respectively. From Table 3, we can see $\Xi_k = \text{Cond.}|_{(2^k,2^k,2^k)}/\text{Cond.}|_{(2^{k-1},2^{k-1},2^{k-1})} \approx 2\ (k = 4, 5, 6)$, which show the modified quadrature method has excellent stability.

Fig. 4 shows that isolines of temperature distribution in the subregion Ω_2 by using $n = 3 \times 2^6$ boundary nodes and the errors for points along the line $0.10 \le r \le 4.00;\ z = 0.14$ by modified quadrature method. Fig. 5 and Fig. 6 show the isolines of errors in the subregion Ω_2 using modified quadrature method.

Table 3: The condition number for Example 2

Λ^k	Λ^3	Λ^4	Λ^5	Λ^6
$\lvert\lambda_{\min}\rvert$	2.669E-3	1.302E-3	6.461E-4	2.453E-4
$\lvert\lambda_{\max}\rvert$	1.034	1.034	1.034	1.034
Cond	3.829E+2	7.942E+2	1.601E+3	4.216E+3
Ξ_k	–	2.074	2.016	2.633

Table 4: The errors for Example 2

Λ^k	$(2^3,2^3,2^3)$	$(2^4,2^4,2^4)$	$(2^5,2^5,2^5)$	$(2^6,2^6,2^6)$
$e_n(P_1)$	1.249E-3	5.422E-5	6.052E-6	9.154E-7
$r_n(P_1)$	–	$2^{4.525}$	$2^{3.163}$	$2^{2.775}$
$e_n(P_2)$	6.616E-3	6.621E-4	8.005E-5	1.029E-5
$r_n(P_2)$	–	$2^{3.321}$	$2^{3.048}$	$2^{2.959}$
$e_n(P_3)$	2.378E-2	3.164E-3	3.916E-4	4.948E-5
$r_n(P_3)$	–	$2^{3.113}$	$2^{3.014}$	$2^{2.985}$

5. CONCLUSIONS

In this paper, the modified quadrature method with Sidi transformation are applied for the first kind integral equations of 3D axisymmetric Laplace problems on polygons. The numerical results show that the rate of error convergence for the modified quadrature can achieve $O\ (h^3)$. Furthermore, the excellent stability of the modified

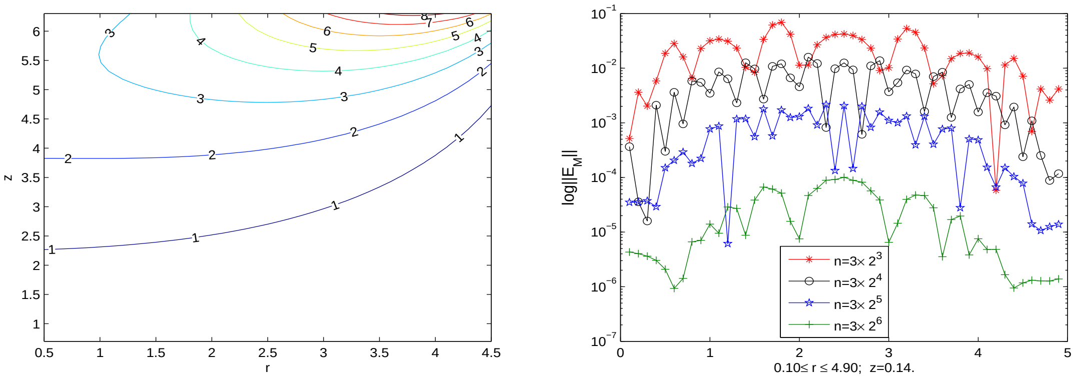

Figure 4: The distributions of isolines temperature by φ_3 transformation when $n = 3 \times 2^6$ and errors for points along the line $0.10 \leq r \leq 4.00$; $z = 0.14$

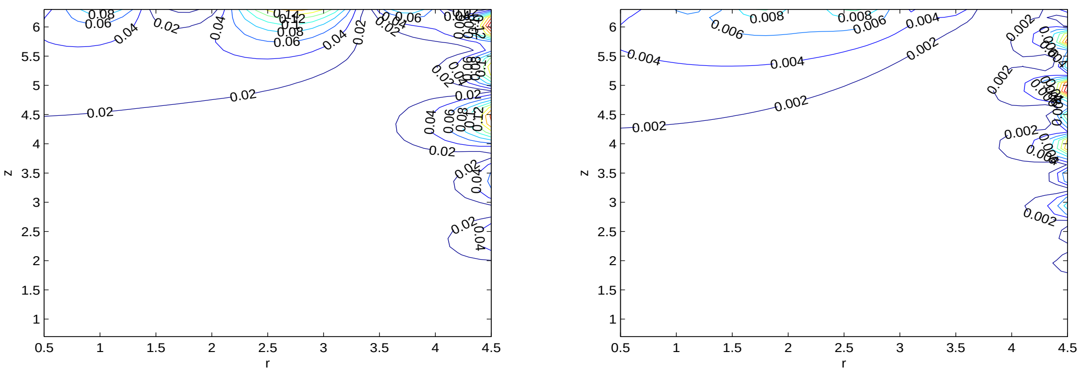

Figure 5: Isolines of errors of temperature by 3×2^3 boundary nodes (left) and Isolines of errors of temperature by 3×2^4 boundary nodes (right)

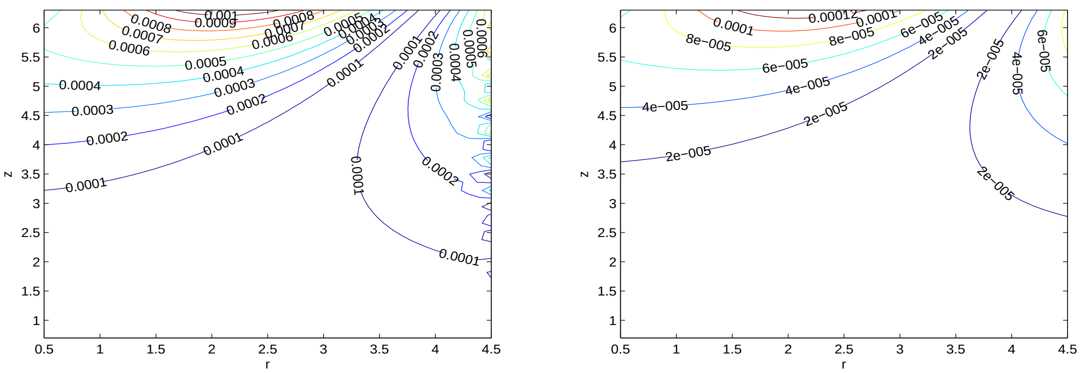

Figure 6: Isolines of errors of temperature by 3×2^5 boundary nodes (left) and Isolines of errors of temperature by 3×2^6 boundary nodes (right)

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quadrature method has been presented, which shows the condition number of the discrete system of the weakly singularity problems is of the order $O\ (h^{-1})$.

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STABILITY OF THE GENERALIAZED VERSION OF CAUCHY TYPE EQUATION FROM FUNCTIONAL OPERATOR VIEWPOINT

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Abstract. We introduce a viewpoint to analyze functional inequalities for the generalized version of Hyers-Ulam stability problems. Especially, this article focuses the following generalized version of Cauchy functional inequality containing an general extra term G_f.

$$\|f(x+y)-f(x)-f(y)-G_f(x,y)\| \leq \phi(x,y)$$

We investigate the functional operator properties of G_f. And by virtue of this observation, we provide a method to produce lots of interesting additive functional equations as applications.

Key Words : Generalized Hyres-Ulam stability, Cauchy type additive functional equation, functional operator, Banach space, iterative technique

2010 Mathematics Subject Classification : Primary 39B, Secondary 26D

1. Introduction

In 1940, S. M. Ulam proposed the following stability problem (See [9]) :

"Let G_1 be a group and G_2 a metric group with the metric d. Given a constant $\delta > 0$, does there exist a constant $c > 0$ such that if a mapping $f : G_1 \longrightarrow G_2$ satisfies $d(f(xy), f(x)f(y)) < c$ for all $x, y \in G_1$, then a unique homomorphism $h : G_1 \longrightarrow G_2$ exists with $d(f(x), h(x)) < \delta$ for all $x \in G_1$?"

In 1941, D. H. Hyers [2] answered this problem under the assumption that the groups G_1 and G_2 are Banach spaces. Later T. Aoki [1] and Th. M. Rassias [7] generalized the result of Hyers. Following these results, many interesting stability problems have been studied by various mathematicians.

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Among these, for additive equations, Th. M. Rassias [7] solved the generalized Hyers-Ulam stability of the functional inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p)$$

for some $\epsilon \geq 0$ and p with $p < 1$ and all $x, y \in X$, where $f : X \longrightarrow Y$ is a function between Banach spaces.

In 1982-1989, J. M. Rassias [3, 4, 5] solved the generalized Hyers-Ulam stability of the functional inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon\|x\|^p\|y\|^q$$

for some $\epsilon \geq 0$ and $p, q \in R$ with $p + q \neq 1$ and for all $x, y \in X$, where $f : X \longrightarrow Y$ is a function from a real normed space X to a real Banach space Y.

Also K. Ravi, M. Arunkumar and J. M. Rassias introduced the product-sum function in [8] and M. J. Rassias proved in [6] the generalized Hyers-Ulam product-sum stability of the Cauchy type additive functional inequality

$$\|f(x+y)+f(x-y)+f(y-x)-f(x)-f(y)\| \leq \epsilon(\|x\|^{\alpha/2}\|y\|^{\alpha/2}+\|x\|^{\alpha}+\|y\|^{\alpha}) \tag{1.1}$$

for every $x, y \in X$ with $\epsilon \geq 0$ and $\alpha \neq 1$, where $f : X \longrightarrow Y$ is a function from a real normed space X to a real Banach space Y. Here, we look at the functional inequality (1.1) as the following.

$$\|f(x+y) - f(x) - f(y) + G_f(x,y)\| \leq \epsilon(\|x\|^{\alpha/2}\|y\|^{\alpha/2} + \|x\|^{\alpha} + \|y\|^{\alpha}),$$

where $G_f(x,y) = f(x-y) + f(y-x)$. In this inequality, $G_f(x,y)$ can be considered as a functional operator depending not only on $x, y \in X$ but also on $f : X \longrightarrow Y$. We can easily know that $G_f(x,y) = f(x-y)+f(y-x) = 0$ for all additive function f.

With this point of view, in general, for a given functional inequality

$$\|F(f,x,y)\| \leq \phi(x,y) \tag{1.2}$$

we split the functional operator F into two parts as

$$F(f,x,y) = S(f,x,y) + G_f(x,y).$$

Here $S(f,x,y)$ is a standard form which characterize the function f satisfying (1.2), and G_f is the extra term. Especially in this paper, we deal with the case of $S(f,x,y) = f(x+y) - f(x) - f(y)$, and are interested in the conditions on G_f as functional operator in this case. In other words, we are interested in what kind of terms can be added to the standard additive functional equation

$$f(x+y) = f(x) + f(y)$$

while the generalized Hyers-Ulam stability still holds for the new functional equation

$$f(x+y) = f(x) + f(y) + G_f(x,y). \tag{1.3}$$

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The precise definition of G_f is given in section 2. Finally we remark that in order to the stability holds for the generalized version of the Cauchy type functional equation (1.3), a function f satisfying (1.3) should be additive. This would be an indispensable necessary condition on G_f in the general context. So we have to impose this fact as a condition in Theorem 2.1. However we eliminate this condition in some practical problems in section 3.

Furthermore in section 3, by our new observations, we illustrate methods which are easier than tedious calculations to solve the stability problems. And some of the interesting examples made from our observations are listed.

2. Cauchy type additive functional inequalities with general terms

Let X be a real normed linear space and Y a real Banach space. For given $l \in \mathbb{N}$ and any $i \in \{1, 2, \cdots, l\}$, let $\sigma_i : X \times X \longrightarrow X$ be a binary operation such that

$$\sigma_i(rx, ry) = r\sigma_i(x, y)$$

for all $x, y \in X$ and all $r \in \mathbb{R}$. It is clear that $\sigma_i(0, 0) = 0$.

Also let $F : Y^l \longrightarrow Y$ be a linear, continuous function. For a map $f : X \longrightarrow Y$, define

$$G_f(x, y) = F(f(\sigma_1(x, y)), f(\sigma_2(x, y)), \cdots, f(\sigma_l(x, y))).$$

As mentioned in section 1, we consider the following condition on G_f which should be proved when specific expression for G_f is given.

Condition 1. If f satisfies the equation (1.3), then $G_f(x, y) = 0$ for all $x,\, y \in X$.

We remark that Condition 1 implies $G_f(x, y) = 0$ for all $x,\, y \in X$ when f is additive and satisfies (1.3). Now we prove the following main stability theorem.

Theorem 2.1. *Let G_f be a functional operator satisfying Condition 1, and assume that*

$$\|G_f(x, x)\| \le \sum_{i=1}^{s} \|a_i G_f(\lambda_i x, 0)\| + \sum_{j=1}^{t} \|b_j G_f(0, \delta_j x)\| \tag{2.1}$$

for some $s, t \in \mathbb{N}$, some real numbers $a_i, b_j, \lambda_i, \delta_j$ and for all $x \in X$ and all $f : X \longrightarrow Y$.

And let $f : X \longrightarrow Y$ be a mapping such that the inequality

$$\|f(x + y) - f(x) - f(y) - G_f(x, y)\| \le \epsilon(\|x\|^p\|y\|^p + \|x\|^{2p} + \|y\|^{2p}) \tag{2.2}$$

holds for all $x, y \in X$, $\epsilon \ge 0$ and $p \ne \frac{1}{2}$.

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Then there is a unique additive mapping $A : X \longrightarrow Y$ such that

$$\|f(x) - A(x)\| \leq \frac{3 + \sum_{i=1}^{s} |a_i||\lambda_i|^{2p} + \sum_{j=1}^{t} |b_i||\delta_i|^{2p}}{|2 - 4^p|} \epsilon \|x\|^{2p} \tag{2.3}$$

for all $x \in X$.

Proof. First we show that $f(0) = 0$. Since

$$G_f(x, y) = F\big(f(\sigma_1(x, y)), f(\sigma_2(x, y)), \cdots, f(\sigma_l(x, y))\big)$$

for a continuous linear map F, we can write

$$G_f(x, y) = \sum_{1 \leq i \leq l} c_i f(\sigma_i(x, y))$$

for some $c_i \in \mathbb{R}$. Since $\sigma_i(0, 0) = 0$, $G_f(0, 0) = cf(0)$, where $c = \sum_{1 \leq i \leq l} c_i$.

We claim that $c \neq -1$. If $c = -1$, then it is easily checked that a non-zero constant function f satisfies (1.3). But this contradicts to Condition 1. So, $c \neq -1$.

Setting $x = 0$, $y = 0$ in (2.2), we have

$$0 = f(0) + G_f(0, 0) = (1 + c)f(0).$$

Since $c \neq -1$, we have $f(0) = 0$.

Now we analyze the condition on G_f. Setting $y = 0$ ($x = 0$, $y = x$, resp.) in (2.2), we get

$$\|G_f(x, 0)\| \leq \epsilon \|x\|^{2p} \ (\|G_f(0, x)\| \leq \epsilon \|x\|^{2p}, \text{ resp.}) \tag{2.4}$$

for all $x \in X$. Replacing y by x in (2.2), we get

$$\|f(2x) - 2f(x) - G_f(x, x)\| \leq 3\epsilon \|x\|^{2p}$$

for all $x \in X$. By (2.1) and (2.4), we have

$$\begin{aligned} &\|f(2x) - 2f(x)\| \\ &\leq 3\epsilon \|x\|^{2p} + \|G_f(x, x)\| \\ &\leq 3\epsilon \|x\|^{2p} + \sum_{i=1}^{s} \|a_i G_f(\lambda_i x, 0)\| + \sum_{j=1}^{t} \|b_j G_f(0, \delta_j x)\| \\ &\leq \left(3 + \sum_{i=1}^{s} |a_i||\lambda_i|^{2p} + \sum_{j=1}^{t} |b_j||\delta_j|^{2p}\right) \epsilon \|x\|^{2p} \end{aligned} \tag{2.5}$$

for all $x \in X$. Let

$$M = 3 + \sum_{i=1}^{s} |a_i||\lambda_i|^{2p} + \sum_{j=1}^{t} |b_j||\delta_j|^{2p}$$

Suppose that $p < \frac{1}{2}$. Then by (2.5), for any $n \in \mathbb{N} \cup \{0\}$, we have

$$
\begin{aligned}
&\|f(x) - 2^{-n} f(2^n x)\| \\
&= \|f(x) - 2^{-1} f(2x)\| + \|2^{-1} f(2x) - 2^{-2} f(2^2 x)\| + \\
&\quad \cdots + \|2^{-n+1} f(2^{n-1} x) - 2^{-n} f(2^n x)\| \\
&\le M\epsilon \|x\|^{2p} \frac{1 - 2^{(2p-1)n}}{2 - 4^p}
\end{aligned} \tag{2.6}
$$

for all $x \in X$. For any $n \in \mathbb{N} \cup \{0\}$, let $f_n(x) = 2^{-n} f(2^n x)$. Then for $n > m$ in $\mathbb{N} \cup \{0\}$, by (2.6), we have

$$
\begin{aligned}
&\|f_n(x) - f_m(x)\| \\
&= \|2^{-n} f(2^n x) - 2^{-m} f(2^m x)\| \\
&= 2^{-m} \|f(2^m x) - 2^{-(n-m)} f(2^{n-m}(2^m x))\| \\
&\le 2^{-m} M\epsilon \|2^m x\|^{2p} \frac{1 - 2^{(2p-1)(n-m)}}{2 - 4^p} \\
&\le M\epsilon \|x\|^{2p} \frac{1 - 2^{(2p-1)(n-m)}}{2 - 4^p} 2^{m(2p-1)}
\end{aligned}
$$

for all $x \in X$. Since $2p - 1 < 0$, $\{f_n(x)\}$ is a Cauchy sequence in Y. Since Y is a Banach space, there exists a mapping $A : X \longrightarrow Y$ such that

$$A(x) = \lim_{n \to \infty} f_n(x)$$

for all $x \in X$. By (2.6), we have

$$\|f(x) - A(x)\| \le \frac{M}{2 - 4^p} \epsilon \|x\|^{2p}$$

for all $x \in X$. Replacing x by $2^n x$ and y by $2^n y$ in (2.2), we have

$$
\begin{aligned}
&\|2^{-n} f(2^n(x+y)) - 2^{-n} f(2^n x) - 2^{-n} f(2^n y) - 2^{-n} G_f(2^n x, 2^n y)\| \\
&\le \epsilon (\|x\|^p \|y\|^p + \|x\|^{2p} + \|y\|^{2p}) 2^{n(2p-1)}
\end{aligned} \tag{2.7}
$$

for all $x, y \in X$. Since $2p - 1 < 0$, $\lim_{n\to\infty} 2^{n(2p-1)} = 0$ and by (2.7), we have

$$A(x+y) = A(x) + A(y) + \lim_{n \to \infty} 2^{-n} G_f(2^n x, 2^n y)$$

for all $x, y \in X$. Since F is linear and continuous, by the conditions of σ_i, we have

$$
\begin{aligned}
&\lim_{n\to\infty} 2^{-n} G_f(2^n x, 2^n y) \\
&= \lim_{n\to\infty} 2^{-n} F(f(\sigma_1(2^n x, 2^n y)), f(\sigma_2(2^n x, 2^n y)), \cdots, f(\sigma_l(2^n x, 2^n y))) \\
&= \lim_{n\to\infty} F(2^{-n} f(2^n \sigma_1(x,y)), 2^{-n} f(2^n \sigma_2(x,y)), \cdots, 2^{-n} f(2^n \sigma_l(x,y))) \\
&= F(A(\sigma_1(x,y)), A(\sigma_2(x,y)), \cdots, A(\sigma_l(x,y))) \\
&= G_A(x,y),
\end{aligned}
$$

and we have

$$A(x+y) = A(x) + A(y) + G_A(x,y)$$

for all $x, y \in X$. Hence A satisfies (1.3) and so $G_A(x,y) = 0$ for all x, $y \in X$ by Condition 1. Thus A is additive.

Now, we will show that A is unique. Suppose that $C : X \longrightarrow Y$ is an additive mapping satisfying (2.3). Since both A and C are additive,

$$A(2^n x) = 2^n A(x), \quad C(2^n x) = 2^n C(x)$$

for all $x \in X$ and all $n \in \mathbb{N} \cup \{0\}$. By (2.3), we have

$$\begin{aligned}
&\|A(x) - C(x)\| \\
&\le 2^{-n}\|A(2^n x) - f(2^n x)\| + 2^{-n}\|f(2^n x) - C(2^n x)\| \\
&\le \frac{M}{2-4^p}\epsilon\|x\|^{2p}2^{(2p-1)n+1} \longrightarrow 0
\end{aligned}$$

as $n \longrightarrow \infty$, because $2p < 1$. Hence we can conclude

$$A(x) = C(x)$$

for all $x \in X$. Thus we prove Theorem 2.1 for the case of $p < \frac{1}{2}$.

Suppose that $p > \frac{1}{2}$. Replacing x by $2^{-1}x$ in (2.5), we get

$$\|f(x) - 2f(2^{-1}x)\| \le M\epsilon\|x\|^{2p}2^{-2p}$$

for all $x \in X$ and by similar calculation as (2.6), we get

$$\begin{aligned}
&\|f(x) - 2^n f(2^{-n}x)\| \\
&\le \frac{M}{4^p - 2}\epsilon\|x\|^{2p}(1 - 2^{(1-2p)n})
\end{aligned}$$

for all $x \in X$ and all $n \in \mathbb{N}$. Then $\{2^n f(2^{-n}x)\}$ is a Cauchy sequence in Y. Since Y is a Banach space, there exists a mapping $A : X \longrightarrow Y$ such that

$$A(x) = \lim_{n\to\infty} 2^n f(2^{-n}x)$$

for all $x \in X$. Moreover, we have

$$\|f(x) - A(x)\| \le \frac{M}{4^p - 2}\epsilon\|x\|^{2p}$$

for all $x \in X$. Replacing x by $2^{-n}x$ and y by $2^{-n}y$ in (2.2), we have

$$\begin{aligned}
&\|2^n f(2^{-n}(x+y)) - 2^n f(2^{-n}x) - 2^n f(2^{-n}y) - 2^n G_f(2^{-n}x, 2^{-n}y)\| \\
&\le \epsilon(\|x\|^p\|y\|^p + \|x\|^{2p} + \|y\|^{2p})2^{n(1-2p)}
\end{aligned}$$

for all $x, y \in X$. Since $1 - 2p < 0$, $\lim_{n\to\infty} 2^{n(1-2p)} = 0$ and hence

$$A(x+y) = A(x) + A(y) + G_A(x,y)$$

for all $x, y \in X$.

The rest of the proof for $p > \frac{1}{2}$ is similar to the above proof for $p < \frac{1}{2}$. $\square$

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Surprisingly, if we confine G_f to somewhat special cases, the Condition 1 in Theorem 2.1 is automatically satisfied just after checking some additional conditions on G_f, without considering full equation (1.3). For example, among G_f satisfying

$$G_f(x,y) = \sum_{1\le i\le l} c_i f(m_i x + n_i y) \tag{2.8}$$

for m_i, $n_i \in \mathbb{Q}$ and $c_i \in \mathbb{R}$ with

$$\sum c_i m_i = \sum c_i n_i = 0, \ \sum c_i |m_i| \neq 0 \text{ or } \sum c_i |n_i| \neq 0, \tag{2.9}$$

we can find a class of examples of Cauchy type functional equations satisfying the generalized Hyers-Ulam stability. The following corollary gives an answer to this problem, which contains many interesting examples.

Corollary 2.2. *Assume that G_f satisfies (2.8), (2.9) and all of the conditions in Theorem 2.1 hold except Condition 1. Further, suppose that for all $k \in \mathbb{N}$, there exists $\mu_k \in \mathbb{R}$ such that*

$$\|G_f(x,(k+1)x)\| \le \|G_f(\mu_k x, k\mu_k x)\| \tag{2.10}$$

and

$$G_f(-x,-y) = G_f(x,y), \tag{2.11}$$

for all x, $y \in X$ and all $f : X \longrightarrow Y$. Then there is a unique additive mapping $A : X \longrightarrow Y$ such that

$$\|f(x) - f(0) - A(x)\| \le \frac{3 + \sum_{i=1}^{s} |a_i||\lambda_i|^{2p} + \sum_{j=1}^{t} |b_i||\delta_i|^{2p}}{|2-4^p|}\epsilon\|x\|^{2p}$$

for all $x \in X$.

Proof. When G_f has the form of (2.8), if f satisfies (1.3) then $f(0) = 0$ or $\sum_i c_i = -1$. But in the case of $\sum_i c_i = -1$, it is easily verified that $f(x) - f(0)$ also satisfies (1.3) and (2.2). Hence it is enough to prove with the assumption $f(0) = 0$.

First, we claim that if f satisfies (1.3) then f is additive. Set $y = 0$ ($x = 0$, *resp.*) in (1.3), then we have

$$G_f(x,0) = 0 \quad (G_f(0,y) = 0, \ resp.). \tag{2.12}$$

Setting $y = x$ in (1.3), we have

$$G_f(x,x) = f(2x) - 2f(x).$$

So by (2.1) and (2.12), we have

$$f(2x) - 2f(x) = 0. \tag{2.13}$$

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From (2.10), we have

$$\begin{aligned}\|G_f(x, kx)\| \leq& \|G_f\big(\mu_{k-1}x,\ (k-1)\mu_{k-1}x\big)\| \\ &\vdots \\ \leq& \|G_f\big(\mu_{k-1}\mu_{k-2}\cdots\mu_1 x,\ \mu_{k-1}\mu_{k-2}\cdots\mu_1 x\big)\|\end{aligned} \tag{2.14}$$

for $k \geq 2$. By (2.1) and (2.12) we have $G_f(x, x) = 0$ for all $x \in X$. So from (2.14) we have

$$G_f(x, kx) = 0 \tag{2.15}$$

for all $k \in \mathbb{N}$ and all $x \in X$. Setting $y = kx$ in (1.3), we have

$$G_f(x, kx) = f\big((k+1)x\big) - f(x) - f(kx),$$

hence by (2.15),

$$f\big((k+1)x\big) - f(x) - f(kx) = 0. \tag{2.16}$$

So, from (2.13) and (2.16) the standard induction argument gives that

$$f(nx) = nf(x)$$

for all positive integer n, and by the same argument in the proof of Theorem 2.3 below, we also have

$$f(rx) = rf(x) \tag{2.17}$$

for all positive rational number r.

Now, without loss of generality, we may assume that not all m_i's are zeroes and $\sum c_i|m_i| \neq 0$. Denote by I^+, I^- the set of all i's satisfying m_i to be positive, negative respectively. Then by (2.17), we have

$$\begin{aligned}0 = G_f(x, 0) = \sum c_i f(m_i x) &= \sum_{j \in I^+} c_j f(m_j x) + \sum_{k \in I^-} c_k f(m_k x) \\ &= \sum_{j \in I^+} c_j m_j f(x) + \sum_{k \in I^-} c_k(-m_k) f(-x).\end{aligned} \tag{2.18}$$

By (2.9), we have

$$\sum_{j \in I^+} c_j m_j = -\sum_{k \in I^-} c_k m_k \neq 0, \tag{2.19}$$

hence by (2.18) and (2.19), we have

$$f(-x) = -f(x).$$

It means that $G_f(-x, -y) = -G_f(x, y)$, so from (2.11) we have $G_f(x, y) = 0$ for all x, $y \in X$, that is, f is additive.

In fact, we showed that

$$f(rx) = rf(x)$$

for all rational number r.

Now the rest part of the proof is the same as that of Theorem 2.1. $\square$

Note that we don't need continuity condition on f to prove Theorem 2.1. As in [7], if we impose the continuity condition we can also get the similar conclusion in our cases. So we can get the following theorem.

Theorem 2.3. *Let $f : X \longrightarrow Y$ be a mapping satisfying the conditions of Theorem 2.1. Suppose that $f(tx)$ is continuous on $\mathbb{R}$ for each $x \in X$. Then there is a unique linear mapping $A : X \longrightarrow Y$ satisfying (2.3).*

Proof. By Theorem 2.1, there is a unique additive mapping $A : X \longrightarrow Y$ satisfying (2.3). Let $r \in \mathbb{R}$. If $r \in \mathbb{N}$, then $A(rx) = rA(x)$ for all $x \in X$, because A is additive. Since A is additive,

$$A(-x) = -A(x)$$

for all $x \in X$ and hence if $r = 0$ or r is a negative integer, then

$$A(rx) = rA(x)$$

for all $x \in X$. For any integer $m(\neq 0)$,

$$A(x) = A\left(m\frac{1}{m}x\right) = mA\left(\frac{1}{m}x\right)$$

and so

$$\frac{1}{m}A(x) = A\left(\frac{1}{m}x\right)$$

If $r = \frac{n}{m}$ for integers $n, m(m \neq 0)$, then

$$A\left(\frac{n}{m}x\right) = \frac{n}{m}A(x) \tag{2.20}$$

for all $x \in X$. Suppose that $r \in \mathbb{R}$. Then there is a sequence $\{r_n\}$ in $\mathbb{Q}$ such that $r = \lim_{n\to\infty} r_n$ and since $f(tx)$ is continuous on $\mathbb{R}$ for a fixed $x \in X$, by (2.20), we have

$$\begin{aligned}
A(rx) &= \lim_{n\to\infty} 2^{-n} f(2^n rx) \\
&= \lim_{n\to\infty} 2^{-n}\left(\lim_{k\to\infty} f(2^n r_k x)\right) \\
&= \lim_{k\to\infty} A(r_k x) \\
&= \lim_{k\to\infty} r_k A(x) \\
&= rA(x)
\end{aligned}$$

for all $x \in X$. Hence A is a linear mapping. □

Remark 2.4. In Corollary 2.2, if we impose the continuity condition on $f(tx)$ as in Theorem 2.3, the functional G_f can be extended to

$$G_f(x, y) = \sum_{1\leq i\leq l} c_i f(m_i x + n_i y)$$

for c_i, m_i, $n_i \in \mathbb{R}$,

3. Applications

In this section, by virtue of the theorems in section 2, we prove the generalized Hyers-Ulam stability theorem for some selected functional equations. The first example can be proved by slightly modified application of Corollary 2.2. Consider the following functional equation :

$$\begin{aligned} &f(x+y) - f(2x-3y) - f(3y-2x) - f(6y-4x) + 4f\left(\frac{3}{2}y - x\right) \\ =&f(x) + f(y) \end{aligned} \tag{3.1}$$

Theorem 3.1. *Let $f : X \longrightarrow Y$ be a mapping such that*

$$\begin{aligned} &\|f(x+y) - f(2x-3y) - f(3y-2x) - f(6y-4x) + 4f\left(\frac{3}{2}y - x\right) \\ &- f(x) - f(y)\| \le \epsilon(\|x\|^p\|y\|^p + \|x\|^{2p} + \|y\|^{2p}) \end{aligned}$$

for all $x, y \in X$, some $\epsilon \ge 0$, and $p \ne \frac{1}{2}$. Then there is a unique additive mapping $A : X \longrightarrow Y$ such that

$$\|f(x) - f(0) - A(x)\| \le \frac{3 \cdot 4^p + 1}{4^p|2 - 4^p|}\epsilon\|x\|^{2p}$$

for all $x \in X$.

Proof. In this case,

$$G_f(x, y) = f(2x-3y) + f(3y-2x) + f(6y-4x) - 4f\left(\frac{3}{2}y - x\right).$$

Let $\widetilde{f}(x) = f(x) - f(0)$. Then

$$\begin{aligned} G_{\widetilde{f}}(x, 0) =&\widetilde{f}(2x) + \widetilde{f}(-2x) + \widetilde{f}(-4x) - 4\widetilde{f}(-x) \\ G_{\widetilde{f}}(x, x) =&\widetilde{f}(-x) + \widetilde{f}(x) + \widetilde{f}(2x) - 4\widetilde{f}\left(\frac{1}{2}x\right) \\ G_{\widetilde{f}}(x, kx) =&\widetilde{f}((2-3k)x) + \widetilde{f}((3k-2)x) \\ &+ \widetilde{f}(2(3k-2)x) - 4\widetilde{f}\left(\left(\frac{3k-2}{2}\right)x\right) \\ G_{\widetilde{f}}(x, (k+1)x) =&\widetilde{f}((-1-3k)x) + \widetilde{f}((3k+1)x) \\ &+ \widetilde{f}(2(3k+1)x) - 4\widetilde{f}\left(\left(\frac{3k+1}{2}\right)x\right) \end{aligned}$$

Now simple calculations give the following.

$$\begin{aligned} G_{\widetilde{f}}(x, x) &= G_{\widetilde{f}}\left(-\frac{1}{2}x, 0\right) \\ G_{\widetilde{f}}(x, (k+1)x) &= G_{\widetilde{f}}(\mu_k x,\ k\mu_k x), \end{aligned}$$

where $\mu_k = \frac{3k+1}{3k-2}$.

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By the same argument in the proof of Corollary 2.2, we have

$$\widetilde{f}(rx) = r\widetilde{f}(x) \tag{3.2}$$

for positive rational number r. So, if $\widetilde{f}$ satisfies (3.1), we have

$$\begin{aligned} 0 =& G_{\widetilde{f}}(x,0) = \widetilde{f}(2x) + \widetilde{f}(-2x) + \widetilde{f}(-4x) - 4\widetilde{f}(-x) \\ =& 2\left(\widetilde{f}(x) + \widetilde{f}(-x)\right). \end{aligned} \tag{3.3}$$

By (3.2) and (3.3), if $\widetilde{f}$ satisfies (3.1) then $G_{\widetilde{f}}(x,y) = 0$ for all x, $y \in X$, hence $\widetilde{f}$ is additive.

So the proof is completed by Corollary 2.2. □

Second example does not satisfy the conditions in Corollary 2.2, but Theorem 2.1 works well for it. Consider the following functional equation :

$$f(x+y) = f(2x-4y) - f(x) + 5f(y) \tag{3.4}$$

Theorem 3.2. *Let $f : X \longrightarrow Y$ be a mapping such that*

$$\begin{aligned} &\|f(x+y) - f(2x-4y) + f(x) - 5f(y)\| \\ &\leq \epsilon(\|x\|^p\|y\|^p + \|x\|^{2p} + \|y\|^{2p}) \end{aligned}$$

for all $x, y \in X$, some $\epsilon \geq 0$, and $p \neq \frac{1}{2}$. Then there is a unique additive mapping $A : X \longrightarrow Y$ such that

$$\|f(x) - A(x)\| \leq \frac{3(4^p+1)}{4^p|2-4^p|}\epsilon\|x\|^{2p},$$

for all $x \in X$.

Proof. Setting $y = 0$ in (3.4), we have

$$f(2x) = 2f(x). \tag{3.5}$$

Setting $y = x$ in (3.4) and using (3.5), we have

$$f(-x) = -f(x). \tag{3.6}$$

Setting $x = -2y$, $y = x$ in (3.4), and using (3.5), (3.6), we have

$$f(x-2y) = -4f(x+y) - 2f(y) + 5f(x). \tag{3.7}$$

Using (3.5) and calculating $\{(3.4) + 2 \times (3.7)\}$, we have

$$f(x+y) = f(x) + f(y),$$

hence f is additive.

Now

$$G_f(x,y) = f(2x-4y) - 2f(x) + 4f(y)$$

in this case, so we have the followings.

$$\begin{aligned} G_f(x,x) &= f(-2x) + 2f(x) \\ G_f(x,0) &= f(2x) - 2f(x) \\ G_f(0,x) &= f(-4x) + 4f(x) \end{aligned}$$

So, we have

$$G_f(x,x) = 2G_f\left(\frac{x}{2}, 0\right) + G_f\left(0, \frac{x}{2}\right),$$

hence

$$\|G_f(x,x)\| \le 2\left\|G_f\left(\frac{x}{2}, 0\right)\right\| + \left\|G_f\left(0, \frac{x}{2}\right)\right\|.$$

Thus the proof is completed by Theorem 2.1. □

Next two examples have some nondetermined constants in the functional equations. Consider the following equation :

$$f(x+y) + f(mx-ny) + f(ny-mx) = f(x) + f(y) \tag{3.8}$$

for real numbers m, n. If $m = n = 0$, then we have the result trivially, so without loss of generality we assume that $m \neq 0$.

Theorem 3.3. *Let $f : X \longrightarrow Y$ be a mapping such that*

$$\begin{aligned} &\|f(x+y) + f(mx-ny) + f(ny-mx) - f(x) - f(y)\| \\ &\le \epsilon(\|x\|^p\|y\|^p + \|x\|^{2p} + \|y\|^{2p}) \end{aligned}$$

for all $x, y \in X$, some $\epsilon \ge 0$, and $p \neq \frac{1}{2}$. Then there is a unique additive mapping $A : X \longrightarrow Y$ such that

$$\|f(x) - A(x)\| \le \frac{3m^{2p} + |m-n|^{2p}}{m^{2p}|2-4^p|}\epsilon\|x\|^{2p},$$

for all $x \in X$.

Proof. Suppose that f satisfies (3.8). Replacing y by 0 in (3.8), we have

$$f(mx) = -f(-mx) \tag{3.9}$$

for all $x \in X$. Replacing x by $\frac{x}{m}$ in (3.9), we have

$$f(x) = f(-x) \tag{3.10}$$

for all $x \in X$. Hence by (3.10), f is additive.

For any $x, y \in X$, let

$$\sigma_1(x,y) = mx - ny, \quad \sigma_2(x,y) = ny - mx$$

and for any $s, t \in Y$, let

$$F(s,t) = s + t.$$

Then $\sigma_i(rx, ry) = r\sigma_i(x,y)$ for all $x, y \in X$ and all $r \in R$ and $F : Y^2 \longrightarrow Y$ is a linear, continuous function. Since

$$G_f(x,y) = -f(mx-ny) - f(ny-mx),$$

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we have

$$\begin{aligned} G_f(x,x) &= -f((m-n)x) - f((n-m)x) \\ G_f(x,0) &= -f(mx) - f(-mx) \end{aligned}$$

for all $x \in X$. Hence we have

$$G_f(x,x) = G_f(\lambda x, 0)$$

for all $x \in X$, where $\lambda = \frac{m-n}{m}$.

So the proof is completed by Theorem 2.1. □

Remark 3.4. When m and n are rational numbers in Theorem 3.3, in virtue of Corollary 2.2, we can get the results by checking the followings.

$$\begin{aligned} G_f(-x,-y) &= -f(-mx+ny) - f(-ny+mx) \\ G_f(x,0) &= -f(mx) - f(-mx) \\ G_f(x,x) &= -f((m-n)x) - f((n-m)x) \\ G_f\big(x,(k+1)x\big) &= -f\Big(\big(m-n(k+1)\big)x\Big) - f\Big(\big(n(k+1)-m\big)x\Big) \\ G_f(x,kx) &= -f\Big(\big(m-nk\big)x\Big) - f\Big(\big(nk-m\big)x\Big) \end{aligned}$$

Now simple calculations give the followings.

$$\begin{aligned} \sum c_i|m_i| &= -|m| - |m| \neq 0 \\ G_f(-x,-y) &= G_f(x,y) \\ G_f(x,x) &= G_f(\lambda x, 0) \\ G_f\big(x,(k+1)x\big) &= G_f\big(\mu_k x, k\mu_k x\big) \end{aligned}$$

with $\lambda = \frac{m-n}{m}$, $\mu_k = \frac{m-n(k+1)}{m-nk}$ if $m \neq nk$ for any $k \in \mathbb{N}$.

If $m = nk_0$ for some $k_0 \in \mathbb{N}$, then $G_f(x, k_0 x) = 0$. So, we can still have $G_f(x,kx) = 0$ for all $k \in \mathbb{N}$ by restarting induction argument in Corollary 2.2 from k_0. Now Corollary 2.2 can be applied.

In this case, using Corollary 2.2 looks like more complicated than checking Condition 1. But Corollary 2.2 provides a systematic way to prove the stability theorem. Also, it is more effective to use Corollary 2.2 when G_f has many terms. Furthermore Corollary 2.2 gives a class of functional equations which satisfy generalized Hyers-Ulam stability theorem just by calculating the properties of G_f.

Theorem 3.3 can be regarded as a general version of the following stability problem appeared in [6].

Theorem 3.5. ([6]) *Let $f : X \longrightarrow Y$ be a mapping such that*

$$\begin{aligned} &\|f(x+y) + f(x-y) + f(y-x) - f(x) - f(y)\| \\ &\leq \epsilon(\|x\|^p\|y\|^p + \|x\|^{2p} + \|y\|^{2p}) \end{aligned}$$

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for all $x, y \in X$, some $\epsilon \geq 0$, and $p \neq \frac{1}{2}$. Then there is a unique additive mapping $A : X \longrightarrow Y$ such that

$$\|f(x) - A(x)\| \leq \frac{3}{|2 - 4^p|}\epsilon\|x\|^{2p}$$

for all $x \in X$.

The last example in this paper looks like a little bit complicated, but it can be proved by Theorem 2.1. Consider the following equation :

$$2f(x + y) + f(mx) + mf(-x) + f(-x - y) = f(x) + f(y) \tag{3.11}$$

for real number $m \neq -2$.

Theorem 3.6. *Let $f : X \longrightarrow Y$ be a mapping such that*

$$\begin{aligned}&\|2f(x + y) + f(mx) + mf(-x) + f(-x - y) - f(x) - f(y)\|\\ &\leq \epsilon(\|x\|^p\|y\|^p + \|x\|^{2p} + \|y\|^{2p})\end{aligned}$$

for all $x, y \in X$, some $\epsilon \geq 0$, and $p \neq \frac{1}{2}$. Then there is a unique additive mapping $A : X \longrightarrow Y$ such that

$$\|f(x) - A(x)\| \leq \frac{5 + 4^p}{|2 - 4^p|}\epsilon\|x\|^{2p} \tag{3.12}$$

for all $x \in X$.

Proof. Suppose that f satisfies (3.11). Putting $x = 0$, $y = 0$ in (3.11), we have $f(0) = 0$. Putting $y = 0$ in (3.11) we have

$$f(mx) + mf(-x) + f(x) + f(-x) = 0 \tag{3.13}$$

for all $x \in X$. Putting $x = 0, y = x$ in (3.11), we have

$$f(x) + f(-x) = 0 \tag{3.14}$$

for all $x \in X$. Plugging (3.14) into (3.13), we have

$$f(mx) + mf(-x) = 0 \tag{3.15}$$

for all $x \in X$. Applying (3.14) and (3.15) to (3.11), we have

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in X$. Thus f is additive.

Since

$$G_f(x, y) = -f(mx) - mf(-x) - f(x + y) - f(-x - y),$$

we have

$$\begin{aligned}G_f(x, x) &= -f(mx) - mf(-x) - f(2x) - f(-2x)\\ G_f(x, 0) &= -f(mx) - mf(-x) - f(x) - f(-x)\\ G_f(0, x) &= -f(x) - f(-x)\end{aligned}$$

for all $x \in X$. Hence

$$\begin{aligned}
&\|G_f(x,x)\| \\
=&\|f(mx) + mf(-x) + f(2x) + f(-2x)\| \\
\le&\|f(mx) + mf(-x) + f(x) + f(-x)\| + \|f(x) + f(-x)\| + \|f(2x) + f(-2x)\| \\
=&\|G_f(x,0)\| + \|G_f(0,x)\| + \|G_f(0,2x)\|
\end{aligned}$$

for all $x \in X$.

Now the proof is completed by Theorem 2.1. $\square$

As Remark 3.4, we can show that if m is a rational number in Theorem 3.6, then the result can be obtained after straightforward checking the properties of G_f as in Corollary 2.2, instead of after proving Condition 1. Furthermore, even though the condition $m \neq -2$ is needed for using Theorem 2.1 directly, if $m = -2$ then we can actually conclude that there exists a unique additive mapping $A : X \longrightarrow Y$ satisfying

$$\|f(x) - f(0) - A(x)\| \le \frac{5 + 4^p}{|2 - 4^p|}\epsilon\|x\|^{2p}$$

for all $x \in X$, by the argument in the proof of Corollary 2.2.

And, in all of the theorems in this section, if $f(tx)$ is continuous on $\mathbb{R}$ for each $x \in X$ then A is linear by Theorem 2.3.

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Inequalities of Simpson Type for Functions Whose Third Derivatives Are Extended s-Convex Functions and Applications to Means

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Received on April 4, 2014; Accepted on August 26, 2014

Abstract

In the paper, the authors establish some new inequalities of Simpson type for functions whose third derivatives are extended s-convex functions, and apply these inequalities to derive some inequalities of special means.

2010 Mathematics Subject Classification: Primary 26D15, 26D20; Secondary 41A55, 41A99.

Key words and phrases: Simpson type inequality; extended s-convex functions; Hölder inequality.

1 Introduction

Throughout this paper, we use the following notation

$$\mathbb{R} = (-\infty, \infty), \quad \mathbb{R}_0 = [0, \infty), \quad \text{and} \quad \mathbb{R}_+ = (0, \infty).$$

The following definitions are well known in the literature.

Definition 1.1. A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

In [9], H. Hudzik and L. Maligranda considered, among others, the class of functions which are called s-convex in the second sense.

Definition 1.2 ([9]). Let s be a real number $s \in (0,1]$. A function $f : \mathbb{R}_0 \to \mathbb{R}_0$ is said to be s-convex (in the second sense), or say, f belongs to the class K_s^2, if

$$f(\lambda x + (1-\lambda)y) \le \lambda^s f(x) + (1-\lambda)^s f(y)$$

for all $x, y \in I$ and $\lambda \in [0,1]$.

In [18], the concept of extended s-convex functions was introduced as follows.

Definition 1.3 ([18]). For some $s \in [-1,1]$, a function $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ is said to be extended s-convex if

$$f(\lambda x + (1-\lambda)y) \le \lambda^s f(x) + (1-\lambda)^s f(y)$$

is valid for all $x, y \in I$ and $\lambda \in (0,1)$.

If $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function on $[a,b] \subseteq I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x \le \frac{f(a)+f(b)}{2}. \tag{1.1}$$

In the literature, Simpson's inequality may be recited as Theorem 1.1 below.

Theorem 1.1 ([6]). *If $f : [a,b] \to \mathbb{R}$ is a four times continuously differentiable function on (a,b) and its fourth derivative on (a,b) is bounded by $\left\|f^{(4)}\right\| = \sup_{x\in(a,b)}\left|f^{(4)}(x)\right| < \infty$, then*

$$\left|\int_a^b f(x)\,\mathrm{d}\,x - \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right]\right| \le \frac{(b-a)^5\left\|f^{(4)}\right\|}{2880}. \tag{1.2}$$

In recent decades, many inequalities of Hermite-Hadamard type and Simpson type for various kind of convex functions have been established. Some of them may be recited as follows.

Theorem 1.2 ([5, Theorems 2.2 and 2.3]). *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and $a, b \in I^\circ$ with $a < b$.*

1. *If $|f'(x)|$ is convex on $[a,b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{(b-a)(|f'(a)|+|f'(b)|)}{8}. \tag{1.3}$$

2. *If the mapping $|f'(x)|^{p/(p-1)}$ is convex on $[a,b]$ for $p > 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b-a}{2(p+1)^{1/p}}\left[\frac{|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}}{2}\right]^{1-1/p}. \tag{1.4}$$

Theorem 1.3 ([12, Theorems 1 and 2]). *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable on I° and $a, b \in I$ with $a < b$. If $|f'|^q$ is convex on $[a,b]$ for $q \ge 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b-a}{4}\left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{1/q} \tag{1.5}$$

and

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b-a}{4}\left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{1/q}. \tag{1.6}$$

Theorem 1.4 ([1, Theorems 2]). *Let $f'': I \subseteq \mathbb{R} \to \mathbb{R}$ be an absolutely continuous function on I° such that $f''' \in L([a,b])$ and $a,b \in I^\circ$ with $a<b$. If $|f'''|$ is quasi-convex on $[a,b]$, then*

$$\begin{aligned}\left|\int_a^b f(x)\,\mathrm{d}\,x - \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right| \\ \le \frac{(b-a)^4}{1152}\left[\max\left\{|f'''(a)|, \left|f'''\left(\frac{a+b}{2}\right)\right|\right\}+\max\left\{\left|f'''\left(\frac{a+b}{2}\right)\right|, |f'''(b)|\right\}\right]. \quad (1.7)\end{aligned}$$

Theorem 1.5 ([3, Theorem 3.1]). *Let $f: I \subseteq \mathbb{R}_0 \to \mathbb{R}_0$ be differentiable on I°, $a,b \in I^\circ$ with $a<b$, and $f''' \in L([a,b])$. If $|f'''|^q$ is s-convex on$[a,b]$ for $s \in (0,1]$ and $q \ge 1$, then*

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x - \frac{b-a}{12}\big[f'(b)-f'(a)\big]\right| \\ \le \frac{(b-a)^3}{192}\left[\frac{2^{2-s}(s+6+2^{s+2}s)}{(s+2)(s+3)(s+4)}\right]^{1/q}\big[|f'''(a)|^q+|f'''(b)|^q\big]^{1/q}. \quad (1.8)\end{aligned}$$

Theorem 1.6 ([11, Theorems 3]). *Let $f: I \subseteq \mathbb{R}_0 \to \mathbb{R}_0$ be a differentiable functions on I° such that $f''' \in L([a,b])$ and $a,b \in I^\circ$ with $a<b$. If $|f'''|^q$ is s-convex in the second sense on $[a,b]$ for some fixed $s \in (0,1]$, then*

$$\begin{aligned}\left|\int_a^b f(x)\,\mathrm{d}\,x - \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right| \le \frac{(b-a)^4}{6}\big[|f'''(a)|+|f'''(b)|\big] \\ \times\left\{\frac{2^{-4-s}[(1+s)(2+s)+34+2^{s+4}(-2+s)+11s+s^2]}{(s+1)(s+2)(s+3)(s+4)}\right\}.\end{aligned}$$

Theorem 1.7 ([18, Theorem 3.1]). *Let $f: I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be a differentiable function on I°, $a,b \in I$ with $a<b$, $f' \in L([a,b])$, and $0 \le \lambda, \mu \le 1$, such that $|f'|^q$ for $q \ge 1$ is extended s-convex on $[a,b]$ for some fixed $s \in [-1,1]$.*

1. *If $-1 < s \le 1$, then*

$$\begin{aligned}&\left|\frac{\lambda f(a)+\mu f(b)}{2} + \frac{2-\lambda-\mu}{2}f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b-a}{2^{s/q+2}}\left[\frac{1}{(s+1)(s+2)}\right]^{1/q} \\ &\times\left\{\left(\frac{1}{2}-\lambda+\lambda^2\right)^{1-1/q}\Big[\big\{2(2-\lambda)^{s+2}+[(s+2)\lambda-2]2^{s+1}+(s+2)\lambda-s-3\big\}|f'(a)|^q\right. \\ &+\big[2\lambda^{s+2}-(s+2)\lambda+s+1\big]|f'(b)|^q\Big]^{1/q}+\left(\frac{1}{2}-\mu+\mu^2\right)^{1-1/q}\Big[\big(2\mu^{s+2}-(s+2)\mu+s \\ &\left.+1\big)|f'(a)|^q+\big[2(2-\mu)^{s+2}+((s+2)\mu-2)2^{s+1}+(s+2)\mu-s-3\big]|f'(b)|^q\Big]^{1/q}\right\}. \quad (1.9)\end{aligned}$$

2. *If $s=-1$, we have*

$$\begin{aligned}\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b-a}{2^{3-2/q}} \\ \times\left\{\big[(2\ln 2-1)|f'(a)|^q+|f'(b)|^q\big]^{1/q}+\big[|f'(a)|^q+(2\ln 2-1)|f'(b)|^q\big]^{1/q}\right\}. \quad (1.10)\end{aligned}$$

For more information on Simpson type inequalities in recent years, please refer to [2, 4, 7, 8, 10, 13, 14, 15, 16, 17, 19] and closely related references therein.

In this paper, we will establish some new integral inequalities of Simpson type for extended s-convex functions.

2 An integral identity

In order to prove our main results, we need the following integral identity.

Lemma 2.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a third differentiable mappings on I°, $a, b \in I$ with $a < b$, and $f''' \in L([a, b])$. Then*

$$\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x \\ = \frac{(b-a)^3}{96}\int_0^1 t(1-t)^2\left[f'''\left(tb + (1-t)\frac{a+b}{2}\right) - f'''\left(ta + (1-t)\frac{a+b}{2}\right)\right]\mathrm{d}\,t. \tag{2.1}$$

Proof. Integrating by parts and changing variable of definite integral yield

$$\begin{aligned}
&\int_0^1 t(1-t)^2\left[f'''\left(tb + (1-t)\frac{a+b}{2}\right) - f'''\left(ta + (1-t)\frac{a+b}{2}\right)\right]\mathrm{d}\,t \\
&= \frac{2}{b-a}\left\{\left[t(1-t)^2 f''\left(tb + (1-t)\frac{a+b}{2}\right)\Big|_0^1 - \int_0^1 (3t^2 - 4t + 1)f''\left(tb + (1-t)\frac{a+b}{2}\right)\mathrm{d}\,t\right]\right. \\
&\quad \left. + \left[t(1-t)^2 f''\left(ta + (1-t)\frac{a+b}{2}\right)\Big|_0^1 - \int_0^1 (3t^2 - 4t + 1)f''\left(ta + (1-t)\frac{a+b}{2}\right)\mathrm{d}\,t\right]\right\} \\
&= -\frac{4}{(b-a)^2}\left\{\left[\int_0^1 (3t^2 - 4t + 1)\,\mathrm{d}\,f'\left(tb + (1-t)\frac{a+b}{2}\right)\right]\right. \\
&\quad \left. - \left[\int_0^1 (3t^2 - 4t + 1)\,\mathrm{d}\,f'\left(ta + (1-t)\frac{a+b}{2}\right)\right]\right\} \\
&= -\frac{4}{(b-a)^2}\left[-f'\left(\frac{a+b}{2}\right) - \frac{2}{b-a}\int_0^1 (6t-4)\,\mathrm{d}\,f\left(tb + (1-t)\frac{a+b}{2}\right)\right. \\
&\quad \left. + f'\left(\frac{a+b}{2}\right) - \frac{2}{b-a}\int_0^1 (6t-4)\,\mathrm{d}\,f\left(ta + (1-t)\frac{a+b}{2}\right)\right] \\
&= \frac{8}{(b-a)^3}\left[2f(a) + 8f\left(\frac{a+b}{2}\right) + 2f(b)\right] - \frac{96}{(b-a)^4}\int_a^b f(x)\,\mathrm{d}\,x.
\end{aligned}$$

Multiplying by $\frac{(b-a)^3}{96}$ on both sides of the above equations leads to the identity (2.1). Lemma 2.1 is proved. □

3 Simpson type inequalities for extended s-convex functions

Now we are in a position to establish some new integral inequalities of Simpson type for functions whose third derivatives are extended s-convex functions.

4

Theorem 3.1. *Let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be a third differentiable mappings on I, $a, b \in I^\circ$ with $a < b$, and $f''' \in L([a, b])$ such that $|f'''|^q$ is extended s-convex on $[a, b]$ for $q \ge 1$ and some fixed $s \in [-1, 1]$. Then*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \left(\frac{1}{12}\right)^{1-1/q}\left[\frac{1}{(s+2)(s+3)(s+4)}\right]^{1/q} \times \frac{(b-a)^3}{96}\left\{\left[2|f'''(a)|^q+(s+2)\left|f'''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}+\left[(s+2)\left|f'''\left(\frac{a+b}{2}\right)\right|^q+2|f'''(b)|^q\right]^{1/q}\right\}.$$

Proof. By virtue of Lemma 2.1, Hölder's integral inequality, and the extended s-convexity of $|f'''|^q$, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \\
&\le \frac{(b-a)^3}{96}\int_0^1 t(1-t)^2\left\{\left|f'''\left(ta+(1-t)\frac{a+b}{2}\right)\right|+\left|f'''\left((1-t)\frac{a+b}{2}+tb\right)\right|\right\}\mathrm{d}\,t \\
&\le \frac{(b-a)^3}{96}\left(\int_0^1 t(1-t)^2\,\mathrm{d}\,t\right)^{1-1/q}\left\{\left[\int_0^1 t(1-t)^2\left(t^s|f'''(a)|^q+(1-t)^s\left|f'''\left(\frac{a+b}{2}\right)\right|^q\right)\mathrm{d}\,t\right]^{1/q}\right. \\
&\quad\left.+\left[\int_0^1 t(1-t)^2\left((1-t)^s\left|f'''\left(\frac{a+b}{2}\right)\right|^q+t^s|f'''(b)|^q\right)\mathrm{d}\,t\right]^{1/q}\right\} \\
&= \frac{(b-a)^3}{96}\left(\frac{1}{12}\right)^{1-1/q}\left[\frac{1}{(s+2)(s+3)(s+4)}\right]^{1/q}\left\{\left[2|f'''(a)|^q+(s+2)\left|f'''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}\right. \\
&\quad\left.+\left[(s+2)\left|f'''\left(\frac{a+b}{2}\right)\right|^q+2|f'''(b)|^q\right]^{1/q}\right\}.
\end{aligned}$$

Theorem 3.1 is proved. □

Corollary 3.1.1. *Under conditions of Theorem 3.1,*

1. *if $q = 1$, we have*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{(b-a)^3}{48(s+2)(s+3)(s+4)}\left[|f'''(a)|+(s+2)\left|f'''\left(\frac{a+b}{2}\right)\right|+|f'''(b)|\right];$$

2. *if $q = 1$ and $s = 1$, then*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{(b-a)^3}{2880}\left[|f'''(a)|+3\left|f'''\left(\frac{a+b}{2}\right)\right|+|f'''(b)|\right];$$

3. *if $q=1$ and $s=0$, we have*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{(b-a)^3}{1152}\left[|f'''(a)|+2\left|f'''\left(\frac{a+b}{2}\right)\right|+|f'''(b)|\right];$$

4. *if $q=1$ and $s=-1$, we have*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{(b-a)^3}{288}\left[|f'''(a)|+\left|f'''\left(\frac{a+b}{2}\right)\right|+|f'''(b)|\right].$$

Corollary 3.1.2. *Under conditions of Theorem 3.1,*

1. *if $s=1$, then*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{(b-a)^3}{1152}\left(\frac{1}{5}\right)^{1/q} \times\left\{\left[2|f'''(a)|^q+3\left|f'''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}+\left[3\left|f'''\left(\frac{a+b}{2}\right)\right|^q+2|f'''(b)|^q\right]^{1/q}\right\};$$

2. *if $s=0$, we have*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{(b-a)^3}{1152}\left\{\left[|f'''(a)|^q+\left|f'''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}+\left[\left|f'''\left(\frac{a+b}{2}\right)\right|^q+|f'''(b)|^q\right]^{1/q}\right\};$$

3. *if $s=-1$, we have*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{(b-a)^3}{1152}2^{1/q}\left\{\left[2|f'''(a)|^q+\left|f'''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}+\left[\left|f'''\left(\frac{a+b}{2}\right)\right|^q+2|f'''(b)|^q\right]^{1/q}\right\}.$$

Theorem 3.2. *Let $f: I\subseteq\mathbb{R}_0\to\mathbb{R}$ be a third differentiable mappings on I, $a,b\in I^\circ$ with $a<b$, and $f'''\in L([a,b])$ such that $|f'''|^q$ is extended s-convex on $[a,b]$ for $q\ge1$ and some fixed $s\in[-1,1]$. Then*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{(b-a)^3}{96}\left(\frac{1}{2}\right)^{1-1/q}$$

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$$\times \left\{ \left[B(s+2,2q+1)|f'''(a)|^q + \frac{1}{(2q+s+1)(2q+s+2)} \left| f'''\left(\frac{a+b}{2}\right) \right|^q \right]^{1/q} + \left[\frac{1}{(2q+s+1)(2q+s+2)} \left| f'''\left(\frac{a+b}{2}\right) \right|^q + B(s+2,2q+1)|f'''(b)|^q \right]^{1/q} \right\},$$

where $B(\alpha, \beta)$ is the classical beta function defined by

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,\mathrm{d}\,t, \quad \alpha,\beta > 0.$$

Proof. Since $|f'''|^q$ is extended s-convex on $[a, b]$, by Lemma 2.1 and Hölder's inequality, we obtain

$$\begin{aligned}
&\left| \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x \right| \\
&\le \frac{(b-a)^3}{96}\int_0^1 t(1-t)^2 \left\{ \left| f'''\left(ta + (1-t)\frac{a+b}{2}\right) \right| + \left| f'''\left((1-t)\frac{a+b}{2} + tb\right) \right| \right\} \mathrm{d}\,t \\
&\le \frac{(b-a)^3}{96}\left(\int_0^1 t\,\mathrm{d}\,t\right)^{1-1/q} \left\{ \left[\int_0^1 t(1-t)^{2q}\left(t^s|f'''(a)|^q + (1-t)^s \left| f'''\left(\frac{a+b}{2}\right) \right|^q \right) \mathrm{d}\,t \right]^{1/q} \right. \\
&\quad \left. + \left[\int_0^1 t(1-t)^{2q}\left((1-t)^s \left| f'''\left(\frac{a+b}{2}\right) \right|^q + t^s|f'''(b)|^q \right) \mathrm{d}\,t \right]^{1/q} \right\} \\
&= \frac{(b-a)^3}{96}\left(\frac{1}{2}\right)^{1-1/q} \left\{ \left[B(s+2,2q+1)|f'''(a)|^q + \frac{1}{(2q+s+1)(2q+s+2)} \left| f'''\left(\frac{a+b}{2}\right) \right|^q \right]^{1/q} \right. \\
&\quad \left. + \left[\frac{1}{(2q+s+1)(2q+s+2)} \left| f'''\left(\frac{a+b}{2}\right) \right|^q + B(s+2,2q+1)|f'''(b)|^q \right]^{1/q} \right\}.
\end{aligned}$$

Theorem 3.2 is thus proved. ☐

Corollary 3.2.1. *Under conditions of Theorem 3.2,*

1. *if $s = 1$, then*

$$\left| \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x \right| \le \frac{(b-a)^3}{192}\left[\frac{1}{(2q+1)(q+1)(2q+3)} \right]^{1/q}$$
$$\times \left\{ \left[2|f'''(a)|^q + (2q+1)\left| f'''\left(\frac{a+b}{2}\right) \right|^q \right]^{1/q} + \left[(2q+1)\left| f'''\left(\frac{a+b}{2}\right) \right|^q + 2|f'''(b)|^q \right]^{1/q} \right\};$$

2. *if $s = 0$, we have*

$$\left| \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x \right| \le \frac{(b-a)^3}{192}\left[\frac{1}{(2q+1)(q+1)} \right]^{1/q}$$
$$\times \left\{ \left[|f'''(a)|^q + \left| f'''\left(\frac{a+b}{2}\right) \right|^q \right]^{1/q} + \left[\left| f'''\left(\frac{a+b}{2}\right) \right|^q + |f'''(b)|^q \right]^{1/q} \right\};$$

3. if $s=-1$, we have

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\le\frac{(b-a)^3}{192}\left[\frac{1}{q(2q+1)}\right]^{1/q}$$
$$\times\left\{\left[2q|f'''(a)|^q+\left|f'''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}+\left[\left|f'''\left(\frac{a+b}{2}\right)\right|^q+2q|f'''(b)|^q\right]^{1/q}\right\}.$$

Theorem 3.3. *Let $q\ge r,\ell,m,n>0$, $f:I\subseteq\mathbb{R}_0\to\mathbb{R}$ be a third differentiable mappings on I, $a,b\in I^\circ$ with $a<b$, and $f'''\in L([a,b])$ such that $|f'''|^q$ is extended s-convex on $[a,b]$ for $q>1$ and some fixed $s\in[-1,1]$. Then*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\le\frac{(b-a)^3}{96}\left\{\left[B\left(\frac{2q-r-1}{q-1},\frac{3q-\ell-1}{q-1}\right)\right]^{1-1/q}\right.$$
$$\times\left[B(r+s+1,\ell+1)|f'''(a)|^q+B(r+1,\ell+s+1)\left|f'''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}+\left[B\left(\frac{2q-m-1}{q-1},\right.\right.$$
$$\left.\left.\frac{3q-n-1}{q-1}\right)\right]^{1-1/q}\left.\left[B(m+1,n+s+1)\left|f'''\left(\frac{a+b}{2}\right)\right|^q+B(m+s+1,n+1)|f'''(b)|^q\right]^{1/q}\right\}.$$

Proof. Since $|f'''|^q$ is extended s-convex on $[a,b]$, by Lemma 2.1 and Hölder's inequality, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\\
&\le\frac{(b-a)^3}{96}\int_0^1 t(1-t)^2\left\{\left|f'''\left(ta+(1-t)\frac{a+b}{2}\right)\right|+\left|f'''\left((1-t)\frac{a+b}{2}+tb\right)\right|\right\}\mathrm{d}\,t\\
&\le\frac{(b-a)^3}{96}\left\{\left[\int_0^1 t^{(q-r)/(q-1)}(1-t)^{(2q-\ell)/(q-1)}\,\mathrm{d}\,t\right]^{1-1/q}\left[\int_0^1 t^r(1-t)^\ell\left(t^s|f'''(a)|^q\right.\right.\right.\\
&\quad\left.\left.+(1-t)^s\left|f'''\left(\frac{a+b}{2}\right)\right|^q\right)\mathrm{d}\,t\right]^{1/q}+\left[\int_0^1 t^{(q-m)/(q-1)}(1-t)^{(2q-n)/(q-1)}\,\mathrm{d}\,t\right]^{1-1/q}\\
&\quad\left.\times\left[\int_0^1 t^m(1-t)^n\left((1-t)^s\left|f'''\left(\frac{a+b}{2}\right)\right|^q+t^s|f'''(b)|^q\right)\mathrm{d}\,t\right]^{1/q}\right\}\\
&=\frac{(b-a)^3}{96}\left\{\left[B\left(\frac{2q-r-1}{q-1},\frac{3q-\ell-1}{q-1}\right)\right)^{1-1/q}\left[B(r+s+1,\ell+1)|f'''(a)|^q\right.\right.\\
&\quad\left.+B(r+1,\ell+s+1)\left|f'''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}+\left[B\left(\frac{2q-m-1}{q-1},\frac{3q-n-1}{q-1}\right)\right]^{1-1/q}\\
&\quad\left.\times\left[B(m+1,n+s+1)\left|f'''\left(\frac{a+b}{2}\right)\right|^q+B(m+s+1,n+1)|f'''(b)|^q\right]^{1/q}\right\}.
\end{aligned}$$

The proof of Theorem 3.3 is thus completed. □

Corollary 3.3.1. *Under conditions of Theorem 3.3, if $r=m$ and $\ell=n$, we have*

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$$\begin{aligned}\left|\frac{f(a)+4f\left(\frac{a+b}{2}\right)+f(b)}{6}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| &\le \frac{(b-a)^3}{96}\left[B\left(\frac{2q-r-1}{q-1},\frac{3q-\ell-1}{q-1}\right)\right]^{1-1/q}\\ &\times\left\{\left[B(r+s+1,\ell+1)|f'''(a)|^q+B(r+1,\ell+s+1)\left|f'''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}\right.\\ &\left.+\left[B(r+1,\ell+s+1)\left|f'''\left(\frac{a+b}{2}\right)\right|^q+B(r+s+1,\ell+1)|f'''(b)|^q\right]^{1/q}\right\}.\end{aligned}$$

Corollary 3.3.2. *Under conditions of Theorem 3.3,*

1. *if* $r=\ell=m=n$, *we have*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\\ &\le\frac{(b-a)^3}{96}\left[B\left(\frac{2q-r-1}{q-1},\frac{3q-r-1}{q-1}\right)\right]^{1-1/q}[B(r+1,r+s+1)]^{1/q}\\ &\times\left\{\left[|f'''(a)|^q+\left|f'''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}+\left[\left|f'''\left(\frac{a+b}{2}\right)\right|^q+|f'''(b)|^q\right]^{1/q}\right\};\end{aligned}$$

2. *if* $r=\ell=m=n$ *and* $s=1$, *then*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\\ &\le\frac{(b-a)^3}{96}[B(r+2,r+1)]^{1/q}\left[B\left(\frac{2q-r-1}{q-1},\frac{3q-r-1}{q-1}\right)\right]^{1-1/q}\\ &\times\left\{|f'''(a)|^q+\left|f'''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}+\left[\left|f'''\left(\frac{a+b}{2}\right)\right|^q+|f'''(b)|^q\right]^{1/q}\right\};\end{aligned}$$

3. *if* $r=\ell=m=n$ *and* $s=0$, *we have*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\\ &\le\frac{(b-a)^3}{96}\left[B\left(\frac{2q-r-1}{q-1},\frac{3q-r-1}{q-1}\right)\right]^{1-1/q}[B(r+1,r+1)]^{1/q}\\ &\times\left\{|f'''(a)|^q+\left|f'''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}+\left[\left|f'''\left(\frac{a+b}{2}\right)\right|^q+|f'''(b)|^q\right]^{1/q}\right\};\end{aligned}$$

4. *if* $r=\ell=m=n$ *and* $s=-1$, *we have*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\\ &\le\frac{(b-a)^3}{96}[B(r,r+1)]^{1/q}\left[B\left(\frac{2q-r-1}{q-1},\frac{3q-r-1}{q-1}\right)\right]^{1-1/q}\end{aligned}$$

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$$\times\left\{|f'''(a)|^q+\left|f'''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}+\left[\left|f'''\left(\frac{a+b}{2}\right)\right|^q+|f'''(b)|^q\right]^{1/q}\right\}.$$

Corollary 3.3.3. *Under conditions of Theorem 3.3,*

1. *if* $r=\ell=m=n=1$, *then*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\le\frac{(b-a)^3}{96}\left[\frac{(q-1)^2}{(3q-2)(4q-3)}\right]^{1-1/q}$$
$$\times\left[\frac{1}{(s+2)(s+3)}\right]^{1/q}\left\{\left[|f'''(a)|^q+\left|f'''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}+\left[\left|f'''\left(\frac{a+b}{2}\right)\right|^q+|f'''(b)|^q\right]^{1/q}\right\};$$

2. *if* $r=\ell=m=n=1$ *and* $s=1$, *then*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\le\frac{(b-a)^3}{96}\left[\frac{(q-1)^2}{(3q-2)(4q-3)}\right]^{1-1/q}$$
$$\times\left(\frac{1}{12}\right)^{1/q}\left\{\left[|f'''(a)|^q+\left|f'''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}+\left[\left|f'''\left(\frac{a+b}{2}\right)\right|^q+|f'''(b)|^q\right]^{1/q}\right\};$$

3. *if* $r=\ell=m=n=1$ *and* $s=0$, *we have*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\le\frac{(b-a)^3}{96}\left[\frac{(q-1)^2}{(3q-2)(4q-3)}\right]^{1-1/q}$$
$$\times\left(\frac{1}{6}\right)^{1/q}\left\{\left[|f'''(a)|^q+\left|f'''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}+\left[\left|f'''\left(\frac{a+b}{2}\right)\right|^q+|f'''(b)|^q\right]^{1/q}\right\};$$

4. *if* $r=\ell=m=n=1$ *and* $s=-1$, *we have*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\le\frac{(b-a)^3}{96}\left[\frac{(q-1)^2}{(3q-2)(4q-3)}\right]^{1-1/q}$$
$$\times\left(\frac{1}{2}\right)^{1/q}\left\{\left[|f'''(a)|^q+\left|f'''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}+\left[\left|f'''\left(\frac{a+b}{2}\right)\right|^q+|f'''(b)|^q\right]^{1/q}\right\}.$$

Corollary 3.3.4. *Under conditions of Theorem 3.3,*

1. *if* $r=\ell=m=n=q$, *we have*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\le\frac{(b-a)^3}{96}\left(\frac{q-1}{2q-1}\right)^{1-1/q}$$
$$\times[B(q+1,q+s+1)]^{1/q}\left\{|f'''(a)|^q+\left|f'''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}+\left[\left|f'''\left(\frac{a+b}{2}\right)\right|^q+|f'''(b)|^q\right]^{1/q}\right\};$$

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2. *if* $r=\ell=m=n=q$ *and* $s=1$, *then*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\le\frac{(b-a)^3}{96}\left(\frac{q-1}{2q-1}\right)^{1-1/q}$$
$$\times\left[B(q+1,q+2)\right]^{1/q}\left\{\left[|f'''(a)|^q+\left|f'''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}+\left[\left|f'''\left(\frac{a+b}{2}\right)\right|^q+|f'''(b)|^q\right]^{1/q}\right\};$$

3. *if* $r=\ell=m=n=q$ *and* $s=0$, *we have*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\le\frac{(b-a)^3}{96}\left(\frac{q-1}{2q-1}\right)^{1-1/q}$$
$$\times[B(q+1,q+1)]^{1/q}\left\{\left[|f'''(a)|^q+\left|f'''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}+\left[\left|f'''\left(\frac{a+b}{2}\right)\right|^q+|f'''(b)|^q\right]^{1/q}\right\};$$

4. *if* $r=\ell=m=n=q$ *and* $s=-1$, *we have*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\le\frac{(b-a)^3}{96}\left(\frac{q-1}{2q-1}\right)^{1-1/q}$$
$$\times[B(q,q+1)]^{1/q}\left\{\left[|f'''(a)|^q+\left|f'''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}+\left[\left|f'''\left(\frac{a+b}{2}\right)\right|^q+|f'''(b)|^q\right]^{1/q}\right\}.$$

Similar to the proof of Theorem 3.3, we may also establish the following theorem.

Theorem 3.4. *Let* $q\ge r,\ell,m,n\ge 0$, $f:I\subseteq\mathbb{R}_0\to\mathbb{R}$ *be a third differentiable mappings on* I, $a,b\in I^\circ$ *with* $a<b$, *and* $f'''\in L([a,b])$ *such that* $|f'''|^q$ *is extended* s*-convex on* $[a,b]$ *for* $q>1$ *and some fixed* $s\in(-1,1]$. *Then*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\le\frac{(b-a)^3}{96}\left\{\left[B\left(\frac{2q-r-1}{q-1},\frac{3q-\ell-1}{q-1}\right)\right]^{1-1/q}\right.$$
$$\times\left[B(r+s+1,\ell+1)|f'''(a)|^q+B(r+1,\ell+s+1)\left|f'''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}+\left[B\left(\frac{2q-m-1}{q-1},\right.\right.$$
$$\left.\left.\frac{3q-n-1}{q-1}\right)\right]^{1-1/q}\left[B(m+1,n+s+1)\left|f'''\left(\frac{a+b}{2}\right)\right|^q+B(m+s+1,n+1)|f'''(b)|^q\right]^{1/q}\right\}.$$

Corollary 3.4.1. *Under conditions of Theorem 3.4,*

1. *if* $r=m=0$, *then*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\le\frac{(b-a)^3}{96}$$
$$\times\left\{\left[B\left(\frac{2q-1}{q-1},\frac{3q-\ell-1}{q-1}\right)\right]^{1-1/q}\left[B(s+1,\ell+1)|f'''(a)|^q+\frac{1}{\ell+s+1}\left|f'''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}\right.$$
$$\left.+\left[B\left(\frac{2q-1}{q-1},\frac{3q-n-1}{q-1}\right)\right]^{1-1/q}\left[\frac{1}{n+s+1}\left|f'''\left(\frac{a+b}{2}\right)\right|^q+B(s+1,n+1)|f'''(b)|^q\right]^{1/q}\right\};$$

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2. if $\ell = n = 0$, we have

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{(b-a)^3}{96}$$
$$\times\left\{\left[B\left(\frac{2q-r-1}{q-1},\frac{3q-1}{q-1}\right)\right]^{1-1/q}\left[\frac{1}{r+s+1}|f'''(a)|^q+B(r+1,s+1)\left|f'''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}\right.$$
$$\left.+\left[B\left(\frac{2q-m-1}{q-1},\frac{3q-1}{q-1}\right)\right]^{1-1/q}\left[B(m+1,s+1)\left|f'''\left(\frac{a+b}{2}\right)\right|^q+\frac{1}{m+s+1}|f'''(b)|^q\right]^{1/q}\right\};$$

3. if $r = \ell = m = n = 0$, we have

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{(b-a)^3}{96}\left[B\left(\frac{2q-1}{q-1},\frac{3q-1}{q-1}\right)\right]^{1-1/q}$$
$$\times\left(\frac{1}{s+1}\right)^{1/q}\left\{\left[|f'''(a)|^q+\left|f'''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}+\left[\left|f'''\left(\frac{a+b}{2}\right)\right|^q+|f'''(b)|^q\right]^{1/q}\right\}.$$

4 Applicaions to means

We now consider for positive numbers $b > a > 0$ the arithmetic mean $A(a,b) = \frac{a+b}{2}$ and the generalized logarithmic mean

$$L_r(a,b) = \left[\frac{b^{r+1}-a^{r+1}}{(r+1)(b-a)}\right]^{1/r}, \quad r \ne 0,-1. \tag{4.1}$$

Let $s > 0$, $q \ge 1$ and define $f(x) = \frac{x^{s+2}}{(s+1)(s+2)}$ for $x \in \mathbb{R}_+$. If $1 \le s \le 2$ and $(s-1)q \le 1$, we have

$$\left|f'''(tx+(1-t)y)\right|^q \le s^q\left[t^{(s-1)q}x^{(s-1)q}+(1-t)^{(s-1)q}y^{(s-1)q}\right] \le t^{s-1}|f'''(x)|^q+(1-t)^{s-1}|f'''(y)|^q.$$

If $0 < s \le 1$ and $-1 < (s-1)q$, we have

$$\left|f'''(tx+(1-t)y)\right|^q \le t^{s-1}|f'''(x)|^q+(1-t)^{s-1}|f'''(y)|^q$$

for $x, y \in \mathbb{R}_+$ and $t \in (0,1)$. If $0 < s \le 2$ and $-1 < (s-1)q \le 1$, then the function $|f'''(x)|^q = |s|^q x^{(s-1)q}$ is extended $(s-1)$-convex on $\mathbb{R}_+$.

Applying Theorem 3.1 to the function $|f'''(x)|^q = |s|^q x^{(s-1)q}$ yields the following conclusions.

Theorem 4.1. *Let $b > a > 0$, $q \ge 1$, $0 < s \le 2$, and $-1 < (s-1)q \le 1$. Then*

$$\left|\frac{1}{3}\left[A\left(a^{s+2},b^{s+2}\right)+2A^{s+2}(a,b)\right]-L_{s+2}^{s+2}(a,b)\right| \le \frac{(b-a)^3 s(s+1)(s+2)}{96}\left[\frac{1}{(s+2)(s+3)(s+4)}\right]^{1/q}$$
$$\times\left(\frac{1}{12}\right)^{1-1/q}\left\{\left[2a^{(s-1)q}+(s+2)A^{(s-1)q}(a,b)\right]^{1/q}+\left[(s+2)A^{(s-1)q}(a,b)+2b^{(s-1)q}\right]^{1/q}\right\}. \tag{4.2}$$

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Specially, if $q=1$, then

$$\left|\frac{1}{3}\left[A\left(a^{s+2},b^{s+2}\right)+2A^{s+2}(a,b)\right]-L_{s+2}^{s+2}(a,b)\right| \le \frac{(b-a)^3 s(s+1)}{48(s+3)(s+4)} \times \left[2A\left(a^{s-1},b^{s-1}\right)+(s+2)A^{s-1}(a,b)\right]. \quad (4.3)$$

Using Corollary 3.3.3, we have

Theorem 4.2. *Let $b>a>0$, $q>1$, $0<s\le 2$, and $-1<(s-1)q\le 1$. Then*

$$\left|\frac{A\left(a^{s+2},b^{s+2}\right)+2A^{s+2}(a,b)}{3}-L_{s+2}^{s+2}(a,b)\right| \le \frac{(b-a)^3 s(s+1)(s+2)}{96}\left[\frac{(q-1)^2}{(3q-2)(4q-3)}\right]^{1-1/q} \times \left[\frac{1}{(s+2)(s+3)}\right]^{1/q}\left\{\left[a^{(s-1)q}+A^{(s-1)q}(a,b)\right]^{1/q}+\left[A^{(s-1)q}(a,b)+b^{(s-1)q}\right]^{1/q}\right\}. \quad (4.4)$$

Letting $q=2$ and $r=\ell=m=n=0$ in Corollary 3.4.1 leads to the following theorem.

Theorem 4.3. *Let $b>a>0$ and $\frac{1}{2}\le s\le\frac{3}{2}$. Then*

$$\left|\frac{1}{3}\left[A\left(a^{s+2},b^{s+2}\right)+2A^{s+2}(a,b)\right]-L_{s+2}^{s+2}(a,b)\right| \le \frac{(b-a)^3 s(s+2)}{96}\left(\frac{s+1}{105}\right)^{1/2} \times \left\{\left[a^{2(s-1)}+A^{2(s-1)}(a,b)\right]^{1/2}+\left[A^{2(s-1)}(a,b)+b^{2(s-1)}\right]^{1/2}\right\}. \quad (4.5)$$

Acknowledgements

This work was partially supported by the NNSF under Grant No. 11361038 of China and the Foundation of Research Program of Science and Technology at Universities of Inner Mongolia Autonomous Region under Grant No. NJZY13159, China.

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REPRESENTATION OF HIGHER-ORDER EULER NUMBERS USING THE SOLUTION OF BERNOULLI EQUATION

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Abstract. In this paper, we derive the several representations of the solution of the special Bernoulli differential equation. And then we have the relations between Euler numbers and higher-order Euler numbers.

1. Introduction

A differential equation is a relationship between a function of time and its derivatives. Differential equations appear naturally in many areas of science and the humanities. In mathematics, an ordinary differential equation of the form

$$\frac{dy}{dt} + P(t)y = Q(t)y^{\alpha} \ (\alpha \neq 0, 1) \tag{1}$$

is called a Bernoulli equation which is named after Jacob Bernoulli, who discussed it in 1695. Bernoulli equations are special because they are nonlinear differential equations with known exact solutions.

When $\alpha = 2$, the Bernoulli equation has the solution which is the functions of the exponential generating function of the Euler numbers. It is well known that the Euler polynomials, $E_n(x), n = 0, 1, \cdots$ are defined by means of the following generating function;

$$\frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \tag{2}$$

[1, 3, 7, 17, 23, 24] We note that, by substituting $x = 0$ into (2), $E_n(0) = E_n, n = 0, 1, \cdots$ are the familiar Euler numbers which is defined by

$$\frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \tag{3}$$

From the definition of $E_n(x)$ we know that

$$\sum_{k=0}^{n} \binom{n}{k} E_k(x) + E_n(x) = 2x^n, \ n = 0, 1, \cdots, \tag{4}$$

Key words and phrases. Bernoulli equation, Euler numbers, higher-order Euler numbers.
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where $E_k = E_k(0)$ is the kth Euler number. These can be rewritten as follows:

$$(E(x)+1)^n + E_n(x) = 2x^n, \; n = 0, 1, \cdots \tag{5}$$

by using the symbolic convention about replacing $E_n(x)$ by $(E(x))^n$. When $x = 0$, these relations are given by

$$(E+1)^n + E_n = 2\delta_{0,\delta}, \; n = 0, 1, \cdots, \tag{6}$$

where $\delta_{0,n}$ is Kronecker symbol and $(E+1)^n$ is interpreted as $\sum_{k=0}^{n} \binom{n}{k} E_k$. [2-6, 9-15]

This notion can be expanded to the multiple case. The Euler polynomials of order $r \in \mathbb{N}$ are defined by the generating function to be

$$\left(\frac{2}{e^t+1}\right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x) \frac{t^n}{n!}. \tag{7}$$

When $x = 0$, $E_n^{(r)} = E_n^{(r)}(0), n = 0, 1, \cdots$ are also called the Euler numbers of order r. [8, 10, 18, 19-22]

Recently, several authors have studied the identities of Euler numbers and polynomials.[3-12, 18,28] In this paper, we derive the several representations of the solution of the special Bernoulli differential equation. And then we have the relations between Euler numbers and higher-order Euler numbers. We indebted this idea to the paper [19].

2. Representation of Euler numbers

As is well known that when $\alpha = 2$ a special Bernoulli equation

$$\frac{dy}{dt} + y = y^2 \tag{8}$$

has the solution

$$y(t) = \frac{1}{e^t+1}. \tag{9}$$

Differentiating the both side of (8) is

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} = 2y\frac{dy}{dt}. \tag{10}$$

From (8) and (10), we see that

$$y^3 = \frac{1}{2!}\left(\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y\right). \tag{11}$$

Continuing this process, we have the following lemma.

Lemma 2.1. *The solution $y(t)$ of the Bernoulli equation for $\alpha = 2$ in (8) has the representation as the following; for some real coefficients a_m, $m = 0, 1, \cdots$*

$$y^{r+1}(t) = \frac{1}{r!}\sum_{m=0}^{r} a_{m,r}\frac{d^m}{dt^m}y(t), \; r = 0, 1, \cdots. \tag{12}$$

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Proof. Suppose that it holds for any $k \in \mathbb{N}$. Then for some real constants $a_{m,k}, m = 0, n1, \cdots$

$$y^{k+1} = \frac{1}{k!} \sum_{m=0}^{k} a_{m,k} \frac{d^m y}{dt^m}. \tag{13}$$

Differentiating the both side of (12) is

$$(k+1) y^k \frac{dy}{dt} = \frac{1}{k!} \sum_{m=0}^{k} a_{m,k} \frac{d^{m+1} y}{dt^{m+1}}. \tag{14}$$

Since $dy/dt = y^2 - y$, this implies that

$$y^{k+2} = \frac{1}{(k+1)!} \sum_{m=0}^{k} a_{m,k} \frac{d^{m+1} y}{dt^{m+1}} + \sum_{m=0}^{k} \frac{d^m y}{dt^m}. \tag{15}$$

Take

$$a_{m,k+1} = \begin{cases} a_{0,k}, & m = 0 \\ a_{m-1,k} + (k+1) a_{m,k}, & 1 \leq m \leq k \\ a_{k,k}, & m = k+1. \end{cases} \tag{16}$$

Therefore, we obtain that

$$y^{k+2} = \frac{1}{(k+1)!} \sum_{m=0}^{k+1} a_{m,k+1} \frac{d^m y}{dt^m}. \tag{17}$$

By the mathematical induction, we have the desired result. □

In the proof of Lemma 2.1 we know that for any given $r \in \mathbb{N}$

$$a_{m+1,r+1} = a_{m,r} + r a_{m+1,r}, \ 0 \leq m \leq r-1. \tag{18}$$

In order to determine the coefficients $a_{m,r}, m = 0, 1, \cdots$, we consider the function of t and μ as follows; for $|t| < 1$

$$f(t, \mu) = \sum_{r=1}^{\infty} \left(\sum_{m=0}^{r-1} a_{m,r} \mu^m \right) \frac{t^r}{r!}. \tag{19}$$

From (17), we know that

$$\sum_{r=1}^{\infty} \left(\sum_{m=0}^{r-1} a_{m+1,r+1} \frac{t^r}{r!} \right) \mu^m = f(t, \mu) + \sum_{r=1}^{\infty} \left(\sum_{m=0}^{r-1} r a_{m+1,r} \frac{t^r}{r!} \right) \mu^m. \tag{20}$$

Then we have the following theorem.

Lemma 2.2. *Let* $f(t, \mu)$ *be defined in* (18). *Then we have*

$$f(t, \mu) = \frac{1}{\mu}(1-t)^{\mu} - \frac{1}{\mu}.$$

Proof. Observe that

$$\begin{aligned}\sum_{r=1}^{\infty}\left(\sum_{m=0}^{r-1} r a_{m+1,r}\frac{t^r}{r!}\right)\mu^m =&\frac{1}{\mu}\sum_{r=1}^{\infty}\sum_{m=1}^{r} r a_{m,r}\frac{t^r}{r!}\mu^m \\ =&\frac{1}{\mu}\sum_{r=1}^{\infty}\left\{\sum_{m=0}^{r} a_{m,r}\frac{t^r}{(r-1)!}\mu^m - a_{0,r}\frac{t^r}{(r-1)!}\right\} \\ =&\frac{t}{\mu}\sum_{r=1}^{\infty}\left\{\sum_{m=0}^{r-1} a_{m,r}\frac{t^{r-1}}{(r-1)!}\mu^m - \frac{1}{1-t}\right\} \\ =&\frac{t}{\mu}\left(f'(t,\mu)-\frac{1}{1-t}\right).\end{aligned} \tag{21}$$

From (19) we have

$$\sum_{r=1}^{\infty}\left(\sum_{m=0}^{r-1} a_{m+1,r+1}\frac{t^r}{r!}\right)\mu^m = f(t,\mu)+\frac{t}{\mu}\left(f'(t,\mu)-\frac{1}{1-t}\right). \tag{22}$$

In (22), the left hand side is

$$\begin{aligned}\sum_{r=1}^{\infty}\left(\sum_{m=0}^{r-1} a_{m+1,r+1}\frac{t^r}{r!}\right)\mu^m =&\frac{1}{\mu}\sum_{r=2}^{\infty}\left\{\sum_{m=1}^{r-1} a_{m,r}\frac{t^{r-1}}{(r-1)!}\mu^m - a_{0,r}\frac{t^{r-1}}{(r-1)!}\right\} \\ =&\frac{1}{\mu}\left\{\sum_{r=2}^{\infty}\sum_{m=0}^{r-1} a_{m,r}\frac{t^{r-1}}{(r-1)!}\mu^m - \frac{t}{1-t}\right\}.\end{aligned} \tag{23}$$

This implies that

$$\sum_{r=1}^{\infty}\left(\sum_{m=0}^{r-1} a_{m+1,r+1}\frac{t^r}{r!}\right)\mu^m = \frac{1}{\mu}\left\{\frac{\partial}{\partial t}f(t,\mu)-\frac{1}{1-t}\right\}. \tag{24}$$

By (22) and (24), we obtain a first order linear differential equation about t

$$\frac{d}{dt}f(t,\mu)-\frac{\mu}{1-t}f(t,\mu)=\frac{1}{1-t}. \tag{25}$$

Multiplying the integration factor $e^{\int \mu/(1-t)dt}=1/(1-t)^{\mu}$ in the both side of (25) we obtain the solution of (25)

$$f(t,\mu)=\frac{1}{\mu}(1-t)^{\mu}-\frac{1}{\mu} \tag{26}$$

because $f(0,\mu)=0$. This is completion of the proof. □

As is well known that the Taylor expansion of $\ln x$ at $x=1$ is

$$\ln x=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(x-1)^n,\ 0<x<2. \tag{27}$$

From (27), Lemma 2.1 and Lemma 2.2, the coefficients, $a_{m,r}, m=0,1,\cdots$, in Lemma 2.1 are determined in the following theorem.

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Theorem 2.3. *The solution $y(t)$ of the Bernoulli equation for $\alpha = 2$ in (8) is satisfied the following equations; for $r = 1, 2, \cdots$*

$$y^r(t) = (-1)^r r \sum_{m=0}^{r-1} \frac{1}{(m+1)!} \sum_{\substack{l_1+\cdots+l_{m+1}=r \\ l_1,\cdots,l_{m+1}\geq 1}} \frac{1}{l_1 \cdots l_{m+1}} \frac{d^m}{dt^m} y(t). \tag{28}$$

Proof. From the definition of $f(t,\mu)$ and Lemma 2.2, we know that

$$\sum_{r=1}^{\infty} \left(\sum_{m=0}^{r-1} a_{m,r} \mu^m \right) \frac{t^r}{r!} = \sum_{n=1}^{\infty} \mu^{n-1} \frac{(\ln(1-t))^n}{n!}. \tag{29}$$

Since

$$\ln(1-t) = \sum_{k=1}^{\infty} \frac{(-1)^{2k+1}}{k} t^k, \; -1 < t < 1, \tag{30}$$

this implies that for $-1 < t < 1$

$$\sum_{r=1}^{\infty} \sum_{m=0}^{r-1} a_{m,r} \mu^m \frac{t^r}{r!} = \sum_{n=1}^{\infty} \mu^{n-1} \left\{ \sum_{k_1,\cdots,k_n \geq 1} \frac{(-1)^{k_1+\cdots+k_n}}{k_1 \cdots k_n} t^{k_1+k_2+\cdots+k_n} \right\} \frac{1}{n!}. \tag{31}$$

In the right side of (30), taking $s = k_1 + \cdots + k_n$ and changing the order of sums, thus we have

$$\begin{aligned} \sum_{r=1}^{\infty} \sum_{m=0}^{r-1} a_{m,r} \mu^m \frac{t^r}{r!} &= \sum_{n=1}^{\infty} \sum_{s=n}^{\infty} \sum_{\substack{l_1,\cdots,l_{m+1}\geq 1 \\ l_1+\cdots+l_{m+1}=s}} \frac{s!}{n!} \frac{(-1)^s}{l_1 \cdots l_n} \mu^{n-1} \frac{t^s}{s!} \\ &= \sum_{s=1}^{\infty} \sum_{n=1}^{s} \sum_{\substack{l_1,\cdots,l_{m+1}\geq 1 \\ l_1+\cdots+l_{m+1}=s}} \frac{s!}{n!} \frac{(-1)^s}{l_1 \cdots l_n} \mu^{n-1} \frac{t^s}{s!} \\ &= \sum_{s=1}^{\infty} \sum_{n=0}^{s-1} \sum_{\substack{l_1,\cdots,l_{m+1}\geq 1 \\ l_1+\cdots+l_{m+1}=s}} \frac{s!}{(n+1)!} \frac{(-1)^s}{l_1 \cdots l_{n+1}} \mu^{n} \frac{t^s}{s!}. \end{aligned} \tag{32}$$

Consequently, we obtain

$$\sum_{r=1}^{\infty} \sum_{m=0}^{r-1} a_{m,r} \mu^m \frac{t^r}{r!} = \sum_{r=1}^{\infty} \sum_{m=0}^{r-1} \sum_{\substack{l_1,\cdots,l_{m+1}\geq 1 \\ l_1+\cdots+l_{m+1}=r}} \frac{r!}{(m+1)!} \frac{(-1)^r}{l_1 \cdots l_{m+1}} \mu^m \frac{t^r}{r!}. \tag{33}$$

comparing the both sides in (32), the coefficients $a_{m,r}, m = 0, 1, \cdots, r-1 (r \in \mathbb{N})$ are determined as following

$$a_{m,r} = (-1)^r \frac{r!}{(m+1)!} \sum_{\substack{l_1,\cdots,l_{m+1}\geq 1 \\ l_1+\cdots+l_{m+1}=r}} \frac{1}{l_1 \cdots l_{m+1}}. \tag{34}$$

From Lemma 2.1, we have the desired result in the theorem. □

By definition of the Euler numbers, $E_n, n = 0, 1, \cdots$, we know that

$$\frac{d^m}{dt^m} y(t) = \frac{1}{2} \sum_{l=0}^{\infty} E_{l+m} \frac{t^l}{l!}. \tag{35}$$

Therefore, from Theorem 2.3 we have the following corollary.

Corollary 2.4. *For the Euler numbers, $E_n, n = 0, 1, \cdots$, the solution $y(t)$ of the Bernoulli equation for $\alpha = 2$ in (8) is represented as following; for $r = 1, 2, \cdots$*

$$y^r(t) = \sum_{n=0}^{\infty} \left\{ (-1)^r \frac{r}{2} \sum_{m=0}^{r-1} \frac{1}{(m+1)!} \sum_{\substack{l_1, \cdots, l_{m+1} \geq 1 \\ l_1 + \cdots + l_{m+1} = r}} \frac{1}{l_1 \cdots l_{m+1}} E_{n+m} \right\} \frac{t^n}{n!},$$

where E_k is k-th Euler number.

In the case of higher-order Euler numbers, $E_n^{(r)}, n = 0, 1, \cdots (r \in \mathbb{N})$, we know that

$$y^r(t) = 2^r \sum_{n=0}^{\infty} E_n^{(r)} \frac{t^n}{n!}. \tag{36}$$

From Corollary 2.4 we have the following corollary.

Corollary 2.5. *The Euler numbers with order r, $E_n^{(r)}, n \in \mathbb{N}$, are represented as following formula; for $r = 1, 2, \cdots$*

$$E_n^{(r)} = (-1)^r r 2^{r-1} \sum_{m=0}^{r-1} \frac{1}{(m+1)!} \sum_{\substack{l_1, \cdots, l_{m+1} \geq 1 \\ l_1 + \cdots + l_{m+1} = r}} \frac{1}{l_1 \cdots l_{m+1}} E_{n+m},$$

where E_k is k-th Euler number.

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Intuitionistic fuzzy stability of an Euler-Lagrange type quartic functional equation

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Abstract. In this paper, we investigate the Hyers-Ulam stability of the following Euler-Lagrange type quartic functional equation

$$f(ax+y)+f(x+ay)+\frac{1}{2}a(a-1)^2f(x-y)=(a^2-1)^2(f(x)+f(y))+\frac{1}{2}a(a+1)^2f(x+y)$$

in intuitionistic fuzzy normed spaces, where $a\neq 0, a\neq \pm 1$. Furthermore, we investigate intuitionistic fuzzy continuity through the existence of a certain solution of a fuzzy stability problem for approximately Euler-Lagrange quartic functional equation.

1. Introduction

Fuzzy set theory is a powerful hand set for modeling uncertainty and vagueness in various problems arising in the field of science and engineering. The fuzzy topology proves to be a very useful tool to deal with such situations where the use of classical theories breaks down. In 1984, Katsaras [17] introduced an idea of a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. In the same year Wu and Fang [41] introduced a notion fuzzy normed apace to give a generalization of the Kolmogoroff normalized theorem for fuzzy topological vector spaces. In 1992, Felbin [8] introduced an alternative definition of a fuzzy norm on a vector space with an associated metric of Kaleva and Seikkala type [15]. Some mathematics have define fuzzy normed on a vector form various point of view [19, 29, 42]. In particular, Bang and Samanta [4] following Cheng and Mordeson [6] gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric of Kramosil and Michalek type [18]. They established a decomposition theorem of fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [5]. The notion of intuitionistic fuzzy set introduced by Atanassov [3] has been used extensively in many areas of mathematics and sciences. The notion of intuitionistic fuzzy norm [24, 28, 35] is also useful one to deal with the inexactness and vagueness arising in modeling. There are many situations where the norm of a vector is not possible to find and the concept of intuitionistic fuzzy norm seems to be more suitable in such cases, that is, we can deal with such situations by modeling the inexactness through the intuitionistic fuzzy norm. The stability problem for Pexiderized quadratic functional equation, Jensen functional equation, cubic functional equation, functional equations associated with inner product spaces, and additive functional equation was considered in [21, 22, 23, 25, 26, 40], respectively, in the intuitionistic fuzzy normed spaces.

[0] **Mathematics Subject Classification 2010**: Primary 39B82; 39B52; 46S40; 47S40; 54A40.
[0]**Keywords**: Hyers-Ulam stability; Euler-Lagrange type quartic functional equation; intuitionistic fuzzy normed space; intuitionistic fuzzy continuity.

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One of the most interesting questions in the theory of functional analysis concerning the Ulam stability problem of functional equations is as follows. If the problem accepts a solution, we say that the equation is stable. The first problem concerning group homomorphisms was raised by Ulam [39] in 1940. In the next year Hyers [10] gave a first affirmative answer to the question of Ulam in context of Banach spaces. Subsequently, the result of Hyers was generalized by Aoki [2] for additive mapping and by Rassias [33] for linear mapping by considering an unbounded Cauchy difference. The result of Rassias has provided a lot of influence during the last three decades in the development of generalization of Hyers-Ulam stability concept. Furthermore, in 1994, Găvruţa [9] provided a further generalization of Rassias' theorem in which he replaced the bound $\varepsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$. Recently several stability results have been obtained for various equations and mappings with more general domains and ranges have been investigated by a number of authors and there are many interesting results concerning this problem [1, 7, 11, 12, 13, 30, 31, 34, 36, 37]. In particular, Rassias [32] introduced the following Euler-Lagrange type quadratic functional equation

$$f(ax + by) + f(ax - by) = (a^2 + b^2)(f(x) + f(y)),$$

for fixed reals a, b with $a \neq 0$, $b \neq 0$. Jun and Kim [14] proved the Hyers-Ulam stability of the following Euler-Lagrange type cubic functional equation

$$f(ax + by) + f(bx + ay) = (a + b)(a - b)^2(f(x) + f(y)) + ab(a + b)f(x + y),$$

where $a \neq 0$, $b \neq 0$, $a \pm b \neq 0$, for all $x, y \in X$. Also Lee et al. [20] considered the following functional equation

$$f(2x + y) + f(2x - y) = 4(f(x + y) + f(x - y)) + 24f(x) - 6f(y)$$

and established the general solution and the Hyers-Ulam stability for this functional equation. This functional equation is called quartic functional equation and every solution of this quartic equation is said to be a quartic function. Obviously, the function $f(x) = x^4$ satisfies the quartic functional equation. Kang [16] investigated the solution and the Hyers-Ulam stability for the quartic functional equation

$$f(ax + by) + f(ax - by) = a^2b^2(f(x + y) + f(x - y)) + 2a^2(a^2 - b^2)f(x) - 2b^2(a^2 - b^2)f(y),$$

where $a, b \in \mathbb{Z} \setminus \{0\}$ with $a \neq \pm 1, \pm b$. Several Euler-Lagrange type functional equations have been investigated by numerous mathematicians [43, 44, 45].

In this paper, we consider the following Euler-Lagrange type quartic functional equation

$$f(ax+y)+f(x+ay)+\frac{1}{2}a(a-1)^2f(x-y) = (a^2-1)^2(f(x)+f(y))+\frac{1}{2}a(a+1)^2f(x+y) \quad (1.1)$$

for a fixed integer $a \neq 0, \pm 1$ and all $x, y \in X$. We determine some stability results concerning the above Euler-Lagrange quartic functional equation in the setting of intuitionistic fuzzy normed spaces (IFNS). Further, we study intuitionistic fuzzy continuity through the existence of a certain solution of a fuzzy stability problem for approximately Euler-Lagrange type quartic functional equation.

Throughout this paper, assume that the symbol $\mathbb{N}$ denotes the set of all natural numbers.

2. **Preliminaries**

In this section we recall some notations and basic definitions used in this paper.

A binary operation $* : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is said to be a continuous t-norm if it satisfies the following conditions:

(i) $*$ is associative and commutative, (ii) $*$ is continuous, (iii) $a * 1 = a$ for all $a \in [0,1]$, (iv) $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for each $a, b, c, d \in [0,1]$.

A binary operation $\diamond : [0,1] \times [0,1] \longrightarrow [0,1]$ is said to be a continuous t-conorm if it satisfies the following conditions:

(i) $\diamond$ is associative and commutative, (ii) $*$ is continuous, (iii) $a \diamond 0 = a$ for all $a \in [0,1]$, (iv) $a \diamond b \le c \diamond d$ whenever $a \le c$ and $b \le d$ for each $a, b, c, d \in [0,1]$.

The five-tuple $(\mathcal{X}, \mu, \nu, *, \diamond)$ is said to be intuitionistic fuzzy normed spaces (for short, INFS) if $\mathcal{X}$ is a vector space, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm, and μ, ν are fuzzy sets on $\mathcal{X} \times (0, \infty)$ satisfying the following conditions. For every $x, y \in \mathcal{X}$ and $s, t > 0$, (i) $\mu(x,t) + \nu(x,t) \le 1$, (ii) $\mu(x,t) > 0$, (iii) $\mu(x,t) = 1$ if and only if $x = 0$, (iv) $\mu(\alpha x, t) = \mu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$, (v) $\mu(x,t) * \mu(y,s) \le \mu(x+y, t+s)$, (vi) $\mu(x,.) : (0,\infty) \to [0,1]$ is continuous, (vii) $\lim_{t\to\infty} \mu(x,t) = 1$ and $\lim_{t\to 0} \mu(x,t) = 0$, (viii) $\nu(x,t) < 1$, (ix) $\nu(x,t) = 0$ if and only if $x = 0$, (x) $\nu(\alpha x, t) = \nu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$, (v) $\nu(x,t) \diamond \nu(y,s) \ge \nu(x+y, t+s)$, (vi) $\nu(x,.) : (0,\infty) \to [0,1]$ is continuous, (vii) $\lim_{t\to\infty} \nu(x,t) = 0$ and $\lim_{t\to 0} \nu(x,t) = 1$.

In this case (μ, ν) is called an intuitionistic fuzzy norm. For simplicity in notation, we denote the intuitionistic fuzzy normed space by $(\mathcal{X}, \mu, \nu)$ instead of $(\mathcal{X}, \mu, \nu, *, \diamond)$. For example, let $(\mathcal{X}, \|.\|)$ be a normed space, and let $a * b = ab$ and $a \diamond b = \min\{a+b, 1\}$ for all $a, b \in [0,1]$. For all $x \in \mathcal{X}$ and every $t \in \mathbb{R}$, consider

$$\mu(x,t) = \begin{cases} \frac{t}{t+\|x\|}, & t > 0 \\ 0, & t \le 0 \end{cases}$$

and

$$\nu(x,t) = \begin{cases} \frac{\|x\|}{t+\|x\|}, & t > 0 \\ 0, & t \le 0, \end{cases}$$

then $(\mathcal{X}, \mu, \nu)$ is an IFNS.

The concept of convergence and Cauchy sequences in the setting of IFNS were introduced by Saadati and Park [35] and further studied by Mursaleen and Mohiuddine [27]. Let $(\mathcal{X}, \mu, \nu)$ be an IFNS. Then a sequence $\{x_n\}$ is said to be:

(i) convergent to $L \in \mathcal{X}$ with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $\varepsilon > 0$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $\mu(x_n - L, t) > 1 - \varepsilon$ and $\nu(x_n - L, t) < \varepsilon$ for all $n \ge n_0$. In this case, we write $(\mu, \nu) - \lim x_n = L$ or $x_n \to L$ as $n \to \infty$.

(ii) a Cauchy sequence with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $\varepsilon > 0$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $\mu(x_n - x_m, t) > 1 - \varepsilon$ and $\nu(x_n - x_m, t) < \varepsilon$ for all $n, m \ge n_0$.

An IFNS $(\mathcal{X}, \mu, \nu)$ is said to be complete if each intuitionistic fuzzy Cauchy sequence in $(\mathcal{X}, \mu, \nu)$ is intuitionistic fuzzy convergent in $(\mathcal{X}, \mu, \nu)$. In this case, $(\mathcal{X}, \mu, \nu)$ is called an intuitionistic fuzzy Banach space.

3. Approximation on intuitionistic fuzzy normed spaces

Theorem 3.1. *Let $\mathcal{X}$ be a linear space and let $(\mathcal{Z}, \mu', \nu')$ be an IFNS. Let $\varphi : \mathcal{X} \times \mathcal{X} \to \mathcal{Z}$ be a mapping such that, for some $|\alpha| > |a|^4$,*

$$\begin{aligned} \mu'(\varphi(\frac{x}{a}, 0), t) &\ge \mu'(\varphi(x, 0), |\alpha| t), \\ \nu'(\varphi(\frac{x}{a}, 0), t) &\le \nu'(\varphi(x, 0), |\alpha| t) \end{aligned}$$

and let $(\mathcal{Y}, \mu, \nu)$ be an intuitionistic fuzzy Banach space and let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping satisfying $f(0) = 0$ and a φ-approximately quartic mapping in the sense that

$$\begin{aligned} \mu(f(ax+y) &+ f(x+ay) + \frac{1}{2}a(a-1)^2 f(x-y) - (a^2-1)^2(f(x)+f(y)) \\ &- \frac{1}{2}a(a+1)^2 f(x+y), t) \geq \mu'(\varphi(x,y), t), \\ \nu(f(ax+y) &+ f(x+ay) + \frac{1}{2}a(a-1)^2 f(x-y) - (a^2-1)^2(f(x)+f(y)) \\ &- \frac{1}{2}a(a+1)^2 f(x+y), t) \leq \nu'(\varphi(x,y), t) \end{aligned} \tag{3.1}$$

for all $x, y \in \mathcal{X}$ and all $t > 0$. Then there exists a unique quartic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ such that

$$\begin{aligned} \mu(\mathcal{Q}(x) - f(x), t) &\geq \mu'(\varphi(x,0), \frac{(|\alpha| - |a|^4)t}{2}), \\ \nu(\mathcal{Q}(x) - f(x), t) &\leq \nu'(\varphi(x,0), \frac{(|\alpha| - |a|^4)t}{2}) \end{aligned} \tag{3.2}$$

for all $x \in \mathcal{X}$ and all $t > 0$.

Proof. Letting $y = 0$ in (3.1), we get

$$\begin{aligned} \mu(f(ax) - a^4 f(x), t) &\geq \mu'(\varphi(x,0), t), \\ \nu(f(ax) - a^4 f(x), t) &\leq \nu'(\varphi(x,0), t) \end{aligned} \tag{3.3}$$

for all $x \in \mathcal{X}$ and all $t > 0$, which implies that

$$\begin{aligned} \mu(a^4 f(\frac{x}{a}) - f(x), t) &\geq \mu'(\varphi(x,0), |\alpha| t), \\ \nu(a^4 f(\frac{x}{a}) - f(x), t) &\leq \nu'(\varphi(x,0), |\alpha| t). \end{aligned} \tag{3.4}$$

Replacing x by $\frac{x}{a^n}$ in (3.4), we obtain

$$\begin{aligned} \mu(a^{4(n+1)} f(\frac{x}{a^{n+1}}) - a^{3n} f(\frac{x}{a^n}), |a|^{4n} t) &\geq \mu'(\varphi(x,0), |\alpha|^{n+1} t), \\ \nu(a^{4(n+1)} f(\frac{x}{a^{n+1}}) - a^{3n} f(\frac{x}{a^n}), |a|^{4n} t) &\leq \nu'(\varphi(x,0), |\alpha|^{n+1} t). \end{aligned} \tag{3.5}$$

Replacing t by $\frac{t}{|\alpha|^{n+1}}$ in (3.5), we get

$$\begin{aligned} \mu(a^{4(n+1)} f(\frac{x}{a^{n+1}}) - a^{3n} f(\frac{x}{a^n}), \frac{|a|^{4n}}{|\alpha|^{n+1}} t) &\geq \mu'(\varphi(x,0), t), \\ \nu(a^{4(n+1)} f(\frac{x}{a^{n+1}}) - a^{3n} f(\frac{x}{a^n}), \frac{|a|^{4n}}{|\alpha|^{n+1}} t) &\leq \nu'(\varphi(x,0), t). \end{aligned} \tag{3.6}$$

It follows from

$$a^{4n} f(\frac{x}{a^n}) - f(x) = \sum_{j=0}^{n-1} (a^{4(j+1)} f(\frac{x}{a^{j+1}}) - a^{3j} f(\frac{x}{a^j}))$$

and (3.6) that

$$\begin{aligned}
\mu(a^{4n}f(\frac{x}{a^n})-f(x),\sum_{j=0}^{n-1}\frac{|a|^{4j}}{|\alpha|^{j+1}}t) &\geq \prod_{j=0}^{n-1}\mu(a^{4(j+1)}f(\frac{x}{a^{j+1}})-a^{3j}f(\frac{x}{a^j}),\frac{|a|^{4j}}{|\alpha|^{j+1}}t)\\
&\geq \mu'(\varphi(x,0),t),\\
\nu(a^{4n}f(\frac{x}{a^n})-f(x),\sum_{j=0}^{n-1}\frac{|a|^{4j}}{|\alpha|^{j+1}}t) &\leq \coprod_{j=0}^{n-1}\nu(a^{4(j+1)}f(\frac{x}{a^{j+1}})-a^{3j}f(\frac{x}{a^j}),\frac{|a|^{4j}}{|\alpha|^{j+1}}t)\\
&\leq \nu'(\varphi(x,0),t),
\end{aligned}\tag{3.7}$$

for all $x\in\mathcal{X}$, all $t>0$ and $n>0$, where

$$\prod_{j=0}^{n-1}\omega_j=\omega_0*\omega_2*\ldots*\omega_{n-1},\quad \coprod_{j=0}^{n-1}\omega_j=\omega_0\diamond\omega_2\diamond\ldots\diamond\omega_{n-1}.$$

Replacing x by $\frac{x}{a^m}$ in (3.7), we obtain

$$\mu(a^{4(n+m)}f(\frac{x}{a^{n+m}})-a^{4m}f(\frac{x}{a^m}),\sum_{j=0}^{n-1}\frac{|a|^{4(j+m)}}{|\alpha|^{j+m+1}}t)\geq\mu'(\varphi(x,0),t),$$
$$\nu(a^{4(n+m)}f(\frac{x}{a^{n+m}})-a^{4m}f(\frac{x}{a^m}),\sum_{j=0}^{n-1}\frac{|a|^{4(j+m)}}{|\alpha|^{j+m+1}}t)\leq\nu'(\varphi(x,0),t).$$

Hence

$$\mu(a^{4(n+m)}f(\frac{x}{a^{n+m}})-a^{4m}f(\frac{x}{a^m}),\sum_{j=m}^{n+m-1}\frac{|a|^{4j}}{|\alpha|^{j+1}}t)\geq\mu'(\varphi(x,0),t),$$
$$\nu(a^{4(n+m)}f(\frac{x}{a^{n+m}})-a^{4m}f(\frac{x}{a^m}),\sum_{j=m}^{n+m-1}\frac{|a|^{4j}}{|\alpha|^{j+1}}t)\leq\nu'(\varphi(x,0),t)$$

for all $x\in\mathcal{X}$, all $t>0$, $n\geq 0$ and $m\geq 0$. Thus

$$\begin{aligned}
\mu(a^{4(n+m)}f(\frac{x}{a^{n+m}})-a^{4m}f(\frac{x}{a^m}),t)&\geq\mu'(\varphi(x,0),\frac{t}{\sum_{j=m}^{n+m-1}\frac{|a|^{4j}}{|\alpha|^{j+1}}}),\\
\nu(a^{4(n+m)}f(\frac{x}{a^{n+m}})-a^{4m}f(\frac{x}{a^m}),t)&\leq\nu'(\varphi(x,0),\frac{t}{\sum_{j=m}^{n+m-1}\frac{|a|^{4j}}{|\alpha|^{j+1}}})
\end{aligned}\tag{3.8}$$

for all $x\in\mathcal{X}$, all $t>0$, $n\geq 0$ and $m\geq 0$. Since $|\alpha|>|a|^4$ and $\Sigma_{j=0}^{\infty}\frac{|a|^{4j}}{|\alpha|^j}<\infty$, the sequence $a^{4n}f(\frac{x}{a^n})$ is a Cauchy sequence in $(\mathcal{Y},\mu,\nu)$. Since $(\mathcal{Y},\mu,\nu)$ is complete, this sequence converges to some point $\mathcal{Q}(x)\in\mathcal{Y}$. Fix $x\in\mathcal{X}$ and put $m=0$ in (3.8). Then we obtain

$$\mu(a^{4n}f(\frac{x}{a^n})-f(x),t)\geq\mu'(\varphi(x,0),\frac{t}{\sum_{j=0}^{n-1}\frac{|a|^{4j}}{|\alpha|^{j+1}}}),$$
$$\nu(a^{4n}f(\frac{x}{a^n})-f(x),t)\leq\nu'(\varphi(x,0),\frac{t}{\sum_{j=0}^{n-1}\frac{|a|^{4j}}{|\alpha|^{j+1}}})$$

for all $x \in \mathcal{X}$, all $t > 0$ and $n \geq 0$. Thus we get

$$\begin{aligned}
\mu(\mathcal{Q}(x) - f(x), t) &\geq \mu(\mathcal{Q}(x) - a^{4n} f(\frac{x}{a^n}), \frac{t}{2}) * \mu(a^{4n} f(\frac{x}{a^n}) - f(x), \frac{t}{2}) \\
&\geq \mu'(\varphi(x, 0), \frac{t}{2\sum_{j=0}^{n-1} \frac{|a|^{4j}}{|\alpha|^{j+1}}}), \\
\nu(\mathcal{Q}(x) - f(x), t) &\leq \nu(\mathcal{Q}(x) - a^{4n} f(\frac{x}{a^n}), \frac{t}{2}) \diamond \nu(a^{4n} f(\frac{x}{a^n}) - f(x), \frac{t}{2}) \\
&\leq \nu'(\varphi(x, 0), \frac{t}{2\sum_{j=0}^{n-1} \frac{|a|^{4j}}{|\alpha|^{j+1}}})
\end{aligned}$$

for large n. By taking the limit as $n \to \infty$, we obtain

$$\begin{aligned}
\mu(\mathcal{Q}(x) - f(x), t) \geq \mu'(\varphi(x, 0), \frac{(|\alpha| - |a|^4)t}{2}), \\
\nu(\mathcal{Q}(x) - f(x), t) \leq \nu'(\varphi(x, 0), \frac{(|\alpha| - |a|^4)t}{2})
\end{aligned}$$

for all $x \in \mathcal{X}$ and all $t > 0$. Replacing x and y by $\frac{x}{a^n}$ and $\frac{y}{a^n}$ in (3.1), respectively, we obtain

$$\begin{aligned}
&\mu(a^{4n} f(\frac{ax+y}{a^n}) + f(x+ay) + \frac{1}{2}a(a-1)^2 a^{4n} f(\frac{x-y}{a^n}) - (a^2-1)^2 a^{4n}(f(\frac{x}{a^n}) + f(\frac{y}{a^n})) \\
&\qquad - \frac{1}{2}a(a+1)^2 a^{4n} f(\frac{x+y}{a^n}), t) \geq \mu'(\varphi(\frac{x}{a^n}, \frac{y}{a^n}), \frac{t}{|a|^{4n}}), \\
&\nu(a^{4n} f(\frac{ax+y}{a^n}) + f(x+ay) + \frac{1}{2}a(a-1)^2 a^{4n} f(\frac{x-y}{a^n}) - (a^2-1)^2 a^{4n}(f(\frac{x}{a^n}) + f(\frac{y}{a^n})) \\
&\qquad - \frac{1}{2}a(a+1)^2 a^{4n} f(\frac{x+y}{a^n}), t) \leq \nu'(\varphi(\frac{x}{a^n}, \frac{y}{a^n}), \frac{t}{|a|^{4n}})
\end{aligned}$$

for all $x \in \mathcal{X}$ and all $t > 0$. Note that

$$\begin{aligned}
\lim_{n\to\infty} \mu'(a^{4n}\varphi(\frac{x}{a^n}, \frac{y}{a^n}), t) = \lim_{n\to\infty} \mu'(\varphi(x, y), \frac{|\alpha|^n t}{|a|^{4n}}) = 1, \\
\lim_{n\to\infty} \nu'(a^{4n}\varphi(\frac{x}{a^n}, \frac{y}{a^n}), t) = \lim_{n\to\infty} \nu'(\varphi(x, y), \frac{|\alpha|^n t}{|a|^{4n}}) = 0
\end{aligned}$$

for all $x \in \mathcal{X}$ and all $t > 0$. We observe that $\mathcal{Q}$ fulfills (1.1). Therefore, $\mathcal{Q}$ is an Euler-Lagrange type quartic mapping.

It is left to show that the quartic mapping $\mathcal{Q}$ is unique. Assume that there is another Euler-Lagrange type quartic mapping $\mathcal{Q}' : \mathcal{X} \to \mathcal{Y}$ satisfying (3.2). For each $x \in \mathcal{X}$, clearly

$a^{4n}\mathcal{Q}(\frac{x}{a^n}) = \mathcal{Q}(x)$ and $a^{4n}\mathcal{Q}'(\frac{x}{a^n}) = \mathcal{Q}'(x)$ for all $n \in \mathbb{N}$. It follows from (3.3) that

$$\begin{aligned}\mu(\mathcal{Q}(x) - \mathcal{Q}'(x), t) &= \mu(a^{4n}\mathcal{Q}(\frac{x}{a^n}) - a^{4n}\mathcal{Q}'(\frac{x}{a^n}), t) \\ &= \mu(\mathcal{Q}(\frac{x}{a^n}) - \mathcal{Q}'(\frac{x}{a^n}), \frac{t}{|a|^{4n}}) \\ &= \mu(\mathcal{Q}(\frac{x}{a^n}) - f(\frac{x}{a^n}), \frac{t}{2|a|^{4n}}) \\ &* \mu(f(\frac{x}{a^n}) - \mathcal{Q}'(\frac{x}{a^n}) - \frac{t}{2|a|^{4n}}) \\ &\geq \mu'(\varphi(x,0), \frac{|\alpha|^n(|\alpha| - |a|^4)t}{2|a|^{4n}}),\end{aligned}$$

and similarly

$$\nu(\mathcal{Q}(x) - \mathcal{Q}'(x), t) \leq \nu'(\varphi(x,0), \frac{\alpha^n(|\alpha| - |k|^4)t}{2|a|^{4n}})$$

for all $x \in \mathcal{X}$ and all $t > 0$. Since $|\alpha| > |a|^4$, we get

$$\begin{aligned}\lim_{n\to\infty} \mu'(\varphi(x,0), \frac{|\alpha|^n(|\alpha| - |a|^4)t}{2|a|^{4n}}) = 1, \\ \lim_{n\to\infty} \nu'(\varphi(x,0), \frac{|\alpha|^n(|\alpha| - |a|^4)t}{2|a|^{4n}}) = 0.\end{aligned}$$

Therefore, $\mu(\mathcal{Q}(x) - \mathcal{Q}'(x), t) = 1$ and $\nu(\mathcal{Q}(x) - \mathcal{Q}'(x), t) = 0$ for all $x \in \mathcal{X}$ and all $t > 0$, that is, the mapping $\mathcal{Q}(x)$ is unique, as desired. □

Theorem 3.2. *Let $\mathcal{X}$ be a linear space and let $(\mathcal{Z}, \mu', \nu')$ be an IFNS. Let $\varphi : \mathcal{X} \times \mathcal{X} \to \mathcal{Z}$ be a mapping such that, for some $0 < |\alpha| < |a|^4$,*

$$\begin{aligned}\mu'(\varphi(ax,0), t) \geq \mu'(\alpha\varphi(x,0), t), \\ \nu'(\varphi(ax,0), t) \leq \nu'(\alpha\varphi(x,0), t).\end{aligned}$$

Let $(\mathcal{Y}, \mu, \nu)$ be an intuitionistic fuzzy Banach space and let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping satisfying $f(0) = 0$ and a φ-approximately quartic mapping in the sense that

$$\begin{aligned}\mu(f(ax+y) &+ f(x+ay) + \frac{1}{2}a(a-1)^2 f(x-y) - (a^2-1)^2(f(x)+f(y)) \\ &- \frac{1}{2}a(a+1)^2 f(x+y), t) \geq \mu'(\varphi(x,y), t), \\ \nu(f(ax+y) &+ f(x+ay) + \frac{1}{2}a(a-1)^2 f(x-y) - (a^2-1)^2(f(x)+f(y)) \\ &- \frac{1}{2}a(a+1)^2 f(x+y), t) \leq \nu'(\varphi(x,y), t)\end{aligned}$$

for all $x, y \in \mathcal{X}$ and all $t > 0$. Then there exists a unique quartic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ such that

$$\begin{aligned}\mu(\mathcal{Q}(x) - f(x), t) \geq \mu'(\varphi(x,0), \frac{|a|^4 - |\alpha|}{2}t), \\ \nu(\mathcal{Q}(x) - f(x), t)\nu\nu'(\varphi(x,0), \frac{|a|^4 - |\alpha|}{2}t)\end{aligned} \tag{3.9}$$

for all $x \in \mathcal{X}$ and all $t > 0$.

Proof. The techniques are similar to that of Theorem 3.1. Hence we present a sketch of the proof. Letting $y = 0$ in (3.9), we get

$$\begin{aligned} &\mu(\frac{f(ax)}{a^4} - f(x), t) \geq \mu'(\varphi(x,0), t), \\ &\nu(\frac{f(ax)}{a^4} - f(x), t) \leq \nu'(\varphi(x,0), t) \end{aligned} \tag{3.10}$$

for all $x \in \mathcal{X}$ and all $t > 0$. Replacing x by $a^n x$ in (3.10), we obtain

$$\begin{aligned} &\mu(\frac{f(a^{(n+1)}x)}{a^4} - f(a^n x), t) \geq \mu'(\varphi(x,0), \frac{t}{|\alpha|^n}), \\ &\nu(\frac{f(a^{(n+1)}x)}{a^4} - f(a^n x), t) \leq \nu'(\varphi(x,0), \frac{t}{|\alpha|^n}) \end{aligned}$$

for all $x \in \mathcal{X}$ and all $t > 0$. For positive integers n and m,

$$\begin{aligned} &\mu(\frac{f(a^{(n+m)}x)}{a^{4(n+m)}} - \frac{f(a^m x)}{a^{4m}}, t) \geq \mu'(\varphi(x,0), \frac{t}{\sum_{j=m}^{n+m-1} \frac{|\alpha|^j}{a^{4(j+1)}}}), \\ &\nu(\frac{f(a^{(n+m)}x)}{a^{4(n+m)}} - \frac{f(a^m x)}{a^{4m}}, t) \leq \nu'(\varphi(x,0) \frac{t}{\sum_{j=m}^{n+m-1} \frac{|\alpha|^j}{a^{4(j+1)}}}), \end{aligned} \tag{3.11}$$

for all $x \in \mathcal{X}$ and all $t > 0$. Hence $\{\frac{f(a^n x)}{a^{4n}}\}$ is a Cauchy sequence in intuitionistic fuzzy Banach space. Therefore, there is a mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ defined by $\mathcal{Q}(x) = \lim_{n\to\infty} \frac{f(a^n x)}{a^{4n}}$ and hence (3.11) with $m = 0$ implies

$$\begin{aligned} &\mu(\mathcal{Q}(x) - f(x), t) \geq \mu'(\varphi(x,0), \frac{|a|^4 - |\alpha|}{2} t), \\ &\nu(\mathcal{Q}(x) - f(x), t) \nu \nu'(\varphi(x,0), \frac{|a|^4 - |\alpha|}{2} t) \end{aligned}$$

for all $x \in \mathcal{X}$ and all $t > 0$. This complete the proof. □

4. **Intuitionistic fuzzy continuity**

In this section we establish intuitionistic fuzzy continuity by using continuous approximately quartic mappings.

Definition 4.1. *Let $f : \mathbb{R} \to \mathcal{X}$ be a mapping, where $\mathbb{R}$ is endowed with the Euclidean topology and $\mathcal{X}$ is an intuitionistic fuzzy normed space equipped with intuitionistic fuzzy norm (μ, ν). Then f is called intuitionistic fuzzy continuous at a point $t_0 \in \mathbb{R}$ if for all $\varepsilon > 0$ and all $0 < \alpha < 1$ there exists $\delta > 0$ such that for each t with $0 < |t - t_0| < \delta$*

$$\mu(f(tx) - f(t_0 x), \varepsilon) \geq \alpha,$$

$$\nu(f(tx) - f(t_0 x), \varepsilon) \leq 1 - \alpha.$$

Theorem 4.1. *Let $\mathcal{X}$ be a normed space and let $(\mathcal{Z}, \mu', \nu')$ be an IFNS. Let $(\mathcal{Y}, \mu, \nu)$ be an intuitionistic fuzzy Banach space and let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping satisfying $f(0) = 0$ and an*

(r,s)-approximately quartic mapping in the sense that for some r,s and some $z_0 \in \mathcal{Z}$

$$\begin{aligned}\mu(f(ax+y) &+ f(x+ay)+\frac{1}{2}a(a-1)^2f(x-y)-(a^2-1)^2(f(x)+f(y))\\ &- \frac{1}{2}a(a+1)^2f(x+y),t) \geq \mu'((\|x\|^r+\|y\|^s)z_0,t),\\ \nu(f(ax+y) &+ f(x+ay)+\frac{1}{2}a(a-1)^2f(x-y)-(a^2-1)^2(f(x)+f(y))\\ &- \frac{1}{2}a(a+1)^2f(x+y),t) \leq \nu'((\|x\|^r+\|y\|^s)z_0,t)\end{aligned}$$

for all $x,y \in \mathcal{X}$ and all $t>0$. If $r,s<4$, then there exists a unique quartic mapping $\mathcal{Q}:\mathcal{X}\to\mathcal{Y}$ such that

$$\begin{aligned}&\mu(\mathcal{Q}(x)-f(x),t) \geq \mu'(\|x\|^r z_0, \frac{(|a|^4-|a|^r)t}{2}),\\ &\nu(\mathcal{Q}(x)-f(x),t) \leq \nu'(\|x\|^r z_0, \frac{(|a|^4-|a|^r)t}{2})\end{aligned}\tag{4.1}$$

for all $x \in \mathcal{X}$ and all $t>0$. Furthermore, if for some $x \in \mathcal{X}$ and all $n \in \mathbb{N}$, the mapping $\mathcal{H}:\mathbb{R}\to\mathcal{Y}$ defined by $\mathcal{H}(t)=f(a^n tx)$ is intuitionistic fuzzy continuous. Then the mapping $t \to \mathcal{Q}(tx)$ from $\mathbb{R}$ to $\mathcal{Y}$ is intuitionistic fuzzy continuous.

Proof. Define $\varphi:\mathcal{X}\times\mathcal{X}\to\mathcal{Z}$ by $\varphi(x,y)=(\|x\|^r+\|y\|^s)z_0$. By Theorem 3.2, there exists a unique quartic mapping $\mathcal{Q}$ satisfying (4.1). We have

$$\begin{aligned}&\mu(\mathcal{Q}(x)-\frac{f(a^nx)}{a^{4n}},t)=\mu(\frac{\mathcal{Q}(a^nx)}{a^{4n}}-\frac{f(a^nx)}{a^{4n}},t)\\ &=\mu(\mathcal{Q}(a^nx)-f(a^nx),|a|^{4n}t) \geq \mu'(|a|^{4rn}\|x\|^r z_0, \frac{|a|^{4n}(|a|^4-|a|^r)t}{2})\\ &\geq \mu'(\|x\|^r z_0, \frac{|a|^{4n}(|a|^4-|a|^r)t}{2|a|^{rn}}),\\ &\nu(\mathcal{Q}(x)-\frac{f(a^nx)}{a^{4n}},t)=\nu(\frac{\mathcal{Q}(a^nx)}{a^{4n}}-\frac{f(a^nx)}{a^{4n}},t),\\ &=\nu(\mathcal{Q}(a^nx)-f(a^nx),|a|^{4n}t) \leq \nu'(|a|^{4rn}\|x\|^r z_0, \frac{|a|^{4n}(|a|^4-|a|^r)t}{2}),\\ &\leq \nu'(\|x\|^r z_0, \frac{|a|^{4n}(|a|^4-|a|^r)t}{2|a|^{rn}})\end{aligned}\tag{4.2}$$

for all $x \in \mathcal{X}$, all $t>0$ and $n \in \mathbb{N}$. Fix $x \in \mathcal{X}$ and $t_0 \in \mathbb{R}$. Given $\varepsilon>0$ and $0<\alpha<1$. By (4.2), we obtain

$$\begin{aligned}&\mu(\mathcal{Q}(tx)-\frac{f(a^ntx)}{a^{4n}},t) \geq \mu'(\|x\|^r z_0, \frac{|a|^{4n}(|a|^4-|a|^r)t}{2|t|^r|a|^{rn}})\\ &\geq \mu'(\|x\|^r z_0, \frac{|a|^{4n}(|a|^4-|a|^r)t}{2|1+t_0|^r|a|^{rn}}),\\ &\mu(\mathcal{Q}(tx)-\frac{f(a^ntx)}{a^{4n}},t) \geq \mu'(\|x\|^r z_0, \frac{|a|^{4n}(|a|^4-|a|^r)t}{2|t|^r|a|^{rn}})\\ &\geq \mu'(\|x\|^r z_0, \frac{|a|^{4n}(|a|^4-|a|^r)t}{2|1+t_0|^r|a|^{rn}})\end{aligned}\tag{4.3}$$

for all $t \in \mathbb{R}$ and all $|t - t_0| < 1$. Since $r < 3$, we have $\lim_{n\to\infty} \frac{|a|^{4n}(|a|^4-|a|^r)t}{2|1+t_0|^r|a|^{rn}} = \infty$, and hence there exists $n_0 \in \mathbb{N}$ such that

$$\mu(\mathcal{Q}(tx) - \frac{f(a^{n_0}tx)}{a^{4n_0}}, \frac{\varepsilon}{3}) \geq \alpha,$$
$$\nu(\mathcal{Q}(tx) - \frac{f(a^{n_0}tx)}{a^{4n_0}}, \frac{\varepsilon}{3}) \leq 1-\alpha$$

for all $t \in \mathbb{R}$ and all $|t - t_0| < 1$. By the intuitionistic fuzzy continuity of the mapping $s \to f(a^{n_0}tx)$, there exists $\delta < 1$ such that for each t with $0 < |t - t_0| < \delta$, we have

$$\mu(\frac{f(a^{n_0}tx)}{a^{4n_0}} - \frac{f(a^{n_0}t_0x)}{a^{4n_0}}, \frac{\varepsilon}{3}) \geq \alpha,$$
$$\nu(\frac{f(a^{n_0}tx)}{a^{4n_0}} - \frac{f(a^{n_0}t_0x)}{a^{4n_0}}, \frac{\varepsilon}{3}) \leq 1-\alpha.$$

It follows that

$$\begin{aligned}
&\mu(\mathcal{Q}(tx) - \mathcal{Q}(t_0x), \varepsilon) \geq \mu(\mathcal{Q}(tx) - \frac{f(a^{n_0}tx)}{a^{4n_0}}, \frac{\varepsilon}{3})* \\
&\mu(\frac{f(a^{n_0}tx)}{a^{4n_0}} - \frac{f(a^{n_0}t_0x)}{a^{4n_0}}, \frac{\varepsilon}{3}) * \mu(\frac{f(a^{n_0}t_0x)}{a^{4n_0}} - \mathcal{Q}(t_0x), \frac{\varepsilon}{3}) \geq \alpha \\
&\mu(\mathcal{Q}(tx) - \mathcal{Q}(t_0x), \varepsilon) \geq \mu(\mathcal{Q}(tx) - \frac{f(a^{n_0}tx)}{a^{4n_0}}, \frac{\varepsilon}{3})* \\
&\mu(\frac{f(a^{n_0}tx)}{a^{4n_0}} - \frac{f(a^{n_0}t_0x)}{a^{4n_0}}, \frac{\varepsilon}{3}) * \mu(\frac{f(a^{n_0}t_0x)}{a^{4n_0}} - \mathcal{Q}(t_0x), \frac{\varepsilon}{3}) \geq \alpha, \\
&\nu(\mathcal{Q}(tx) - \mathcal{Q}(t_0x), \varepsilon) \leq \nu(\mathcal{Q}(tx) - \frac{f(a^{n_0}tx)}{a^{4n_0}}, \frac{\varepsilon}{3})\diamond \\
&\nu(\frac{f(a^{n_0}tx)}{a^{4n_0}} - \frac{f(a^{n_0}t_0x)}{a^{4n_0}}, \frac{\varepsilon}{3}) \diamond \nu(\frac{f(a^{n_0}t_0x)}{a^{4n_0}} - \mathcal{Q}(t_0x), \frac{\varepsilon}{3}) \leq \alpha \\
&\nu(\mathcal{Q}(tx) - \mathcal{Q}(t_0x), \varepsilon) \leq \nu(\mathcal{Q}(tx) - \frac{f(a^{n_0}tx)}{a^{4n_0}}, \frac{\varepsilon}{3})\diamond \\
&\nu(\frac{f(a^{n_0}tx)}{a^{4n_0}} - \frac{f(a^{n_0}t_0x)}{a^{4n_0}}, \frac{\varepsilon}{3}) \diamond \nu(\frac{f(a^{n_0}t_0x)}{a^{4n_0}} - \mathcal{Q}(t_0x), \frac{\varepsilon}{3}) \leq 1-\alpha
\end{aligned}$$

for all $t \in \mathbb{R}$ and all $0 < |t - t_0| < \delta$. Therefore, the mapping $t \to \mathcal{Q}(tx)$ is intuitionistic fuzzy continuous. $\square$

Theorem 4.2. *Let $\mathcal{X}$ be a normed space and let $(\mathcal{Z}, \mu', \nu')$ be an IFNS. Let $(\mathcal{Y}, \mu, \nu)$ be an intuitionistic fuzzy Banach space and let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping satisfying $f(0) = 0$ and a (r, s)-approximately quartic mapping in the sense that for some r, s and some $z_0 \in \mathcal{Z}$*

$$\begin{aligned}
\mu(f(ax+y) &+ f(x+ay) + \frac{1}{2}a(a-1)^2 f(x-y) - (a^2-1)^2(f(x)+f(y)) \\
&- \frac{1}{2}a(a+1)^2 f(x+y), t) \geq \mu'((\|x\|^r + \|y\|^s)z_0, t), \\
\nu(f(ax+y) &+ f(x+ay) + \frac{1}{2}a(a-1)^2 f(x-y) - (a^2-1)^2(f(x)+f(y)) \\
&- \frac{1}{2}a(a+1)^2 f(x+y), t) \leq \nu'((\|x\|^r + \|y\|^s)z_0, t)
\end{aligned}$$

for all $x, y \in \mathcal{X}$ *and all* $t > 0$*. If* $r, s < 4$*, then there exists a unique quartic mapping* $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ *such that*

$$\mu(\mathcal{Q}(x) - f(x), t) \geq \mu'(\|x\|^r z_0, \frac{(|a|^r - |a|^4)t}{2}),$$
$$\nu(\mathcal{Q}(x) - f(x), t) \leq \nu'(\varphi(x, 0), \frac{(|a|^r - |a|^4)t}{2})$$

for all $x \in \mathcal{X}$ *and all* $t > 0$*. Furthermore, if for some* $x \in \mathcal{X}$ *and all* $n \in \mathbb{N}$*, the mapping* $\mathcal{H} : \mathbb{R} \to \mathcal{Y}$ *defined by* $\mathcal{H}(t) = f(a^n tx)$ *is intuitionistic fuzzy continuous. Then the mapping* $t \to \mathcal{Q}(tx)$ *from* $\mathbb{R}$ *to* $\mathcal{Y}$ *is intuitionistic fuzzy continuous.*

Proof. Define $\varphi : \mathcal{X} \times \mathcal{X} \to \mathcal{Z}$ by $\varphi(x, y) = (\|x\|^r + \|y\|^s)z_0$ for all $x \in \mathcal{X}$. Since $r > 4$, we have $\alpha = |a|^r > |a|^4$. By Theorem 3.1, there exists a unique quartic mapping $\mathcal{Q}$ satisfying (4.3).

The rest of the proof can be done by the same line as in Theorem 4.1. □

Acknowledgments

C. Park and D. Y. Shin were supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2012R1A1A2004299) and (NRF-2010-0021792), respectively.

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Volume 19, Number 4 October 2015
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

An international publication of Eudoxus Press, LLC
(twelve times annually)

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Journal of Computational Analysis and Applications

A quartely international publication of Eudoxus Press, LLC, of TN.

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Applications of sampling Kantorovich operators to thermographic images for seismic engineering

Federico Cluni, **Danilo Costarelli**,
Anna Maria Minotti and **Gianluca Vinti***

Abstract

In this paper, we present some applications of the multivariate sampling Kantorovich operators S_w to seismic engineering. The mathematical theory of these operators, both in the space of continuous functions and in Orlicz spaces, show how it is possible to approximate/reconstruct multivariate signals, such as images. In particular, to obtain applications for thermographic images a mathematical algorithm is developed using MATLAB and matrix calculus. The setting of Orlicz spaces is important since allow us to reconstruct not necessarily continuous signals by means of S_w. The reconstruction of thermographic images of buildings by our sampling Kantorovich algorithm allow us to obtain models for the simulation of the behavior of structures under seismic action. We analyze a real world case study in term of structural analysis and we compare the behavior of the building under seismic action using various models.

AMS 2010 Subject Classification: 41A35, 46E30, 47A58,47B38, 94A12

Key words and phrases: Sampling Kantorovich operators, Orlicz spaces, Image Processing, thermographic images, structural analysis.

1 Introduction

The sampling Kantorovich operators have been introduced to approximate and reconstruct not necessarily continuous signals. In [3], the authors introduced this operators starting from the well-known generalized sampling operators (see e.g. [10, 12, 9, 22, 4]) and replacing, in their definition, the sample values $f(k/w)$ with $w \int_{k/w}^{(k+1)/w} f(u)\, du$. Clearly, this is the most natural mathematical modification to obtain operators which can be well-defined also for general measurable, locally integrable functions, not necessarily continuous. Moreover, this situation very often occur in Signal Processing, when

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one cannot match exactly the sample at the point k/w: this represents the so-called "time-jitter" error. The theory of sampling Kantorovich operators allow us to reduces the time-jitter error, calculating the information in a neighborhood of k/w rather that exactly in the node k/w. These operators, as the generalized sampling operators, represent an approximate version of classical sampling series, based on the Whittaker-Kotelnikov-Shannon sampling theorem (see e.g. [1]).

Subsequently, the sampling Kantorovich operators have been studied in various settings. In [15, 16] the multivariate version of these operators were introduced. Results concerning the order of approximation are shown in [17]. Extensions to more general contexts are presented in [23, 24, 25, 6].

The multivariate sampling Kantorovich operators considered in this paper are of the form:

$$(S_w f)(\underline{x}) \ := \ \sum_{\underline{k}\in\mathbb{Z}^n} \chi(w\underline{x} - t_{\underline{k}}) \left[\frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} f(\underline{u})\ d\underline{u}\right], \qquad (\underline{x} \in \mathbb{R}^n), \tag{I}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a locally integrable function such that the above series is convergent for every $\underline{x} \in \mathbb{R}^n$. The symbol $t_{\underline{k}} = (t_{k_1}, ..., t_{k_n})$ denotes vectors where each $(t_{k_i})_{k_i\in\mathbb{Z}}$, $i = 1, ..., n$ is a certain strictly increasing sequence of real numbers with $\Delta_{k_i} = t_{k_{i+1}} - t_{k_i} > 0$. Note that, the sequences $(t_{k_i})_{k_i\in\mathbb{Z}}$ are not necessary equally spaced. We denote by $R^w_{\underline{k}}$ the sets:

$$R^w_{\underline{k}} \ := \ \left[\frac{t_{k_1}}{w}, \frac{t_{k_1+1}}{w}\right] \times \left[\frac{t_{k_2}}{w}, \frac{t_{k_2+1}}{w}\right] \times ... \times \left[\frac{t_{k_n}}{w}, \frac{t_{k_n+1}}{w}\right], \tag{II}$$

$w > 0$ and $A_{\underline{k}} = \Delta_{k_1} \cdot \Delta_{k_2} \cdot ... \cdot \Delta_{k_n}$, $\underline{k} \in \mathbb{Z}^n$. Moreover, the function $\chi : \mathbb{R}^n \to \mathbb{R}$ is a kernel satisfying suitable assumptions.

For the operators in (I) we recall some convergence results. We have that the family $(S_w f)_{w>0}$ converges pointwise to f, when f is continuous and bounded and $(S_w f)_{w>0}$ converges uniformly to f, when f is uniformly continuous and bounded. Moreover, to cover the case of not necessarily continuous signal, we study our operators in the general setting of Orlicz spaces $L^\varphi(\mathbb{R}^n)$. For functions f belonging to $L^\varphi(\mathbb{R}^n)$ and generated by the convex φ-function φ, the family of sampling Kantorovich operators is "modularly" convergent to f, being the latter the natural concept of convergence in this setting.

The latter result, allow us to apply the theory of the sampling Kantorovich operators to approximate and reconstruct images. In fact, static gray scale images are characterized by jumps of gray levels mainly concentrated in their contours or edges and this can be translated, from a mathematical point of view, by discontinuities (see e.g. [18]).

Here, we introduce and analyze in detail some practical applications of the sampling Kantorovich algorithm to thermographic images, very useful

for the analysis of buildings in seismic engineering. The thermography is a remote sensing technique, performed by the image acquisition in the infrared. Thermographic images are widely used to make non-invasive investigations of structures, to analyze the story of the building wall, to make diagnosis and monitoring buildings, and to make structural measurements. A further important use, is the application of the texture algorithm for the separation between the bricks and the mortar in masonries images. Through this procedure the civil engineers becomes able to determine the mechanical parameters of the structure under investigation. Unfortunately, the direct application of the texture algorithm to the thermographic images, can produce errors, as an incorrect separation between the bricks and the mortar.

Then, we use the sampling Kantorovich operators to process the thermographic images before to apply the texture algorithm. In this way, the result produced by the texture becomes more refined and therefore we can apply structural analysis after the calculation of the various parameters involved. In order to show the feasibility of our applications, we present in detail a real-world case-study.

2 Preliminaries

In this section we recall some preliminaries, notations and definitions.

We denote by $C(\mathbb{R}^n)$ (resp. $C^0(\mathbb{R}^n)$) the space of all uniformly continuous and bounded (resp. continuous and bounded) functions $f : \mathbb{R}^n \to \mathbb{R}$ endowed with the usual sup-norm $\|f\|_\infty := \sup_{\underline{u}\in\mathbb{R}^n} |f(\underline{u})|$, $\underline{u} = (u_1, ..., u_n)$, and by $C_c(\mathbb{R}^n) \subset C(\mathbb{R}^n)$ the subspace of the elements having compact support. Moreover, $M(\mathbb{R}^n)$ will denote the linear space of all (Lebesgue) measurable real functions defined on $\mathbb{R}^n$.

We now recall some basic fact concerning Orlicz spaces, see e.g. [20, 19, 21, 8, 7].
The function $\varphi : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ is said to be a φ-function if it satisfies the following assumptions: (i) $\varphi(0) = 0$, and $\varphi(u) > 0$ for every $u > 0$; (ii) φ is continuous and non decreasing on $\mathbb{R}^+_0$; (iii) $\lim_{u\to\infty} \varphi(u) = +\infty$.

The functional $I^\varphi : M(\mathbb{R}^n) \to [0, +\infty]$ (see e.g. [19, 7]) defined by

$$I^\varphi[f] := \int_{\mathbb{R}^n} \varphi(|f(\underline{x})|)\, d\underline{x}, \qquad (f \in M(\mathbb{R}^n)),$$

is a modular in $M(\mathbb{R}^n)$. The Orlicz space generated by φ is given by

$$L^\varphi(\mathbb{R}^n) := \{f \in M(\mathbb{R}^n) : \ I^\varphi[\lambda f] < +\infty, \text{ for some } \lambda > 0\}.$$

The space $L^\varphi(\mathbb{R}^n)$ is a vector space and an important subspace is given by

$$E^\varphi(\mathbb{R}^n) := \{f \in M(\mathbb{R}^n) : \ I^\varphi[\lambda f] < +\infty, \text{ for every } \lambda > 0\}.$$

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$E^\varphi(\mathbb{R}^n)$ is called the space of all finite elements of $L^\varphi(\mathbb{R}^n)$. It is easy to see that the following inclusions hold: $C_c(\mathbb{R}^n) \subset E^\varphi(\mathbb{R}^n) \subset L^\varphi(\mathbb{R}^n)$. Clearly, functions belonging to $E^\varphi(\mathbb{R}^n)$ and $L^\varphi(\mathbb{R}^n)$ are not necessarily continuous. A norm on $L^\varphi(\mathbb{R}^n)$, called Luxemburg norm, can be defined by

$$\|f\|_\varphi := \inf\{\lambda > 0 : I^\varphi[f/\lambda] \le \lambda\}, \qquad (f \in L^\varphi(\mathbb{R}^n)).$$

We will say that a family of functions $(f_w)_{w>0} \subset L^\varphi(\mathbb{R}^n)$ is norm convergent to a function $f \in L^\varphi(\mathbb{R}^n)$, i.e., $\|f_w - f\|_\varphi \to 0$ for $w \to +\infty$, if and only if $\lim_{w\to+\infty} I^\varphi[\lambda(f_w - f)] = 0$, for every $\lambda > 0$. Moreover, an additional concept of convergence can be studied in Orlicz spaces: the "modular convergence". The latter induces a topology (modular topology) on the space $L^\varphi(\mathbb{R}^n)$ ([19, 7]).

We will say that a family of functions $(f_w)_{w>0} \subset L^\varphi(\mathbb{R}^n)$ is modularly convergent to a function $f \in L^\varphi(\mathbb{R}^n)$ if $\lim_{w\to+\infty} I^\varphi[\lambda(f_w - f)] = 0$, for some $\lambda > 0$. Obviously, norm convergence implies modular convergence, while the converse implication does not hold in general. The modular and norm convergence are equivalent if and only if the φ-function φ satisfies the Δ_2-condition, see e.g., [19, 7]. Finally, as last basic property of Orlicz spaces, we recall the following.

Lemma 2.1 ([5]). *The space $C_c(\mathbb{R}^n)$ is dense in $L^\varphi(\mathbb{R}^n)$ with respect to the modular topology, i.e., for every $f \in L^\varphi(\mathbb{R}^n)$ and for every $\varepsilon > 0$ there exists a constant $\lambda > 0$ and a function $g \in C_c(\mathbb{R}^n)$ such that $I^\varphi[\lambda(f-g)] < \varepsilon$.*

3 The sampling Kantorovich operators

In this section we recall the definition of the operators with which we will work. We will denote by $t_{\underline{k}} = (t_{k_1}, ..., t_{k_n})$ a vector where each element $(t_{k_i})_{k_i\in\mathbb{Z}}$, $i = 1, ..., n$ is a sequence of real numbers with $-\infty < t_{k_i} < t_{k_{i+1}} < +\infty$, $\lim_{k_i\to\pm\infty} t_{k_i} = \pm\infty$, for every $i = 1, ..., n$, and such that there exists Δ, $\delta > 0$ for which $\delta \le \Delta_{k_i} := t_{k_{i+1}} - t_{k_i} \le \Delta$, for every $i = 1, ..., n$. Note that, the elements of $(t_{k_i})_{k_i\in\mathbb{Z}}$ are not necessary equally spaced. In what follows, we will identify with the symbol Π^n the sequence $(t_{\underline{k}})_{\underline{k}\in\mathbb{Z}^n}$.

A function $\chi : \mathbb{R}^n \to \mathbb{R}$ will be called a kernel if it satisfies the following properties:

(χ1) $\chi \in L^1(\mathbb{R}^n)$ and is bounded in a neighborhood of $\underline{0} \in \mathbb{R}^n$;

(χ2) for every $\underline{u} \in \mathbb{R}^n$, $\displaystyle\sum_{\underline{k}\in\mathbb{Z}^n} \chi(\underline{u} - t_{\underline{k}}) = 1$;

(χ3) for some $\beta > 0$,

$$m_{\beta,\Pi^n}(\chi) = \sup_{\underline{u}\in\mathbb{R}^n} \sum_{\underline{k}\in\mathbb{Z}^n} \left|\chi(\underline{u} - t_{\underline{k}})\right| \cdot \left\|\underline{u} - t_{\underline{k}}\right\|_2^\beta < +\infty,$$

where $\|\cdot\|_2$ denotes the usual Euclidean norm.

4

We now recall the definition of the linear multivariate sampling Kantorovich operators introduced in [15]. Define:

$$(S_w f)(\underline{x}) \;:=\; \sum_{\underline{k}\in\mathbb{Z}^n} \chi(w\underline{x}-t_{\underline{k}}) \left[\frac{w^n}{A_{\underline{k}}}\int_{R^w_{\underline{k}}} f(\underline{u})\, d\underline{u}\right], \qquad (\underline{x}\in\mathbb{R}^n), \tag{1}$$

where $f:\mathbb{R}^n\to\mathbb{R}$ is a locally integrable function such that the above series is convergent for every $\underline{x}\in\mathbb{R}^n$, where

$$R^w_{\underline{k}} \;:=\; \left[\frac{t_{k_1}}{w},\frac{t_{k_1+1}}{w}\right]\times\left[\frac{t_{k_2}}{w},\frac{t_{k_2+1}}{w}\right]\times\ldots\times\left[\frac{t_{k_n}}{w},\frac{t_{k_n+1}}{w}\right],$$

$w>0$ and $A_{\underline{k}}=\Delta_{k_1}\cdot\Delta_{k_2}\cdot\ldots\cdot\Delta_{k_n}$, $\underline{k}\in\mathbb{Z}^n$. The operators in (1) have been introduced in [3] in the univariate setting.

Remark 3.1. (a) Under conditions $(\chi1)$ and $(\chi3)$, the following properties for the kernel χ can be proved:

$$m_{0,\Pi^n}(\chi) \;:=\; \sup_{\underline{u}\in\mathbb{R}^n}\sum_{\underline{k}\in\mathbb{Z}^n}\left|\chi(\underline{u}-t_{\underline{k}})\right| \;<\; +\infty,$$

and, for every $\gamma>0$

$$\lim_{w\to+\infty}\sum_{\left\|w\underline{u}-t_{\underline{k}}\right\|_2>\gamma w}\left|\chi(w\underline{u}-t_{\underline{k}})\right| \;=\; 0, \tag{2}$$

uniformly with respect to $\underline{u}\in\mathbb{R}^n$, see [15].

(b) By (a), we obtain that $S_w f$ with $f\in L^\infty(\mathbb{R}^n)$ are well-defined. Indeed,

$$|(S_w f)(\underline{x})| \;\le\; m_{0,\Pi^n}(\chi)\,\|f\|_\infty \;<+\infty,$$

for every $\underline{x}\in\mathbb{R}^n$, i.e. $S_w: L^\infty(\mathbb{R}^n)\to L^\infty(\mathbb{R}^n)$.

Remark 3.2. Note that, in the one-dimensional setting, choosing $t_k=k$, for every $k\in\mathbb{Z}$, condition $(\chi2)$ is equivalent to

$$\widehat{\chi}(k) := \begin{cases} 0, & k\in\mathbb{Z}\setminus\{0\}, \\ 1, & k=0, \end{cases}$$

where $\widehat{\chi}(v) := \int_{\mathbb{R}}\chi(u)e^{-ivu}\,du$, $v\in\mathbb{R}$, denotes the Fourier transform of χ; see [11, 3, 15, 17].

4 Convergence results

In this section, we show the main approximation results for the multivariate sampling Kantorovich operators. In [15], the following approximation theorem for our operators has been proved.

Theorem 4.1. *Let $f \in C^0(\mathbb{R}^n)$. Then, for every $\underline{x} \in \mathbb{R}^n$,*

$$\lim_{w \to +\infty} (S_w f)(\underline{x}) \;=\; f(\underline{x}).$$

In particular, if $f \in C(\mathbb{R})$, then

$$\lim_{w \to +\infty} \|S_w f - f\|_\infty \;=\; 0.$$

Proof. Here we highlight the main points of the proof.
Let $f \in C^0(\mathbb{R}^n)$ and $\underline{x} \in \mathbb{R}^n$ be fixed. By the continuity of f we have that for every fixed $\varepsilon > 0$ there exists $\gamma > 0$ such that $|f(\underline{x}) - f(\underline{u})| < \varepsilon$ for every $\|\underline{x} - \underline{u}\|_2 \leq \gamma$, $\underline{u} \in \mathbb{R}^n$. Then, by $(\chi 2)$ we obtain:

$$\begin{aligned} |(S_w f)(\underline{x}) - f(\underline{x})| \;&\leq\; \sum_{\underline{k} \in \mathbb{Z}^n} \left|\chi(w\underline{x} - t_{\underline{k}})\right| \frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} |f(\underline{u}) - f(\underline{x})| \, d\underline{u} \\ = \;\Big(\sum_{\|w\underline{x} - t_{\underline{k}}\|_2 \leq w\gamma/2} &+ \sum_{\|w\underline{x} - t_{\underline{k}}\|_2 > w\gamma/2} \Big) \left|\chi(w\underline{x} - t_{\underline{k}})\right| \frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} |f(\underline{u}) - f(\underline{x})| \; d\underline{u} \\ = \;\; I_1 + I_2. & \end{aligned}$$

For $\underline{u} \in R^w_{\underline{k}}$ and $\|w\underline{x} - t_{\underline{k}}\|_2 \leq w\gamma/2$ we have $\|\underline{u} - \underline{x}\|_2 \leq \gamma$ for $w > 0$ sufficiently large, then by the continuity of f we obtain $I_1 \leq m_{0,\Pi^n}(\chi)\varepsilon$ (see Remark 3.1 (a)). Moreover, by the boundedness of f and (2) we obtain $I_2 \leq 2\|f\|_\infty \varepsilon$ for $w > 0$ sufficiently large, then the first part of the theorem follows since $\varepsilon > 0$ is arbitrary. The second part of the theorem follows similarly replacing $\gamma > 0$ with the parameter of the uniform continuity of f. □

In order to obtain a modular convergence result, the following norm-convergence theorem for the sampling Kantorovich operators (see [15]) can be formulated.

Theorem 4.2. *Let φ be a convex φ-function. For every $f \in C_c(\mathbb{R}^n)$ we have*

$$\lim_{w \to +\infty} \|S_w f - f\|_\varphi \;=\; 0.$$

Now, we recall the following modular continuity property for S_w, useful to prove the modular convergence for the above operators in Orlicz spaces.

Theorem 4.3. *Let φ be a convex φ-function. For every $f \in L^\varphi(\mathbb{R}^n)$ there holds*

$$I^\varphi[\lambda S_w f] \;\leq\; \frac{\|\chi\|_1}{\delta^n \cdot m_{0,\Pi^n}(\chi)} I^\varphi[\lambda m_{0,\Pi^n}(\chi) f],$$

for some $\lambda > 0$. In particular, S_w maps $L^\varphi(\mathbb{R}^n)$ in $L^\varphi(\mathbb{R}^n)$.

Now, the main result of this section follows (see [15]).

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Theorem 4.4. *Let φ be a convex φ-function. For every $f \in L^\varphi(\mathbb{R}^n)$, there exists $\lambda > 0$ such that*

$$\lim_{w\to+\infty} I^\varphi[\lambda(S_w f - f)] \;=\; 0.$$

Proof. Let $f \in L^\varphi(\mathbb{R}^n)$ and $\varepsilon > 0$ be fixed. By Lemma 2.1, there exists $\overline{\lambda} > 0$ and $g \in C_c(\mathbb{R}^n)$ such that $I^\varphi[\overline{\lambda}(f-g)] < \varepsilon$. Let now $\lambda > 0$ such that $3\lambda(1 + m_{0,\Pi^n}(\chi)) \leq \overline{\lambda}$. By the properties of φ and Theorem 4.3, we have

$$\begin{aligned}
I^\varphi[\lambda(S_w f - f)] \;&\leq\; I^\varphi[3\lambda(S_w f - S_w g)] + I^\varphi[3\lambda(S_w g - g)] + I^\varphi[3\lambda(f-g)] \\
\leq\; &\frac{1}{m_{0,\Pi^n}(\chi)\cdot\delta^n}\,\|\chi\|_1\, I^\varphi[\overline{\lambda}(f-g)] + I^\varphi[3\lambda(S_w g - g)] + I^\varphi[\overline{\lambda}(f-g)] \\
<\; &\left(\frac{1}{m_{0,\Pi^n}(\chi)\cdot\delta^n}\,\|\chi\|_1 + 1\right)\varepsilon + I^\varphi[3\lambda(S_w g - g)].
\end{aligned}$$

The assertion follows from Theorem 4.2. □

The setting of Orlicz spaces allows us to give a unitary approach for the reconstruction since we may obtain convergence results for particular cases of Orlicz spaces. For instance, choosing $\varphi(u) = u^p$, $1 \leq p < +\infty$, we have that $L^\varphi(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $I^\varphi[f] = \|f\|_p^p$, where $\|\cdot\|_p$ is the usual L^p-norm. Then, from Theorem 4.3 and Theorem 4.4 we obtain the following corollary.

Corollary 4.5. *For every $f \in L^p(\mathbb{R}^n)$, $1 \leq p < +\infty$, the following inequality holds:*

$$\|S_w f\|_p \;\leq\; \delta^{-n/p}\,[m_{0,\Pi^n}(\chi)]^{(p-1)/p}\;\|\chi\|_1^{1/p}\;\|f\|_p\,.$$

Moreover, we have:

$$\lim_{w\to+\infty} \|S_w f - f\|_p \;=\; 0.$$

The corollary above, allows us to reconstruct L^p-signals (in L^p-sense), therefore signals/images not necessarily continuous. Other examples of Orlicz spaces for which the above theory can be applied can be found e.g., in [20, 19, 7, 3, 15]. The theory of sampling Kantorovich operators in the general setting of Orlicz spaces allows us to obtain, by means of a unified treatment, several applications in many different contexts.

5 Examples of special kernels

One important fact in our theory is the choice of the kernels, which influence the order of approximation that can be achieved by our operators (see e.g. [17] in one-dimensional setting).

For instance, one can take into consideration *radial kernels*, i.e., functions for which the value depends on the Euclidean norm of the argument only. Example of such a kernel can be given, for example, by the Bochner-Riesz

kernel, defined as follows $b^{\alpha}(\underline{x}) := 2^{\alpha}\Gamma(\alpha+1)\|\underline{x}\|_2^{-(n/2)+\alpha}\mathcal{B}_{(n/2)+\alpha}(\|\underline{x}\|_2)$, for $\underline{x} \in \mathbb{R}^n$, where $\alpha > (n-1)/2$, $\mathcal{B}_\lambda$ is the Bessel function of order λ and Γ is the Euler function. For more details about this matter, see e.g. [10].

To construct, in general, kernels satisfying all the assumptions (χ_i), $i = 1, 2, 3$ is not very easy.

For this reason, here we show a procedure useful to construct examples using product of univariate kernels, see e.g. [10, 15, 16]. In this case, we consider the case of uniform sampling scheme, i.e., $t_{\underline{k}} = \underline{k}$.

Denote by $\chi_1, ..., \chi_n$, the univariate functions $\chi_i : \mathbb{R} \to \mathbb{R}$, $\chi_i \in L^1(\mathbb{R})$, satisfying the following assumptions:

$$m_{\beta,\Pi^1}(\chi_i) \ := \ \sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi_i(u-k)|\,|u-k|^{\beta} \ < \ +\infty, \tag{3}$$

for some $\beta > 0$, χ_i is bounded in a neighborhood of the origin and

$$\sum_{k\in\mathbb{Z}} \chi_i(u-k) = 1, \tag{4}$$

for every $u \in \mathbb{R}$, for $i = 1, ..., n$. Now, setting $\chi(\underline{x}) := \prod_{i=1}^n \chi_i(x_i)$, $\underline{x} = (x_1, ..., x_n) \in \mathbb{R}^n$, it is easy to prove that χ is a multivariate kernel for the operators S_w satisfying all the assumptions of our theory, see e.g., [10, 15].

As a first example, consider the univariate Fejér's kernel defined by $F(x) \ := \ \frac{1}{2}\text{sinc}^2\left(\frac{x}{2}\right)$, $x \in \mathbb{R}$, where the sinc function is given by

$$\text{sinc}(x) \ := \ \begin{cases} \dfrac{\sin \pi x}{\pi x}, & x \in \mathbb{R} \setminus \{0\}, \\ 1, & x = 0. \end{cases}$$

Clearly, F is bounded, belongs to $L^1(\mathbb{R})$ and satisfies the moment conditions (3) for $\beta = 1$, as shown in [11, 3, 15]. Furthermore, taking into account that the Fourier transform of F is given by (see [11])

$$\widehat{F}(v) := \begin{cases} 1 - |v/\pi|, & |v| \leq \pi, \\ 0, & |v| > \pi, \end{cases}$$

we obtain by Remark 3.2 that condition (4) is fulfilled. Then, we can define $\mathcal{F}_n(\underline{x}) = \prod_{i=1}^n F(x_i)$, $\underline{x} = (x_1, ..., x_n) \in \mathbb{R}^n$, the multivariate Fejér's kernel, satisfying the condition upon a multivariate kernel. The Fejér's kernel F and the bivariate Fejér's kernel $F(x) \cdot F(y)$ are plotted in Figure 1.

The Fejér's kernel $\mathcal{F}_n$ is an example of kernel with unbounded support, then to evaluate our sampling Kantorovich series at any given $\underline{x} \in \mathbb{R}^n$, we need of an infinite number of mean values $w^n \int_{R^w_{\underline{k}}} f(\underline{u})\ d\underline{u}$. However, if the function f has compact support, this problem does not arise. In case of function having unbounded support the infinite sampling series must be

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Figure 1: The Fejér's kernel F (left) and the bivariate Fejér's kernel $\mathcal{F}(x, y)$ (right).

Figure 2: The B-spline M_3 (left) and the bivariate B-spline $\mathcal{M}_3^2(x, y)$ (right).

truncated to a finite one, which leads to the so-called truncation error. In order to avoid the truncation error, one can take kernels χ with bounded support. Remarkable examples of kernels with compact support, can be constructed using the well-known central B-spline of order $k \in \mathbb{N}$, defined by

$$M_k(x) := \frac{1}{(k-1)!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} \left(\frac{k}{2} + x - i \right)_+^{k-1},$$

where the function $(x)_+ := \max\{x, 0\}$ denotes the positive part of $x \in \mathbb{R}$ (see [3, 23, 15]). The central B-spline M_3 and the bivariate B-spline kernel $M_3(x) \cdot M_3(y)$ are plotted in Figure 2. We have that, the Fourier transform of M_k is given by $\widehat{M_k}(v) := \operatorname{sinc}^n\left(\frac{v}{2\pi}\right)$, $v \in \mathbb{R}$, and then, if we consider the case of the uniform spaced sampling scheme, condition (4) is satisfied by Remark 3.2. Clearly, M_k are bounded on $\mathbb{R}$, with compact support $[-n/2, n/2]$, and hence $M_k \in L^1(\mathbb{R})$, for all $k \in \mathbb{N}$. Moreover, it easy to deduce that condition (3) is fulfilled for every $\beta > 0$, see [3]. Hence $\mathcal{M}_k^n(\underline{x}) := \prod_{i=1}^n M_k(x_i)$, $\underline{x} = (x_1, ..., x_n) \in \mathbb{R}^n$, is the multivariate B-spline kernel of order $k \in \mathbb{N}$.

Finally, other important examples of univariate kernels are given by the Jackson-type kernels, defined by $J_k(x) = c_k \operatorname{sinc}^{2k}\left(\frac{x}{2k\pi\alpha}\right)$, $x \in \mathbb{R}$, with $k \in \mathbb{N}$, $\alpha \geq 1$, where the normalization coefficients c_k are given by $c_k := \left[\int_{\mathbb{R}} \operatorname{sinc}^{2k}\left(\frac{u}{2k\pi\alpha}\right) du\right]^{-1}$. Since J_k are band-limited to $[-1/\alpha, 1/\alpha]$, i.e., their Fourier transform vanishes outside this interval, condition $(\chi 2)$ is satisfied by Remark 3.2 and (3) is satisfied since $J_k(x) = \mathcal{O}(|x|^{-2k})$, as $x \to \pm\infty$, $k \in \mathbb{N}$. In similar manner, by the previous procedure we can construct a multivariate version of Jackson kernels. For more details about Jackson-type kernels, and for others useful examples of kernels see e.g. [11, 7, 2, 3].

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6 The sampling Kantorovich algorithm for image reconstruction

In this section, we show applications of the multivariate sampling Kantorovich operators to image reconstruction. First of all, we recall that every bi-dimensional gray scale image A is represented by a suitable matrix and can be modeled as a step function I, with compact support, belonging to $L^p(\mathbb{R}^2)$, $1 \leq p < +\infty$. The definition of I arise naturally as follows:

$$I(x, y) \; := \; \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij} \cdot \mathbf{1}_{ij}(x, y), \quad (x, y) \in \mathbb{R}^2,$$

where $\mathbf{1}_{ij}(x, y)$, $i, j = 1, 2, ..., m$, are the characteristics functions of the sets $(i-1,\ i] \times (j-1,\ j]$ (i.e. $\mathbf{1}_{ij}(x, y) = 1$, for $(x, y) \in (i-1,\ i] \times (j-1,\ j]$ and $\mathbf{1}_{ij}(x, y) = 0$ otherwise).

The above function $I(x, y)$ is defined in such a way that, to every pixel (i, j) it is associated the corresponding gray level a_{ij}.

We can now consider approximation of the original image by the bivariate sampling Kantorovich operators $(S_w I)_{w>0}$ based upon some kernel χ.

Then, in order to obtain a new image (matrix) that approximates in L^p-sense the original one, it is sufficient to sample $S_w I$ (for some $w > 0$) with a fixed sampling rate. In particular, we can reconstruct the approximating images (matrices) taking into consideration different sampling rates and this is possible since we know the analytic expression of $S_w I$.

If the sampling rate is chosen higher than the original sampling rate, one can get a new image that has a better resolution than the original one's. The above procedure has been implemented by using MATLAB and tools of matrix computation in order to obtain an optimized algorithm based on the multivariate sampling Kantorovich theory.

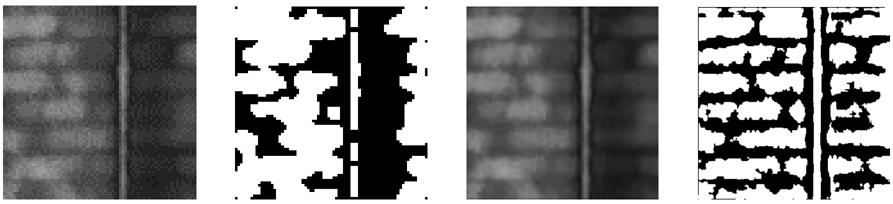
In the next sections, examples of thermographic images reconstructed by the sampling Kantorovich operators will be given to show the main applications of the theory to civil engineering. In particular, a real world case-study is analyzed in term of modal analysis to obtain a model useful to study the response of the building under seismic action.

7 An application of thermography to civil engineering

In the present application, thermographic images will be used to locate the resisting elements and to define their dimensions, and moreover to investigate the actual texture of the masonry wall, i.e., the arrangement of blocks (bricks and/or stones) and mortar joints. In general, thermographic images have

a resolution too low to accurately analyze the texture of the masonries, therefore it has to be increased by means of suitable tools.

In order to obtain a consistent separation of the phases, that is a correct identification of the pixel which belong to the blocks and those who belong to mortar joints, the image is converted from gray-scale representation to black-and-white (binary) representation by means of an image texture algorithm, which employs techniques belonging to the field of digital image processing. The image texture algorithm, described in details in [13], leads to areas of white pixels identified as blocks and areas of black pixels identified as mortar joints. However, the direct application of the image texture algorithm to the thermographic images (see Figure 3(a)), can produce errors, as an incorrect separation between the bricks and the mortar (see Figure 3(b)). Therefore, we can use the sampling Kantorovich operators to process the thermographic images. In particular, here we used the operators S_w based upon the bivariate Jackson-type kernel with $k = 12$ (see Section 5) and the parameter $w = 40$ (see Figure 3(c)). The application of the image texture algorithm produces a consistent separation of the phases (see Figure 3(d)).



(a) (b) (c) (d)

Figure 3: (a) Original thermographic image (75×75 pixel resolution) and (b) its texture; (c) Reconstructed thermographic image (450×450 pixel resolution) and (d) its texture.

In order to perform structural analysis, the mechanical characteristics of an homogeneous material equivalent to the original heterogeneous material are sought. The equivalence is in the sense that, when subjected to the same boundary conditions (b.c.), the overall responses in terms of mean values of stresses and deformations are the same. The equivalent elastic properties are estimated by means of the "test-window" method [14].

8 Case-study

The image texture algorithm described previously has been used to analyze a real-world case-study: a building consisting of two levels and an attic, with a very simple architectural plan, a rectangle with sides 11 m and 11.4 m (see Figure 4). The vertical structural elements consist of masonry walls. At first level the masonry walls have thickness of 40 cm with blocks made of stones,

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while at second level the masonry is made of squared tuff block and it has thickness of about 35 cm. Both surfaces of the walls are plastered. The slab can be assumed to be rigid in the horizontal plane and the building is placed in a medium risk seismic area. According to a preliminary visual survey, the structure does not have strong asymmetries between principal directions and the distribution of the masses are quite uniform.

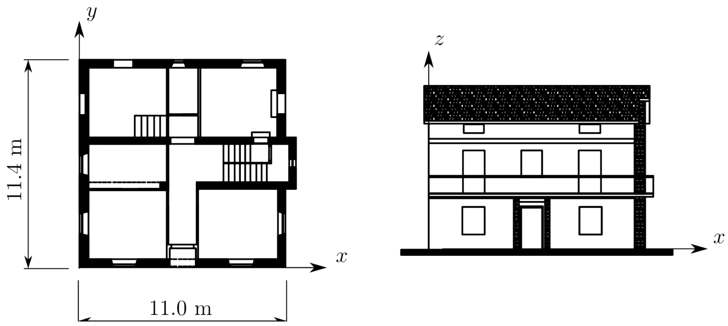

Figure 4: Case-study building: plant (left) and main facade (right).

The response of the building is evaluated by means of three different models. In the first model, only information that can be gathered by the visual survey have been used, and the characteristics of material has been assume according to Italian Building Code. In the second model information concerning the actual geometry of vertical structural elements acquired by means of thermographic survey have been used; for example, the survey showed that the openings at ground level in the main facade were once larger and have been subsequently partially filled with non-structural masonry, and thus the dimensions of all the masonry walls in the facade have to be reduced. Eventually, in the third model information about the actual texture of masonries of ground level (chaotic masonry, Fig. 5(a)) and of second level (periodic masonry, Fig. 5(b)) have been used. In particular, the actual textures where established using the reconstructed thermographic images.

(a) (b)

Figure 5: Texture of (a) ground level and (b) second level masonries.

The main characteristics of the models are reported in Table 1. For the mechanical characteristic the Young's modulus E and the shear modulus

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G are shown. In the third model, the following mechanical characteristics (Young's modulus E and Poisson's ratio ν) of the constituent phases have been used: for the stones of the ground level, $E = 25000\,\mathrm{N\,mm^{-2}}, \nu = 0.2$, for the bricks of the second level $E = 1700\,\mathrm{N\,mm^{-2}}, \nu = 0.2$, for the mortar of both levels $E = 2500\,\mathrm{N\,mm^{-2}}, \nu = 0.2$.

	geometry	ground level E $\mathrm{N\,mm^{-2}}$	ground level G $\mathrm{N\,mm^{-2}}$	second level E $\mathrm{N\,mm^{-2}}$	second level G $\mathrm{N\,mm^{-2}}$
Model #1	visual survey	3346	1115	1620	540
Model #2	thermogr. survey	3346	1115	1620	540
Model #3	thermogr. survey	7050	2957	1996	833

Table 1: Geometry and masonries' mechanical characteristics in the models.

The behavior under seismic actions is estimated by means of modal analysis, using a commercial code based on the Finite Element Method. The periods of the first three modes and the corresponding mass participating ratio for the two principal direction of seismic action are reported in Tab. 2. For all the models, the first mode is along the y axis, the second is along the x axis, and the third is mainly torsional. Anyway, the second mode is not a pure translational one since the asymmetry in the walls distribution produces also a torsional component.

	Periods			Mass partecipating ratio					
				x direction			y direction		
	1st mode	2nd mode	3rd mode	1st mode	2nd mode	3rd mode	1st mode	2nd mode	3rd mode
Model #1	0.22	0.15	0.13	0.00	0.64	0.13	0.73	0.00	0.00
Model #2	0.22	0.14	0.13	0.00	0.51	0.27	0.73	0.01	0.00
Model #3	0.18	0.13	0.11	0.00	0.46	0.26	0.68	0.01	0.00

Table 2: Periods and mass participating ratio for two directions of seismic action of the first three modes .

As can be noted, the first two models have the same period for the fundamental mode in y direction; nevertheless, in x direction there is a slight difference due to the reduced dimensions of masonry walls width discovered by means of thermographic survey. In fact, the reduction of dimensions leads to a decrease of global stiffness while the total mass is almost the same. The periods of third model show a reduction of about 20% due to the greater value of equivalent elastic moduli estimated by means of homogenized texture. As already noted, for seismic action in x direction the structure response is dominated by second mode, with also a significant contribution from the third mode (which is torsional); for seismic action in y the response is dominated by first mode. It is also worth noting that Model #3 shows a reduced mass participating ratio for the fundamental mode in each direction, and therefore

a greater number of modes should be considered in the evaluation of seismic response in order to achieve a suitable accuracy.

9 Concluding remarks

The sampling Kantorovich operators and the corresponding MATLAB algorithm for image reconstruction are very useful to enhance the quality of thermographic images of portions of masonry walls. In particular, after the processing by sampling Kantorovich algorithm, the thermographic image has higher definition with respect to the original one and therefore, it was possible to estimate the mechanical characteristics of homogeneous materials equivalent to actual masonries, taking into account the texture (i.e., the arrangement of blocks and mortar joints). These materials were used to model the behavior of a case study under seismic action. This model has been compared with others constructed by well-know methods, using "naked eyes" survey and the mechanical parameters for materials taken from the Italian Building Code. Our method based on the processing of thermographic images by sampling Kantorovich operators enhances the quality of the model with respect to that based on visual survey only.
In particular the proposed approach allows to overcome some difficulties that arise when dealing with the vulnerability analysis of existing structures, which are: i) the knowledge of the actual geometry of the walls (in particular the identification of hidden doors and windows); ii) the identification of the actual texture of the masonry and the distribution of inclusions and mortar joints, and from this iii) the estimation of the elastic characteristics of the masonry. It is noteworthy that, for item i) the engineer has usually limited knowledge, due to the lack of documentation, while for items ii) and iii) he usually use tables proposed in technical manuals and standards which however give large bounds in order to encompass the generality of the real masonries. Instead, the use of reconstruction techniques on thermographic images coupled with homogenization permits to reduce these latter uncertainty on the estimation of the mechanical characteristics of the masonry.

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ON THE STOLARSKY-TYPE INEQUALITY FOR THE CHQOUET INTEGRAL AND THE GENERATED CHOQUET INTEGRAL

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Abstract. Roman-Fores (2008) have studied Stolarsky-type inequality for the fuzzy-integral with respect to a fuzzy measure. Recently, Daraby(2012) proved the Stolarsky-type inequality for the pseudo-integral with respect to the Lebesgue measure and gave an open problem: "Does Stolarsky's inequality hold for the Choquet integral?" In this paper, we prove the Stolarsky-type inequality for the Choquet integral with respect to a fuzzy measure. Furthermore, we investigate the Stolarsky-type inequality for the generated Choquet integral with respect to a fuzzy measure.

1. Introduction

Choquet [1], Couse *et al.* [2], Mihailovic and Pap [10], Murofushi *et al.* [11], Narukaw *et al.* [12], Pedrycz *et al.*[13], Rebille [14], Shieh *et al.* [17], Torra and Narukawa [20], and Tsai and Lu [21] have studied the Choquet integral with respect to a fuzzy measure, for examples, convergence theorems for the Choquet integral, some properties of the generated Choquet integral, and some applications of the Choquet integral criterion, etc. Furthermore, the authors in [15,16,19] have been studied various inequalities, for examples, Jensen-type inequality, Hardy-type inequality, and give an open problem: Does Stolarsky's inequality hold for the Choquet integral?

Jang *et al.* [5], Jang and Kwon [6], Jang [7-9], Schjaer-Jacobsen [18] have studied the Choquet integral of measurable interval-valued functions which are used for representing uncertain functions. Roman-Fores [4] also have studied Stolarsky-type inequality for the fuzzy-integral with respect to a fuzzy measure. Recently, Daraby [3] and Flores-Franulic proved the Stolarsky-type inequality for the pseudo-integral with respect to a σ-$\oplus$-decomposable measure.

In this paper, we prove a Stolarsky-type inequality for the Choquet integral with respect to a fuzzy measure and for the generated Choquet integral with respect to a fuzzy measure.

1991 *Mathematics Subject Classification.* 28E10, 28E20, 03E72, 26E50 11B68.

Key words and phrases. fuzzy measure, Choquet integral, generated Choquet integral, Stolarsky-type inequality, Lebesgue inetgral, measurable function.

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The paper is organized in five sections. In section 2, we list definitions and some properties of the Choquet integral with respect to a fuzzy measure and introduce the two types fuzzy Stolarsky's inequality for the fuzzy integral with respect to a continuous fuzzy measure. In section 3, we prove the Stolarsky-type inequality for the Choquet integral with respect to a fuzzy measure which is the solution of an open problem : Does Stolarsky's inequality hold for the Choquet integral? In section 4, we prove the Stolarsky-type inequality for the generated Choquet integral with respect to a fuzzy measure which is the generalized solution of an open problem : Does Stolarsky's inequality hold for the Choquet integral? In section 5, we give a brief summary results and some conclusions.

2. Definitions and Preliminaries

In this section, we introduce a fuzzy measure, the fuzzy integral with respect to a fuzzy measure, and the Choquet integral with respect to a fuzzy measure, the generated Choquet integral defined by Mihailovic and Pap [10], and the Stolarsky-type inequality for the fuzzy integral with respect to a continuous fuzzy measure of a measurable nonnegative function. Let $([0,1],\mathcal{A})$ be a measurable space and $\mathfrak{F}([0,1])$ be the set of all measurable functions from $[0,1]$ to $[0,1]$.

Definition 2.1. ([11-13]) (1) A fuzzy measure $\mu : \mathcal{A} \longrightarrow [0,\infty]$ on a measurable space $(X,\mathcal{A})$ is a real-valued set function satisfying

$$\begin{array}{ll} \text{(i)} & \mu(\emptyset)=0 \\ \text{(ii)} & \mu(A)\le\mu(B) \quad \text{whenever} \quad A,B\in\mathcal{A} \quad \text{and} \quad A\subset B. \end{array}$$

(2) A fuzzy measure μ is said to be finite if $\mu(X)<\infty$.

(3) A fuzzy measure μ is said to be continuous from below if for any sequence $\{A_n\}\subset\mathcal{A}$ and $A\in\mathcal{A}$, such that

$$A_n \uparrow A, \quad \text{then} \quad \lim_{n\to\infty}\mu(A_n)=\mu(A). \tag{1}$$

(4) A fuzzy measure μ is said to be continuous from above if for any sequence $\{A_n\}\subset\mathcal{A}$ and $A\in\mathcal{A}$ such that

$$\mu(A_1)<\infty \quad \text{and} \quad A_n\downarrow A, \quad \text{then} \quad \lim_{n\to\infty}\mu(A_n)=\mu(A). \tag{2}$$

(5) A fuzzy measure μ is said to be continuous if it is continuous from below and continuous from above.

Definition 2.2. ([4]) (1) Let μ be a fuzzy measure on $([0,1],\mathcal{A})$, $f\in\mathfrak{F}([0,1])$, and $A\in\mathcal{A}$. The fuzzy integral with respect to a fuzzy measure μ of f on A is defined by

$$(F)\int_A f d\mu = \sup_{\alpha\ge 0}\min\{\alpha\wedge\mu_{f,A}(\alpha)\}, \tag{3}$$

where

$$\mu_{f,A}(\alpha)=\mu\left(A\cap\{x\in X|f(x)\ge\alpha\}\right) \tag{4}$$

for all $\alpha\in[0,\infty)$.

(2) A measurable function f is said to be integrable if $(F)\int_A f d\mu$ is finite.

Definition 2.3. ([13]) (1) Let μ be a fuzzy measure on $([0,1],\mathcal{A})$, $f \in \mathfrak{F}([0,1])$, and $A \in \mathcal{A}$. The Choquet integral with respect to a fuzzy measure μ of f is defined by

$$(C)\int_A f d\mu = \int_0^\infty \mu_f(\alpha) d\alpha \tag{5}$$

where

$$\mu_{f,A}(\alpha) = \mu\left(A \cap \{x \in X | f(x) \geq \alpha\}\right) \tag{6}$$

for all $\alpha \in [0,\infty)$ and the integral on the right-hand side is the Lebesgue integral of $\mu_{f,A}$.

(2) A measurable function f is said to be integrable if $(C)\int_A f d\mu$ is finite.

Flores-Franulie *et al.* [4] proved the two types fuzzy Stolarsky's inequality for the fuzzy integral with respect to a continuous fuzzy measure.

Theorem 2.1. ([4]) (Fuzzy Stolarsky's inequality: decreasing case) *Let $a,b > 0$. If $f \in \mathfrak{F}([0,1])$ is a continuous and strictly decreasing function and μ is the Lebesgue measure on $\mathcal{A}$, then the inequality*

$$(F)\int_{[0,1]} f\left(x^{\frac{1}{a+b}}\right) d\mu \geq \left((F)\int_{[0,1]} f\left(x^{\frac{1}{a}}\right) d\mu\right)\left((F)\int_{[0,1]} f\left(x^{\frac{1}{b}}\right) d\mu\right) \tag{7}$$

holds.

Theorem 2.2. ([4]) (Fuzzy Stolarsky's inequality: increasing case) *Let $a,b > 0$. If $f \in \mathfrak{F}([0,1])$ is a continuous and strictly increasing function and μ is the Lebesgue measure on $\mathcal{A}$, then the inequality*

$$(F)\int_{[0,1]} f\left(x^{\frac{1}{a+b}}\right) d\mu \geq \left((F)\int_{[0,1]} f\left(x^{\frac{1}{a}}\right) d\mu\right)\left((F)\int_{[0,1]} f\left(x^{\frac{1}{b}}\right) d\mu\right) \tag{8}$$

holds.

3. Stolarsky-type inequality for the Choqeut integral

In this section, we consider an open problem: Does Stolarsky's inequality hold for the Choquet integral? in Daraby [3] and prove a Stolarsky-type inequality for the Choquet integral with respect to a fuzzy measure.

Theorem 3.1. *Let $\mu : \mathcal{A} \longrightarrow [0,\infty)$ be a continuous fuzzy measure. If $f \in \mathfrak{F}([0,1])$ is a continuous function, then we have*

$$(C)\int_{[0,1]} f d\mu \in [0,1]. \tag{9}$$

Proof. Since μ is continuous and f is continuous, $\mu_{f,[0,1]}$ is increasing continuous on $[0,1]$ and hence we have $0 \leq \mu_{f,[0,1]}(\alpha) \leq 1$ for all $\alpha \in [0,1]$. Thus, we have

$$(C)\int_{[0,1]} f d\mu = \int_0^1 \mu_{f,[0,1]}(\alpha) d\alpha \leq 1. \tag{10}$$

That is,

$$(C)\int_{[0,1]} f d\mu \in [0,1]. \tag{11}$$

Theorem 3.2. *Let $\mu : \mathcal{A} \longrightarrow [0,\infty)$ be a continuous fuzzy measure and $a, b, a+b \in (0,\infty)$. If $f \in \mathfrak{F}([0,1])$ is a continuous and increasing function, then the inequality*

$$(C)\int_{[0,1]} f\left(x^{\frac{1}{a+b}}\right) d\mu \geq (C)\int_{[0,1]} f\left(x^{\frac{1}{a}}\right) d\mu (C)\int_{[0,1]} f\left(x^{\frac{1}{b}}\right) d\mu \tag{12}$$

holds.

Proof. Since $a, b, a+b \in (0,\infty)$, $x^{\frac{1}{a+b}} > x^{\frac{1}{a}}$ and $x^{\frac{1}{a+b}} > x^{\frac{1}{b}}$ for all $x \in [0,1]$. Since f is a continuous and increasing function, $f\left(x^{\frac{1}{a+b}}\right) \geq f\left(x^{\frac{1}{a}}\right)$ and $f\left(x^{\frac{1}{a+b}}\right) \geq f\left(x^{\frac{1}{b}}\right)$ for all $x \in [0,1]$. Hence we have

$$\mu\left(\{x|f\left(x^{\frac{1}{a+b}}\right)\}\right) \geq \mu\left(\{x|f\left(x^{\frac{1}{a}}\right)\}\right) \tag{13}$$

and

$$\mu\left(\{x|f\left(x^{\frac{1}{a+b}}\right)\}\right) \geq \mu\left(\{x|f\left(x^{\frac{1}{b}}\right)\}\right). \tag{14}$$

By (13) and (14), we have

$$(C)\int_{[0,1]} f\left(x^{\frac{1}{a+b}}\right) d\mu \geq (C)\int_{[0,1]} f\left(x^{\frac{1}{a}}\right) d\mu \tag{15}$$

and

$$(C)\int_{[0,1]} f\left(x^{\frac{1}{a+b}}\right) d\mu \geq (C)\int_{[0,1]} f\left(x^{\frac{1}{b}}\right) d\mu. \tag{16}$$

By Theorem 3.1, (15), and (16), we have

$$\begin{aligned} (C)\int_{[0,1]} f\left(x^{\frac{1}{a+b}}\right) d\mu &\geq \left((C)\int_{[0,1]} f\left(x^{\frac{1}{a+b}}\right) d\mu\right)^2 \\ &\geq (C)\int_{[0,1]} f\left(x^{\frac{1}{a}}\right) d\mu (C)\int_{[0,1]} f\left(x^{\frac{1}{b}}\right) d\mu. \end{aligned} \tag{17}$$

Let $\mathfrak{F}([0,\infty))$ be the set of all measurable function from $[0,\infty)$ to $[0,\infty)$.

Theorem 3.3. *Let $\mu : \mathcal{A} \longrightarrow [0,\infty)$ be a continuous fuzzy measure and $a > 0, b > 0$. If $f \in \mathfrak{F}([0,\infty))$ is a continuous function, and it is increasing on $[0,1]$ and decreasing on $(1,\infty)$, then the inequality*

$$(C)\int_{[0,1]} f\left(x^{\frac{1}{a+b}}\right) d\mu \geq (C)\int_{[0,1]} f\left(x^{\frac{1}{a}}\right) d\mu (C)\int_{[0,1]} f\left(x^{\frac{1}{b}}\right) d\mu \tag{18}$$

holds.

Proof. If $a > 0, b > 0$, then we have

$$x^{\frac{1}{a+b}} > x^{\frac{1}{a}} \text{ and } x^{\frac{1}{a+b}} > x^{\frac{1}{b}} \text{ for all } x \in [0,1] \tag{19}$$

and

$$x^{\frac{1}{a+b}} < x^{\frac{1}{a}} \text{ and } x^{\frac{1}{a+b}} < x^{\frac{1}{b}} \text{ for all } x \in (1,\infty). \tag{20}$$

Since f is increasing on $[0,1]$ and decreasing on $(1,\infty)$, by (19) and (20), we have

$$f\left(x^{\frac{1}{a+b}}\right) \geq f\left(x^{\frac{1}{a}}\right) \text{ and } f\left(x^{\frac{1}{a+b}}\right) \geq f\left(x^{\frac{1}{b}}\right) \text{ for all } x \in [0,\infty). \tag{21}$$

By (21), we have

$$\mu\left(\{x|f\left(x^{\frac{1}{a+b}}\right)\}\right) \geq \mu\left(\{x|f\left(x^{\frac{1}{a}}\right)\}\right) \tag{22}$$

and

$$\mu\left(\{x|f\left(x^{\frac{1}{a+b}}\right)\}\right) \geq \mu\left(\{x|f\left(x^{\frac{1}{b}}\right)\}\right). \tag{23}$$

By (22) and (23), we have

$$(C)\int_{[0,\infty)} f\left(x^{\frac{1}{a+b}}\right) d\mu \geq (C)\int_{[0,\infty)} f\left(x^{\frac{1}{a}}\right) d\mu \tag{24}$$

and

$$(C)\int_{[0,\infty)} f\left(x^{\frac{1}{a+b}}\right) d\mu \geq (C)\int_{[0,\infty)} f\left(x^{\frac{1}{b}}\right) d\mu. \tag{25}$$

By Theorem 3.1, (24), and (25), we have

$$\begin{aligned}(C)\int_{[0,\infty)} f\left(x^{\frac{1}{a+b}}\right) d\mu &\geq \left((C)\int_{[0,\infty)} f\left(x^{\frac{1}{a+b}}\right) d\mu\right)^2 \\ &\geq (C)\int_{[0,\infty)} f\left(x^{\frac{1}{a}}\right) d\mu (C)\int_{[0,\infty)} f\left(x^{\frac{1}{b}}\right) d\mu. \qquad (26)\end{aligned}$$

4. Stolarsky-type inequality for the generated Choquet integral

In this section, we consider the generated Choquet integral with respect to a fuzzy measure and investigate Stolarsky-type inequality for the generated Choquet integral.

Definition 4.1. [9] Let μ be a fuzzy measure and $g : [0,\infty] \longrightarrow [0,\infty]$, $g(0) = 0$ be an odd, strictly increasing, continuous function.

(1) The generated Choquet integral with respect to μ of $f \in \mathfrak{F}([0,\infty))$ is defined by

$$(GC)\int_{[0,\infty]} f d\mu = g^{-1}\left((C)\int_{[0,\infty)} g \circ f dg \circ \mu\right) \tag{27}$$

where $g \circ f$ is the composition of g and f.

(2) A measurable function f is said to be integrable if $(GC)\int_{[0,\infty)} f d\mu \in [0,\infty)$.

Theorem 4.1. *Let $\mu : \mathcal{A} \longrightarrow [0,\infty)$ be a continuous fuzzy measure, $g : [0,\infty] \longrightarrow [0,\infty]$, $g(0) = 0$ be an odd, strictly increasing, continuous function, and $a > 0, b > 0$. If $f \in \mathfrak{F}([0,\infty))$ is integrable function with $(GC)\int_{[0,\infty)} f d\mu \in [0,1]$, and it is increasing on $[0,1]$ and decreasing on $(1,\infty)$, and if $(GC)\int_{[0,\infty)} f\left(x^{\frac{1}{a+b}}\right) d\mu \in (0,1]$, then the inequality*

$$(GC)\int_{[0,1]} f\left(x^{\frac{1}{a+b}}\right) d\mu \geq (GC)\int_{[0,1]} f\left(x^{\frac{1}{a}}\right) d\mu (GC)\int_{[0,1]} f\left(x^{\frac{1}{b}}\right) d\mu \tag{28}$$

holds.

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Proof. Since $a > 0, b > 0$ and f is increasing on $[0,1]$ and decreasing on $(1,\infty)$, by (19) and (20), we have

$$f\left(x^{\frac{1}{a+b}}\right) \geq f\left(x^{\frac{1}{a}}\right) \text{ and } f\left(x^{\frac{1}{a+b}}\right) \geq f\left(x^{\frac{1}{b}}\right) \text{ for all } x \in [0,\infty). \tag{29}$$

Since g is strictly increasing, by (29), we have

$$g\circ f\left(x^{\frac{1}{a+b}}\right) \geq g\circ f\left(x^{\frac{1}{a}}\right) \text{ and } g\circ f\left(x^{\frac{1}{a+b}}\right) \geq g\circ f\left(x^{\frac{1}{b}}\right) \text{ for all } x \in [0,\infty). \tag{30}$$

Note that if μ is a continuous fuzzy measure and $g : [0,\infty] \longrightarrow [0,\infty]$, $g(0) = 0$ be an increasing continuous function, then $g\circ\mu$ is a continuous fuzzy measure. Thus, by (30), we have

$$g\circ\mu\left(\left\{x|f\left(x^{\frac{1}{a+b}}\right)\right\}\right) \geq g\circ\mu\left(\left\{x|f\left(x^{\frac{1}{a}}\right)\right\}\right) \tag{31}$$

and

$$g\circ\mu\left(\left\{x|f\left(x^{\frac{1}{a+b}}\right)\right\}\right) \geq g\circ\mu\left(\left\{x|f\left(x^{\frac{1}{b}}\right)\right\}\right). \tag{32}$$

By (31) and (32), we have

$$(C)\int_{[0,\infty)} g\circ f\left(x^{\frac{1}{a+b}}\right) dg\circ\mu \geq (C)\int_{[0,\infty)} g\circ f\left(x^{\frac{1}{a}}\right) dg\circ\mu \tag{33}$$

and

$$(C)\int_{[0,\infty)} g\circ f\left(x^{\frac{1}{a+b}}\right) dg\circ\mu \geq (C)\int_{[0,\infty)} g\circ f\left(x^{\frac{1}{b}}\right) dg\circ\mu. \tag{34}$$

We note that if g is an odd, strictly increasing continuous function, then its inverse g^{-1} is an odd, strictly increasing continuous function. Thus, by (34), we have

$$\begin{aligned}(GC)\int_{[0,\infty)} f\left(x^{\frac{1}{a+b}}\right) d\mu &= g^{-1}\left((C)\int_{[0,\infty)} g\circ f\left(x^{\frac{1}{a+b}}\right) dg\circ\mu\right)\\ &\geq g^{-1}\left((C)\int_{[0,\infty)} g\circ f\left(x^{\frac{1}{a}}\right) dg\circ\mu\right)\\ &= (GC)\int_{[0,\infty)} f\left(x^{\frac{1}{a}}\right) d\mu \end{aligned} \tag{35}$$

and

$$\begin{aligned}(GC)\int_{[0,\infty)} f\left(x^{\frac{1}{a+b}}\right) d\mu &= g^{-1}\left((C)\int_{[0,\infty)} g\circ f\left(x^{\frac{1}{a+b}}\right) dg\circ\mu\right)\\ &\geq g^{-1}\left((C)\int_{[0,\infty)} g\circ f\left(x^{\frac{1}{b}}\right) dg\circ\mu\right)\\ &= (GC)\int_{[0,\infty)} f\left(x^{\frac{1}{b}}\right) d\mu \end{aligned} \tag{36}$$

By Theorem 3.1, (35), and (36), we have

$$\begin{aligned}(GC)\int_{[0,\infty)} f\left(x^{\frac{1}{a+b}}\right) d\mu &\geq \left((GC)\int_{[0,\infty)} f\left(x^{\frac{1}{a+b}}\right) d\mu\right)^2\\ &\geq (GC)\int_{[0,\infty)} f\left(x^{\frac{1}{a}}\right) d\mu (GC)\int_{[0,\infty)} f\left(x^{\frac{1}{b}}\right) d\mu. \end{aligned} \tag{37}$$

5. Conclusions

This study was to solve an open problem: Does Stolarsky's inequality for the Choquet integral? Thus, we solved Stolarsky-type inequality for the Choquet integral with respect to a continuous fuzzy integral under some sufficient conditions (see Theorems 3.2 and 3.3). We also proved Stolarsky-type inequality for the generated Choquet integral with respect to a continuous fuzzy integral under some sufficient conditions (see Theorem 4.1).

Furthermore, we give another open problem: Does Stolarsky's inequality for the interval-valued generated Choquet integral?

Acknowledgement: This paper was supported by Konkuk University in 2014.

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Extended Cesáro operator from weighted Bergman space to Zygmund-type space

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[1]

Abstract

We discuss the boundedness and compactness of extended cesáro operator from weighted Bergman space to Zygmund-type space on the unit ball of C^n.

1 Introduction

Let $H(B_n)$ be the class of all holomorphic functions on B_n, where B_n is the unit ball in the n-dimensional complex space C^n. Let dv denote the Lebesegue measure on B_n normalized so that $v(B_n) = 1$. For $f \in H(B_n)$, let

$$\Re f(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z)$$

be the radial derivative of f. We write $\Re^m f = \Re^{m-1}(\Re f)$.

The Bloch space $\mathcal{B} = \mathcal{B}(B_n)$ is defined as the space of all $f \in H(B_n)$ such that

$$\|f\|_{\mathcal{B}} = \sup\{(1-|z|^2)|\Re f(z)| : z \in B_n\} < \infty.$$

The little Bloch space $\mathcal{B}_0 = \mathcal{B}_0(B_n)$ consists of those $f \in \mathcal{B}$ satisfying

$$(1-|z|^2)\,|\Re f(z)| \to 0, \text{ as } |z| \to 1.$$

For $p > 0, \alpha > -1$, the weighted Bergman space $A^p_\alpha = A^p_\alpha(B_n)$ consists of all $f \in H(B_n)$ such that

$$\|f\|^p_{A^p_\alpha} = \int_{B_n} |f(z)|^p (1-|z|^2)^\alpha dv(z). \tag{1}$$

When $p \geq 1$, the weighted Bergman space with the norm $\|.\|_{A^p_\alpha}$ becomes a Banach space. If $p \in (0,1)$, it is a Fréchet space with the translation invariant metric $d(f,g) = \|f-g\|^p_{A^p_\alpha}$.

The Zygmund space $\mathcal{Z} = \mathcal{Z}(B_n)$ consists of those functions whose first order partial derivatives are in the Bloch space. As we all know $f \in \mathcal{Z}$ if and only if $\|f\|_{\mathcal{Z}} = |f(0)| + \sup_{z\in B_n}(1-|z|^2)|\Re^2 f(z)| < \infty$.

[1]The authors were supported in part by the National Natural Science Foundation of China (Grant Nos. 11371276; 11301373; 1120133)

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2010 Mathematics Subject Classification. Primary: 47B38; Secondary: 32A37, 32A38, 32H02, 47B33.
Key words and phrases.Extended Cesáro operator, weighted Bergman space, Zygmund-type space, bounded, compact.

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Let $\mathcal{Z}_0 = \mathcal{Z}_0(B_n)$ denote the subspace of $\mathcal{Z}$ consisting of those $f \in \mathcal{Z}$ for which

$$\lim_{|z|\to 1}(1-|z|^2)|\Re^2 f(z)| = 0.$$

Both $\mathcal{Z}$ and $\mathcal{Z}_0$ are Banach spaces under the norm $\|f\|_{\mathcal{Z}}$.

A positive continuous function μ on [0,1) is called *normal*, if there exist three positive constants $0 \le \delta < 1$, and $0 < s < t < \infty$, such that, for $r \in [\delta, 1)$

$$\frac{\mu(r)}{(1-r)^s} \downarrow 0, \quad \frac{\mu(r)}{(1-r)^t} \uparrow \infty, \quad \text{as } r \to 1.$$

In the rest of this paper we always assume that μ is normal on $[0,1)$ and if $\mu : B_n \to [0,\infty)$ is normal, we will also suppose μ is radial on B_n, that is, $\mu(z) = \mu(|z|), z \in B_n$.

For a normal function μ on $[0,1)$, the Zymund-type space $\mathcal{Z}_\mu = \mathcal{Z}_\mu(B_n)$, consists of all $f \in H(B_n)$ satisfying the norm:

$$\|f\|_{\mathcal{Z}_\mu} = |f(0)| + \sup_{z\in B_n} \mu(z)|\Re^2 f(z)| < \infty.$$

Moreover, the little Zymund-type space $\mathcal{Z}_{\mu,0} = \mathcal{Z}_{\mu,0}(B_n)$ contains $f \in \mathcal{Z}_\mu$ such that

$$\lim_{|z|\to 1} \mu(z)|\Re^2 f(z)| = 0.$$

When $\mu(r) = 1 - r^2$, the (little) Zymund-type space is the (little) Zymund space.

For $g \in H(B_n)$, we consider the extended cesáro operator T_g defined on $H(B_n)$ by

$$T_g f(z) = \int_0^1 f(tz)\Re g(tz)\frac{dt}{t}.$$

Fang and Hu, respectively, gave some sufficient and necessary conditions for the extended cesáro operator to be bounded and compact on Zygmund spaces, mixed norm spaces in papers [3, 4, 5, 6]. Other related results can be found in [1, 9, 10, 13]. Building on those foundations, the present paper focuses on the boundedness and compactness of extended cesáro operator from the weighted Bergman space to Zygmund-type space on the unit ball of C^n. The paper is constructed as follows: some lemmas are given in section 2, section 3 and section 4 devote to the main results.

Throughout the remainder of this paper, C will denote a positive constant, the exact value of which will vary from one appearance to the next.

2 Some Lemmas

Lemma 1. *Let $g \in H(B_n)$, then $\Re(T_g f)(z) = f(z)\Re g(z)$ for any $f \in H(B_n)$ and $z \in B_n$.*

Lemma 2. *[12, Theorem 2.1] Suppose that $p \in (0,\infty)$ and $\alpha > -1$, then we have that*

$$|f(z)| \le \frac{\|f\|_{A^p_\alpha}}{(1-|z|^2)^{(n+1+\alpha)/p}}, \quad f \in A^p_\alpha, \quad z \in B_n.$$

Lemma 3. *[8] Suppose that $0 < p < \infty$, $\alpha > -1$, then for all $f \in A^p_\alpha$,*

$$\|f\|^p_{A^p_\alpha} \asymp |f(0)|^p + \int_{B_n} |\Re f(z)|^p (1-|z|^2)^{p+\alpha} dv(z).$$

From Lemma 3, if $f \in A^p_\alpha$ then $\Re f \in A^p_{\alpha+p}$ and $\|\Re f\|_{A^p_{\alpha+p}} \le C\|f\|_{A^p_\alpha}$.

The following criterion follows from an easy modification of [2, Proposition 3.11].

Lemma 4. *Assume that $0 < p < \infty, \alpha > -1$, μ is normal and $g \in H(B_n)$. Then $T_g : A^p_\alpha \to \mathcal{Z}_\mu$ is compact if and only if $T_g : A^p_\alpha \to \mathcal{Z}_\mu$ is bounded and for any bounded sequence $\{f_k\}_{k\in\mathbb{N}}$ in A^p_α which converges to zero uniformly on compact subsets of B_n as $k \to \infty$, then $\|T_g f_k\|_{\mathcal{Z}_\mu} \to 0, \ \ k \to \infty$.*

The next lemma similarly follows the proof of [7, Lemma 1] with minor modifications.

Lemma 5. *A close set $\mathcal{K}$ in $\mathcal{Z}_{\mu,0}$ is compact if and only if it is bounded and satisfies $\lim\limits_{|z|\to 1} \sup\limits_{f\in\mathcal{K}} \mu(z)|\Re^2 f(z)| = 0$.*

3 The boundedness of $T_g : A^p_\alpha \to \mathcal{Z}_\mu(Z_{\mu,0})$.

Theorem 1. *Suppose that $p > 1$ and $\alpha > -1$, μ is normal and $g \in H(B_n)$. Then $T_g : A^p_\alpha \to \mathcal{Z}_\mu$ is bounded if and only if*

$$M_1 := \sup_{z\in B_n} \frac{\mu(z)|\Re^2 g(z)|}{(1-|z|^2)^{(n+\alpha+1)/p}} < \infty, \tag{2}$$

$$M_2 := \sup_{z\in B_n} \frac{\mu(z)|\Re g(z)|}{(1-|z|^2)^{(n+\alpha+p+1)/p}} < \infty. \tag{3}$$

Proof. Sufficiency. Suppose (2) and (3) hold. By Lemma 3, $\Re f \in A^p_{\alpha+p}$ and $\|\Re f\|_{A^p_{\alpha+p}} \le C\|f\|_{A^p_\alpha}$. Since $T_g f(0) = 0$, by Lemma 1, Lemma 2 and Lemma 3, it follows that

$$\begin{aligned}
\|T_g f(z)\|_{\mathcal{Z}_\mu} &= |T_g f(0)| + \sup_{z\in B_n} \mu(z)|\Re^2(T_g f)(z)| \\
&= \sup_{z\in B_n} \mu(z)|\Re(f\Re g)(z)| \\
&= \sup_{z\in B_n} \mu(z)|\Re f(z)\Re g(z) + f(z)\Re^2 g(z)| \\
&\le \sup_{z\in B_n} \mu(z)|\Re f(z)\Re g(z)| + \sup_{z\in B_n} \mu(z)|f(z)\Re^2 g(z)| \\
&\le C \sup_{z\in B_n} \frac{\mu(z)|\Re g(z)|}{(1-|z|^2)^{(n+\alpha+p+1)/p}} \|\Re f\|_{A^p_{\alpha+p}} + C \sup_{z\in B_n} \frac{\mu(z)|\Re^2 g(z)|}{(1-|z|^2)^{(n+\alpha+1)/p}} \|f\|_{A^p_\alpha} \\
&\le C\|f\|_{A^p_\alpha} < \infty.
\end{aligned} \tag{4}$$

From above we obtain the boundedness of $T_g : A^p_\alpha \to \mathcal{Z}_\mu$.

Necessity. Suppose that $T_g : A^p_\alpha \to \mathcal{Z}_\mu$ is bounded. Choose

$$t > n\max(1, \frac{1}{p}) + \frac{\alpha+1}{p}. \tag{5}$$

Thus $c = pt - (n+1+\alpha) > 0$. For $a \in B_n$, set

$$f_a(z) = \frac{(1-|a|^2)^{t-\frac{n+1+\alpha}{p}}}{(1-\langle z, a\rangle)^t}, \ \ z \in B_n. \tag{6}$$

By [12, Theorem 1.12], we obtain that $f_a \in A^p_\alpha$ and $\sup\limits_{a\in B_n} \|f_a\|_{A^p_\alpha} \le C$. Moreover,

$$f_a(a) = \frac{1}{(1-|a|^2)^{\frac{n+1+\alpha}{p}}} \quad \text{and} \quad |\Re f_a(a)| = \frac{t|a|^2}{(1-|a|^2)^{\frac{n+1+\alpha}{p}+1}}.$$

Using the boundedness of $T_g : A^p_\alpha \to \mathcal{Z}_\mu$ and Lemma 1, it is clear that

$$\begin{aligned}\|T_g f_a(z)\|_{\mathcal{Z}_\mu} &= |T_g f_a(0)| + \sup_{z\in B_n} \mu(z)|\Re^2(T_g f_a)(z)| \\ &= \sup_{z\in B_n} \mu(z)|\Re(f_a \Re g)(z)| \\ &= \sup_{z\in B_n} \mu(z)|\Re f_a(z)\Re g(z) + f_a(z)\Re^2 g(z)| \\ &\geq \mu(a)|\Re f_a(a)||\Re g(a)| - \mu(a)|f_a(a)||\Re^2 g(a)| \\ &= \frac{t\mu(a)|\Re g(a)||a|^2}{(1-|a|^2)^{\frac{n+1+\alpha}{p}+1}} - \frac{\mu(a)|\Re^2 g(a)|}{(1-|a|^2)^{\frac{n+1+\alpha}{p}}}. \end{aligned} \tag{7}$$

For $a \in B_n$, define the function

$$h_a(z) = 2\frac{(1-|a|^2)^{t-\frac{n+1+\alpha}{p}}}{(1-\langle z,a\rangle)^t} - (1-|a|^2)^{\frac{n+1+\alpha}{p}}\Big(\frac{(1-|a|^2)^{t-\frac{n+1+\alpha}{p}}}{(1-\langle z,a\rangle)^t}\Big)^2,$$

where t also satisfies (5). By [12, Theorem 1.12], we also get $h_a \in A^p_\alpha$ and $\sup_{a\in B_n} \|h_a\|_{A^p_\alpha} \leq C$. Moreover, $\Re h_a(a) = 0$ and $h_a(a) = 1/(1-|a|^2)^{\frac{n+1+\alpha}{p}}$. Hence

$$\begin{aligned}\|T_g h_a(z)\|_{\mathcal{Z}_\mu} &= |T_g h_a(0)| + \sup_{z\in B_n} \mu(z)|\Re^2(T_g h_a)(z)| \\ &= \sup_{z\in B_n} \mu(z)|\Re h_a(z)\Re g(z) + h_a(z)\Re^2 g(z)| \\ &\geq \mu(a)|\Re h_a(a)\Re g(a) + h_a(a)\Re^2 g(a)| \\ &= \frac{\mu(a)|\Re^2 g(a)|}{(1-|a|^2)^{\frac{n+1+\alpha}{p}}}. \end{aligned} \tag{8}$$

Since a is an arbitrary element in B_n, then from (8) we get (2). Combining (7) and (2), it follows that

$$\sup_{z\in B_n} \frac{\mu(z)|\Re g(z)||z|^2}{(1-|z|^2)^{\frac{n+1+\alpha}{p}+1}} < \infty. \tag{9}$$

If we take $f(z) = 1 \in A^p_\alpha$ and obtain $\Re f(z) = 0$. Then

$$\begin{aligned}\sup_{z\in B_n} \mu(z)|\Re^2(T_g f)(z)| &= \sup_{z\in B_n} \mu(z)|\Re f(z)\Re g(z) + f(z)\Re^2 g(z)| \\ &= \sup_{z\in B_n} \mu(z)|\Re^2 g(z)| < \infty. \end{aligned} \tag{10}$$

On the other hand, we take the functions $f_j(z) = z_j \in A^p_\alpha, j \in \{1,...,n\}$ and have $\Re f_j(z) = z_j,\ j \in \{1,...,n\}$. Thus

$$\begin{aligned}\sup_{z\in B_n} \mu(z)|\Re^2(T_g f_j)(z)| &= \sup_{z\in B_n} \mu(z)|\Re f_j(z)\Re g(z) + f_j(z)\Re^2 g(z)| \\ &= \sup_{z\in B_n} \mu(z)|z_j||\Re g(z) + \Re^2 g(z)| < \infty. \end{aligned} \tag{11}$$

Further by (11) it is clear that

$$\sup_{z\in B_n} \mu(z)|z||\Re g(z) + \Re^2 g(z)| \leq \sup_{z\in B_n} \mu(z)\Big(\sum_{j=1}^n |z_j|\Big)|\Re g(z) + \Re^2 g(z)| < \infty. \tag{12}$$

Employing (12) we have that

$$\sup_{|z|>1/2} \mu(z)|\Re g(z)+\Re^2 g(z)| \le \sup_{z\in B_n} \mu(z)2|z||\Re g(z)+\Re^2 g(z)| < \infty.$$

It is obvious that $\sup_{|z|\le 1/2} \mu(z)|\Re g(z)+\Re^2 g(z)| < \infty$. Therefore,

$$\sup_{z\in B_n} \mu(z)|\Re g(z)+\Re^2 g(z)| < \infty. \tag{13}$$

Combining (10) and (13) we get that

$$M_3 := \sup_{z\in B_n} \mu(z)|\Re g(z)| < \infty. \tag{14}$$

Employing (14) and (9), respectively, we obtain that

$$\sup_{|z|\le 1/2} \frac{\mu(z)|\Re g(z)|}{(1-|z|^2)^{(n+\alpha+p+1)/p}} \le C \sup_{z\in B_n} \mu(z)|\Re g(z)| < \infty. \tag{15}$$

$$\sup_{|z|>1/2} \frac{\mu(z)|\Re g(z)|}{(1-|z|^2)^{(n+\alpha+p+1)/p}} \le \sup_{z\in B_n} \frac{\mu(z)|\Re g(z)|(4|z|^2)}{(1-|z|^2)^{(n+\alpha+p+1)/p}} < \infty. \tag{16}$$

Combining (15) and (16), (3) follows. The proof is completed. □

Theorem 2. *Suppose that $p>1$ and $\alpha>-1$, μ is normal and $g\in H(B_n)$. Then $T_g : A^p_\alpha \to \mathcal{Z}_{\mu,0}$ is bounded if and only if $T_g : A^p_\alpha \to \mathcal{Z}_\mu$ is bounded and*

$$\lim_{|z|\to 1} \mu(z)|\Re g(z)| = 0, \tag{17}$$

$$\lim_{|z|\to 1} \mu(z)|\Re^2 g(z)| = 0. \tag{18}$$

Proof. Sufficiency. Suppose that $T_g : A^p_\alpha \to \mathcal{Z}_\mu$ is bounded, (17) and (18) hold. For any polynomial P, by (17) and (18), we have that

$$\begin{aligned} \mu(z)|\Re^2(T_g P(z))| &\le \mu(z)|\Re P(z)\Re g(z)| + \mu(z)|P(z)\Re^2 g(z)| \\ &\le \mu(z)|\Re g(z)|\|\Re P\|_\infty + \mu(z)|\Re^2 g(z)|\|P\|_\infty \to 0,\ |z|\to 1, \end{aligned}$$

where $\|\Re P\|_\infty := \sup_{z\in B_n} |\Re P(z)| < \infty$ and $\|P\|_\infty := \sup_{z\in B_n} |P(z)| < \infty$. Since the set of all polynomials is dense in A^p_α, then for any $f\in A^p_\alpha$, there is a polynomial sequence P_n such that $\|P_n - f\|_{A^p_\alpha} \to 0,\ n\to\infty$. By the boundedness of $T_g : A^p_\alpha \to \mathcal{Z}_\mu$,

$$\|T_g P_n - T_g f\|_{\mathcal{Z}_\mu} \le \|T_g\|\|P_n - f\|_{A^p_\alpha} \to 0,\ n\to\infty.$$

Hence $T_g(A^p_\alpha) \subseteq \mathcal{Z}_{\mu,0}$. That is, $T_g : A^p_\alpha \to \mathcal{Z}_{\mu,0}$ is bounded.

Necessity. Suppose that $T_g : (A^p_\alpha) \to \mathcal{Z}_{\mu,0}$ is bounded. Choosing $f(z)=1$, we obtain (18). Taking $f_j(z)=z_j,\ j\in\{1,...,n\}$, we have that

$$\mu(z)|\Re^2(T_g f_j)(z)| = \mu(z)|z_j||\Re g(z)+\Re^2 g(z)| \to 0,\ |z|\to 1.$$

From the above inequality,

$$\mu(z)|z||\Re g(z)+\Re^2 g(z)| \le \mu(z)\Big(\sum_{j=1}^n |z_j|\Big)|\Re g(z)+\Re^2 g(z)| \to 0,\ |z|\to 1.$$

That is,

$$\mu(z)|\Re g(z)+\Re^2 g(z)| \to 0,\ |z|\to 1. \tag{19}$$

Combing (19) and (18), (17) follows. The proof is completed. □

4 The compactness of $T_g : A^p_\alpha \to \mathcal{Z}_\mu(Z_{\mu,0})$.

Theorem 3. *Suppose that $p > 1$ and $\alpha > -1$, μ is normal and $g \in H(B_n)$. Then the following statements are equivalent:*
(a) $T_g : A^p_\alpha \to \mathcal{Z}_\mu$ is compact.
(b) $T_g : A^p_\alpha \to \mathcal{Z}_{\mu,0}$ is compact.
(c) $T_g : A^p_\alpha \to \mathcal{Z}_\mu$ is bounded and

$$\lim_{|z|\to 1} \frac{\mu(z)|\Re^2 g(z)|}{(1-|z|^2)^{(n+\alpha+1)/p}} = 0, \tag{20}$$

$$\lim_{|z|\to 1} \frac{\mu(z)|\Re g(z)|}{(1-|z|^2)^{(n+\alpha+p+1)/p}} = 0. \tag{21}$$

Proof. (b)⇒(a). This implication is obvious.

(a)⇒(c). Suppose that $T_g : A^p_\alpha \to \mathcal{Z}_\mu$ is compact. It is obvious that $T_g : A^p_\alpha \to \mathcal{Z}_\mu$ is bounded. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in B_n satisfying $|z_k| \to 1$ as $k \to \infty$. Consider the function sequence $(f_k)_{k\in\mathbb{N}}$ defined by

$$f_k(z) = \frac{(1-|z_k|^2)^{t-\frac{n+\alpha+1}{p}}}{(1-\langle z, z_k\rangle)^t}, \tag{22}$$

where t satisfies (5). From Theorem 1 we know $f_k \in A^p_\alpha$ and $\sup_{k\in\mathbb{N}} \|f_k\|_{A^p_\alpha} \le C$. It is easy to check that $f_k \to 0$ uniformly on compact subsets of B_n as $k \to \infty$. By Lemma 4, $\lim_{k\to\infty} \|T_g f_k\|_{\mathcal{Z}_\mu} = 0$. Using the same proof of (7), it follows that

$$\|T_g f_k(z)\|_{\mathcal{Z}_\mu} \ge \frac{t\mu(z_k)|\Re g(z_k)||z_k|^2}{(1-|z_k|^2)^{\frac{n+1+\alpha}{p}+1}} - \frac{\mu(z_k)|\Re^2 g(z_k)|}{(1-|z_k|^2)^{\frac{n+1+\alpha}{p}}}. \tag{23}$$

Next take the function sequence

$$h_k(z) = 2\frac{(1-|z_k|^2)^{t-\frac{n+1+\alpha}{p}}}{(1-\langle z, z_k\rangle)^t} - (1-|z_k|^2)^{\frac{n+1+\alpha}{p}}\Big(\frac{(1-|z_k|^2)^{t-\frac{n+1+\alpha}{p}}}{(1-\langle z, z_k\rangle)^t}\Big)^2,$$

where t also satisfies (5). Then $(h_k)_{k\in\mathbb{N}}$ is a bounded sequence in A^p_α and $h_k \to 0$ uniformly on compact subsets of B_n as $k \to \infty$. By Lemma 4, $\lim_{k\to\infty} \|T_g h_k\|_{\mathcal{Z}_\mu} = 0$. By the same proof of (8) we obtain that

$$\|T_g h_k(z)\|_{\mathcal{Z}_\mu} \ge \frac{\mu(z_k)|\Re^2 g(z_k)|}{(1-|z_k|^2)^{\frac{n+1+\alpha}{p}}}. \tag{24}$$

Letting $k \to \infty$ in (24), it follows (20). Combining (24) and (23), (21) follows.

(c)⇒(b). Suppose $T_g : A^p_\alpha \to \mathcal{Z}_\mu$ is bounded, (20) and (21) hold. By the similar proof of (4), we obtain that

$$\mu(z)|\Re^2(T_g f)(z)| \quad \le \quad C\Big(\frac{\mu(z)|\Re g(z)|}{(1-|z|^2)^{(n+\alpha+p+1)/p}} + \frac{\mu(z)|\Re^2 g(z)|}{(1-|z|^2)^{(n+\alpha+1)/p}}\Big)\|f\|_{A^p_\alpha}.$$

From the above inequality , (20) and (21) we have that

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$$\begin{aligned}
&\lim_{|z|\to 1} \sup_{\|f\|_{A^p_\alpha}\le 1} \mu(z)|\Re^2(T_g f)(z)| \\
&\le \lim_{|z|\to 1} \sup_{\|f\|_{A^p_\alpha}\le 1} C\Big(\frac{\mu(z)|\Re g(z)|}{(1-|z|^2)^{(n+\alpha+p+1)/p}} + \frac{\mu(z)|\Re^2 g(z)|}{(1-|z|^2)^{(n+\alpha+1)/p}}\Big)\|f\|_{A^p_\alpha} \\
&\le C \lim_{|z|\to 1} \Big(\frac{\mu(z)|\Re g(z)|}{(1-|z|^2)^{(n+\alpha+p+1)/p}} + \frac{\mu(z)|\Re^2 g(z)|}{(1-|z|^2)^{(n+\alpha+1)/p}}\Big) = 0. \qquad (25)
\end{aligned}$$

Employing Lemma 5, the compactness of the operator $T_g : A^p_\alpha \to \mathcal{Z}_{\mu,0}$ follows. The proof is completed. □

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The numerical solution for the interval-valued differential equations†

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Abstract A parallel algorithm is developed for the numerical solution of interval-valued differential equation. In this paper, we investigated the fuzzy Euler rule, Simpson rule, composite Simpson rule and improved Simpson rule of interval-valued differential equation and integral equation, at the same time, the truncation errors and convergence theorem are discussed. Finally, an example is given to illustrate the methods.
Keywords: Fuzzy Euler rule; Simpson rule; composite Simpson rule; improved Simpson rule; fuzzy integral equation

1. Introduction

Introduction to fuzzy differential equations and fuzzy integral equations are presented by Kandel and Byatt [1, 2] and later applied in fuzzy processes and fuzzy dynamical systems [3, 4]. The study of fuzzy differential equation forms a suitable setting for mathematical modeling of real-world problems in which uncertainties or vagueness pervade. Thinking about a physical problem which is transformed into a deterministic problem of partial differential equations we cannot usually be sure that this modeling is perfect. Especially, if the data (e.g. initial value) are not known precisely but only through some measurements the intervals which cover the data are determined. For example, mathematical models in science and engineering often contain parameters that are uncertain. These parameters are usually represented by random numbers, fields or processes. Therefore there appear problems of differential equations with uncertainty. However, when the stochastic characteristics of these parameters are not precisely known, an interval representation, or, more generally, a fuzzy representation may be more appropriate. Therefore, such uncertainty can be expressed in terms of intervals and ordinary differential equations. It is also clear that uncertainty expressed by a fuzzy representation is more appropriate.

The concept of fuzzy sets which was originally introduced by Zadeh [5] led to the definition of the fuzzy number and its implementation in fuzzy control [6] and approximate reasoning problems [7, 8]. The fuzzy mapping function was introduced by Chang and Zadeh [6]. Later, Dubois and Prade [9] presented an elementary fuzzy calculus based on the extension principle [5]. Puri and Ralescu [10] suggested two definitions for fuzzy derivative of fuzzy functions. The first method was based on the H-difference notation and was further investigated by Kaleva [11]. The second method was derived from the embedding technique and was followed by Goetschel and Voxman [12] who gave it a more applicable representation. The concept of integration of fuzzy functions was first introduced by Dubois and Prade [9]. Alternative approaches were later suggested by Goetschel and Voxman [12], Kaleva [11] and others. While Goetschel and Voxman [12] preferred a Riemann integral type approach, Kaleva [11] chose to define the integral of fuzzy function, using the Lebesgue type concept for integration.

Knowledge about dynamical systems modeled by differential equations is often incomplete or vague. For example, for parametric quantities, functional relationships, or initial conditions, the well-known methods of solving fuzzy differential equation analytically or numerically can only be used for finding the selected system behavior, e.g., by fixing unknown parameters to some plausible values. As a new

†This work is supported by National Social Science Fund of China (12XJY006) and the Key Project of Teaching Research of Lanzhou University of Finance and Economics (20120107).
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and powerful mathematical tool, fuzzy differential equations have been studied by several approaches [13, 14, 15].

In this work we concentrate on numerical procedures for solving fuzzy differential equations and fuzzy integral equations, whenever these equations possess unique fuzzy solutions. In Section 2 we briefly present the basic notations of fuzzy number, fuzzy function, fuzzy derivative, and fuzzy integral. In Section 3 a general Cauchy problem is defined and a numerical algorithm for solving it, the fuzzy Euler method, is proposed. In Section 4, the Simpson rule, composite Simpson rule, improved Simpson rule of fuzzy integral equations and their truncation errors are discussed. In section 5, an example is given to illustrate our methods.

2. Preliminaries

Let $\mathbb{I}$ denote a family of all nonempty, compact and convex subsets of $\mathbb{R}$ (intervals). The addition and scalar multiplication in $\mathbb{I}$ we define as usual (cf. [1, 2]), i.e. for $A, B \in \mathbb{I}, A = [a^-, a^+], B = [b^-, b^+]$, and $\lambda \geq 0$,

$$A + B = [a^- + b^-, a^+ + b^+], \quad \lambda A = [\lambda a^-, \lambda a^+], \quad (-\lambda)A = [(-\lambda)a^+, (-\lambda)a^-].$$

Also, for $A \in \mathbb{I}, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}, \lambda_3\lambda_4 \geq 0$ it holds

$$\lambda_1(\lambda_2 A) = (\lambda_1\lambda_2)A, \quad (\lambda_3 + \lambda_4)A = \lambda_3 A + \lambda_4 A.$$

The Hausdorff metric in $\mathbb{I}$ is defined as follows:

$$H(A, B) = \max\{|a^- - b^-|, |a^+ - b^+|\}.$$

It is known (cf. [16, 17]) that $(\mathbb{I}, H)$ is a complete, separable and locally compact metric space, also it becomes a semilinear metric space with algebraic operations of addition and non-negative scalar multiplication.

For the metric H the following properties hold (cf. [16]):

$$H(A + C, B + C) = H(A, B),$$

$$H(A + B, C + D) \leq H(A, C) + H(B, D),$$

$$H(\lambda A, \lambda B) = |\lambda| H(A, B),$$

for every $A, B, C, D \in \mathbb{I}$, and every $\lambda \in \mathbb{R}$.

Let $A, B \in \mathbb{I}$. If there exists an interval $C \in \mathbb{I}$ such that $A = B + C$, then we call C Hukuhara difference of A and B. The interval C we denote by $A \ominus B$. Notice, for example, that $[1, 2] \ominus [1, 3]$ does not exist, and $A \ominus B \neq A + (-)B$.

For $A = [a^-, a^+] \in \mathbb{I}$ denote the diameter and the magnitude of A by

$$\mathrm{diam}(A) := a^+ - a^- \quad \text{and} \quad |||A||| := H(A, \{0\}) = \max\{|a^-|, |a^+|\},$$

respectively. It is known that $A \ominus B$ exists in the case $\mathrm{diam}(A) \geq \mathrm{diam}(B)$ (cf. [16, 17]). Also one can verify the following properties for $A, B, C, D \in \mathbb{I}$:

$$\text{if } \ A \ominus B \ \text{ exists, then } \ |||A \ominus B||| = H(A, B);$$

$$\text{if } \ A \ominus B, A \ominus C \ \text{ exist, then } \ H(A \ominus B, A \ominus C) = H(B, C);$$

$$\text{if } \ A \ominus B, C \ominus D \ \text{ exist, then } \ H(A \ominus B, C \ominus D) = H(A + D, B + C);$$

$$\text{if } \ A \ominus B, A \ominus (B + C) \ \text{ exist, then there exists } \ (A \ominus B) \ominus C \ \text{ and } \ (A \ominus B) \ominus C = A \ominus (B + C);$$

$$\text{if } \ A \ominus B, A \ominus C, C \ominus D \ \text{ exist, then there exists } \ (A \ominus B) \ominus (A \ominus C) \ \text{ and } \ (A \ominus B) \ominus (A \ominus C) = C \ominus B.$$

Definition 2.1 We say that the interval-valued mapping $F : [\alpha, \beta] \to \mathbb{I}$ is continuous at the point $t \in [\alpha, \beta]$, if for every $\varepsilon > 0$ there exists $\delta = \delta(t, \varepsilon) > 0$, such that for all $s \in [\alpha, \beta]$, if $|t - s| < \delta$, one has $H(F(t), F(s)) < \varepsilon$.

If $F : [\alpha, \beta] \to \mathbb{I}$ is continuous at every point $t \in [\alpha, \beta]$, then we will say that F is continuous on $[\alpha, \beta]$.

Further we want to introduce the notions of differentiability and integrability which will be used in the paper.

Definition 2.2 ([18, 19]) Let $F : [\alpha, \beta] \to \mathbb{I}$ be given. We define the nth-order differential of F as follows: Let $F : [\alpha, \beta] \to \mathbb{I}$ and $t_0 \in (\alpha, \beta)$. We say that F is strongly generalized differentiable of the nth-order at t_0. If there exists elements $F^{(s)}(t_0) \in \mathbb{I}, s = 1, 2, \cdots, n$. For all $h > 0$ sufficiently small, there exists $F^{(s-1)}(t_0) \ominus F^{(s-1)}(t_0 + h), F^{(s-1)}(t_0 - h) \ominus F^{(s-1)}(t_0)$ and the following limits hold (in the metric H):

$$\lim_{h\to 0^+} (-\frac{1}{h})(F^{(s-1)}(t_0 - h) \ominus F^{(s-1)}(t_0)) = \lim_{h\to 0^+} (-\frac{1}{h})(F^{(s-1)}(t_0) \ominus F^{(s-1)}(t_0 + h)) = F^{(s)}(t_0).$$

If $s = 1$, we get the following definition.

Definition 2.3 ([18, 19]) A mapping $F : [\alpha, \beta] \to \mathbb{I}$ is second type Hukuhara differentiable at $t_0 \in [\alpha, \beta]$, if there exists $F^{'}(t_0) \in \mathbb{I}$ such that the limits

$$\lim_{h\to 0^+} (-\frac{1}{h})(F(t_0 - h) \ominus F(t_0)), \quad \lim_{h\to 0^+} (-\frac{1}{h})(F(t_0) \ominus F(t_0 + h))$$

exist and are equal to $F^{'}(t_0)$. The interval $F^{'}(t_0)$ is said to be the second type Hukuhara derivative of interval-valued mapping F at the point t_0.

The limits are taken in the metric space $(\mathbb{I}, H)$, and at the boundary points one considers only the one-sided derivatives. The function $F : [\alpha, \beta] \to \mathbb{I}$ is called second type Hukuhara differentiable on $[\alpha, \beta]$ if F is second type Hukuhara differentiable at every point $t_0 \in [\alpha, \beta]$.

Remark 2.1 ([23]) Let $F : [\alpha, \beta] \to \mathbb{I}$ be given. Denote $F(t) = [F^-(t), F^+(t)]$, where $F^-, F^+ : [\alpha, \beta] \to \mathbb{R}$. If the mapping F is nth-order differentiable at $t_0 \in [\alpha, \beta]$, then the real-valued functions F^-, F^+ are differentiable at t_0 and $F^{'}(t_0) = [(F^+)^{'}(t_0), (F^-)^{'}(t_0)]$.

Remark 2.2 Let $F : [\alpha, \beta] \to \mathbb{I}$ be given. Denote $F(t) = [F^-(t), F^+(t)]$, where $F^-, F^+ : [\alpha, \beta] \to \mathbb{R}$. If the mapping F is nth-order differentiable at $t_0 \in [\alpha, \beta]$, then the real-valued functions F^-, F^+ are differentiable at t_0 and $F^{(2k-1)}(t_0) = [(F^+)^{(2k-1)}(t_0), (F^-)^{(2k-1)}(t_0)], F^{(2k)}(t_0) = [(F^-)^{(2k)}(t_0), (F^+)^{(2k)}(t_0)]$, $k = 1, 2, \cdots, \frac{n}{2}, k \in \mathbf{Z}$.

Remark 2.3 ([23]) Let Let $F : [\alpha, \beta] \to \mathbb{I}$ be second type Hukuhara differentiable on $[\alpha, \beta]$. Then

(i) F is continuous on $[\alpha, \beta]$,

(ii) the function $\mathrm{diam}(F) : [\alpha, \beta] \to [0, \infty)$ is nonincreasing on $[\alpha, \beta]$.

For convenience of the numerical calculations, We now follow Goetschel and Voxman [5] and define the integral of a interval-valued function using the Riemann integral concept.

Definition 2.4 Let $F : [\alpha, \beta] \to \mathbb{I}$ is an an interval-valued function, $F(t) = [F^-(t), F^+(t)]$. For each partition $P = \{t_0, t_1, \cdots, t_n\}$ of $[\alpha, \beta]$ and for arbitrary $\xi_i : t_{i-1} \le \xi_i \le t_i, 1 \le i \le n$, let

$$R_P = \sum_{i=1}^{n} F(\xi_i)(t_i - t_{i-1}).$$

The definite integral of $F(t)$ over $[\alpha, \beta]$ is

$$\int_\alpha^\beta F(t)dt = \lim_{\|T\|\to 0} R_P,$$

where

$$\|T\| = \max_{0\le i\le n} |t_i - t_{i-1}|,$$

provided that this limit exists in the metric H.

If the interval-valued function $F(t)$ is continuous in the metric H, its definite integral exists [5]. Furthermore,

$$\int_\alpha^\beta F(t)dt := [\int_\alpha^\beta F^-(t)dt, \int_\alpha^\beta F^+(t)dt].$$

By the Newton-Leibniz formula (see Theorem 30, Corollary 31, [6]) one can write: if an interval-valued function F is second type Hukuhara differentiable on $[\alpha, \beta]$, then $F(\alpha) = F(\beta) + (-1) \int_\alpha^\beta F^{'}(t)dt$.

It should be noted that the fuzzy integral can be also defined using the Lebesgue-type approach [22, 23]. However, if $F(t)$ is continuous, both approaches yield the same value. Moreover, the representation of the fuzzy integral using Definition 2.3 is more convenient for numerical calculations.

3. Interval initial value problem

In this paper, we consider the Cauchy problem for interval differential equations with the second type Hukuhara derivative, i.e.,

$$\begin{cases} X'(t) = F(t, X(t)), \\ X(t_0) = X_0, \end{cases} \tag{1}$$

where $F : [\alpha, \beta] \times \mathbb{I} \to \mathbb{I}, X_0 = [X_0^-, X_0^+] \in \mathbb{I}$ with X_0^-, X_0^+ are the data of the equation, and the symbol $'$ denotes the derivative from Definition 2.2.

A more thorough discussion of the interval-valued function $F(t, X)$ which is necessary before proceeding with numerical examples will be given later. Sufficient conditions for the existence of a unique solution to Eq. (1) are that F is continuous, nontrivial and that a Lipschitz condition

$$H(F(t, A), F(t, B)) \leq L \cdot H(A, B), A, B \in \mathbb{I}$$

is satisfied for some $L > 0$. These conditions are given by Marek T. Malinowski [24]. Due to Remark 2.1 we may replace Eq. (1) by the equivalent system

$$\begin{cases} (X^+)'(t) = F^-(t, X^-(t), X^+(t)), X^-(t_0) = X_0^-, \\ (X^-)'(t) = F^+(t, X^-(t), X^+(t)), X^+(t_0) = X_0^+, \end{cases} \tag{2}$$

which thus possesses a unique solution $X(t) = [X^-(t), X^+(t)] \in \mathbb{I}$, which is second type Hukuhara differentiable. Eq.(2) represents an ordinary Cauchy problem for which any converging classical numerical procedure may be applied. The most elementary numerical scheme is the Euler and Runge－Kutta method. Here, we only discuss the fuzzy Euler method.

To integrate the system given in Eq. (2) from $[\alpha, \beta]$, we replace the interval $[\alpha, \beta]$ by a set of discrete equally spaced grid points

$$\alpha = t_0 < t_1 < \cdots < t_N = \beta,$$

at which the exact solution $X(t) = [X^-(t), X^+(t)]$ is approximated by some $Y(t) = [Y^-(t), Y^+(t)]$. The exact and approximate solutions at $t_n, 0 \leq n \leq N$ are denoted by $X_n = [X_n^-, X_n^+]$ and $Y_n = [Y_n^-, Y_n^+]$, respectively. The grid points at which the solution is calculated are

$$t_n = t_0 + nh, h = \frac{\beta - \alpha}{N}.$$

The Euler method is based on the first-order approximation of $(X^-)'(t)$ and $(X^+)'(t)$, given by

$$(X^-)'(t_n) = \frac{X^-(t_{n+1}) - X^-(t_n)}{h},$$

$$(X^+)'(t_n) = \frac{X^+(t_{n+1}) - X^+(t_n)}{h},$$

then, we obtain

$$X_{n+1}^- \approx X_n^- + hF_n^+,$$

$$X_{n+1}^+ \approx X_n^+ + hF_n^-,$$

where

$$F_n^- = F^-(t_n, X_n^-, X_n^+),$$

$$F_n^+ = F^+(t_n, X_n^-, X_n^+).$$

Following above equality, we define

$$Y_{n+1}^- = Y_n^- + hF^+(t_n, Y_n^-, Y_n^+),$$

$$Y_{n+1}^+ = Y_n^+ + hF^-(t_n, Y_n^-, Y_n^+),$$

where $Y_0^- = X_0^-, Y_0^+ = X_0^+$. The polygon curves

$$Y^-(t, h) \triangleq \{[t_0, Y_0^-], [t_1, Y_1^-], \cdots, [t_N, Y_N^-]\},$$

$$Y^+(t,h) \triangleq \{[t_0, Y_0^+], [t_1, Y_1^+], \cdots, [t_N, Y_N^+]\},$$

are the Euler approximates to $X^-(t)$ and $X^+(t)$, respectively, over the interval $t_0 \le t \le t_N$. In what follows, we will present a convergence theorem, i.e.,

$$\lim_{h\to 0} Y^-(t,h) = X^-(t),$$

$$\lim_{h\to 0} Y^+(t,h) = X^+(t).$$

Let $F^-(t,u,v)$ and $F^+(t,u,v)$ be the functions F^- and F^+ of Eq. (2) where u and v are constants and $u \le v$. In other words, $F^-(t,u,v)$ and $F^+(t,u,v)$ are obtained by substituting $X = (u,v)$ in Eq. (2). The domain where F^- and F^+ are defined is therefore $K = \{(t,u,v)|t_0 \le t \le t_N, -\infty < v < \infty, -\infty < u < v\}$.

Theorem 3.1 Let $F^-(t,u,v), F^+(t,u,v)$ belong to $C^1(K)$ and let the derivatives of F^- and F^+ be bounded over K. Then, the Euler approximates of Eq. (2) converge to the exact solutions $X^-(t)$ and $X^+(t)$ uniformly in t.

Proof As in ordinary differential equations [25], it is sufficient to show By using Taylor's theorem we get

$$X_{n+1}^- \approx X_n^- + hF_n^+ + \frac{(X^-(\xi_n^-))''}{2}h^2, \xi_n^- \in (t_n, t_{n+1}),$$

$$X_{n+1}^+ \approx X_n^- + hF_n^- + \frac{(X^+(\xi_n^-))''}{2}h^2, \xi_n^+ \in (t_n, t_{n+1}).$$

Consequently

$$X_{n+1}^- - Y_{n+1}^- = X_n^- - Y_n^- + hF^+(t_n, X_n^-, X_n^+) - hF^+(t_n, Y_n^-, Y_n^+) + \frac{(X^-)''(\xi_n^-)}{2}h^2, \xi_n^- \in (t_n, t_{n+1}),$$

$$X_{n+1}^+ - Y_{n+1}^+ = X_n^+ - Y_n^+ + hF^-(t_n, X_n^-, X_n^+) - hF^-(t_n, Y_n^-, Y_n^+) + \frac{(X^+)''(\xi_n^+)}{2}h^2, \xi_n^+ \in (t_n, t_{n+1}),$$

Denote $W_n = X_n^- - Y_n^-, V_n = X_n^+ - Y_n^+$. Then

$$|W_{n+1}| \le |W_n| + h|F^+(t_n, X_n^-, X_n^+) - F^+(t_n, Y_n^-| + \frac{h^2}{2}M^-,$$

$$|V_{n+1}| \le |V_n| + h|F^-(t_n, X_n^-, X_n^+) - F^-(t_n, Y_n^-| + \frac{h^2}{2}M^+,$$

where

$$M^- = \max_{1\le n\le N}(X^-)''(\xi_n^-), \quad M^+ = \max_{1\le n\le N}(X^+)''(\xi_n^+).$$

By applying the Lipschitz condition, we obtain

$$|W_{n+1}| \le |W_n| + Lh\max\{|W_n|, |V_n|\} + \frac{h^2}{2}M^-,$$

$$|V_{n+1}| \le |V_n| + Lh\max\{|W_n|, |V_n|\} + \frac{h^2}{2}M^+,$$

denote $U_n = |W_n| + |V_n|$, since $\max\{a,b\} = \frac{1}{2}(a + b + |a-b|)$, then

$$|W_{n+1}| + |V_{n+1}| \le (1+2Lh)(|W_n| + |V_n|) + \frac{h^2}{2}M^- + \frac{h^2}{2}M^+$$

then

$$U_n \le (1+2Lh)U_{n-1} + \frac{h^2}{2}M^- + \frac{h^2}{2}M^+ \le (1+2Lh)^n U_0 + \frac{(1+2Lh)^n - 1}{2Lh}(\frac{h^2}{2}M^- + \frac{h^2}{2}M^+).$$

then

$$\lim_{h\to 0}(|W_n| + |V_n|) = |W_0| + |V_0|,$$

Since $W_0 = X_0^- - Y_0^- = 0, V_0 = X_0^+ - Y_0^+ = 0$, we obtain

$$\lim_{n\to\infty} W_n = 0, \quad \lim_{n\to\infty} V_n = 0.$$

4. Fuzzy integral equations

The notation of fuzzy integral equation may be introduced indirectly by replacing the fuzzy Cauchy problem

$$\begin{cases} X'(t) = F(t, X(t)), t \in [\alpha, \beta] \\ X(t_0) = X_0, \end{cases} \tag{3}$$

by a fuzzy integral equation

$$X(t) = X_0 \ominus (-1) \int_{t_0}^{t} F(s, X(s))ds.$$

Due to the form of integral of interval-valued function $F(t)$, we may replace Eq.(3) by the equivalent system

$$X(t) = X_0 \ominus (-1)[\int_{t_0}^{t} F^-(s, X^-(s), X^+(s))ds, \int_{t_0}^{t} F^+(s, X^-(s), X^+(s))ds],$$

which thus possesses a unique solution $X(t) = [X^-(t), X^+(t)] \in \mathbb{I}$, which is second type Hukuhara differentiable. Eq.(3) represents an ordinary integral for which any converging classical numerical integration may be applied.

However, this equation may not be easily used for approximating $X(t)$ because of the possible complexity of the integrand $F(t, X(t))$. Generally, it would be quite tiresome to determine $F^-(s, X(s))$ and $F^+(s, X(s))$ at arbitrary stage of the computation. A more applicable approach is presented in this section.

Prior to representing a numerical procedure for calculating the integral of an interval-valued function, we will further restrict our discussion to $X(t) = (X^-(t), X^+(t))$ interval-valued function for which $X(t)$ is continuous.

4.1. The numerical method for integration

To calculate the Riemann integrals $\int_\alpha^\beta F^-(t)dt$ and $\int_\alpha^\beta F^+(t)dt$, we apply the Simpson rule(SR) and composite Simpson's rule(CSR). The interval $[\alpha, \beta]$ is partitioned by equally spaced points:

$$\alpha = t_0 < t_1 < \cdots < t_N = \beta,$$

$$t_i - t_{i-1} = \frac{\beta - \alpha}{N}, 1 \le i \le N,$$

and the integral of an arbitrary crisp function $F^-(s, X^-(s), X^+(s)), F^+(s, X^-(s), X^+(s))$ over $[\alpha, \beta]$ is approximated by

$$\begin{aligned}\int_\alpha^\beta F^-(s, X^-(s), X^+(s))ds \approx \frac{\beta - \alpha}{6}(&F^-(\alpha, X^-(\alpha), X^+(\alpha)) + 4F^-(\frac{\alpha + \beta}{2}, X^-(\frac{\alpha + \beta}{2}), X^+(\frac{\alpha + \beta}{2})) \\ &+F^-(\beta, X^-(\beta), X^+(\beta))) \\ \int_\alpha^\beta F^+(s, X^-(s), X^+(s))ds \approx \frac{\beta - \alpha}{6}(&F^+(\alpha, X^-(\alpha), X^+(\alpha)) + 4F^+(\frac{\alpha + \beta}{2}, X^-(\frac{\alpha + \beta}{2}), X^+(\frac{\alpha + \beta}{2})) \\ &+F^+(\beta, X^-(\beta), X^+(\beta)))\end{aligned}$$

Furthermore, if the interval $[\alpha, \beta]$ is partitioned by equally spaced points:

$$\alpha = t_0 < t_1 < \cdots < t_{2N} = \beta.$$

The grid points at which the solution is calculated are

$$t_i - t_{i-1} = \frac{\beta - \alpha}{2N}, t_i = a + ih, 0 \le i \le 2N,$$

Let $F^-(t) = F^-(t, X^-(t), X^+(t)), F^+(t) = F^+(t, X^-(t), X^+(t))$, according to the Simpson rule over $[t_{2i}, t_{2i+2}]$, we get

$$\int_{t_{2i}}^{t_{2i+2}} F^-(t)dt = \frac{h}{3}(F^-(t_{2i}) + 4F^-(t_{2i+1}) + F^-(t_{2i+2}) - \frac{h^5}{90}(F^-)^4(\xi_i),$$

$$\int_{t_{2i}}^{t_{2i+2}} F^+(t)dt = \frac{h}{3}(F^+(t_{2i}) + 4F^+(t_{2i+1}) + F^+(t_{2i+2}) - \frac{h^5}{90}(F^+)^4(\xi_i),$$

then we get the composite Simpson rule is

$$\int_{\alpha}^{\beta} F^-(t)dt = \sum_{i=0}^{2N}\int_{t_{2i}}^{t_{2i+2}} F^-(t)dt \approx \frac{h}{3}(F^-(t_0) + F^-(t_{2i}) + 4\sum_{i=0}^{2N} F^-(t_{2i+1}) + 2\sum_{i=0}^{2N} F^-(t_{2i})),$$

$$\int_{\alpha}^{\beta} F^+(t)dt = \sum_{i=0}^{2N}\int_{t_{2i}}^{t_{2i+2}} F^+(t)dt \approx \frac{h}{3}(F^+(t_0) + F^+(t_{2i}) + 4\sum_{i=0}^{2N} F^+(t_{2i+1}) + 2\sum_{i=0}^{2N} F^+(t_{2i})),$$

Theorem 4.1 Let $F^-(s), F^+(s)$ belong to $C^4[\alpha,\beta]$, Then, the truncation error of the Simpson rule is

$$R_1(F) \in [\min(R_1(F^-), R_1(F^+)), \max(R_1(F^-), R_1(F^+))],$$

where $R_1(F^-) = -\frac{(\beta-\alpha)^5}{2880}(F^-)^{(4)}(\eta), R_1(F^+) = -\frac{(\beta-\alpha)^5}{2880}(F^+)^{(4)}(\xi), \eta, \xi \in [\alpha,\beta]$.

Proof For the $F(t) = [F^-(t), F^+(t)]$, constructing the polynomial $P_3^-(t), P_3^+(t)$ that their power are less than 3, which satisfy

$$P_3^-(\alpha) = F^-(\alpha), \quad P_3^+(\alpha) = F^+(\alpha),$$

$$P_3^-(\frac{\alpha+\beta}{2}) = F^-(\frac{\alpha+\beta}{2}), \quad P_3^+(\frac{\alpha+\beta}{2}) = F^+(\frac{\alpha+\beta}{2}),$$

$$(P_3^-)'(\frac{\alpha+\beta}{2}) = (F^-)'(\frac{\alpha+\beta}{2}), \quad (P_3^+)'(\frac{\alpha+\beta}{2}) = (F^+)'(\frac{\alpha+\beta}{2}),$$

Since the algebraic accuracy of the Simpson rule is 3, then

$$F^-(t) - P_3^-(t) = \frac{(F^-)^{(4)}(\eta)}{4!}(t-a)(t-\frac{\alpha+\beta}{2})(t-b), \eta \in [\alpha,\beta],$$

$$F^+(t) - P_3^+(t) = \frac{(F^+)^{(4)}(\xi)}{4!}(t-a)(t-\frac{\alpha+\beta}{2})(t-b), \xi \in [\alpha,\beta],$$

thus, we obtain

$$R_1(F^-) = \frac{1}{4!}\int_{\alpha}^{\beta} \frac{(F^-)^{(4)}(\eta)}{4!}(t-a)(t-\frac{\alpha+\beta}{2})(t-b)dt,$$

$$R_1(F^+) = \frac{1}{4!}\int_{\alpha}^{\beta} \frac{(F^+)^{(4)}(\xi)}{4!}(t-a)(t-\frac{\alpha+\beta}{2})(t-b)dt.$$

It is easily to see that $(t-a)(t-\frac{\alpha+\beta}{2})(t-b) < 0$ for arbitrary $t \in [\alpha,\beta], F(t)$ is continuous over $[\alpha,\beta]$, applying the integral mean value theorem, then, there exist $\eta_1, \xi_1 \in [\alpha,\beta]$, such that

$$R_1(F^-) = -\frac{(\beta-\alpha)^5}{2880}(F^-)^{(4)}(\eta_1),$$

$$R_1(F^+) = -\frac{(\beta-\alpha)^5}{2880}(F^+)^{(4)}(\xi_1).$$

Then,

$$R_1(F) \in [\min(R_1(F^-), R_1(F^+), \max(R_1(F^-), R_1(F^+))].$$

Theorem 4.2 Let $F^-(s), F^+(s)$ belong to $C^4[\alpha,\beta]$. Then, the truncation error of the composite Simpson rule is

$$R_2(F) \in [\min(R_2(F^-), R_2(F^+), \max(R_2(F^-), R_2(F^+))],$$

where $R_2(F^-) = -\frac{(\beta-\alpha)^5}{2880n^4}(F^-)^{(4)}(\eta), R_2(F^+) = -\frac{(\beta-\alpha)^5}{2880n^4}(F^+)^{(4)}(\xi), \eta, \xi \in [\alpha,\beta]$.

Proof The proof is similar to Theorem 4.1.

4.2. The improved Simpson rule(ISR)

To improve the order of the accuracy of the classical Simpson rule, we propose a new numerical calculation formula, we call it the improved Simpson rule. The interval $[\alpha, \beta]$ is partitioned by equally spaced points:

$$\alpha = t_0 < t_1 < \cdots < t_N = \beta,$$

$$h = t_i - t_{i-1} = \frac{\beta - \alpha}{N}, 1 \le i \le N,$$

and the integral of an arbitrary crisp function $F^-(s, X^-(s), X^+(s)), F^+(s, X^-(s), X^+(s))$ over $[\alpha, \beta]$ is approximated by

$$\int_\alpha^\beta F^-(s, X^-(s), X^+(s))ds \approx h(F^-(\frac{\alpha + \beta}{2} + \sum_{i=1}^{N-1} F^-(\frac{t_i + t_{i+1}}{2})),$$

$$\int_\alpha^\beta F^+(s, X^-(s), X^+(s))ds \approx h(F^+(\frac{\alpha + \beta}{2}) + \sum_{i=1}^{N-1} F^+(\frac{t_i + t_{i+1}}{2})).$$

Define

$$S_N^- = h(F^-(\frac{\alpha + \beta}{2}) + \sum_{i=1}^{N-1} F^-(\frac{t_i + t_{i+1}}{2})),$$

$$S_N^+ = h(F^+(\frac{\alpha + \beta}{2}) + \sum_{i=1}^{N-1} F^-(\frac{t_i + t_{i+1}}{2})).$$

Then, we have

$$\lim_{N \to \infty} S_N^- = X^- = \int_\alpha^\beta F^-(s, X^-(s), X^+(s))ds,$$

$$\lim_{N \to \infty} S_N^+ = X^+ = \int_\alpha^\beta F^+(s, X^-(s), X^+(s))ds.$$

Theorem 4.3 If $F(t)$ is continuous (in the metric H) over $[\alpha, \beta]$, the convergence of S_N^-, S_N^+ to X^-, X^+, respectively, is uniform in t.

Proof The continuity of $F(t)$ guarantees the existence of the definite integral of $F(t)$. Thus $R_P = \sum_{i=1}^{N} F(\xi_i)(t_i - t_{i-1})$ converges to this integral in the metric H if

$$\lim_{N \to \infty} \{ \max_{1 \le i \le N} (t_i - t_{i-1}) \} = 0.$$

It is easily seen that the improved Simpson rule using 2N+1 integration nodes can be represented in the form of R_P where S_N^-, S_N^+ holds. For arbitrary $R_P = (R_P^-, R_P^+)$ and $X = (X^-, X^+)$ we have

$$H(R_P, X) = \max\{|R_p^- - X^-|, |R_p^+ - X^+|\},$$

and since

$$\lim_{N \to \infty} H(R_P, X) = 0, \quad \max_{1 \le i \le N} (t_i - t_{i-1}) \to 0$$

we obtain that R_P^-, R_P^+ converge uniformly to X^-, X^+, respectively. Consequently, S_N^- and S_N^+ (which are a particular case of R_P, R_P) converge uniformly to X^-, X^+ as well. Thus, it concludes the proof of Theorem 4.3.

5. Example

$$\begin{cases} X'(t) = (1-t)[0,4], t \in [0,1] \\ X(t_0) = [0,3], \end{cases} \tag{4}$$

In [12], it has proved (4) possesses a unique second type Hukuhara differentiable solution expressed by

$$X(t) = [4t - 2t^2, 3] \quad \text{for } t \in [0,1].$$

Table 1 Relative values at various points for $X(t)$

h	FER	SR	CSR	ISR	$Exact$
0.01	$[1.9800, 3.0000]$	$[2.0000, 3.0000]$	$[1.9956, 3.0000]$	$[1.9802, 3.0000]$	$[2.0000, 3.0000]$
0.02	$[1.9600, 3.0000]$	$[2.0000, 3.0000]$	$[1.9911, 3.0000]$	$[1.9608, 3.0000]$	$[2.0000, 3.0000]$
0.05	$[1.9000, 3.0000]$	$[2.0000, 3.0000]$	$[1.9778, 3.0000]$	$[1.9050, 3.0000]$	$[2.0000, 3.0000]$
0.10	$[1.8000, 3.0000]$	$[2.0000, 3.0000]$	$[1.9556, 3.0000]$	$[1.8200, 3.0000]$	$[2.0000, 3.0000]$
0.20	$[1.6000, 3.0000]$	$[2.0000, 3.0000]$	$[1.9022, 3.0000]$	$[1.6800, 3.0000]$	$[2.0000, 3.0000]$

The numerical results of the exact solutions and approximate solutions of $X(t)$ are shown with Table 1. Except the the Simpson rule, then the composite Simpson rule is the better approximation for the exact solution. In our example, the approximation solution is equal to exact solution by using Simpson rule, indeed, this is a coincidence, because from Theorem 4.1 and 4.2 we know that composite Simpson rule is better than Simpson rule in approximating $X(t)$.

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Controllability for semilinear integrodifferential control systems with unbounded operators

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Abstract

In this paper we deal with approximate controllability for semilinear integrodifferential control systems with unbounded operators and nonlinear integral terms by using the homotopy property of topological degree theory. Our method is to apply for the compactness of the solution mapping under the natural assumptions on inclusion relations of state spaces by using the known Sobolev's imbedding theorem and a variation of solutions of the given system.

Keywords: semilinear integrodifferential control system, approximate controllability, compactness, topological degree theory, reachable set,

AMS Classification Primary 35B37; Secondary 93C20

1 Introduction

In this paper, we are interested in the following semilinear functional integrodifferential control system on Hilbert spaces:

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This work was supported by a Research Grant of Pukyong National University(2014Year).

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$$\begin{cases} x'(t) = Ax(t) + g(t, x(t), \int_0^t k(t,s,x(s))ds) + (Bu)(t), \quad 0 < t, \\ x(0) = x_0. \end{cases} \tag{1.1}$$

Let H be a Hilbert space and V be another Hilbert space as a dense subspace of H. Here, $A : D(A) \subset H \to H$ is an unbounded operator and the controller B is a linear bounded operator from a Banach space $L^2(0, T; U)$ to $L^2(0, T; H)$ for any $T > 0$, where U is a Banach space. the nonlinear term g is a semilinear version of the quasilinear form.

The problem of existence for solutions of semilinear evolution equations in Banach spaces has been established by several authors [1, 2] and references and therein.

As for control problems for the semilinear control systems, Naito [3] and [2, 4, 5] proved the approximate controllability under the range conditions of the controller B. In recent years, Carrasco and Lebia [6, 7] discussed sufficient conditions for approximate controllability for semilinear differential equations with delay. The previous results on the approximate controllability of semilinear control systems have been proved as a particular case of sufficient conditions for the approximate solvability of semilinear equations by assuming either the compactness of the semigroup generated by A or the approximate controllability of the corresponding linear system.

The purpose of this paper is to establish the approximate controllability for semilinear integrodifferential control systems with unbounded operators and nonlinear integral terms under more general conditions on the nonlinear term and the controller. We no longer require the strict range condition on B. First, provided that the injection $V \subset H$ is compact, we will obtain the compactness of the solution mapping of initial data to state space by using the known Sobolev's imbedding theorem and a variation of solutions of the given system, thereafter, we investigate approximate controllability of (1.1) by using the homotopy property of topological degree theory, which is applicable for control problems and the optimal control problem of systems governed by nonlinearity.

2 Nonlinear functional equations

If H is identified with its dual space we may write $V \subset H \subset V^*$ densely and the corresponding injections are continuous, where V^* stands for the dual space of V. The norm on V(resp. H) will be denoted by $||\cdot||$ (resp. $|\cdot|$). For the sake of simplicity, we may consider

$$||u||_* \leq |u| \leq ||u||, \quad u \in V, \tag{2.1}$$

where $||\cdot||_*$ is the norm of the element of V^*.

We need to impose the following conditions:

Assumption (A). Let $A : D(A) \subset H \to H$ be given a linear operator such that

(A1) $A(0) = 0, \quad (Au, u) \geq \omega_1 ||u||^2, \quad \forall v \in V,$

(A2) $|Au| \leq \omega_3(||u|| + 1)$, where $\omega_1,\ \omega_3$ are some positive constants.

Assumption (K). Let $k : \mathbb{R}^+ \times \mathbb{R}^+ V \to H$ be a nonlinear mapping satisfying the following:

(K1) For any $x \in V$ the mapping $k(\cdot, \cdot, x)$ is measurable;

(K2) There exist positive constants K_0, K_1 such that

$$|k(t, s, 0)| \leq K_0, \quad |k(t, s, x) - k(t, s, y)| \leq K_1 ||x - y||$$

for all $(t, s) \in \mathbb{R}^+ \times [-h, 0]$ and $x, y \in V$.

Assumption (G). Let $g : \mathbb{R}^+ \times V \times H \to H$ be a nonlinear mapping satisfying the following:

(G1) For any $x \in V$, $y \in H$ the mapping $g(\cdot, x, y)$ is measurable;

(G2) There exist positive constants L_0, L_1, L_2 such that

$$|g(t, 0, 0)| \leq L_0, \quad |g(t, x, y) - g(t, \hat{x}, \hat{y})| \leq L_1 ||x - \hat{x}|| + L_2 |y - \hat{y}|$$

for all $t \in \mathbb{R}^+$, $x, \hat{x} \in V$, and $y, \hat{y} \in H$.

From the following inequalities

$$\omega_1 ||u||^2 \leq \mathrm{Re}\,(Au, u) + \omega_2 |u|^2 \leq C|Au|\,|u| + \omega_2 |u|^2 \leq \max\{C, \omega_2\} ||u||_{D(A)} |u|,$$

where

$$||u||_{D(A)} = (|Au|^2 + |u|^2)^{1/2}$$

is the graph norm of $D(A)$, it follows that there exists a constant $C_0 > 0$ such that

$$||u|| \leq C_0 ||u||_{D(A)}^{1/2} |u|^{1/2}. \tag{2.2}$$

Thus we have the following sequence

$$D(A) \subset V \subset H \subset V^* \subset D(A)^*,$$

where each space is dense in the next one which continuous injection.

Lemma 2.1. *With the notations (2.1), (2.2), we have*

$$(V, V^*)_{1/2,2} = H, \quad \text{and } (D(A), H)_{1/2,2} = V,$$

where $(V, V^*)_{1/2,2}$ denotes the real interpolation space between V and V^*(Section 1.3.3 of [8]).

For $x \in L^2(-h, T; V)$, $T > 0$ we set

$$G(t, x) = g(t, x(t), \int_0^t k(t, s, x(s))ds).$$

The above operator g is the semilinear case of the nonlinear part of quasilinear equations considered by Yong and Pan [9]. Here as in [9] we consider the Borel measurable corrections of $x(\cdot)$.

Lemma 2.2. *Let $x \in L^2(0, T; V)$, $T > 0$. Then $G(\cdot, x) \in L^2(0, T; H)$ and*

$$||G(\cdot, x)||_{L^2(0,T;H)} \le L_0\sqrt{T} + (L_1\sqrt{T} + L_2K_1T)||x||_{L^2(0,T;V)}. \tag{2.3}$$

Moreover if $x_1,\ x_2 \in L^2(0, T; V)$, then

$$||G(\cdot, x_1) - G(\cdot, x_2)||_{L^2(0,T;H)} \le (L_1\sqrt{T} + L_2K_1T)||x_1 - x_2||_{L^2(0,T;V)}. \tag{2.4}$$

Proof. From (G2), (2.1) and the above inequality it is easily seen that

$$\begin{aligned}
||G(\cdot, x)||_{L^2(0,T;H)} &\le ||G(\cdot, 0)|| + ||G(\cdot, x) - G(\cdot, 0)|| \\
&\le L_0\sqrt{T} + L_1||x||_{L^2(0,T;V)} + L_2||\int_0^{\cdot} k(\cdot, s, x(s))ds||_{L^2(0,T;H)} \\
&\le L_0\sqrt{T} + (L_1\sqrt{T} + L_2K_1T)||x||_{L^2(0,T;V)}.
\end{aligned}$$

Similarly, we can prove (2.4). □

Now, consider the following semilinear control system with $B = I$:

$$\begin{cases} x'(t) = Ax(t) + G(t, x(t)) + h(t), \quad t > 0, \\ x(0) = x_0. \end{cases} \tag{2.5}$$

Referring to the result by Jeong and Rho [10], the result on the solvability of the equation (1.1) can be given as follows.

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Proposition 2.1. *(1) Let $x_0 \in H$ and $h \in L^2(0,T;V^*)$, $T > 0$. Then the nonlinear equation (2.5) has a unique solution x belonging to*

$$L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H)$$

and satisfying there exists a constant C_1 such that

$$||x||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \le C_1(1 + |x_0| + ||h||_{L^2(0,T;V^*)}). \tag{2.6}$$

(2) For any $x_0 \in V$ and $h \in L^2(0,T;H)$. Then the nonlinear equation (2.5) has a unique solution x belonging to

$$L^2(0,T;D(A)) \cap W^{1,2}(0,T;H) \subset C([0,T];V)$$

and satisfying there exists a constant C_1 such that

$$||x||_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)} \le C_1(1 + ||x_0|| + ||h||_{L^2(0,T;H)}). \tag{2.7}$$

(3) If $(x_0, h) \in H \times L^2(0,T;V^*)$, then the mapping

$$H \times L^2(0,T;V^*) \ni (x_0, h) \mapsto x \in L^2(-h,T;V)) \cap W^{1,2}(0,T;V^*)$$

is Lipschitz continuous.

3 Approximate controllability

In this section we make the natural assumption that the embedding $V \subset H$ is compact. Let A and G be the operators satisfying Assumption (A) and (G), respectively. Let x_h be the solution of the equation (2.5) corresponding to $h \in L^2(0,T;H)$. We define the solution mapping S from $L^2(0,T;H)$ to $L^2(0,T;V)$ by

$$(Sh)(t) = x_h(t), \quad h \in L^2(0,T;H). \tag{3.1}$$

Let $\mathcal{A}$ and $\mathcal{G}$ be the operators corresponding to the maps A and G, which are defined by $\mathcal{A}(x)(\cdot) = Ax(\cdot)$ and $\mathcal{G}(h)(\cdot) = G(\cdot, x_h)$, respectively. Then since the solution x is represented by

$$x_h(t) = x_0 + \int_0^t ((I + \mathcal{G} - \mathcal{A}S)h)(s)ds. \tag{3.2}$$

The following is from the Theorem 2 of Au bin [12]

Lemma 3.1. *Under the compactness of the imbedding $V \subset H$, we have that the imbedding $L^2(0,T;D(A)) \cap W^{1,2}(0,T;H) \subset L^2(0,T;V)$ is compact.*

Theorem 3.1. *The mapping $S : h \mapsto x_h$ is compact from $L^2(0,T;H)$ to $L^2(0,T;V)$ where x_h is the solution of (2.5) corresponding to h. Therefore, $\mathcal{G} - \mathcal{A}S$ is a compact mapping from $L^2(0,T;H)$ to itself.*

Proof. If $h \in L^2(0,T;H)$ then form (2.6) it follows that $x_h \in L^2(0,T;V)$ satisfying

$$||x_h||_{L^2(0,T;V)} \le C_1(1 + |x_0| + ||h||_{L^2(0,T;V^*)}), \tag{3.3}$$

and so $G(\cdot, x_h) \in L^2(0,T;H)$. Consequently, according to (2) of Proposition 2.3 we have $x_h \in L^2(0,T;D(A)) \cap W^{1,2}(0,T;H)$ and from (2.3), (2.7), and (3.3)

$$\begin{aligned} &||x_h||_{L^2(0,T;D(A)) \cap W^{1,2}(0,T;H)} \le C_1||1 + G(\cdot, x_h) + h||_{L^2(0,T;H)} \\ &\le C_1\big\{1 + L_0\sqrt{T} + (L_1\sqrt{T} + L_2K_1T)||x||_{L^2(0,T;V)} + ||h||_{L^2(0,T;H)}\big\}. \end{aligned}$$

Thus, from (3.3) it follows that if h is bounded in $L^2(0,T;H)$, then so is x_h in $L^2(0,T;D(A)) \cap W^{1,2}(0,T;H)$. Invoking Lemma 3.1, since $L^2(0,T;D(A)) \cap W^{1,2}(0,T;H) \subset L^2(0,T;V)$ is compact, the mapping $h \mapsto x_h$ is compact from from $L^2(0,T;H)$ to $L^2(0,T;V)$. Hence, the mapping $h \mapsto Sh = x_h$ is compact from $L^2(0,T;H)$ to $L^2(0,T;H)$. Therefore, by Assumption (G), $\mathcal{G}$ is also a compact mapping from $L^2(0,T;H)$ to $L^2(0,T;H)$ and so is $\mathcal{G} - \mathcal{A}$ from $L^2(0,T;H)$ to itself. □

Lemma 3.2. *Put $\mathcal{F} = \mathcal{G} - \mathcal{A}S$. Then we have*

$$||\tilde{\mathcal{F}}u||_{L^2(0,T;H)} \le N_1(T)||u||_{L^2(0,T;V^*)} + 1) + N_2(T), \tag{3.4}$$

where

$$N_1(T) = (1 + {\omega_1}^{-1}\omega_3\sqrt{T})(L_1\sqrt{T} + L_2K_1T), \tag{3.5}$$

$$N_2(T) = \big\{\omega_3\sqrt{T} + C_1(L_1\sqrt{T} + L_2K_1T)\big\}(|x_0| + 1) + 1 + {\omega_1}^{-1}\omega_3L_0T. \tag{3.6}$$

Proof. Taking scalar product on both sides of (2.5) by $x(t)$

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|x(t)|^2 + \omega_1||x(t)||^2 &\le ||G(t,x(t)) + h(t)||_*||x(t)|| \\ &\le \frac{1}{2\omega_1}|G(t,x(t)) + h(t)|^2 + \frac{\omega_1}{2}||x(t)||^2 \end{aligned}$$

Integrating on $[0,t]$, we get

$$|x(t)|^2 + \frac{\omega_1}{2}\int_0^t ||x(s)||^2 ds \le |x_0|^2 + \frac{1}{\omega_1}\int_0^t ||G(t,x(t)) + h(s)|^2 ds.$$

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By using H/"older inequality, it follows that

$$||Sh||_{L^2(0,T;V)} = ||x_h||_{L^2(0,T;V)} \leq \sqrt{T}(|x_0| + \frac{1}{\omega_1}||G(\cdot, x(\cdot)) + h||_{L^2(0,T;H)}, \tag{3.7}$$

From (2.3), (3.3), (3.7) and Assumption (A) it follows that

$$\begin{aligned} ||\mathcal{F}u||_{L^2(0,T;H)} &\leq ||\mathcal{G}u||_{L^2(0,T;H)} + \omega_3(||Su||_{L^2(0,T;V)} + 1) \\ &\leq ||\mathcal{G}u||_{L^2(0,T;H)} + \omega_3\sqrt{T}\big(|x_0| + {\omega_1}^{-1}||\mathcal{G}u + u||_{L^2(0,T;V^*)} + 1\big) \\ &\leq \omega_3\sqrt{T}\big(|x_0| + {\omega_1}^{-1}||u||_{L^2(0,T;H)} + 1) \\ &\quad + \big\{1 + {\omega_1}^{-1}\omega_3\sqrt{T}(L_0\sqrt{T} + (L_1\sqrt{T} + L_2K_1T)||x_u||_{L^2(0,T;V)}\big\}, \end{aligned}$$

therefore, (3.4) derives from (3.3). □

The solution of (1,1) is denoted by $x(T; g, u)$ associated with the nonlinear term g and control u at time T.

Definition 3.1. *The system (1.1) is said to be approximately controllable at time T if $Cl\{x(T; g, u) : u \in L^2(0, T; U)\} = H$ where Cl denotes the closure in H.*

Theorem 3.2. *Let Assumptions (A), (K), and (G) be satisfied and let the embedding $V \subset H$ be compact. Then the system (1.1) is approximately controllable at time T.*

Proof. We denote the range of the operator B by H_B. Then $L^2(0, T; H) = \overline{H}_B + \overline{H}_B^{\perp}$, where $\overline{H}_B$ is the closure of H_B in $L^2(0, T; H)$. Then for each $y \in L^2(0, T; H)$, there exists a unique $(y^1, y^2) \in \overline{H}_B \times \overline{H}_B^{\perp}$ such that $y = y^1 + y^2$. Let P be the projection of $L^2(0, T; H)$ onto $\overline{H}_B$ defined by $P(y) = y^1$. Let

$$\mathcal{F} = \mathcal{G} - \mathcal{A}S, \quad \text{and } \tilde{\mathcal{F}}u = \mathcal{F}(Pu)$$

for $u \in L^2(0, T; H)$. Then $\tilde{\mathcal{F}}$ is also a compact mapping from $L^2(0, T; H)$ to itself. We define the reachable sets for the system (1.1) as follows:

$$R_T = \{x(T; g, u) : u \in L^2(0, T; U)\}.$$

Actually we are going to show that R_T is a dense subset of H. Let us fix $T_0 > 0$ so that

$$N_1(T_0) = (1 + {\omega_1}^{-1}\omega_3\sqrt{T_0})(L_1\sqrt{T_0} + L_2K_1T_0) < 1, \tag{3.8}$$

Let $z \in H$. Put $z_0 = (z + x_0)/T_0 \in L^2(0, T_0; H)$, where x_0 is the given initial data. Then there exists a unique representation $z_0 = z_0^1 + z_0^2$ with $z_0^1 \in \overline{H}_B$ and $z_0^2 \in \overline{H}_B^{\perp}$. Let r be such that $||z_0||_{L^2(0,T_0;H)} < r$ and

$$z_0 \in V_r = \{y \in L^2(0, T_0; H) : ||y||_{L^2(0,T_0;H)} < r\}.$$

Take a constant $d > 0$ such that

$$(r + N_2(T_0))(1 - N_1(T_0))^{-1} < d, \tag{3.9}$$

where N_2 is the constant defined by (3.6). Let us consider the equation

$$z_0 = \lambda\tilde{\mathcal{F}}u + u, \quad 0 \le \lambda \le 1. \tag{3.10}$$

Let u be the solution of (3.10). Since $z_0 \in V_d$ and from (3.4) and Assumption (A) it follows that

$$||u|| \le ||z_0|| + ||\tilde{\mathcal{F}}u|| \le r + N_1(T_0)||u||_{L^2(0,T_0;H)} + N_2(T_0),$$

and hence, by (3.8) and (3.9)

$$||u|| \le (r + N_2(T_0))(1 - N_1(T_0))^{-1} < d.$$

It follows that $u \notin \partial V_d$ where ∂V_d stands for the boundary of V_d. Thus by using the homotopy property of topological degree theory, there exists $u_B \in L^2(0, T_0; H)$ such that

$$z = x_0 + \int_0^t ((I + \mathcal{G} - \mathcal{A}S)u_B)(s)ds.$$

Since $u_B \in \overline{H}_B$, there exists a sequence $\{u_n\} \in L^2(0, T_0; U)$ such that $Bu_n \mapsto u$ in $L^2(0, T_0; H)$. Then by (3) of Proposition 2.1, we have that $x(\cdot; g, u_n) \mapsto x_{u_B} = z$ in $L^2(0, T_0; V) \cap W^{1,2}(0, T_0; V^*) \subset C([0, T_0]; H)$. Thus we conclude $z \in \overline{R_{T_0}}$. Therefore, the system (1.1) is approximately controllable at time T_0. Since the condition (3.8) is independent of initial values, Considering the control system (1.1) with the initial value $x(T_0) = x_{T_0}$, we can extend the result to the time $2T_0$. Hence, repeating this process we conclude that the system (1.1) is approximately controllable at time T. □

Example 3.1. *Let*

$$H = L^2(0, \pi),\ V = H_0^1(0, \pi),\ V^* = H^{-1}(0, \pi),$$
$$a(u, v) = \int_0^\pi \frac{du(x)}{dx}\overline{\frac{dv(x)}{dx}}dx,$$

and

$$(Ax)(\xi) = -\frac{d^2x(\xi)}{d\xi^2} \quad \text{with} \quad D(A) = \{y \in H^2(0, \pi) : y(0) = y(\pi) = 0\}.$$

Then it is easily seen that the system (1.1) with the operator A mentioned above is approximately controllable at time T.

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STABILITY OF POSITIVE-ADDITIVE FUNCTIONAL EQUATIONS IN FUZZY C^*-ALGEBRAS

AFRAH A.N. ABDOU*, FATMA S. AL-SIREHY AND YEOL JE CHO

ABSTRACT. In this paper, we introduce a positive-additive functional equation in fuzzy C^*-algebras. Using fixed point methods, we prove the stability of the positive-additive functional equation in fuzzy C^*-algebras.

1. Introduction

The stability problem of functional equations was originated from a question of Ulam [1] concerning the stability of group homomorphisms:

Given a group G_1, a metric group G_2 with the metric $d(\cdot,\cdot)$ and a positive number ε, does there exist $\delta > 0$ such that, if a mapping $f : G_1 \to G_2$ satisfies

$$d(f(xy), f(x)f(y)) \leq \delta$$

for all $x, y \in G_1$, then a homomorphism $h : G_1 \to G_2$ exists with

$$d(f(x), h(x)) \leq \varepsilon$$

for all $x \in G_1$?

Hyers [2] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [3] for additive mappings and by Th.M. Rassias [4] for linear mappings by considering an unbounded Cauchy difference, respectively. The paper of Th.M. Rassias [4] has provided a lot of influence in the development of what we call *generalized Hyers-Ulam stability* or *Hyers-Ulam-Rassias stability* of functional equations. A generalization of Th.M. Rassias' theorem was obtained by Găvruta [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of the Th.M. Rassias' approach. Rassias [6]–[8] followed the innovative approach of Th.M. Rassias' theorem [4] in which he replaced the factor

$$\|x\|^p + \|y\|^p$$

2010 *Mathematics Subject Classification.* Primary 39B72, 39B62, 47H10, 46L05.

Key words and phrases. Functional equation, fixed point, generalized Hyers-Ulam stability, functional inequality, linear mapping, fuzzy C^*-algebra.

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by

$$\|x\|^p \cdot \|y\|^q$$

for any $p, q \in \mathbb{R}$ with $p + q \neq 1$. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [9], [10], [11], [12], [13]).

2. Preliminaries

We recall a fundamental result in fixed point theory.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies the following conditions:

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 2.1. ([14, 15]) *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for each $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all $n \geq 0$ or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty$ *for all $n \geq n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X : d(J^{n_0} x, y) < \infty\}$;*
(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all $y \in Y$.*

We use the definition of fuzzy normed spaces given in [16, 17, 18, 19, 20, 21, 22] to investigate a fuzzy version of the Hyers-Ulam stability for the Cauchy-Jensen functional equation in the fuzzy normed algebra setting (see also [23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35]).

Definition 2.2. ([16]) Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0, 1]$ is called a *fuzzy norm* on X if, for all $x, y \in X$ and $s, t \in \mathbb{R}$,

(N1) $N(x, t) = 0$ for all $t \leq 0$;
(N2) $x = 0$ if and only if $N(x, t) = 1$ for all $t > 0$;
(N3) $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N4) $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$;
(N5) $N(x, \cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1$;
(N6) for all $x \neq 0$, $N(x, \cdot)$ is continuous on $\mathbb{R}$.

The pair (X, N) is called a *fuzzy normed vector space*.

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Definition 2.3. ([16]) (1) Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to *be convergent* or *converge* if there exists $x \in X$ such that

$$\lim_{n\to\infty} N(x_n - x, t) = 1$$

for all $t > 0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by $N\text{-}\lim_{n\to\infty} x_n = x$.

(2) Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if, for each $\varepsilon > 0$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$ and $p > 0$,

$$N(x_{n+p} - x_n, t) > 1 - \varepsilon.$$

It is well-known that every convergent sequence in a fuzzy normed vector space is a Cauchy sequence. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space.*

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if, for each sequence $\{x_n\}$ converging to x_0 in X, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X (see [16, 22]).

Definition 2.4. ([24]) A *fuzzy normed algebra* (X, N) is a fuzzy normed space (X, N) with the algebraic structure such that

(N7) $N(xy, ts) \geq \min\{N(x, t), N(y, s)\}$ for all $x, y \in X$ and $t, s > 0$.

Every normed algebra $(X, \|\cdot\|)$ defines a fuzzy normed algebra (X, N), where

$$N(x, t) = \frac{t}{t + \|x\|}$$

for all $t > 0$. This space is called the *induced fuzzy normed algebra.*

Definition 2.5. (1) Let (X, N) and (Y, N) be fuzzy normed algebras. An $\mathbb{R}$-linear mapping $f : X \to Y$ is called a *homomorphism* if

$$f(xy) = f(x)f(y)$$

for all $x, y \in X$.

(2) An $\mathbb{R}$-linear mapping $f : X \to X$ is called a *derivation* if

$$f(xy) = f(x)y + xf(y)$$

for all $x, y \in X$.

Definition 2.6. Let $(\mathcal{U}, N)$ be a fuzzy Banach algebra. Then an *involution* on $\mathcal{U}$ is a mapping $u \to u^*$ from $\mathcal{U}$ into $\mathcal{U}$ which satisfies the following conditions:

(a) $u^{**} = u$ for all $u \in \mathcal{U}$;

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(b) $(\alpha u + \beta v)^* = \overline{\alpha} u^* + \overline{\beta} v^*$;
(c) $(uv)^* = v^* u^*$ for all $u, v \in \mathcal{U}$.

If, in addition, $N(u^*u, ts) = \min\{N(u,t), N(u,s)\}$ and $N(u^*, t) = N(u,t)$ for all $u \in \mathcal{U}$ and $t, s > 0$, then $\mathcal{U}$ is a fuzzy C^*–algebra.

In this paper, we use $*$ for min.

Definition 2.7. ([36]) Let (A, N) be a fuzzy C^*-algebra and $x \in A$ be a self-adjoint element, i.e., $x^* = x$. Then x is said to be *positive* if it is of the form yy^* for some $y \in A$.

The set of positive elements of A is denoted by A^+.

Note that A^+ is a closed convex cone (see [36]). It is well-known that, for any positive element x and $n \geq 1$, there exists a unique positive element $y \in A^+$ such that $x = y^n$. We denote y by $x^{\frac{1}{n}}$ (see [37]).

Kenary [38] introduced the following functional equation:

$$f\Big(\big(\sqrt{x} + \sqrt{y}\big)^2\Big) = \big(\sqrt{f(x)} + \sqrt{f(y)}\big)^2$$

in the set of non-negative real numbers.

In this paper, we introduce the following functional equation

$$T\Big(\big(x^{\frac{1}{m}} + y^{\frac{1}{m}}\big)^m\Big) = \Big(T(x)^{\frac{1}{m}} + T(y)^{\frac{1}{m}}\Big)^m \tag{2.1}$$

for all $x, y \in A^+$ and a fixed integer m greater than 1, which is called a *positive-additive functional equation*. Each solution of the positive-additive functional equation is called a *positive-additive mapping*.

Note that the function $f(x) = cx$ for any $c \geq 0$ in the set of non-negative real numbers is a solution of the functional equation (1.1).

In 1996, Isac and Th.M. Rassias [39] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [40], [41], [42]–[45]).

Throughout this paper, let A^+ and B^+ be the sets of positive elements in fuzzy C^*-algebras (A, N) and (B, N), respectively. Assume that m is a fixed integer greater than 1.

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3. Stability of the positive-additive functional equation (1.1): fixed point approach

In this section, we investigate the positive-additive functional equation (1.1) in C^*-algebras.

Lemma 3.1. ([46]) *Let $T : A^+ \to B^+$ be a positive-additive mapping satisfying* (1.1). *Then T satisfies*

$$T(2^{mn}x) = 2^{mn}T(x)$$

for all $x \in A^+$ and $m, n \in \mathbb{N}$.

Using the fixed point method, we prove the Hyers-Ulam stability of the positive-additive functional equation (1.1) in fuzzy C^*-algebras. Note that the fundamental ideas in the proofs of the main results in this section are contained in [14, 40, 41].

Theorem 3.2. *Let $\varphi : A^+ \times A^+ \times (0, \infty) \to [0, 1]$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y, t) \geq \varphi\Big(2^m x, 2^m y, \frac{2^m t}{L}\Big) \tag{3.1}$$

for all $x, y \in A^+$ and $t > 0$. Let $f : A^+ \to B^+$ be a mapping satisfying

$$N\Big(f\Big(\big(x^{\frac{1}{m}} + y^{\frac{1}{m}}\big)^m\Big) - \big(f(x)^{\frac{1}{m}} + f(y)^{\frac{1}{m}}\big)^m, t\Big) \geq \varphi(x, y, t) \tag{3.2}$$

for all $x, y \in A^+$ and $t > 0$. Then there exists a unique positive-additive mapping $T : A^+ \to A^+$ satisfying (1.1) *and*

$$N(f(x) - T(x), t) \geq \varphi\Big(x, x, \frac{(2^m - 2^m L)t}{L}\Big) \tag{3.3}$$

for all $x \in A^+$ and $t > 0$.

Proof. Letting $y = x$ in (2.2), we get

$$N(f(2^m x) - 2^m f(x), t) \geq \varphi(x, x, t) \tag{3.4}$$

for all $x \in A^+$ and $t > 0$. Consider the set

$$X := \{g : A^+ \to B^+\}$$

and introduce the generalized metric on X as follows:

$$d(g, h) = \inf\Big\{\mu \in \mathbb{R}_+ : N(g(x) - h(x), t) \geq \varphi\Big(x, x, \frac{t}{\mu}\Big), \forall x \in A^+, t > 0\Big\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (X, d) is complete (see [47]).

Now, we consider the linear mapping $J : ightarrowX$ such that

$$Jg(x) := 2^m g\Big(\frac{x}{2^m}\Big)$$

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for all $x \in A^+$. Let $g, h \in X$ be given such that $d(g,h) = \varepsilon$. Then we ahve

$$N(g(x) - h(x), t) \geq \varphi(x, x, t)$$

for all $x \in A^+$ and $t > 0$. Hence we have

$$N(Jg(x) - Jh(x), t) = N\Big(2^m g\Big(\frac{x}{2^m}\Big) - 2^m h\Big(\frac{x}{2^m}\Big), t\Big) \geq \varphi\Big(x, x, \frac{t}{L}\Big)$$

for all $x \in A^+$ and $t > 0$ and so $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g,h)$$

for all $g, h \in X$. It follows from (2.4) that

$$N(f(x) - 2^m f\left(\frac{x}{2^m}\right), t) \geq \varphi\Big(x, x, \frac{2^m t}{L}\Big)$$

for all $x \in A^+$ and $t > 0$ and so $d(f, Jf) \leq \frac{L}{2^m}$.

By Theorem 1.1, there exists a mapping $T : A^+ \to B^+$ satisfying the following:

(1) T is a fixed point of J, i.e.,

$$T\Big(\frac{x}{2^m}\Big) = \frac{1}{2^m} T(x) \tag{3.5}$$

for all $x \in A^+$. The mapping T is a unique fixed point of J in the set

$$M = \{g \in X : d(f,g) < \infty\}.$$

This implies that T is a unique mapping satisfying (2.5) such that there exists $\mu \in (0, \infty)$ satisfying

$$N(f(x) - T(x), t) \geq \varphi\Big(x, x, \frac{t}{\mu}\Big)$$

for all $x \in A^+$ and $t > 0$;

(2) $d(J^n f, T) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 2^{mn} f\left(\frac{x}{2^{mn}}\right) = T(x)$$

for all $x \in A^+$;

(3) $d(f, T) \leq \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, T) \leq \frac{L}{2^m - 2^m L}.$$

This implies that the inequality (2.3) holds.

By (2.1) and (2.2), we have

$$\begin{aligned}
& N\Big(f\Big(\frac{\big(x^{\frac{1}{m}} + y^{\frac{1}{m}}\big)^m}{2^{mn}}\Big) - \Big(\Big(2^{mn} f\Big(\frac{x}{2^{mn}}\Big)\Big)^{\frac{1}{m}} + \Big(2^{mn} f\Big(\frac{y}{2^{mn}}\Big)\Big)^{\frac{1}{m}}\Big)^m, \frac{t}{2^{mn}}\Big) \\
\geq\ & \varphi\Big(\frac{x}{2^{mn}}, \frac{y}{2^{mn}}, \frac{t}{2^{mn}}\Big) \\
\geq\ & \varphi\Big(x, y, \frac{t}{L^{mn}}\Big)
\end{aligned}$$

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for all $x, y \in A^+$, $n \in \mathbb{N}$ and $t > 0$ and so

$$N\Big(T\Big(\big(x^{\frac{1}{m}} + y^{\frac{1}{m}}\big)^m\Big) - \Big(T(x)^{\frac{1}{m}} + T(y)^{\frac{1}{m}}\Big)^m, t\Big) = 1$$

for all $x, y \in A^+$ and $t > 0$. Thus the mapping $T : A^+ \to B^+$ is positive-additive. This completes the proof. $\square$

Corollary 3.3. *Let $p > 1$, θ_1, θ_2 be non-negative real numbers and $f : A^+ \to B^+$ be a mapping such that*

$$\begin{aligned} & N\Big(f\Big(\big(x^{\frac{1}{m}} + y^{\frac{1}{m}}\big)^m\Big) - \big(f(x)^{\frac{1}{m}} + f(y)^{\frac{1}{m}}\big)^m, t\Big) \\ \geq \ & \frac{t}{t + \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}}} \end{aligned} \tag{3.6}$$

for all $x, y \in A^+$ and $t > 0$. Then there exists a unique positive-additive mapping $T : A^+ \to B^+$ satisfying (1.1) *and*

$$N(f(x) - T(x), t) \geq \frac{t}{t + \frac{2\theta_1 + \theta_2}{2^{mp} - 2^m}} ||x||^p$$

for all $x \in A^+$ and $t > 0$.

Proof. The proof follows from Theorem 3.2 by taking

$$\varphi(x, y, t) = \frac{t}{t + \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}}}$$

for all $x, y \in A^+$ and $t > 0$ and $L = 2^{m-mp}$. $\square$

Theorem 3.4. *Let $\varphi : A^+ \times A^+ \times (0, \infty)] \to [0, 1]$ be a function such that there exists $L < 1$ with*

$$\varphi(x, y, t) \geq \varphi\Big(\frac{x}{2^m}, \frac{y}{2^m}, \frac{t}{2^m L}\Big)$$

for all $x, y \in A^+$ and $t > 0$. Let $f : A^+ \to B^+$ be a mapping satisfying (2.2)*. Then there exists a unique positive-additive mapping $T : A^+ \to A^+$ satisfying* (1.1) *and*

$$N(f(x) - T(x), t) \geq \varphi(x, x, (2^m - 2^m L)t)$$

for all $x \in A^+$ and $t > 0$.

Proof. Let (X, d) be the generalized metric space defined in the proof of Theorem 3.2. Consider the linear mapping $J : X \to X$ such that

$$Jg(x) := \frac{1}{2^m} g(2^m x)$$

for all $x \in A^+$. It follows from (2.4) that

$$N\Big(f(x) - \frac{1}{2^m} f(2^m x), t\Big) \geq \varphi(x, x, 2^m t)$$

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for all $x \in A^+$ and $t > 0$. So $d(f, Jf) \leq \frac{1}{2^m}$.

The rest of the proof is similar to the proof of Theorem 3.2. □

Corollary 3.5. *Let* $0 < p < 1$, θ_1, θ_2 *be non-negative real numbers and* $f : A^+ \to B^+$ *be a mapping satisfying* (2.6). *Then there exists a unique positive-additive mapping* $T : A^+ \to B^+$ *satisfying* (1.1) *and*

$$N(f(x) - T(x), t) \geq \frac{t}{t + \frac{2\theta_1+\theta_2}{2^m - 2^{mp}}\|x\|^p}$$

for all $x \in A^+$ *and* $t > 0$.

Proof. The proof follows from Theorem 3.4 by taking

$$\varphi(x, y, t) = \frac{t}{t + \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}}}$$

for all $x, y \in A^+$ and $t > 0$ and $L = 2^{mp-m}$. □

Acknowledgement. This project was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, under Grant No. (31-130-35-HiCi). The authors, therefore, acknowledge with thanks DSR technical and financial support.

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Common fixed point theorems for occasionally weakly biased mappings under contractive conditions in Menger PM-spaces

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Abstract
In this paper, utilizing the concepts of weakly biased mappings and occasionally weakly biased mappings for a pair of self-mappings in Menger PM-spaces, some new common fixed point theorems under contractive conditions are obtained. Some examples are also given to exemplify our main results.
Keywords: common fixed point; Menger PM-space; weakly biased mapping; occasionally weakly biased mapping

1 Introduction

The theory of probabilistic metric space was initiated and studied by K. Menger in 1942[1]-[2], and has been applied to many fields, such as cluster analysis, mathematical statistics and chaos theory[3]-[4]. The study on fixed point problems in probabilistic metric spaces has attracted much attention during the past few years (see *e.g.* [5]-[7]).

The concepts of compatible mappings and occasionally weakly compatible mappings in metric spaces were introduced by Jungck [8]-[9]. The concept of compatible mappings in a Menger space was initiated by Mishra [10] and since then many fixed point results are obtained [11]-[13]. In 2009, Fang defined the property (E.A) for two single-valued mappings in Menger PM-spaces and studied the existence of common fixed points[14]. In 2011, Ali *et al.* obtained some common fixed point results for strict contractions in Menger PM-spaces using the common property (E.A) for two pairs of single-valued mappings [15]. In [16], the authors studied the existence of common fixed points for hybrid pairs of mappings satisfying the common property (E.A) in Menger PM-spaces.

Hussain *et al.* introduced the concepts of weakly biased mappings and occasionally weakly biased mappings in metric spaces [17]. Using such concepts, the authors further studied the common fixed points under contractive conditions for two and four single valued mappings[18]. Recently, A. Bhatt and H. Chandra introduced such concepts in Menger PM-spaces[19].

The purpose of this paper is to study the existence of coincidence and common fixed points for weakly biased mappings and occasionally weakly biased mappings under some contractive conditions. Some examples are also given to illustrate our results.

2 Preliminaries

A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is called a *distribution function* if it is nondecreasing left-continuous with $\sup_{t\in\mathbb{R}} F(t) = 1$ and $\inf_{t\in\mathbb{R}} F(t) = 0$. We will denote by $\mathscr{D}$ the set of all distribution functions while H will

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†Supported by the Natural Science Foundation of China (11326099,11461045,11361042,11071108), the Natural Science Foundation of Jiangxi Province of China (20142BAB211016,20132BAB201001,2010GZS0147) and the Youth Foundation of the Education Department of Jiangxi (GJJ13012).

always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, & t \leq 0, \\ 1, & t > 0. \end{cases}$$

Let $F_1, F_2 \in \mathscr{D}$. The algebraic sum $F_1 \oplus F_2$ is defined by $(F_1 \oplus F_2)(t) = \sup_{t_1+t_2=t} \min\{F_1(t_1), F_2(t_2)\}$for all $t \in \mathbb{R}$. Let f and g be two functions defined on $\mathbb{R}$ with positive values. The notation $f > g$ means that $f(t) \geq g(t)$ for all $t \in \mathbb{R}$, and there exists at least one $t_0 \in \mathbb{R}$ such that $f(t_0) > g(t_0)$.

A mapping $\Delta : [0,1] \times [0,1] \to [0,1]$ is called a *triangular norm* (for short, a t-norm) if the following conditions are satisfied:

(1) $\Delta(a, 1) = a$;

(2) $\Delta(a, b) = \Delta(b, a)$;

(3) $\Delta(a, c) \geq \Delta(b, d)$ for $a \geq b, c \geq d$;

(4) $\Delta(a, \Delta(b, c)) = \Delta(\Delta(a, b), c)$.

Definition 2.1[4]**.** A triplet $(X, \mathscr{F}, \Delta)$ is called a *Menger probabilistic metric space* (for short, a *Menger PM-space*) if X is a nonempty set, Δ is a t-norm and $\mathscr{F}$ is a mapping from $X \times X$ into $\mathscr{D}$ satisfying the following conditions (we denote $\mathscr{F}(x, y)$ by $F_{x,y}$):

(MS-1) $F_{x,y}(t) = H(t)$ for all $t \in R$ if and only if $x = y$;

(MS-2) $F_{x,y}(t) = F_{y,x}(t)$ for all $t \in R$;

(MS-3) $F_{x,y}(t + s) \geq \Delta(F_{x,z}(t), F_{z,y}(s))$ for all $x, y, z \in X$ and $t, s \geq 0$.

Remark 2.1[4]**.** A sequence $\{x_n\}$ is said to be $\mathscr{T}$-convergent to $x \in X$(we write $x_n \xrightarrow{\mathscr{T}} x$) if for any given $\epsilon > 0$ and $\lambda \in (0, 1]$, there exists a positive integer $N = N(\epsilon, \lambda)$ such that $F_{x_n,x}(\epsilon) > 1 - \lambda$ whenever $n \geq N$, which is equivalent to $\lim_{n\to\infty} F_{x_n,x}(t) = 1$ for all $t > 0$; $\{x_n\}$ is called a $\mathscr{T}$-Cauchy sequence in $(X, \mathscr{F}, \Delta)$ if for any given $\epsilon > 0$ and $\lambda \in (0, 1]$, there exists a positive integer $N = N(\epsilon, \lambda)$ such that $F_{x_n,x_m}(\epsilon) > 1-\lambda$ whenever $n, m \geq N$; $(X, \mathscr{F}, \Delta)$ is said to be $\mathscr{T}$-complete if each $\mathscr{T}$-Cauchy sequence in X is $\mathscr{T}$-convergent in X.

Remark 2.2 [4]**.** Let (X, d) be a metric space. Define a mapping $\mathscr{F} : X \times X \to \mathscr{D}$ by

$$\mathscr{F}(x, y)(t) = F_{x,y}(t) = H(t - d(x, y)), \ \ \forall x, y \in X, \ \ t \in \mathbb{R}. \tag{2.1}$$

Then $(X, \mathscr{F}, \Delta_{min})$ is a Menger PM-space induced by (X, d) with $\Delta_{min}(a, b) = \min\{a, b\}, \forall a, b \in [0, 1]$. If (X, d) is complete, then $(X, \mathscr{F}, \Delta_{min})$ is $\mathscr{T}$-complete.

The following lemma will play an important role in proving our main results in Section 3.

Lemma 2.1 [4]**.** Let $(X, \mathscr{F}, \Delta)$ be a Menger PM-space with a continuous t-norm Δ on $[0,1] \times [0,1]$, $x, y \in X, \{x_n\}, \{y_n\} \subset X$ and $x_n \xrightarrow{\mathscr{T}} x$, $y_n \xrightarrow{\mathscr{T}} y$. Then $\liminf_{n\to\infty} F_{x_n,y_n}(t) \geq F_{x,y}(t)$ for all $t > 0$. Particularly, if $F_{x,y}(\cdot)$ is continuous at the point t_0, then $\lim_{n\to\infty} F_{x_n,y_n}(t_0) = F_{x,y}(t_0)$.

We denote by $C(f, g)$ the set of all coincidence points of f and g. We recall the definition of occasionally weakly compatibility and (E.A) property for self-mappings in Menger PM-spaces.

Definition 2.2[9]**.** Let $(X, \mathscr{F}, \Delta)$ be a Menger PM-space. Then $f : X \to X$ and $g : X \to X$ are said to be *occasionally weakly compatible* if $fgx = gfx$ for some $x \in C(f, g)$.

Definition 2.3[14]. Let $(X, \mathscr{F}, \Delta)$ be a Menger PM-space. A pair of mappings (f, g) is said to satisfy *the property (E.A)*, if there exists a sequence $\{x_n\}$ in X and some $a \in X$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = a$.

We next recall the definitions of weakly g-biased mappings and occasionally weakly g-biased mappings for a pair of self-mappings in Menger PM-spaces.

Definition 2.4[19]**.** Let $(X, \mathscr{F}, \Delta)$ be a Menger PM-space, $f : X \to X$ be a self-mapping. A pair of mappings f and g are said to be *weakly g-biased*, if for any $x \in C(f, g)$ we have $F_{gfx,gx}(t) \geq F_{fgx,fx}(t), \forall t > 0$.

Definition 2.5[19]**.** Let $(X, \mathscr{F}, \Delta)$ be a Menger PM-space, $f : X \to X$ be a self-mapping. A pair of mappings f and g are said to be *occasionally weakly g-biased*, if there exists some $x \in C(f, g)$, such that $F_{gfx,gx}(t) \geq F_{fgx,fx}(t), \forall t > 0$.

Remark 2.3. It is easy to see that occasionally weakly compatible mappings and weakly g-biased mappings are both occasionally weakly g-biased mapping. But the following examples show that the converse are not true.

Example 2.1. Let $X = [0, 1]$ be endowed with the usual metric, that is, $d(x, y) = |x-y|, \forall x, y \in X$. Define $\mathscr{F} : X \times X \to \mathscr{D}$ as $\mathscr{F}(x, y)(t) = F_{x,y}(t) = H(t - d(x, y)),\ \forall x, y \in X, t \in \mathbb{R}$, then by Remark 2.2, we know that $(X, \mathscr{F}, \Delta_{min})$ is a Menger PM-space. Define $f, g : X \to X$ as follows:

$$fx = \begin{cases} 2x, & \text{if} x \in [0, \frac{1}{2}); \\ 1, & \text{if} x \in [\frac{1}{2}, 1]. \end{cases}$$

$$gx = \begin{cases} 1 - 2x, & \text{if} x \in [0, \frac{1}{2}); \\ 0, & \text{if} x \in [\frac{1}{2}, 1]. \end{cases}$$

It is easy to see that $C(f, g) = \{\frac{1}{4}\}$, and $d(gf\frac{1}{4}, g\frac{1}{4}) = |gf\frac{1}{4} - g\frac{1}{4}| = |0 - \frac{1}{2}| = \frac{1}{2} \leq |1 - \frac{1}{2}| = |fg\frac{1}{4} - f\frac{1}{4}| = d(fg\frac{1}{4}, f\frac{1}{4})$. Therefore, $F_{gf\frac{1}{4},g\frac{1}{4}}(t) \geq F_{fg\frac{1}{4},f\frac{1}{4}}(t), \forall t > 0$, and thus (f, g) are occasionally weakly g-biased mappings on $(X, \mathscr{F}, \Delta_{min})$. Furthermore, $fg\frac{1}{4} \neq gf\frac{1}{4}$, so (f, g) are not occasionally weakly compatible mappings.

Example 2.2. Let $X = [0, 1]$ be endowed with the usual metric, that is, $d(x, y) = |x-y|, \forall x, y \in X$. Define $f, g : X \to X$ as follows:

$$fx = \begin{cases} 2x^2, & \text{if} x \in (0, 1]; \\ \frac{1}{2}, & \text{if} x = 0. \end{cases}$$

$$gx = \begin{cases} 2x, & \text{if} x \in (0, 1]; \\ \frac{1}{2}, & \text{if} x = 0. \end{cases}$$

It is easy to see that $C(f, g) = \{0, 1\}$. For $0 \in C(f, g)$, we have $d(gf0, g0) = |gf0 - g0| = \frac{1}{2} > 0 = |fg0 - f0| = d(fg0, f0)$. So $H(\frac{1}{4} - d(gf0, f0)) < H(\frac{1}{4} - d(fg0, f0))$, that is, there exists $t_0 = \frac{1}{4} > 0$, such that $F_{gf0,f0}(t_0) < F_{fg0,f0}(t_0)$. Thus, (f, g) are neither occasionally weakly compatible mappings nor weakly g-biased mappings on $(X, \mathscr{F}, \Delta_{min})$. But for $1 \in C(f, g)$, we have $d(gf1, g1) = |gf1 - g1| = 2 < 6 = |fg1 - f1| = d(fg1, f1)$. So $H(t - d(gf1, g1)) \geq H(t - d(fg1, f1)), \forall t > 0$, that is, $F_{gf1,g1}(t) \geq F_{fg1,f1}(t), \forall t > 0$. This shows that (f, g) are occasionally weakly g-biased mappings on $(X, \mathscr{F}, \Delta_{min})$.

3 Main results

In this section, we shall give the main results of this paper. We first present the following common fixed point theorem for a pair of self-mappings in Menger PM-spaces.

Theorem 3.1. Let $(X, \mathscr{F}, \Delta_{min})$ be a Menger PM-space, where $\Delta_{min} = \min\{a, b\}$. Suppose that $f, g : X \to X$ are self-mappings satisfying the following conditions:

(i) (f,g) satisfies the property (E.A);

(ii) $g(X)$ is a $\mathscr{T}$-closed subset of X;

(iii) For any $x, y \in X$ with $x \neq y$,

$$F_{fx,fy} > \min\{F_{gx,gy}, {}_2[F_{fx,gx} \oplus F_{fx,gy}], {}_{\frac{2}{k}}[F_{fy,gx} \oplus F_{fx,gy}]\}, \tag{3.1}$$

where ${}_af(t)$ means $f(at)$. Then (f,g) has a coincidence point in X. Moreover, if f and g are occasionally weakly g-biased mappings, then f and g have a unique common fixed point in X.

Proof. Since (f,g) satisfy the property (E.A), there exists $\{x_n\} \subset X$, such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = a \in X.$$

Since $g(X)$ is $\mathscr{T}$-closed, there exists some $u \in X$, such that $a = gu$. Thus

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gu. \tag{3.2}$$

We claim that $fu = gu$. Suppose $fu \neq gu$. Then there exists some $t_0 > 0$, such that

$$F_{fu,gu}(2t_0) > F_{fu,gu}(t_0). \tag{3.3}$$

Without loss of generality, we can assume that t_0 is a continuous point of $F_{fu,gu}(\cdot)$. In fact, by the left continuity of the distribution function, we know that there exists some $\delta > 0$, such that

$$F_{fu,gu}(2t) > F_{fu,gu}(t), \quad \forall t \in (t_0 - \delta, t_0].$$

Since the distribution function is nondecreasing, the discontinuous points are at most a countable set. Thus, when t_0 is not a continuous point of $F_{fu,gu}(\cdot)$, we can always choose a point t_1 in $(t_0 - \delta, t_0]$ to replace t_0.

Noting that $\lim\limits_{n\to\infty} fx_n = gu$ and $fu \neq gu$, there exists $n_0 \in N$ such that for all $n \geq n_0$, $fx_n \neq fu$.

From (3.1) we know that for all $n \geq n_0$,

$$F_{fx_n,fu}(t_0) \geq \min\{F_{gx_n,gu}(t_0), [F_{fx_n,gx_n} \oplus F_{fu,gu}](2t_0), [F_{fu,gx_n} \oplus F_{fx_n,gu}]\}(2t_0)\}. \tag{3.4}$$

It is easy to verify that

$$\liminf_{n\to\infty}[F_{fx_n,gx_n} \oplus F_{fu,gu}](2t_0) \geq F_{fu,gu}(2t_0). \tag{3.5}$$

In fact, for any $\delta \in (0, 2t_0)$, we have

$$[F_{fx_n,gx_n} \oplus F_{fu,gu}](2t_0) \geq \min\{F_{fx_n,gx_n}(\delta), F_{fu,gu}(2t_0 - \delta)\}.$$

Letting $\delta \to 0$, by the left continuity of the distribution function, we obtain (3.5). Similarly, we can prove that

$$\liminf_{n\to\infty}[F_{fu,gx_n} \oplus F_{fx_n,gu}](2t_0) \geq F_{fu,gu}(2t_0). \tag{3.6}$$

Noting that t_0 is the continuous point of $F_{fu,gu}(\cdot)$, by Lemma 2.1, we have

$$\lim_{n\to\infty} F_{fx_n,fu}(t_0) = F_{gu,fu}(t_0).$$

Thus, letting $n \to \infty$ in (3.4) and using (3.5) and (3.6), we obtain

$$F_{gu,fu}(t_0) \geq \min\{1, F_{fu,gu}(2t_0), F_{fu,gu}(2t_0)\} = F_{fu,gu}(2t_0),$$

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which contradicts with (3.3). So we get $fu = gu$, that is, u is a coincidence point of f and g.

Since f and g are occasionally weakly g-biased mappings, we assume that for $v \in C(f,g)$, i.e., $fv = gv$, we have $F_{gfv,gv}(t) \geq F_{fgv,fv}(t), \forall t > 0$. Since $fv = gv$, we have $ffv = fgv$ and $gfv = ggv$.

We next prove that $ffv = fv$. In fact, suppose this is not true, that is, $ffv \neq fv$, then $fv \neq v$. By condition (iii), there exists some $t_* > 0$, such that

$$F_{fv,ffv}(t_*) \geq \min\{F_{gv,gfv}(t_*), [F_{fv,gv} \oplus F_{ffv,gfv}](2t_*), [F_{ffv,gv} \oplus F_{fv,gfv}]\}(2t_*)\}. \tag{3.7}$$

For any $\epsilon \in (0, 2t_*)$, we have

$$[F_{fv,gv} \oplus F_{ffv,gfv}]\}(2t_*) \geq \min\{F_{fv,gv}(\epsilon), F_{ffv,gfv}(2t_* - \epsilon)\} = \min\{1, F_{ffv,gfv}(2t_* - \epsilon)\}.$$

Letting $\epsilon \to 0$, by the left continuity of the distribution function, we have

$$\begin{aligned} [F_{fv,gv} \oplus F_{ffv,gfv}](2t_*) &\geq F_{ffv,gfv}(2t_*) \geq \Delta(F_{ffv,fv}(t_*), F_{gfv,gv}(t_*)) \\ &\geq \Delta(F_{ffv,fv}(t_*), F_{fgv,fv}(t_*)) \\ &= \Delta(F_{ffv,fv}(t_*), F_{ffv,fv}(t_*)) \geq F_{ffv,fv}(t_*). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} [F_{ffv,gv} \oplus F_{fv,gfv}](2t_*) &\geq \min\{[F_{ffv,gv}(t_*), F_{fv,gfv}(t_*)\} = \min\{F_{ffv,fv}(t_*), F_{gv,gfv}(t_*)\} \\ &\geq \min\{F_{ffv,fv}(t_*), F_{fgv,fv}(t_*)\} \\ &= \min\{F_{ffv,fv}(t_*), F_{ffv,fv}(t_*)\} = F_{ffv,fv}(t_*). \end{aligned}$$

Then by (3.7), we obtain $F_{fv,ffv}(t_*) > F_{fv,ffv}(t_*)$, which is a contradiction, and thus $ffv = fv$. Since f and g are occasionally weakly g-biased, we get

$$F_{gfv,gv}(t) \geq F_{fgv,fv}(t) = F_{ffv,fv}(t) = 1, \forall t > 0.$$

This implies that $gfv = gv = fv$. So fv is a common fixed point of f and g.

Finally, we prove the uniqueness of the common fixed point. In fact, suppose x_1 and x_2 are two common fixed points of f and g, and $x_1 \neq x_2$. Then there exists some $t_{**} > 0$, such that

$$\begin{aligned} F_{fx_1,fx_2}(t_{**}) &> \min\{F_{gx_1,gx_2}(t_{**}), [F_{fx_1,gx_1} \oplus F_{fx_2,gx_2}](2t_{**}), [F_{fx_2,gx_1} \oplus F_{fx_1,gx_2}](2t_{**})\} \\ &\geq \min\{F_{fx_1,fx_2}(t_{**}), 1, F_{fx_1,fx_2}(t_{**})\} = F_{fx_1,fx_2}(t_{**}), \end{aligned}$$

which is a contradiction. Hence, f and g have a unique common fixed point in X.

Next, we further study the existence of coincidence points and common fixed points for two pairs of self mappings in Menger PM-spaces by introducing the following class of functions.

Suppose that the function $\psi : [0,1] \to [0,1]$ satisfies the following two conditions:

(1) ψ is nondecreasing;

(2) $\psi(x) > x, \forall x \in (0,1)$ and $\psi(0) = 0$.

Theorem 3.2. Let $(X, \mathscr{F}, \Delta)$ be a Menger PM-space with Δ a continuous t-norm on $[0,1] \times [0,1]$. Suppose that $A, B, S, T : X \to X$ are self-mappings satisfying the following conditions:

(i) (B, T) satisfies the property (E.A);

(ii) $B(X) \subset S(X)$ and $T(X)$ is a $\mathscr{T}$-closed subset of X;

(iii) For any $x, y \in X$ with $Ax \neq By$ and any $t > 0$,

$$F_{Ax,By}(t) \geq \psi(\min\{F_{Sx,Ty}(t), [F_{Ax,Sx} \oplus F_{By,Ty}](2t), [F_{Ax,Ty} \oplus F_{By,Sx}](2t)\}), \tag{3.8}$$

Then (A,S) and (B,T) each has a coincidence point in X. Moreover, if (A,S) and (B,T) are occasionally S-biased mappings and occasionally T-biased mappings, respectively, then A,B,S and T have a unique common fixed point in X.

Proof. By condition (i), we know that there exists a sequence $\{x_n\}$ in X, such that

$$\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = a \in X. \tag{3.9}$$

Since $B(X) \subset S(X)$, for each x_n, there exists $y_n \in X$, such that $Bx_n = Sy_n$. Therefore, $\lim_{n\to\infty} Sy_n = a \in X$.

We next prove that $\lim_{n\to\infty} Ay_n = a$. For this sake, we first prove that $\liminf_{n\to\infty} F_{Ay_n,Bx_n}(t) = 1, \forall t > 0$. In fact, suppose this is not true, then there exist $t_0 > 0$, a subsequence $\{y_{n_k}\} \subset \{y_n\}$ and some $0 < \epsilon_0 < 1$, such that $\liminf_{k\to\infty} F_{Ay_{n_k},Bx_{n_k}}(t) = 1 - \epsilon_0$. Thus, there exists $k_0 \in N$, such that for all $k \geq k_0$, we have $Ay_{n_k} \neq Bx_{n_k}$.

By condition (iii), we know that for all $k \geq k_0$,

$$F_{Ay_{n_k},Bx_{n_k}}(t_0) \geq \psi(\min\{F_{Sy_{n_k},Tx_{n_k}}(t_0), [F_{Ay_{n_k},Sy_{n_k}} \oplus F_{Bx_{n_k},Tx_{n_k}}](2t_0), [F_{Ay_{n_k},Tx_{n_k}} \oplus F_{Bx_{n_k},Sy_{n_k}}](2t)\}), \tag{3.10}$$

It is easy to verify that

$$\liminf_{k\to\infty}[F_{Ay_{n_k},Sy_{n_k}} \oplus F_{Bx_{n_k},Tx_{n_k}}](2t_0) \geq \liminf_{n\to\infty} F_{Ay_{n_k},Bx_{n_k}}(2t_0).$$

and

$$\liminf_{k\to\infty}[F_{Ay_{n_k},Tx_{n_k}} \oplus F_{Bx_{n_k},Sy_{n_k}}](2t) \geq \liminf_{n\to\infty} F_{Ay_{n_k},Bx_{n_k}}(2t_0).$$

Then by (3.10), we have $1 - \epsilon_0 = \liminf_{k\to\infty} F_{Ay_{n_k},Bx_{n_k}}(t_0) \geq \psi(\min\{1, 1-\epsilon_0, 1-\epsilon_0\}) > 1 - \epsilon_0$, which is a contradiction. Therefore, $\liminf_{n\to\infty} F_{Ay_n,Bx_n}(t) = 1, \forall t > 0$.

Note that for all $t > 0$ and any $\delta \in (0,t)$,

$$F_{Ay_n,a}(t) \geq \Delta(F_{Ay_n,By_n}(t-\delta), F_{Bx_n,a}(\delta)).$$

Letting $n \to \infty$, we have

$$\liminf_{n\to\infty} F_{Ay_n,a}(t) \geq \Delta(\liminf_{n\to\infty} F_{Ay_n,By_n}(t-\delta), \liminf_{n\to\infty} F_{Bx_n,a}(\delta)) \geq \Delta(1,1) = 1.$$

Hence, $\liminf_{n\to\infty} F_{Ay_n,a}(t) = 1, \forall t > 0$, and thus $\lim_{n\to\infty} Ay_n = a$.

Since $T(X)$ is $\mathscr{T}$-closed in X, from (3.9) we know that there exists $v \in X$, such that $Tv = a$. We now prove that $Bv = Tv$. Suppose this is not true, that is, $Bv \neq Tv$. Noting that $\lim_{n\to\infty} Ay_n = Tv$, there exists $n_0 \in N$, such that for all $n \geq n_0$, $Ay_n \neq Bv$.

Without loss of generality, we can assume that t_1 is a continuous point of $F_{Tv,Bv}(\cdot)$. By condition (iii), we have

$$F_{Ay_n,Bv}(t_1) \geq \psi(\min\{F_{Sy_n,Tv}(t), [F_{Ay_n,Sy_n} \oplus F_{Bv,Tv}](2t), [F_{Ay_n,Tv} \oplus F_{Bv,Sy_n}](2t)\}), \tag{3.11}$$

Similarly, we can verify that

$$\liminf_{n\to\infty}[F_{Ay_n,Sy_n} \oplus F_{Bv,Tv}](2t_1) \geq F_{Bv,Tv}(2t_1)$$

and

$$\liminf_{n\to\infty}[F_{Ay_n,Tv} \oplus F_{Bv,Sy_n}](2t_1) \geq F_{Bv,Tv}(2t_1).$$

Letting $n \to \infty$ in (3.11), we obtain

$$F_{Tv,Bv}(t_1) \geq \psi(\min\{1, F_{Bv,Tv}(2t_1), F_{Bv,Tv}(2t_1)\}) = \psi(F_{Bv,Tv}(2t_1)) > F_{Bv,Tv}(2t_1),$$

which is a contradiction. Therefore, $Bv = Tv$, and thus v is a coincidence point of (B, T).

Since $B(X) \subset T(X)$, and $Bv = a$, there exists $u \in X$, such that $Bv = Su$. Then $Su = Bv = Tv = a$. We next prove that $Au = Su$. Suppose this is not true, that is, $Au \neq Su$, then by condition (iii), we know that for all $t > 0$,

$$\begin{aligned} F_{Au,Su}(t) = F_{Au,Bv}(t) &\geq \psi(\min\{F_{Su,Tv}(t), [F_{Au,Su} \oplus F_{Bv,Tv}](2t), [F_{Au,Tv} \oplus F_{Bv,Su}](2t)\}) \\ &= \psi(\min\{1, 1, F_{Au,Su}](2t)\}) = \psi(F_{Au,Su}(2t)\}) > F_{Au,Su}(2t), \end{aligned}$$

which is a contradiction. Therefore, $Au = Su$, and thus u is a coincidence point of (A, S).

It follows that $C(A, S) \neq \emptyset$ and $C(B, T) \neq \emptyset$. Since (A, S) and (B, T) are occasionally S-biased mappings and occasionally T-biased mappings, respectively, we can assume that $w \in C(A, S)$ and $z \in C(B, T)$, such that for all $t > 0$, we have $F_{SAw,Sw}(t) \geq F_{ASw,Aw}(t)$ and $F_{TBz,Tz}(t) \geq F_{BTz,Bz}(t)$.

By $Aw = Sw$ and $Bz = Tz$, we have $AAw = ASw, SAw = SSw, BBz = BTz$ and $TBz = TTz$.

We claim that $Aw = Bz$. In fact, suppose this is not true, that is, $Aw \neq Bz$, then by condition (iii) of Theorem 3.2, we obtain

$$\begin{aligned} F_{Aw,Bz}(t) &\geq \psi(\min\{F_{Sw,Tz}(t), [F_{Aw,Sw} \oplus F_{Bz,Tz}](2t), [F_{Aw,Tz} \oplus F_{Bz,Sw}](2t)\}) \\ &= \psi(\min\{F_{Aw,Bz}(t), 1, F_{Aw,Bz}(t)\}) = \psi(F_{Aw,Bz}(t)) > F_{Aw,Bz}(t), \end{aligned}$$

which is a contradiction. Therefore, $Aw = Bz$, and thus $Aw = Sw = Bz = Tz$.

We next prove that $AAw = Aw$. In fact, suppose this is not true, that is, $AAw \neq Aw$, which implies that $AAw \neq Bz$, then by condition (iii), we know that for all $t > 0$,

$$F_{AAw,Aw}(t) = F_{AAw,Bz}(t) \geq \psi(\min\{F_{SAw,Tz}(t), [F_{AAw,SAw} \oplus F_{Bz,Tz}](2t), [F_{AAw,Tz} \oplus F_{Bz,SAw}](2t)\}). \tag{3.12}$$

Noting that

$$F_{SAw,Tz}(t) = F_{SAz,Sz}(t) \geq F_{ASw,Aw}(t) = F_{AAw,Aw}(t),$$

$$\begin{aligned} [F_{AAw,SAw} \oplus F_{Bz,Tz}](2t) &\geq F_{AAw,SAw}(2t) \geq \Delta(F_{AAw,Aw}(t), F_{Aw,SAw}(t)) \\ &= \Delta(F_{AAw,Aw}(t), F_{Sw,SAw}(t)) \geq \Delta(F_{AAw,Aw}(t), F_{ASw,Aw}(t)) \\ &= F_{AAw,Aw}(t), \end{aligned}$$

and

$$\begin{aligned} [F_{AAw,Tz} \oplus F_{Bz,SAw}](2t) &= [F_{AAw,Aw} \oplus F_{Sw,SAw}](2t) \geq \min\{F_{AAw,Aw}(t), F_{Sw,SAw}(t)\} \\ &\geq \min\{F_{AAw,Aw}(t), F_{ASw,Aw}(t)\} = F_{AAw,Aw}(t), \end{aligned}$$

by (3.12), we have

$$F_{AAw,Aw}(t) \geq \psi(F_{AAw,Aw}(t)) > F_{AAw,Aw}(t),$$

which is a contradiction. Therefore, $AAw = Aw$. Since (A, S) are occasionally weakly S-biased mappings, we have $F_{SAw,Sw}(t) \geq F_{ASw,Aw}(t) = 1, \forall t > 0$, and thus $SAw = Sw = Aw$.

Furthermore, $BAw = Aw$. In fact, suppose that $BAw \neq Aw$, that is, $Aw \neq BBz$, then by condition (iii), we know that for all $t > 0$,

$$F_{Aw,BAw}(t) = F_{Aw,BBz}(t) \geq \psi(\min\{F_{Sw,TBz}(t), [F_{Aw,Sw} \oplus F_{BBz,TBz}](2t), [F_{Aw,TBz} \oplus F_{BBz,Sw}](2t)\}). \tag{3.13}$$

Similarly, we can obtain that

$$F_{Aw,BAw}(t) \geq \psi(F_{BAw,Aw}(t)) > F_{BAw,Aw}(t),$$

which is a contradiction. Therefore, $BAw = Aw$. Since (B, T) are occasionally weakly T-biased mappings, we have $F_{TAw,Aw}(t) = F_{TBz,Tz}(t) \geq F_{BTz,Bz}(t) = F_{BAw,Aw}(t) = 1, \forall t > 0$, and thus

$TAw = Aw$. Hence, we have $AAw = BAw = SAw = TAw = Aw$, that is, Aw is a common fixed point of A, B, S and T.

Finally, the uniqueness of the common fixed point of A, B, S and T can be easily obtained by condition (iii).

Remark 3.1. From the proof of the above theorem, it is easy to see that the conclusions still hold if we replace the condition (i) and (ii) by the following ones:

(i)' (A, S) satisfies the property (E.A);

(ii)' $A(X) \subset T(X)$ and $S(X)$ is a $\mathscr{T}$-closed subset of X.

4 Applications

In this section, we will give some examples to illustrate the validity of our main results.

Example 4.1. Let (X, d) be a metric space, where $X = [0, 1]$ and $d(x, y) = |x - y|$, and $(X, \mathscr{F}, \Delta_{min})$ be the Menger PM-space induced by (X, d). Define $f, g : X \to X$ as follows:

$$fx = \begin{cases} \frac{1}{2}, & \text{if} x \in (0, \frac{1}{2}]; \\ \frac{3}{5}, & \text{if} x \in (\frac{1}{2}, 1]. \end{cases}$$

$$gx = \begin{cases} x + \frac{1}{3}, & \text{if} x \in (0, \frac{1}{2}); \\ \frac{1}{2}, & \text{if} x = \frac{1}{2}; \\ \frac{2}{5}, & \text{if} x \in (\frac{1}{2}, 1]. \end{cases}$$

It is obvious that $gX = [\frac{1}{3}, \frac{5}{6}]$ is a $\mathscr{T}$-closed subset of X. Also, (f, g) satisfies the property (E.A). In fact, take $x_n \subset X$, and $x_n = \frac{1}{n+2} + \frac{1}{6}$, then we have $x_n \xrightarrow{\mathscr{T}} \frac{1}{6}$ as $n \to \infty$, and thus $fx_n, gx_n \xrightarrow{\mathscr{T}} \frac{1}{2}$ as $n \to \infty$. Moreover, it is easy to verify that for any $x, y \in X$ with $x \neq y$, (3.1) holds. Besides, noting that $C(f, g) = \{\frac{1}{6}, \frac{1}{2}\}$, and $F_{gf\frac{1}{6}, g\frac{1}{6}}(t) \geq F_{fg\frac{1}{6}, f\frac{1}{6}}(t) \forall t > 0$, we get f and g are occasionally weakly g-biased mappings. Thus, all the conditions of Theorem 3.1 are satisfied and f and g have a unique common fixed point in X. In fact, $\frac{1}{2}$ is the desired fixed point.

Example 4.2. Let (X, d) be a metric space, where $X = [0, 1]$ and $d(x, y) = |x - y|$, and $(X, \mathscr{F}, \Delta_{min})$ be the Menger PM-space induced by (X, d). Define $A, B, S, T : X \to X$ as follows:

$$Ax = Bx = \begin{cases} \frac{x+1}{3}, & \text{if} x \in [0, \frac{1}{3}); \\ \frac{1}{3}, & \text{if} x \in [\frac{1}{3}, 1]. \end{cases}$$

$$Sx = Tx = \begin{cases} x + \frac{1}{3}, & \text{if} x \in [0, \frac{1}{3}); \\ \frac{1}{3}, & \text{if} x = \frac{1}{3}; \\ \frac{2}{3}, & \text{if} x \in (\frac{1}{3}, 1]. \end{cases}$$

It is obvious that $AX = BX = [\frac{1}{3}, \frac{4}{9}], SX = TX = [\frac{1}{3}, \frac{2}{3}]$, $BX \subset SX$ and TX is a $\mathscr{T}$-closed subset of X. Also, (A, S) satisfies the property (E.A). In fact, take $x_n \subset X$, and $x_n = \frac{1}{n}$, then we have $x_n \xrightarrow{\mathscr{T}} 0$ as $n \to \infty$, and thus $Ax_n, Sx_n \xrightarrow{\mathscr{T}} \frac{1}{3}$ as $n \to \infty$. Moreover, it is easy to verify that for any $x, y \in X$ with $x \neq y$, (3.8) holds. Since $C(A, S) = \{0, \frac{1}{3}\}$, and $F_{SA\frac{1}{6}, S\frac{1}{6}}(t) \geq F_{AS\frac{1}{6}, A\frac{1}{6}}(t), \forall t > 0$, we get A and S are occasionally weakly S-biased mappings. Thus, all the conditions of Theorem 3.2 are satisfied and A, B, S and T have a unique common fixed point in X. In fact, $\frac{1}{3}$ is the desired fixed point.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 19, NO.4, 2015, COPYRIGHT 2015 EUDOXUS PRESS, LLC

The Existence of Symmetric Positive Solutions for a Nonlinear Multi-Point Boundary Value Problem on Time Scales

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Abstract

In this paper, we study the existence of symmetric positive solutions for the nonlinear multi-point boundary value problem on time scales. By applying fixed-point index theorem, the existence of at least two or many symmetric positive solutions is obtained. An example is given to illustrate our main results.

2010 Mathematics subject Classification: 34B10; 34B18; 39A10.
Keywords: boundary value problem, fixed point theorem, symmetric positive solution, time scales.

1 Introduction

The theory of dynamic equations on time scales goes back to its founder Hilger [4] and is undergoing repeat development as it provides a unifying structure for the study of differential equations in the continuous case and the study of finite difference equations in the discrete case. We refer to the books by Bohner and Peterson [1, 2] and Lakshmikantham et al. [6].

The existence of positive solutions for second-order nonlinear boundary value problems has been studied by many authors using the fixed point theorem in cones. To identify a few, we refer the reader to [5, 8, 10, 12, 14, 15] and references therein. However, they did not further provide characters of positive solutions, such as symmetry. It is now natural to consider the existence of symmetric positive solutions, see [3, 9, 11, 13].

In [13], Yao considered the following two-point boundary value problem (BVP)

$$\begin{cases} w''(t)+h(t)f(w(t))=0, \quad 0<t<1, \\ \alpha u(0)-\beta w'(0)=0, \quad \alpha u(1)+\beta w'(1)=0. \end{cases}$$

The author obtained the existence of n symmetric positive solutions and established a corresponding iterative scheme by using a monotone iterative technique.

In [9], Sun and Zhang considered the following m-point BVP

$$\begin{cases} u''(t)+a(t)f(t,u(t))=0, \quad 0<t<1, \\ u(0)=u(1)=\displaystyle\sum_{i=1}^{m-2}\alpha_i u(\eta_i). \end{cases}$$

By using Krasnoselskii's fixed point theorem in cones and combining it with an available transformation, they established some simple criteria for the existence of at least one, at least two, or many symmetric positive solutions.

Motivated by the above works, in this study, we consider the following second-order multi-point BVP on time scales

$$\begin{cases} \left(p(t)u^{\triangle}(t)\right)^{\nabla}+f(t,u(t))=0, \quad t\in(a,b)_{\mathbb{T}}, \\ \alpha u(a)-\beta u^{[\triangle]}(a)=\displaystyle\sum_{i=1}^{m-2}\alpha_i u(\xi_i), \\ \alpha u(b)+\beta u^{[\triangle]}(b)=\displaystyle\sum_{i=1}^{m-2}\alpha_i u(\eta_i), \end{cases} \tag{1.1}$$

where $\mathbb{T}\subset\mathbb{R}$ be a symmetric bounded time scale, with $a=\min\mathbb{T}$, $b=\max\mathbb{T}$, $[a,b]_{\mathbb{T}}=[a,b]\cap\mathbb{T}$ and $u^{[\triangle]}=p(t)u^{\triangle}(t)$ is called the quasi $\triangle$-derivative of $u(t)$.

Throughout this paper we assume that following conditions hold:

$(C1)$ $p\in\mathcal{C}^{\nabla}([a,b]_{\mathbb{T}},(0,\infty))$, $f\in\mathcal{C}([a,b]_{\mathbb{T}}\times(0,\infty),(0,\infty))$ and p,f are symmetric functions on $[a,b]_{\mathbb{T}}$ such that $p(b+a-t)=p(t)$, $f(b+a-t,u)=f(t,u)$,

$(C2)$ $\alpha,\beta\in[0,\infty)$, $\alpha_i\in[0,\infty)$, $\xi_i,\eta_i\in(a,b)_{\mathbb{T}}$ such that $\xi_i=b+a-\eta_i$ for $i\in\{1,2,...,m-2\}$.

By using the fixed point index theory in the cone [7], we get the existence of at least two or many symmetric positive solutions for the BVP (1.1). Hence, we improve and generalize the existing results of symmetric positive solutions for second-order m-point BVP to some degree, and so it is interesting and important to study BVP (1.1). In fact, our results are also new when $\mathbb{T} = \mathbb{R}$ (the differential case) and $\mathbb{T} = \mathbb{Z}$ (the discrete case). Therefore, the results can be considered as a contribution to this field.

The organization of this paper is as follows. In Section 2, we provide some definitions and preliminary lemmas which are key tools for our main results. We give and prove our main results in Section 3. Finally, in Section 4, we give an example to illustrate how the main results can be used in practice.

2 Preliminaries

In this section, to state the main results of this paper, we need the following lemmas.

Define by $\theta(t)$ and $\varphi(t)$, the solutions of the corresponding homogeneous equation

$$(p(t)u^{\triangle}(t))^{\nabla} = 0, \quad t \in (a,b)_{\mathbb{T}}, \tag{2.1}$$

under the initial conditions

$$\begin{cases} \theta(a) = \beta, \ \theta^{[\triangle]}(a) = \alpha, \\ \varphi(b) = \beta, \ \varphi^{[\triangle]}(b) = -\alpha. \end{cases} \tag{2.2}$$

Using the initial conditions (2.2), we can deduce, from (2.1) for $\theta(t)$ and $\varphi(t)$, the following equations:

$$\theta(t) = \beta + \alpha \int_a^t \frac{\triangle\tau}{p(\tau)}, \tag{2.3}$$

$$\varphi(t) = \beta + \alpha \int_t^b \frac{\triangle\tau}{p(\tau)}. \tag{2.4}$$

Define the number D

$$D = -p(t) \begin{vmatrix} \theta(t) & \varphi(t) \\ \theta^{\triangle}(t) & \varphi^{\triangle}(t) \end{vmatrix},$$

for each $t \in [a,b]_{\mathbb{T}}$. From (2.3) and (2.4), we have that $D = 2\alpha\beta + \alpha^2 \int_a^b \frac{\triangle\tau}{p(\tau)}$. Define

$$\Delta := \begin{vmatrix} 1 - \frac{1}{D}\sum_{i=1}^{m-2} \alpha_i\varphi(\xi_i) & -\frac{1}{D}\sum_{i=1}^{m-2} \alpha_i\theta(\xi_i) \\ -\frac{1}{D}\sum_{i=1}^{m-2} \alpha_i\varphi(\eta_i) & 1 - \frac{1}{D}\sum_{i=1}^{m-2} \alpha_i\theta(\eta_i) \end{vmatrix}. \tag{2.5}$$

Lemma 2.1 *Let* $(C1)$ *and* $(C2)$ *hold. Assume that*

$(C3)$ $\Delta \neq 0$.

Then the BVP (1.1) *has a unique solution*

$$u(t) = \int_a^b G(t,s)f(s,u(s))\nabla s + \frac{A(f)}{D}\theta(t) + \frac{B(f)}{D}\varphi(t), \tag{2.6}$$

where

$$G(t,s) = \frac{1}{D}\begin{cases} \theta(t)\varphi(s), & a \le t \le s \le b, \\ \theta(s)\varphi(t), & a \le s \le t \le b, \end{cases} \tag{2.7}$$

$$A(f) := \frac{1}{\Delta}\begin{vmatrix} 1 - \frac{1}{D}\sum_{i=1}^{m-2} \alpha_i\varphi(\xi_i) & \sum_{i=1}^{m-2}\alpha_i \int_a^b G(\xi_i,s)f(s,u(s))\nabla s \\ -\frac{1}{D}\sum_{i=1}^{m-2} \alpha_i\varphi(\eta_i) & \sum_{i=1}^{m-2}\alpha_i \int_a^b G(\eta_i,s)f(s,u(s))\nabla s \end{vmatrix}, \tag{2.8}$$

$$B(f) := \frac{1}{\Delta}\begin{vmatrix} \sum_{i=1}^{m-2}\alpha_i \int_a^b G(\xi_i,s)f(s,u(s))\nabla s & -\frac{1}{D}\sum_{i=1}^{m-2} \alpha_i\theta(\xi_i) \\ \sum_{i=1}^{m-2}\alpha_i \int_a^b G(\eta_i,s)f(s,u(s))\nabla s & 1 - \frac{1}{D}\sum_{i=1}^{m-2} \alpha_i\theta(\eta_i) \end{vmatrix}, \tag{2.9}$$

and Δ *is given by* (2.5).

Proof. Let $u(t)=\int_a^b G(t,s)f(s,u(s))\nabla s+\frac{A(f)}{D}\theta(t)+\frac{B(f)}{D}\varphi(t)$ be a solution of (1.1), then we have that

$$\begin{aligned}
u(t) &= \int_a^t \frac{1}{D}\theta(s)\varphi(t)f(s,u(s))\nabla s+\int_t^b \frac{1}{D}\theta(t)\varphi(s)f(s,u(s))\nabla s\\
&+ \frac{A(f)}{D}\theta(t)+\frac{B(f)}{D}\varphi(t),\\
p(t)u^{\triangle}(t) &= p(t)\varphi^{\triangle}(t)\int_a^t \frac{1}{D}\theta(s)f(s,u(s))\nabla s+p(t)\theta^{\triangle}(t)\int_t^b \frac{1}{D}\varphi(s)f(s,u(s))\nabla s\\
&+ \frac{A(f)}{D}p(t)\theta^{\triangle}(t)+\frac{B(f)}{D}p(t)\varphi^{\triangle}(t),
\end{aligned}$$

and

$$\begin{aligned}
\Big(p(t)u^{\triangle}(t)\Big)^{\nabla} &= \Big(p(t)\varphi^{\triangle}(t)\Big)^{\nabla}\int_a^t \frac{1}{D}\theta(s)f(s,u(s))\nabla s+p(\rho(t))\varphi^{\triangle}(\rho(t))\frac{1}{D}\theta(t)h(t)\\
&+ \Big(p(t)\theta^{\triangle}(t)\Big)^{\nabla}\int_t^b \frac{1}{D}\varphi(s)f(s,u(s))\nabla s-p(\rho(t))\theta^{\triangle}(\rho(t))\frac{1}{D}\varphi(t)h(t)\\
&+ \frac{A(f)}{D}\Big(p(t)\theta^{\triangle}(t)\Big)^{\nabla}+\frac{B(f)}{D}\Big(p(t)\varphi^{\triangle}(t)\Big)^{\nabla}\\
&= -\frac{p(\rho(t))}{D}\Big[-\varphi^{\triangle}(\rho(t))\theta(t)+\theta^{\triangle}(\rho(t))\varphi(t)\Big]h(t)\\
&= -h(t).
\end{aligned}$$

Since

$$u(a) = \int_a^b \frac{1}{D}\theta(a)\varphi(s)f(s,u(s))\nabla s+\frac{A(f)}{D}\theta(a)+\frac{B(f)}{D}\varphi(a),$$

$$p(a)u^{\triangle}(a)=p(a)\theta^{\triangle}(a)\int_a^b \frac{1}{D}\varphi(s)f(s,u(s))\nabla s+\frac{A(f)}{D}p(a)\theta^{\triangle}(a)+\frac{B(f)}{D}p(a)\varphi^{\triangle}(a),$$

we get

$$\frac{B(f)}{D}\Big(\alpha\varphi(a)-\beta p(a)\varphi^{\triangle}(a)\Big)=\sum_{i=1}^{m-2}\alpha_i\left[\int_a^b G(\xi_i,s)f(s,u(s))\nabla s+\frac{A(f)}{D}\theta(\xi_i)+\frac{B(f)}{D}\varphi(\xi_i)\right]. \tag{2.10}$$

Since

$$u(b)=\int_a^b \frac{1}{D}\theta(s)\varphi(b)f(s,u(s))\nabla s+\frac{A(f)}{D}\theta(b)+\frac{B(f)}{D}\varphi(b),$$

$$p(b)u^{\triangle}(b)=p(b)\varphi^{\triangle}(b)\int_a^b \frac{1}{D}\theta(s)f(s,u(s))\nabla s+\frac{A(f)}{D}p(b)\theta^{\triangle}(b)+\frac{B(f)}{D}p(b)\varphi^{\triangle}(b),$$

we obtain

$$\frac{A(f)}{D}\Big(\alpha\theta(b)+\beta p(b)\theta^{\triangle}(b)\Big)=\sum_{i=1}^{m-2}\alpha_i\left[\int_a^b G(\eta_i,s)f(s,u(s))\nabla s+\frac{A(f)}{D}\theta(\eta_i)+\frac{B(f)}{D}\varphi(\eta_i)\right]. \tag{2.11}$$

From (2.10) and (2.11) and the fact that

$$D:=-p(t)\begin{vmatrix}\theta(t) & \varphi(t)\\ \theta^{\triangle}(t) & \varphi^{\triangle}(t)\end{vmatrix}\text{for } t\in[a,b]_{\mathbb{T}},$$

we get

$$\begin{cases}\left[1-\frac{1}{D}\sum_{i=1}^{m-2}\alpha_i\varphi(\xi_i)\right]B(f)+\left[-\frac{1}{D}\sum_{i=1}^{m-2}\alpha_i\theta(\xi_i)\right]A(f)=\sum_{i=1}^{m-2}\alpha_i\int_a^b G(\xi_i,s)f(s,u(s))\nabla s\\[2ex] \left[-\frac{1}{D}\sum_{i=1}^{m-2}\alpha_i\varphi(\eta_i)\right]B(f)+\left[1-\frac{1}{D}\sum_{i=1}^{m-2}\alpha_i\theta(\eta_i)\right]A(f)=\sum_{i=1}^{m-2}\alpha_i\int_a^b G(\eta_i,s)f(s,u(s))\nabla s\end{cases},$$

which implies that $A(f)$ and $B(f)$ satisfy (2.8) and (2.9), respectively. □

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Lemma 2.2 *For $t,\ s \in [a,b]_{\mathbb{T}}$, we have $G(b+a-t, b+a-s) = G(t,s)$.*

Proof. In fact, if $t \leq s$, then $b+a-t \geq b+a-s$. In view of (2.7) and the assumption $(C1)$, we get

$$\begin{aligned} G(b+a-t, b+a-s) &= \frac{1}{D}\left(\beta + \alpha \int_a^{b+a-s} \frac{\triangle\tau}{p(\tau)}\right)\left(\beta + \alpha \int_{b+a-t}^{b} \frac{\triangle\tau}{p(\tau)}\right) \\ &= \frac{1}{D}\left(\beta + \alpha \int_b^{s} \frac{\triangle(b+a-\tau)}{p(b+a-\tau)}\right)\left(\beta + \alpha \int_{t}^{a} \frac{\triangle(b+a-\tau)}{p(b+a-\tau)}\right) \\ &= \frac{1}{D}\left(\beta + \alpha \int_s^{b} \frac{\triangle\tau}{p(\tau)}d\tau\right)\left(\beta + \alpha \int_{a}^{t} \frac{\triangle\tau}{p(\tau)}d\tau\right) \\ &= G(t,s),\ a \leq t \leq s \leq b. \end{aligned}$$

Similarly, we can prove that for $a \leq s \leq t \leq b$, $G(b+a-t, b+a-s) = G(t,s)$. So, we have $G(b+a-t, b+a-s) = G(t,s)$ for all $(t,s) \in [a,b]_{\mathbb{T}} \times [a,b]_{\mathbb{T}}$, i.e., G(t,s) is symmetric function on $[a,b]_{\mathbb{T}} \times [a,b]_{\mathbb{T}}$. □

It is easy to see that from the conditions $(C1),\ (C2)$ and Lemma 2.2, $A(f) = B(f)$. So, we obtain that the solution of BVP (1.1)

$$u(t) = \int_a^b G(t,s) f(s,u(s)) \nabla s + \frac{B(f)}{\alpha}.$$

Lemma 2.3 *Let $(C1)$ and $(C2)$ hold. Assume*

(C4) $\Delta > 0,\ 1 - \frac{1}{D}\sum_{i=1}^{m-2} \alpha_i \theta(\eta_i),\ 1 - \frac{1}{D}\sum_{i=1}^{m-2} \alpha_i \varphi(\xi_i) > 0.$

Then the unique solution u of the BVP (1.1) satisfies

$$u(t) \geq 0 \text{ for } t \in [a,b]_{\mathbb{T}}.$$

Proof. It is an immediate subsequence of the facts that $G \geq 0$ on $[a,b]_{\mathbb{T}} \times [a,b]_{\mathbb{T}}$ and $B(f) \geq 0$. □

Lemma 2.4 *Let $(C1), (C2)$ and $(C4)$ hold. Then the unique solution u of the BVP (1.1) satisfies*

$$u(t) \geq \Gamma \|u\|,$$

where $\|u\| = \max_{t \in [a,b]_{\mathbb{T}}} |u(t)|$ and

$$\Gamma = \frac{\beta}{\beta + \alpha \int_a^b \frac{\triangle\tau}{p(\tau)}}. \tag{2.12}$$

Proof. We have from (2.7) that

$$0 \leq G(t,s) \leq G(s,s),\ t \in [a,b]_{\mathbb{T}},$$

which implies

$$u(t) \leq \int_a^b G(s,s) f(s,u(s)) \nabla s + \frac{B(f)}{\alpha} \text{ for all } t \in [a,b]_{\mathbb{T}}.$$

Applying (2.7), we have that for $t \in [a,b]_{\mathbb{T}}$

$$\frac{G(t,s)}{G(s,s)} = \begin{cases} \frac{\theta(t)}{\theta(s)}, & a \leq t \leq s \leq b, \\ \frac{\varphi(t)}{\varphi(s)}, & a \leq s \leq t \leq b, \end{cases} \geq \begin{cases} \frac{\beta}{\theta(b)}, & a \leq t \leq s \leq b, \\ \frac{\beta}{\varphi(a)}, & a \leq s \leq t \leq b, \end{cases} \geq \Gamma,$$

where Γ be given as in equation (2.12). Thus, we have

$$\min_{t \in [a,b]_{\mathbb{T}}} G(t,s) \geq \Gamma G(s,s). \tag{2.13}$$

So, for $t \in [a,b]_{\mathbb{T}}$,

$$\begin{aligned} u(t) &= \int_a^b G(t,s)f(s,u(s))\nabla s + \frac{B(f)}{\alpha} \\ &\geq \int_a^b \Gamma G(s,s)f(s,u(s))\nabla s + \frac{B(f)}{\alpha} \\ &\geq \Gamma\left(\int_a^b G(s,s)f(s,u(s))\nabla s + \frac{B(f)}{\alpha}\right) \\ &\geq \Gamma\left\|u\right\|. \end{aligned}$$

□

Let $\mathbb{B} = \mathcal{C}[a,b]$ is a Banach space with the norm $\|u\| = \max_{t\in[a,b]_{\mathbb{T}}} |u(t)|$. Define the cone $K \subset \mathbb{B}$ by

$$K = \left\{u \in \mathbb{B} : u(t) \text{ is nonnegative, symmetric on } [a,b]_{\mathbb{T}} \text{ and } \min_{t\in[a,b]_{\mathbb{T}}} u(t) \geq \Gamma\left\|u\right\|\right\},$$

where Γ be given as in equation (2.12).

We can define an operator $T : K \to \mathbb{B}$ by

$$Tu(t) = \int_a^b G(t,s)f\left(s,u(s)\right)\nabla s + \frac{B(f)}{\alpha}, \tag{2.14}$$

where G and $B(f)$ are defined as in (2.7) and (2.9), respectively.

Lemma 2.5 *Let $(C1) - (C4)$ hold. Then $T : K \to K$ is completely continuous.*

Proof. For all $u \in K$, by $(C1) - (C4)$, we have from (2.14), $Tu(t) \geq 0$, for all $t \in [a,b]_{\mathbb{T}}$. Furthermore, by (2.7) and (2.13),

$$\begin{aligned} Tu(t) &\geq \int_a^b \Gamma G(s,s)f(s,u(s))\nabla s + \frac{B(f)}{\alpha} \\ &\geq \Gamma\left(\int_a^b G(s,s)f(s,u(s))\nabla s + \frac{B(f)}{\alpha}\right) \\ &\geq \Gamma\left\|Tu\right\|, \end{aligned}$$

i.e., $\min_{t\in[a,b]_{\mathbb{T}}} Tu(t) \geq \Gamma\left\|Tu\right\|.$

Noticing $p(t),\ u(t)$ and $f(t,u)$ are symmetric on $[a,b]_{\mathbb{T}}$ and by Lemma 2.2, we have

$$\begin{aligned} Tu(b+a-t) &= \int_a^b G(b+a-t,s)f\left(s,u(s)\right)\nabla s + \frac{B(f)}{\alpha} \\ &= \int_b^a G(b+a-t,b+a-s)f\left(s,u(s)\right)\nabla(b+a-s) + \frac{B(f)}{\alpha} \\ &= \int_a^b G(t,s)f\left(s,u(s)\right)\nabla(b+a-s) + \frac{B(f)}{\alpha} \\ &= Tu(t), \end{aligned}$$

i.e., $Tu(b+a-t) = Tu(t),\ t \in [a,b]_{\mathbb{T}}$. Therefore $Tu(t)$ is symmetric on $[a,b]_{\mathbb{T}}$.

So, $Tu \in K$ and then $Tu \subset K$. Next, by standard methods and Arzela-Ascoli theorem, one can easily prove that operator T is completely continuous. □

3 Main Results

In this section, we consider the existence of at least two or many symmetric positive solutions of the BVP (1.1). The following fixed point theorem is fundamental and important to the proof of our main results.

Lemma 3.1 *(See [7]). Let K be a cone in a real Banach space $\mathbb{B}$. Let D be an open bounded subset of $\mathbb{B}$ with $D_K = D \cap K \neq \emptyset$ and $\overline{D}_K \neq K$. Assume that $T : \overline{D}_K \to K$ is completely continuous such that $u \neq Tu$ for $u \in \partial D_K$. Then the following results hold:*

(i) If $\|Tu\| \leq u,\ u \in \partial D_K$, then $i_K(T,D_K) = 1$.

(ii) If there exists $e \in K \setminus \{0\}$ such that $u \neq Tu + \lambda e$ for all $u \in \partial D_K$ and all $\lambda > 0$, then $i_K(T,D_K) = 0$.

(iii) *Let U be open in P such that $\overline{U} \subset D_K$. If $i_K(T, D_K) = 1$ and $i_K(T, U_K) = 0$, then T has a fixed point in $D_K \setminus \overline{U}_K$. The same result holds if $i_K(T, D_K) = 0$ and $i_K(T, U_K) = 1$.*

We define

$$\begin{aligned} K_\rho &= \{u \in K : \|u\| < \rho\}, \\ \Omega_\rho &= \{u \in K : \min_{t\in[a,b]_{\mathbb{T}}} u(t) < \Gamma\rho\} = \{u \in K : \Gamma\|u\| \le \min_{t\in[a,b]_{\mathbb{T}}} u(t) < \Gamma\rho\}. \end{aligned}$$

Lemma 3.2 *Ω_ρ has the following properties:*

(a) *Ω_ρ is open relative to K.*

(b) *$K_{\Gamma\rho} \subset \Omega_\rho \subset K_\rho$.*

(c) *$u \in \Omega_\rho$ if and only if $\min_{t\in[a,b]_{\mathbb{T}}} u(t) = \Gamma\rho$.*

(d) *If $u \in \Omega_\rho$, then $\Gamma\rho \le u(t) \le \rho$ for $t \in [a,b]_{\mathbb{T}}$.*

Now for convenience we introduce the following notations. Let

$$\begin{aligned} f^\rho_{\Gamma\rho} &= \min\left\{\min_{t\in[a,b]_{\mathbb{T}}} f(t,u) : u \in [\Gamma\rho, \rho]\right\}, \\ f^\rho_0 &= \max\left\{\max_{t\in[a,b]_{\mathbb{T}}} f(t,u) : u \in [0, \rho]\right\}, \\ \tilde{B} &= \frac{1}{\Delta}\begin{vmatrix} \sum_{i=1}^{m-2}\alpha_i\int_a^b G(s,s)\nabla s & -\frac{1}{D}\sum_{i=1}^{m-2}\alpha_i\theta(\xi_i) \\ \sum_{i=1}^{m-2}\alpha_i\int_a^b G(s,s)\nabla s & 1-\frac{1}{D}\sum_{i=1}^{m-2}\alpha_i\theta(\eta_i) \end{vmatrix}, \\ L &= \left(\int_a^b G(s,s)\nabla s + \frac{\tilde{B}}{\alpha}\right)^{-1}. \end{aligned}$$

Theorem 3.1 *Suppose $(C1)-(C4)$ hold.*

$(C5)$ *There exist $\rho_1,\ \rho_2,\ \rho_3 \in (0,\infty)$ with $\rho_1 < \Gamma\rho_2$ and $\rho_2 < \rho_3$ such that*

$$f_0^{\rho_1} < \rho_1 L,\ f^{\rho_2}_{\Gamma\rho_2} > \rho_2 L,\ f_0^{\rho_3} < \rho_3 L.$$

Then problem (1.1) has at least two symmetric positive solutions $u_1,\ u_2$ with $u_1 \in \Omega_{\rho_2}\setminus\overline{K}_{\rho_1},\ u_2 \in K_{\rho_3}\setminus\overline{\Omega}_{\rho_2}$.

$(C6)$ *There exist $\rho_1,\ \rho_2,\ \rho_3 \in (0,\infty)$ with $\rho_1 < \rho_2 < \Gamma\rho_3 < \rho_3$ such that*

$$f^{\rho_1}_{\Gamma\rho_1} > \rho_1 L,\ f_0^{\rho_2} < \rho_2 L,\ f^{\rho_3}_{\Gamma\rho_3} > \rho_3 L.$$

Then problem (1.1) has at least two symmetric positive solutions $u_1,\ u_2$ with $u_1 \in K_{\rho_2}\setminus\overline{\Omega}_{\rho_1},\ u_2 \in \Omega_{\rho_3}\setminus\overline{K}_{\rho_2}$.

Proof. We only consider the condition $(C5)$. If $(C6)$ holds, then the proof is similar to that of the case when $(C5)$ holds. By Lemma 2.5, we know that the operator $T : K \to K$ is completely continuous.

First, we show that $i_K(T, K_{\rho_1}) = 1$. In fact, by (2.14), $f_0^{\rho_1} < \rho_1 L$, we have for $u \in \partial K_{\rho_1}$,

$$\begin{aligned} Tu(t) &= \int_a^b G(t,s)f(s,u(s))\nabla s + \frac{B(f)}{\alpha} \\ &\le \int_a^b G(s,s)f(s,u(s))\nabla s + \frac{1}{\alpha\Delta}\begin{vmatrix} \sum_{i=1}^{m-2}\alpha_i\int_a^b G(s,s)f(s,u(s))\nabla s & -\frac{1}{D}\sum_{i=1}^{m-2}\alpha_i\theta(\xi_i) \\ \sum_{i=1}^{m-2}\alpha_i\int_a^b G(s,s)f(s,u(s))\nabla s & 1-\frac{1}{D}\sum_{i=1}^{m-2}\alpha_i\theta(\eta_i) \end{vmatrix} \\ &\le \int_a^b G(s,s)\nabla s\rho_1 L + \frac{\tilde{B}}{\alpha}\rho_1 L \\ &= \left(\int_a^b G(s,s)\nabla s + \frac{\tilde{B}}{\alpha}\right)\rho_1 L = \rho_1, \end{aligned}$$

i.e., $\|Tu\| < \|u\|$ for $u \in \partial K_{\rho_1}$. By (i) of Lemma 3.1, we obtain that $i_K(T, K_{\rho_1}) = 1$.

Secondly, we show that $i_K(T, \Omega_{\rho_2}) = 0$. Let $e(t) \equiv 1$. Then $e \in \partial K_1$. We claim that

$$u \ne Tu + \lambda e,\ u \in \partial\Omega_{\rho_2},\ \lambda > 0.$$

Suppose that there exists $u_0 \in \partial\Omega_{\rho_2}$ and $\lambda_0 > 0$ such that

$$u_0 = Tu_0 + \lambda_0 e. \tag{3.1}$$

Then, Lemma 2.4, (2.14) and (3.1) imply that for $t \in [a,b]_{\mathbb{T}}$

$$\begin{aligned} u_0 &= Tu_0 + \lambda_0 e \\ &= \int_a^b G(t,s) f(s, u_0(s)) \nabla s + \frac{B(f)}{\alpha} + \lambda_0 e \\ &> \Gamma \int_a^b G(s,s) \nabla s \rho_2 L + \frac{\Gamma \tilde{B}}{\alpha} \rho_2 L + \lambda_0 \\ &= \Gamma \left(\int_a^b G(s,s) \nabla s + \frac{\tilde{B}}{\alpha} \right) \rho_2 L + \lambda_0 \\ &= \Gamma \rho_2 + \lambda_0, \end{aligned}$$

i.e., $\Gamma\rho_2 > \Gamma\rho_2 + \lambda_0$, which is a contradiction. Hence by (ii) of Lemma 3.1, it follows that $i_K(T, \Omega_{\rho_2}) = 0$.

Finally, similar to the proof of $i_K(T, K_{\rho_1}) = 1$, we can prove that $i_K(T, K_{\rho_3}) = 1$. Since $\rho_1 < \Gamma\rho_2$ and Lemma 3.2 (b), we have $\overline{K}_{\rho_1} \subset K_{\Gamma\rho_2} \subset \Omega_{\rho_2}$. Similarly with $\rho_2 < \rho_3$ and Lemma 3.2 (b), we have $\overline{\Omega}_{\rho_2} \subset K_{\rho_2} \subset K_{\rho_3}$. Therefore (iii) of Lemma 3.1 implies that BVP (1.1) has at least two symmetric positive solutions u_1, u_2 with $u_1 \in \Omega_{\rho_2} \setminus \overline{K}_{\rho_1}$, $u_2 \in K_{\rho_3} \setminus \overline{\Omega}_{\rho_2}$. □

Theorem 3.1 can be generalized to obtain many solutions.

Theorem 3.2 *Suppose $(C1) - (C4)$ hold. Then we have the following assertions.*

$(C7)$ *There exists $\{\rho_i\}_{i=1}^{2m_0+1} \subset (0,\infty)$ with $\rho_1 < \Gamma\rho_2 < \rho_2 < \rho_3 < \Gamma\rho_4 < ... < \Gamma\rho_{2m_0} < \rho_{2m_0} < \rho_{2m_0+1}$ such that*

$$f_0^{\rho_{2m-1}} < \rho_{2m-1} L, \ (m = 1, 2, ..., m_0, m_0+1), \ f_{\Gamma\rho_{2m}}^{\rho_{2m}} > \rho_{2m} L, \ (m = 1, 2, ..., m_0).$$

Then the BVP (1.1) has at least $2m_0$ symmetric positive solutions in K.

$(C8)$ *There exists $\{\rho_i\}_{i=1}^{2m_0} \subset (0,\infty)$ with $\rho_1 < \Gamma\rho_2 < \rho_2 < \rho_3 < \Gamma\rho_4 < ... < \Gamma\rho_{2m_0} < \rho_{2m_0}$ such that*

$$f_0^{\rho_{2m-1}} < \rho_{2m-1} L, \ \ f_{\Gamma\rho_{2m}}^{\rho_{2m}} > \rho_{2m} L, \ (m = 1, 2, ..., m_0).$$

Then the BVP (1.1) has at least $2m_0 - 1$ symmetric positive solutions in K.

Theorem 3.3 *Suppose $(C1) - (C4)$ hold. Then we have the following assertions.*

$(C9)$ *There exists $\{\rho_i\}_{i=1}^{2m_0+1} \subset (0,\infty)$ with $\rho_1 < \rho_2 < \Gamma\rho_3 < \rho_3 < ... < \rho_{2m_0} < \Gamma\rho_{2m_0+1} < \rho_{2m_0+1}$ such that*

$$f_{\Gamma\rho_{2m-1}}^{\rho_{2m-1}} > \rho_{2m-1} L, \ (m = 1, 2, ..., m_0, m_0+1), \ f_0^{\rho_{2m}} < \rho_{2m} L, \ (m = 1, 2, ..., m_0).$$

Then the BVP (1.1) has at least $2m_0$ symmetric positive solutions in K.

$(C10)$ *There exists $\{\rho_i\}_{i=1}^{2m_0} \subset (0,\infty)$ with $\rho_1 < \rho_2 < \Gamma\rho_3 < \rho_3 < ... < \Gamma\rho_{2m_0-1} < \rho_{2m_0-1} < \rho_{2m_0}$ such that*

$$f_{\Gamma\rho_{2m-1}}^{\rho_{2m-1}} > \rho_{2m-1} L, \ f_0^{\rho_{2m}} < \rho_{2m} L, \ (m = 1, 2, ..., m_0).$$

Then the BVP (1.1) has at least $2m_0 - 1$ symmetric positive solutions in K.

4 An Example

Example 4.1 In BVP (1.1), suppose that $\mathbb{T} = \left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right]$, $p(t) = 1$, $m = 3$, $\alpha = \beta = 1$, $\alpha_1 = \frac{1}{2}, \xi_1 = \frac{1}{5}$ and $\eta_1 = \frac{4}{5}$, i.e.,

$$\begin{cases} u^{\triangle\nabla}(t) + f(t,u) = 0, \quad t \in (0,1)_{\mathbb{T}}, \\ u(0) - u^{[\triangle]}(0) = \frac{1}{2} u\left(\frac{1}{5}\right), \\ u(1) + u^{[\triangle]}(1) = \frac{1}{2} u\left(\frac{4}{5}\right), \end{cases} \tag{4.1}$$

where

$$f(t,u)=\begin{cases} \frac{1}{3}u(t)+t(1-t), & (t,u)\in[0,1]_{\mathbb{T}}\times\left[0,\frac{3}{2}\right], \\ 5u(t)-7+t(1-t), & (t,u)\in[0,1]_{\mathbb{T}}\times\left[\frac{3}{2},2\right], \\ -\frac{3}{4}u(t)^2+6u(t)-6+t(1-t), & (t,u)\in[0,1]_{\mathbb{T}}\times[2,\infty). \end{cases}$$

By simple calculation, we get $D=3,\ \theta(t)=1+t,\ \varphi(t)=2-t,\ \Delta=\frac{9}{20},\ \Gamma=\frac{1}{2},\ \tilde{B}=\frac{103}{144},\ L=\frac{72}{103}$ and

$$G(t,s)=\frac{1}{3}\begin{cases} (1+t)(2-s), & 0\le t\le s\le 1, \\ (1+s)(2-t), & 0\le s\le t\le 1. \end{cases}$$

It is clear that conditions $(C1)-(C4)$ are satisfied. Taking $\rho_1=\frac{3}{2},\ \rho_2=4,\ \rho_3=9$, we can obtain that

$$\rho_1<\Gamma\rho_2 \quad \text{and} \quad \rho_2<\rho_3.$$

Now, we show that $(C5)$ is satisfied:

$$\begin{aligned} f_0^{\frac{3}{2}} &\le 0.75<\rho_1 L=1.048, \\ f_2^4 &\ge 3>\rho_2 L=2.796, \\ f_0^9 &\le 6.25<\rho_3 L=6.291. \end{aligned}$$

Then, $(C5)$ condition of Theorem 3.1 hold. Hence, we get the BVP (4.1) has at least two symmetric positive solutions.

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Approximate fuzzy double derivations and fuzzy Lie $*$-double derivations

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Abstract. Using the fixed point method, we prove the Hyers-Ulam stability of fuzzy double derivations and fuzzy Lie $*$-double derivations on fuzzy Banach algebras, fuzzy C^*-algebras and fuzzy Lie C^*-algebras associated with the following additive functional equation of n-Apollonius type

$$\sum_{i=1}^{n} f(z-x_i) = -\frac{1}{n}\sum_{1\le i<j\le n} f(x_i+x_j) + nf(z-\frac{1}{n^2}\sum_{i=1}^{n} x_i)$$

for a fixed integer n with $n \geq 2$.

1. Introduction and preliminaries

Let $\mathcal{A}$ be a subalgebra of an algebra $\mathcal{B}$, $\mathcal{X}$ a $\mathcal{B}$-bimodule and $\delta : \mathcal{A} \to \mathcal{B}$ be a linear mapping. A linear mapping $f : \mathcal{A} \to \mathcal{X}$ is called a *δ-derivation* [26] if $f(ab) = f(a)\delta(b) + \delta(a)f(b)$ for all $a, b \in \mathcal{A}$. Clearly, if $\delta = id$, the identity mapping on $\mathcal{A}$, then a δ-derivation is an ordinary derivation. On the other hand, each homomorphism f is a $\frac{f}{2}$-derivation. Thus, the theory of δ-derivation combines the theory of derivations and homomorphisms. If $g : \mathcal{A} \to \mathcal{A}$ is an ordinary derivation and $\delta : \mathcal{A} \to \mathcal{A}$ is a homomorphism, then $f = g\delta$ is a δ-derivation. Although, a δ-derivation is not necessarily of the form $g\delta$, but it seems that the generalized Leibniz rule, $f(ab) = f(a)\delta(b) + \delta(a)f(b)$, comes from this observation. Mirzavaziri and Omidvar Tehrani [27] took ideas from the above fact, they defined a (g, h)-double derivation as follows: If linear mappings $f, g, h : \mathcal{A} \to \mathcal{A}$ satisfy

$$f(ab) = f(a)b + af(b) + g(a)h(b) + h(a)g(b) \tag{1.1}$$

for all $a, b \in \mathcal{A}$, then the linear mapping $f : \mathcal{A} \to \mathcal{A}$ is called a *(g, h)-double derivation*. Moreover, a (g, g)-double derivation is called a *g-double derivation* and they proved that if $\mathcal{A}$ is a C^*-algebra, $f : \mathcal{A} \to \mathcal{A}$ is a $*$-linear mapping and $g : \mathcal{A} \to \mathcal{A}$ is a continuous f-double derivation then f is continuous.

In 1984, Katsaras [21] introduced an idea of a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. In the same year, Wu and Fang [47] introduced a notion fuzzy normed apace to give a generalization of the Kolmogoroff normalized theorem for fuzzy topological vector spaces. In 1992, Felbin [14] introduced an alternative definition of a fuzzy norm on a vector space with an associated metric of Kaleva and Seikkala type [20]. Some mathematics have defined fuzzy normed on a vector form various point of view [24, 36, 48]. In particular, Bag and Samanta [4] following Cheng and Mordeson [7], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric of Kramosil and Michalek type

[0]**2010 Mathematics Subject Classification**: 39B52, 47S40, 46S40, 47H10

[0]**Keywords**: Hyers-Ulam stability; double derivation; $*$-double derivation; fuzzy Banach $*$-algebra; induced fuzzy Lie C^*-algebra; fixed point method.

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[23]. They established a decomposition theorem of fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [5].

A classical equation in the theory of functional equations is the following: "when is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation?". If the problem accepts a solution, we say that the equation is stable. The first problem concerning group homomorphisms was raised by Ulam [46] in 1940. In the next year, Hyers [16] gave a first affirmative answer to the question of Ulam in context of Banach spaces. Subsequently, the result of Hyers was generalized by Aoki [1] for additive mapping and by Rassias [40] for linear mapping by considering an unbounded Cauchy difference. Furthermore, in 1994, Găvruţa [15] provided a further generalization of Rassias' theorem in which he replaced the bound $\varepsilon(\|x\|^p+\|y\|^p)$ in by a general control function $\varphi(x,y)$. Recently several stability results have been obtained for various equations and mappings with more general domains and ranges have been investigated by a number of authors and there are many interesting results concerning this problem [11, 12, 13, 17, 18, 19, 22, 28, 29, 30, 31, 33, 37, 38, 42, 43, 44, 45].

Bourgin [6] proved the Hyers-Ulam stability of ring homomorphisms in unital Banach algebras and Badora [2] gave a generalization of the Bourgin result concerning derivations in operator algebras, which was first obtained by Šemrl [41]. In [3], Badora proved the Hyers-Ulam stability of the functional equation $f(ab) = af(b) + f(a)b$, where f is a mapping in a unital normed algebra $\mathcal{A}$. In the following, we will give some notations that are needed in this paper.

we recall some notations and basic definitions.

Definition 1.1. *Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0,1]$ is said to be a fuzzy norm on X if for all $x, y \in X$ and all $t, s \in \mathbb{R}$,*
$(N1)$ $N(x,t) = 0$ for $t \leq 0$;
$(N2)$ $N(x,t) = 1$ for all $t > 0$ if and only if $x = 0$;
$(N3)$ $N(cx,t) = P(x, \frac{t}{|c|})$ for each $c \neq 0$;
$(N4)$ $N(x+y, s+t) \geq \min\{N(x,t), N(y,s)\}$;
$(N5)$ $N(x,\cdot)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$;
(N6) for $x \neq 0$,$N(x,\cdot)$ is continuous on $\mathbb{R}$.

The pair (X,N) is called a fuzzy normed linear space.

One may regard $N(x,t)$ as the truth value of the statement "the norm of x is less than or equal to the real number t".

Example 1.2. *Let $(X, \|\cdot\|)$ be a normed linear space and $\alpha, \beta > 0$. Then*

$$N(x,t) = \begin{cases} \frac{\alpha t}{\beta t + \|x\|}, & t > 0, \quad x \in X, \\ 0, & t \leq 0, \quad x \in X \end{cases}$$

is a fuzzy norm on X.

Let (X,N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to be convergent if there exists $x \in X$ such that $lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In that case, x is called the limit of the sequence $\{x_n\}$ and we denote it by $N - lim_{n\to\infty} x_n = x$.

A sequence $\{x_n\}$ in X is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy normed is said to be complete and the fuzzy normed vector space is called a fuzzy Banach space. We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then f is said to be continuous on X [5].

Let X is a Banach algebra. Then an involution on x is a mapping $x \to x^*$ from X into X which satisfies
(1) $(x^*)^* = x$ for $x \in X$;
(2) $(\alpha x + \beta y)^* = \overline{\alpha} x^* + \overline{\beta} y^*$ for $x, y \in X$ and $\alpha, \beta \in \mathbb{C}$;
(3) $(xy)^* = y^* x^*$ for $x, y \in X$.

If, in addition, $\|x^* x\| = \|x\|^2$ for all $x \in X$, then X is a C^*-algebra.

Definition 1.3. *Let X be a $*$-algebra and (X, N) be a fuzzy normed space.*
(1) *The fuzzy normed space (X, N) is called a fuzzy normed $*$-algebra if*

$$N(xy, ts) \geq N(x, t).N(y, s), \qquad N(x^*, t) = N(x, t)$$

for all $x, y \in X$ and all positive real numbers t.
(2) *A complete fuzzy normed $*$-algebra is called a fuzzy Banach $*$-algebra.*

Example 1.4. *Let $(X, \|\cdot\|)$ be a normed $*$-algebra. Let*

$$N(x,t) = \begin{cases} \frac{t}{t+\|x\|}, & t > 0, \quad x \in X, \\ 0, & t \leq 0, \quad x \in X. \end{cases}$$

Then (X, N) is a fuzzy normed $*$-algebra.

Definition 1.5. *Let $(X, \|\cdot\|)$ be a normed $*$-algebra or aC^*-algebra and N be a fuzzy norm on X.*
(1) *The fuzzy normed $*$-algebra (X, N) is called an induced fuzzy normed $*$-algebra.*
(2) *The fuzzy C^*-algebra (X, N) is called an induced fuzzy C^*-algebra.*

Let (X, N) and (Y, N) be induced fuzzy normed $*$-algebras. Then a $\mathbb{C}$-linear mapping $f : (X, N) \to (Y, N)$ is called a fuzzy $*$-homomorphism if $f(xy) = f(x)f(y), \quad f(x^*) = f(x)^*$ and a $\mathbb{C}$-linear mapping $f : (X, N) \to (X, N)$ is called a fuzzy $*$-derivation if $f(xy) = f(x)y + xf(y), \quad f(x^*) = f(x)^*$ for all $x, y \in X$.

Definition 1.6. *Let (X, N) be a fuzzy normed $*$-algebra and $g, h : (X, N) \to (X, N)$ be $\mathbb{C}$-linear mappings. A $\mathbb{C}$-linear mapping $f : (X, N) \to (X, N)$ is called a fuzzy *-(g,h)-double derivation if*

$$f(xy) = f(x)y + xf(y) + g(x)h(y) + h(x)g(y),$$
$$f(x^*) = f(x)^*, \quad g(x^*) = g(x)^*, \quad h(x^*) = h(x)^*,$$

for all $x, y \in X$.

Example 1.7. *Let $H : (X, N) \to (X, N)$ be a fuzzy $*$-homomorphism. Then H is a fuzzy $*$-$(\frac{H}{2} - I)$-double derivation where $I : (X, N) \to (X, N)$ is the identity mapping.*

Let X be a nonempty set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if d satisfies
(1) $d(x, y) = 0$ if and only if $x = y$ for $x, y \in X$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We recall the a fundamental result in fixed point theory.

Theorem 1.8. ([25, 39]) *Let (X, d) be a complete generalized metric space and $J : X \to X$ be a strictly contractive mapping with Lipshitz constant $L < 1$. Then, for each given $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty \quad \textit{for all} \quad n \geq 0,$$

or there exists a natural number n_0 such that
(1) *$d(J^n x, J^{n+1} x) < \infty$ for all $n \geq n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X : d(J^{n_0}, y) < \infty\}$;*
(4) *$d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

In 1996, Isac and Rassias [18] were the first to provide applications of stability theory of functional equations for the proof of new fixed-point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [8, 35]).

Ebadian and Ghobadipour [9] proved the Hyers-Ulam stability of double derivations on Banach algebras and Lie $*$-double derivations on Lie C^*-algebras.

Ebadian et al. [10] considered the following additive functional equation of n-Apollonius type

$$\sum_{i=1}^{n} f(z-x_i) = -\frac{1}{n}\sum_{1\le i<j\le n} f(x_i+x_j) + nf(z-\frac{1}{n^2}\sum_{i=1}^{n} x_i) \tag{1.2}$$

where n is a fixed integer with $n \ge 2$ and proved the Hyers-Ulam stability of homomorphisms and derivations on induced fuzzy Lie C^*-algebras assocated with the functional equation (1.2).

In this paper, we consider a mapping $f : X \to X$ satisfying (1.2) and use the fixed point alternative theorem to establish the Hyers-Ulam stability of fuzzy double derivations on fuzzy Banach algebras and Lie $*$-double derivations on induced fuzzy Lie C^*-algebras associated with the functional equation (1.2).

We define the difference operators $D_\mu f_k : X^{n+1} \to X$ and $\Delta : X^2 \to X$ by

$$D_\mu f(z, x_1, \cdots, x_n) := \sum_{i=1}^{n} \mu f_k(z-x_i) + \frac{1}{n}\sum_{1\le i<j\le n} f_k(\mu x_i+\mu x_j) - nf_k(\mu z-\frac{1}{n^2}\sum_{i=1}^{n} \mu x_i),$$

$$\Delta_{f_1,f_2,f_3}(x, y) := f_1(xy) - f_1(x)y - xf_1(y) - f_2(x)f_3(y) - f_3(x)f_2(y),$$

for all $x, y, z, x_1, \cdots, x_n \in X$, all $\mu \in \mathbb{T}^1 := \{u \in \mathbb{C} : |u| = 1\}$, and $k \in \{1, 2, 3\}$.

Throughout this paper, we assume that $J = \{1, 2, 3\}$.

2. Approximate fuzzy double derivations on fuzzy Banach algebras

In this section, we assume that (X, N) is a fuzzy Banach algebra and prove the Hyers- Ulam stability of fuzzy double derivations on fuzzy Banach algebras related to the additive functional equation of n-Apollonius type (1.2).

Definition 2.1. *Let $g, h : X \to X$ be $\mathbb{C}$-linear mappings. Then a $\mathbb{C}$-linear mapping $f : X \to X$ is called a fuzzy (g, h)-double derivation if f satisfies*

$$f(xy) = f(x)y + xf(y) + g(x)h(y) + h(x)g(y)$$

for all $x, y \in X$.

In the following theorem, we show that under condition an almost fuzzy double derivation is a fuzzy double derivation.

Theorem 2.2. *Let $\lambda > 1$ and $k \in J$. Let $f_k : X \to X$ be mappings such that $f_k(\lambda x) = \lambda f(x)$ for all $x \in X$. If there exists a function $\varphi : X^8 \to [0, \infty)$ such that*

$$\lim_{m\to\infty} \frac{1}{\lambda^m}\varphi(\lambda^m x, \lambda^m y, \lambda^m z, \lambda^m w, \lambda^m u, \lambda^m v, \lambda^m t, \lambda^m s) = 0, \tag{2.1}$$

$$\begin{aligned} N(f_1(\mu x+\mu y+zw) &- \mu f_1(x) - \mu f_1(y) - f_2(\mu u+\mu v) - \mu f_2(u) - \mu f_2(v) - f_3(\mu t+\mu s) \\ &- \mu f_3(t) - \mu f_3(s) - f_1(z)w - zf_1(w) - f_2(z)f_3(w) - f_3(z)f_2(w), t) \\ &\ge \frac{t}{t+\varphi(x, y, z, w, u, v, t, s)} \end{aligned} \tag{2.2}$$

for all $x, y, z, w, u, v, t, s \in X$ and all $\mu \in \mathbb{C}$, then f_1 is a fuzzy (f_2, f_3)-double derivation on X.

Proof. Let $k \in J$. It is clearly $f_k(0)$=0. Let $z = w = u = v = t = s = 0$ in (2.2). Then we have

$$\begin{aligned}
&N(f_1(\mu x + \mu y) - \mu f_1(x) - \mu f_1(y), t)\\
&= N(f_1(\mu\lambda^m x + \mu\lambda^m y) - \mu f_1(\lambda^m x) - \mu f_1(\lambda^m y), \lambda^m t)\\
&\geq \frac{\lambda^m t}{\lambda^m t + \varphi(\lambda^m x, \lambda^m y, 0, 0, 0, 0, 0, 0)}
\end{aligned}$$

for all $x, y \in X$, all $\mu \in \mathbb{C}$ and all $t > 0$. Since

$$\lim_{m\to\infty} \frac{t}{t + \frac{1}{\lambda^m}\varphi(\lambda^m x, \lambda^m y, 0, 0, 0, 0, 0, 0)} = 1,$$

$f_1(\mu x + \mu y) = \mu f_1(x) + \mu f_1(y)$ for all $x, y \in X$ and all $\mu \in \mathbb{C}$.

Similarly, we have $f_2(\mu u + \mu v) = \mu f_2(u) + \mu f_2(v)$ and $f_3(\mu t + \mu s) = \mu f_3(t) + \mu f_3(s)$ for all $x, y \in X$ and all $\mu \in \mathbb{C}$.

Now letting $x = y = u = v = t = s = 0$ and $\mu = 1$ in (2.2), we have

$$\begin{aligned}
N(zw) &- f_1(z)w - zf_1(w) - f_2(z)f_3(w) - f_3(z)f_2(w), t)\\
&= N(\lambda^m z\lambda^m w) - f_1(\lambda^m z)\lambda^m w - \lambda^m z f_1(\lambda^m w)\\
&\quad - f_2(\lambda^m z)f_3(\lambda^m w) - f_3(\lambda^m z)f_2(\lambda^m w), \lambda^{2m} t)\\
&\geq \frac{\lambda^{2m} t}{\lambda^{2m} t + \varphi(0, 0, \lambda^m z, \lambda^m w, 0, 0, 0, 0)}
\end{aligned}$$

for all $z, w \in X$ and all $t > 0$. Since

$$\lim_{m\to\infty} \frac{t}{t + \frac{1}{\lambda^{2m}}\varphi(0, 0, \lambda^m z, \lambda^m w, 0, 0, 0, 0,)} = 1,$$

$$f_1(zw) = f_1(z)w + zf_1(w) + f_2(z)f_3(w) + f_3(z)f_2(w)$$

for all $x, y \in X$, that is, f_1 is a fuzzy (f_2, f_3)-double derivation on X. $\square$

Theorem 2.3. *Let $\varphi : X^{n+1} \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{n^2-1}{n^2}z, \frac{n^2-1}{n^2}x_1, \cdots, \frac{n^2-1}{n^2}x_n\right) \leq \frac{n^2-1}{n^2}L\varphi(z, x_1, \cdots, x_n) \tag{2.3}$$

for all $z, x_1, \cdots, x_n \in X$. Let $k \in J$ and $f_k : X \to X$ be mappings satisfying $f_k(0) = 0$ and

$$N(D_\mu f_k(z, x_1, \cdots, x_n), t) \geq \frac{t}{t + \varphi(z, x_1, \cdots, x_n)}, \tag{2.4}$$

$$N(\Delta_{f_1,f_2,f_3}(x, y), t) \geq \frac{t}{t + \varphi(x, y, 0, \cdots, 0)} \tag{2.5}$$

for all $x, y, z, x_1, \cdots, x_n \in X$, all $\mu \in \mathbb{T}^1$ and all $t > 0$. Then $d_k(x) = N\text{–}\lim_{m\to\infty}(\frac{n^2}{n^2-1})^m f_k((\frac{n^2-1}{n^2})^m x)$ exist for each $x \in X$, and define unique $\mathbb{C}$-linear mappings $d_k : X \to X$ such that

$$N(f_k(x) - d_k(x), t) \geq \frac{(n^2-1)(1-L)t}{(n^2-1)(1-L)t + n\varphi(x, 0, \cdots, 0, \underbrace{x}_{jth}, 0, \cdots, 0)}$$

for all $x \in X$ and all $t > 0$. Moreover, d_1 is a fuzzy (d_2, d_3)-double derivation on X.

Proof. Let $k \in J$. Consider the set $\Omega := \{u : X \to X, u(0) = 0\}$ and introduce the generalized metric

$$d(u,v) = \inf\{\eta \in \mathbb{R}^+ : N(u(x) - v(x), \eta t) \geq \frac{t}{t + \varphi(x, 0, \cdots, 0, \underbrace{x}_{jth}, 0, \cdots, 0)},$$

where $\inf \emptyset = +\infty$. The proof of the fact that (Ω, d) is a complete generalized metric space can be found in [8]. Now we consider the mapping $J : \Omega \to \Omega$ defined by

$$Ju(x) := \frac{n^2}{n^2-1} u\left(\frac{n^2-1}{n^2} x\right)$$

for all $u \in \Omega$ and $x \in X$. Let $\varepsilon > 0$ and $u, v \in \Omega$ be given such that $d(u,v) \leq \varepsilon$. Then

$$N(u(x) - v(x), \varepsilon t) \geq \frac{t}{t + \varphi(x, 0, \cdots, 0, \underbrace{x}_{jth}, 0, \cdots, 0)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned}
N(Ju(x) - Jv(x), L\varepsilon t) &= N(\frac{n^2}{n^2-1} u(\frac{n^2-1}{n^2} x) - \frac{n^2}{n^2-1} v(\frac{n^2-1}{n^2} x), L\varepsilon t) \\
&= N(u(\frac{n^2-1}{n^2} x) - v(\frac{n^2-1}{n^2} x), \frac{n^2-1}{n^2} L\varepsilon t) \\
&\geq \frac{\frac{n^2-1}{n^2} Lt}{\frac{n^2-1}{n^2} Lt + \varphi(\frac{n^2-1}{n^2} x, 0, \cdots, 0, \underbrace{\frac{n^2-1}{n^2} x}_{jth}, 0, \cdots, 0)} \\
&\geq \frac{\frac{n^2-1}{n^2} Lt}{\frac{n^2-1}{n^2} Lt + \frac{n^2-1}{n^2} L\varphi(x, 0, \cdots, 0, \underbrace{x}_{jth}, 0, \cdots, 0)} \\
&= \frac{t}{t + \varphi(x, 0, \cdots, 0, \underbrace{x}_{jth}, 0, \cdots, 0)},
\end{aligned}$$

and so $d(u,v) \leq \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$ for all $u, v \in \Omega$. Letting $\mu = 1$ and $z = x_j = x$ for each $1 \leq p \leq n$ with $p \neq j$, $x_p = 0$ in (2.4) , we have

$$N\left(\frac{n^2-1}{n} f_k(x) - n f_k\left(\frac{n^2-1}{n^2} x\right), t\right) \geq \frac{t}{t + \varphi(x, 0, \cdots, 0, \underbrace{x}_{jth}, 0, \cdots, 0)} \tag{2.6}$$

for all $x \in X$ and all $t > 0$. It follows from (2.6) that $d(f_k, Jf_k) \leq \frac{n}{n^2-1}$. By Theorem 1.8, there exists a mapping $d_k : X \to X$ such that the following holds:
(1) d_k is a fixed point of J, that is,

$$d_k(\frac{n^2-1}{n^2} x) = \frac{n^2-1}{n^2} d_k(x) \tag{2.7}$$

for all $x \in X$. The mapping d_k is a unique fixed point of J in the set $\Lambda = \{v \in \Omega : d(u,v) < \infty\}$. This implies that d_k is a unique mapping satisfying (2.7) such that there exists $\eta \in (0, \infty)$

satisfying $N(f_k(x)-d_k(x),\eta t) \geq \frac{t}{t+\varphi(x,0,\cdots,0,\underbrace{x}_{jth},0,\cdots,0)}$ for all $x \in X$ and all $t>0$.

(2) $d(J^m f_k, d_k) \to 0$ as $m \to \infty$. This implies the equality

$$N-\lim_{m\to\infty}(\frac{n^2}{n^2-1})^m f_k((\frac{n^2-1}{n^2})^m x) = d_k(x)$$

exists for each $x \in X$.

(3) $d(f_k, d_k) \leq \frac{1}{1-L} d(f_k, Jf_k)$, which implies inequality $d(f_k, d_k) \leq \frac{1}{\frac{n^2-1}{n}-\frac{n^2-1}{n}L}$ and so

$$N(f_k(x)-d_k(x),t) \geq \frac{(n^2-1)(1-L)t}{(n^2-1)(1-L)t+n\varphi(x,0,\cdots,0,\underbrace{x}_{jth},0,\cdots,0)}.$$

It follows from (2.3) and (2.4) that

$$\begin{aligned}
&N(D_\mu d_k(z,x_1,\cdots,x_n),t)\\
&= \lim_{m\to\infty}(\frac{n^2}{n^2-1})^m N\left(D_\mu f_k\left((\frac{n^2-1}{n^2})^m z,(\frac{n^2-1}{n^2})^m x_1,\cdots,(\frac{n^2-1}{n^2})^m x_n\right),t\right)\\
&\geq \lim_{m\to\infty}\frac{(\frac{n^2-1}{n^2})^m t}{(\frac{n^2-1}{n^2})^m t+\varphi((\frac{n^2-1}{n^2})^m z,(\frac{n^2-1}{n^2})^m x_1,\cdots,(\frac{n^2-1}{n^2})^m x_n)}\\
&\geq \lim_{m\to\infty}\frac{t}{t+(\frac{n^2}{n^2-1})^m\varphi((\frac{n^2-1}{n^2})^m z,(\frac{n^2-1}{n^2})^m x_1,\cdots,(\frac{n^2-1}{n^2})^m x_n)}\\
&\geq \lim_{m\to\infty}\frac{t}{t+L^m\varphi(z,x_1,\cdots,x_n)} = 1
\end{aligned}$$

for all $z, x_1, \cdots, x_n \in X, t>0$ and all $\mu \in \mathbb{T}^1$. Thus

$$\sum_{i=1}^n \mu d_k(z-x_i) = -\frac{1}{n}\sum_{1\leq i<j\leq n} d_k(\mu x_i+\mu x_j) + n d_k(\mu z - \frac{k}{n^2}\sum_{i=1}^n \mu x_i)$$

for all $z, x_1, \cdots, x_n \in X$. By [32], $d_k : X \to X$ are Cauchy additive, that is, $d_k(x+y) = d_k(x)+d_k(y)$ for all $x, y \in X$.

By a similar method to the proof of [34], one can show that the mappings d_k are $\mathbb{C}$-linear.

By (2.5), we have

$$\begin{aligned}
&N((\frac{n^2}{n^2-1})^{2m}\Delta_{f_1,f_2,f_3}((\frac{n^2-1}{n^2})^m x,(\frac{n^2-1}{n^2})^m y),t)\\
&\geq \frac{(\frac{n^2-1}{n^2})^{2m}t}{(\frac{n^2-1}{n^2})^{2m}t+\varphi((\frac{n^2-1}{n^2})^m x,(\frac{n^2-1}{n^2})^m y,0,\cdots,0)}\\
&\geq \frac{(\frac{n^2-1}{n^2})^{2m}t}{(\frac{n^2-1}{n^2})^{2m}t+(\frac{n^2-1}{n^2})^m L^m\varphi(x,y,0,\cdots,0)}
\end{aligned}$$

for all $x, y \in X$ and all $t>0$. Since

$$\lim_{m\to\infty}\frac{(\frac{n^2-1}{n^2})^{2m}t}{(\frac{n^2-1}{n^2})^{2m}t+(\frac{n^2-1}{n^2})^m L^m\varphi(x,y,0,\cdots,0)} = 1$$

for all $x, y \in X$ and all $t>0$,

$$\Delta_{d_1,d_2,d_3}(x,y) = 0,$$

for all $x, y \in X$. So

$$d_1(xy) = d_1(x)y + xd_1(y) + d_2(x)d_3(y) + d_3(x)d_2(y)$$

for all $x, y \in X$. Hence d_1 is a fuzzy (d_2, d_3)-double derivation on X. $\square$

Corollary 2.4. *Let X be a normed vector space with norm $\|\cdot\|$, $\delta \geq 0$ and p be a real number with $p > 1$. Let $k \in J$ and $f_k : X \to X$ be mappings satisfying*

$$N(D_\mu f_k(z, x_1, \cdots, x_n), t) \geq \frac{t}{t + \delta(\|z\|^p + \sum_{i=1}^n \|x_i\|^p)}, \tag{2.8}$$

$$N(\Delta_{f_1,f_2,f_3}(x, y), t) \geq \frac{t}{t + \delta(\|x\|^p + \|y\|^p)} \tag{2.9}$$

for all $x, y, z, x_1, \cdots, x_n \in X$, all $\mu \in \mathbb{T}^1$ and all $t > 0$. Then there exist unique $\mathbb{C}$-linear mappings $d_k : X \to X$ such that

$$N(f_k(x) - d_k(x), t) \geq \frac{(n^2-1)^{1-p} - n^{2(1-p)}t}{(n^2-1)^{1-p} - n^{2(1-p)}t + 2n\delta(n^2-1)^{-p}\|x\|^p}$$

for all $x \in X$ and all $t > 0$. Moreover, $d_1 : X \to X$ is a fuzzy (d_2, d_3)-double derivation on X.

Proof. The proof follows from Theorem 2.3 by taking

$$\varphi(z, x_1, \cdots, x_n) := \delta(\|z\|^p + \sum_{i=1}^n \|x_i\|^p)$$

for all $x, y, z, x_1, \cdots, x_n \in X$. It follows from (2.8) that $f_k(0) = 0$. Choosing $L = (\frac{n^2}{n^2-1})^{1-p}$, we get the desired result. $\square$

Theorem 2.5. *Let $\varphi : X^{n+1} \to [0, \infty)$ be a function such that there exists an $L < 1$ such that*

$$\varphi\left(\frac{n^2}{n^2-1}z, \frac{n^2}{n^2-1}x_1, \cdots, \frac{n^2}{n^2-1}x_n\right) \leq \frac{n^2}{n^2-1}L\varphi(z, x_1, \cdots, x_n) \tag{2.10}$$

for all $z, x_1, \cdots, x_n \in X$. Let $k \in J$ and $f_k : X \to Y$ be mappings satisfying $f_k(0) = 0$, (2.4) and (2.5). Then the limit $d_k(x) = N - \lim_{m\to\infty}(\frac{n^2-1}{n^2})^m f_k((\frac{n^2}{n^2-1})^m x)$ exist for each $x \in X$, and define unique $\mathbb{C}$-linear mappings $d_k : X \to X$ such that

$$N(f_k(x) - d_k(x), t) \geq \frac{(n^2-1)(1-L)t}{(n^2-1)(1-L)t + nL\varphi(x, 0, \cdots, 0, \underbrace{x}_{jth}, 0, \cdots, 0)}$$

for all $x \in X$ and all $t > 0$. Moreover, $d_1 : X \to X$ is a fuzzy (d_2, d_3)-double derivation on X.

Proof. Let (Ω, d) be the generalized metric space in the proof of Theorem 2.3. Consider the linear mapping $J : \Omega \to \Omega$ defined by $Ju(x) := \frac{n^2-1}{n^2}u\left(\frac{n^2}{n^2-1}x\right)$ for all $u \in \Omega$ and $x \in X$. We can conclude that J is a strictly contractive self-mapping of Ω with the Lipschitz constant L.

Let $k \in J$. It follows from (2.6) that

$$N\left(nf_k\left(\frac{n^2-1}{n^2}x\right) - \frac{n^2-1}{n}f_k(x), t\right) \geq \frac{t}{t + \varphi(x, 0, \cdots, 0, \underbrace{x}_{jth}, 0, \cdots, 0)} \tag{2.11}$$

for all $x \in X$ and all $t > 0$. Replacing x by $\frac{n^2}{n^2-1}x$ in (2.11), we obtain

$$N\left(\frac{n^2-1}{n^2} f_k\left(\frac{n^2}{n^2-1}x\right) - f_k(x), t\right) \geq \frac{nt}{nt + \varphi(\frac{n^2}{n^2-1}x, 0, \cdots, 0, \underbrace{\frac{n^2}{n^2-1}x}_{jth}, 0, \cdots, 0)}$$

$$\geq \frac{nt}{nt + \frac{n^2}{n^2-1}L\varphi(x, 0, \cdots, 0, \underbrace{x}_{jth}, 0, \cdots, 0)}$$

It follows that $d(f_k, Jf_k) \leq \frac{nL}{n^2-1}$.

The rest the proof is similar to the proof of Theorem 2.3. $\square$

Corollary 2.6. *Let X be a normed vector space with norm $\|\cdot\|$, $\delta \geq 0$ and p be a real number with $p < 1$. Let $k \in J$ and $f_k : X \to X$ be mappings satisfying (2.8) and (2.9). Then there exist unique $\mathbb{C}$-linear mappings $d_k : X \to X$ such that*

$$N(f_k(x) - d_k(x), t) \geq \frac{(n^2-1)^{p-1} - n^{2(p-1)}t}{(n^2-1)^{p-1} - n^{2(p-1)}t + 2\delta n^{2p-1}(n^2-1)^{-1}\|x\|^p}$$

for all $x \in X$ and all $t > 0$. Moreover, $d_k : X \to X$ is a fuzzy (d_2, d_3)-double derivation on X.

Proof. The proof follows from Theorem 2.5 by taking

$$\varphi(z, x_1, \cdots, x_n) := \delta(\|z\|^p + \sum_{i=1}^{n} \|x_i\|^p)$$

for all $x, y, z, x_1, \cdots, x_n \in X$. It follows from (2.8) that $f_k(0) = 0$. Choosing $L = (\frac{n^2}{n^2-1})^{p-1}$, we get the desired result. $\square$

3. Approximate fuzzy $*$-double derivations on fuzzy C^*-algebras

Throughout this section, we assume that X is a unital C^*-algebra with unit e and unitary group $U(X) := \{u \in X : u^*u = uu^* = e\}$, and (X, N) is an induced fuzzy C^*-algebra.

Theorem 3.1. *Let $\varphi : X^{n+1} \to [0, \infty)$ be a function such that there exists an $L < 1$ satisfying (2.1). Let $k \in J$ and $f_k : X \to X$ be mappings satisfying $f_k(0) = 0$, (2.4) and*

$$N(\Delta_{f_1,f_2,f_3}(u, v), t) \geq \frac{t}{t + \varphi(u, v, 0, \cdots, 0)}, \tag{3.1}$$

$$N(f_k(u^*) - f_k(u)^*, t) \geq \frac{t}{t + \varphi(u, 0, \cdots, 0)} \tag{3.2}$$

for all $u, v \in U(X)$ and all $t > 0$. Then there exist unique $*$- mappings $d_k : X \to X$ satisfying

$$N(f_k(x) - d_k(x), t) \geq \frac{(n^2-1)(1-L)t}{(n^2-1)(1-L)t + n\varphi(x, 0, \cdots, 0, \underbrace{x}_{jth}, 0, \cdots, 0)}$$

for all $x \in X$ and all $t > 0$. Moreover, $d_1 : X \to X$ is a fuzzy $*$-(d_2, d_3)-double derivation on X.

Proof. Let $k \in J$. By Theorem 2.3, there are $\mathbb{C}$-linear mappings $d_k : X \to X$ such that $N - \lim_{m\to\infty}(\frac{n^2}{n^2-1})^m f_k\left((\frac{n^2-1}{n^2})^m x\right) = d_k(x)$ for all $x \in X$.

By (2.3) and (3.1),

$$\begin{aligned} & N\left((\frac{n^2}{n^2-1})^{2m}\Delta_{f_1,f_2,f_3}((\frac{n^2-1}{n^2})^m u, (\frac{n^2-1}{n^2})^m v), t\right) \\ & \geq \frac{(\frac{n^2-1}{n^2})^{2m} t}{(\frac{n^2-1}{n^2})^{2m} t + \varphi((\frac{n^2-1}{n^2})^m u, (\frac{n^2-1}{n^2})^m v, 0, \cdots, 0)} \\ & \geq \frac{(\frac{n^2-1}{n^2})^{2m} t}{(\frac{n^2-1}{n^2})^{2m} t + (\frac{n^2-1}{n^2})^m L^m \varphi(u, v, 0, \cdots, 0)} \end{aligned}$$

for all $u, v \in U(X)$ and all $t > 0$. Since

$$\lim_{m\to\infty} \frac{(\frac{n^2-1}{n^2})^{2m} t}{(\frac{n^2-1}{n^2})^{2m} t + (\frac{n^2-1}{n^2})^m L^m \varphi(u, v, 0, \cdots, 0)} = 1$$

for all $u, v \in U(X)$ and all $t > 0$,

$$\Delta_{d_1,d_2,d_3}(u, v) = 0$$

for all $u, v \in U(X)$. Hence

$$d_1(uv) = d_1(u)v + ud_1(v) + d_2(u)d_3(v) + d_3(u)d_2(v) \tag{3.3}$$

for all $u, v \in U(X)$. The mapping d_1 is $\mathbb{C}$-linear and each $x \in X$ is a finite linear combination of unitary elements, that is, $x = \sum_{i=1}^n \alpha_i u_i$ for $\alpha_i \in \mathbb{C}$ and $u_i \in U(X)$. It follows from (3.3) that

$$\begin{aligned} d_1(xv) &= d_1(\sum_{i=1}^n \alpha_i u_i v) = \sum_{i=1}^n d_1(\alpha_i u_i v) \\ &= \sum_{i=1}^n \alpha_i (d_1(u_i)v + u_i d_1(v) + d_2(u_i)d_3(v) + d_3(u_i)d_2(v)) \\ &= d_1(\sum_{i=1}^n \alpha_i u_i)v + \sum_{i=1}^n \alpha_i u_i) d_1 v + d_2(\sum_{i=1}^n \alpha_i u_i)d_3(v) + d_3(\sum_{i=1}^n \alpha_i u_i)d_2(v) \end{aligned}$$

for all $x \in X$ and all $v \in U(X)$. So

$$d_1(xv) = d_1(x)v + xd_1(v) + d_2(x)d_3(v) + d_3(x)d_2(v)$$

for all $x \in \mathcal{A}$ and all $v \in U(X)$.

Similarly, we can obtain that

$$d_1(xy) = d_1(x)y + xd_1(y) + d_2(x)d_3(y) + d_3(x)d_2(y)$$

for all $x, y \in X$.

By (2.3) and (3.2),

$$\begin{aligned} N((\frac{n^2}{n^2-1})^k f_k((\frac{n^2-1}{n^2})^m u^*) - (\frac{n^2}{n^2-1})^m f_k((\frac{n^2-1}{n^2})^m u)^*, t) &\geq \frac{(\frac{n^2-1}{n^2})^m t}{(\frac{n^2-1}{n^2})^m t + \varphi((\frac{n^2-1}{n^2})^m u, (\frac{n^2-1}{n^2})^m u, 0, \cdots, 0)} \\ &\geq \frac{(\frac{n^2-1}{n^2})^m t}{(\frac{n^2-1}{n^2})^m t + (\frac{n^2-1}{n^2})^m L^m \varphi(u, 0, \cdots, 0)} \end{aligned}$$

for all $u \in U(X)$ and all $t > 0$. Since

$$\lim_{m\to\infty} \frac{(\frac{n^2-1}{n^2})^m t}{(\frac{n^2-1}{n^2})^m t + (\frac{n^2-1}{n^2})^m L^m \varphi(u, 0, \cdots, 0)} = 1$$

for all $u \in U(X)$ and all $t > 0$,

$$d_k(u^*) = d_k(u)^*.$$

The mapping d_k is $\mathbb{C}$-linear and each $x \in X$ is a finite linear combination of unitary elements, that is, $x = \sum_{i=1}^n \alpha_i u_i$ for $\alpha_i \in \mathbb{C}$ and $u_i \in U(X)$. So

$$d_k(x^*) = d_k(\sum_{i=1}^n \overline{\alpha_i} u_i^*) = \sum_{i=1}^n \overline{\alpha_i} d_k(u_i)^* = d_k(\sum_{i=1}^n (\alpha_i u_i))^* = d_k(x)^*$$

for all $x \in X$. Hence $d_1 : X \to X$ is a fuzzy $*$-(d_2, d_3)-double derivation on X. $\square$

Theorem 3.2. *Let $\varphi : X^{n+1} \to [0, \infty)$ be a function such that there exists an $L < 1$ satisfying (2.3). Let $k \in J$ and $f_k : X \to X$ be mappings satisfying $f_k(0) = 0$, (2.4), (3.1) and (3.2). Then there exist unique $*$-mappings $d_k : X \to X$ satisfying*

$$N(f_k(x) - d_k(x), t) \geq \frac{(n^2-1)(1-L)t}{(n^2-1)(1-L)t + nL\varphi(x, 0, \cdots, 0, \underbrace{x}_{jth}, 0, \cdots, 0)}$$

for all $x \in X$ and all $t > 0$. Moreover, $d_1 : X \to X$ is a fuzzy $*$-(d_2, d_3)-double derivation on X.

4. Approximate fuzzy Lie $*$-double derivations on induced fuzzy Lie C^*-algebras

An induced fuzzy C^*-algebra X endowed with the Lie product $[x, y] = xy - yx$ on X, is called an *induced fuzzy Lie C^*-algebra.*

Throughout this section, we assume that X is a unital Lie C^*-algebra with unit e and unitary group $U(X) := \{u \in X : u^*u = uu^* = e\}$, and (X, N) is an induced fuzzy Lie C^*-algebra.

Definition 4.1. *Let X be an induced fuzzy Lie C^*-algebra. Let $k \in J$ and $f_k : X \to X$ be $\mathbb{C}$-linear mappings. A $\mathbb{C}$-linear mapping f_1 is called a fuzzy Lie (f_2, f_3)-double derivation if $f_1([x, y]) = [f_1(x), y] + [x, f_1(y)] + [f_2(x), f_3(y)] + [f_3(x), f_2(y)]$ for all $x, y \in X$.*

For given mappings $f_k : X \to X$, we define

$$\Gamma_{f_1,f_2,f_3}(x, y) := f_1([x, y]) - [f_1(x), y] - [x, f_1(y)] - [f_2(x), f_3(y)] - [f_3(x), f_2(y)]$$

for all $x, y \in X$.

Theorem 4.2. *Let $\varphi : X^{n+1} \to [0, \infty)$ be a function such that there exists an $L < 1$ satisfying (2.3). Let $k \in J$ and $f_k : X \to X$ be mappings satisfying $f_k(0) = 0$, (2.4) and*

$$N(\Gamma_{f_1,f_2,f_3}(u, v), t) \geq \frac{t}{t + \varphi(u, v, 0, \cdots, 0)}, \tag{4.1}$$

$$N(f_k(u^*) - f_k(u)^*, t) \geq \frac{t}{t + \varphi(u, 0, \cdots, 0)} \tag{4.2}$$

for all $u, v \in U(X)$ and all $t > 0$. Then there exist unique $\mathbb{C}$-linear $*$-mappings $d_k : X \to X$ such that

$$N(f_k(x) - d_k(x), t) \geq \frac{(n^2-1)(1-L)t}{(n^2-1)(1-L)t + n\varphi(x, 0, \cdots, 0, \underbrace{x}_{jth}, 0, \cdots, 0)}$$

for all $x \in X$ and all $t > 0$. Moreover, $d_1 : X \to X$ is a fuzzy Lie $*$-(d_2, d_3)-double derivation on X.

Proof. Let $k \in J$. By the same reasoning as in the proofs of Theorems 2.3 and 3.1, we obtain $\mathbb{C}$-linear $*$-mappings $d_k : X \to X$ such that $d_k(x) = N - \lim_{m\to\infty}(\frac{n^2}{n^2-1})^m f_k((\frac{n^2-1}{n^2})^m x)$ for each $x \in X$. It follows from (2.3) and (4.1) that

$$\begin{aligned} N(\Gamma_{d_1,d_2,d_3}(u, v), t) &= N - \lim_{m\to\infty}((\frac{n^2}{n^2-1})^{2m}\Gamma_{f_1,f_2,f_3}((\frac{n^2-1}{n^2})^m u, (\frac{n^2-1}{n^2})^m v), t) \\ &\geq \lim_{m\to\infty} \frac{(\frac{n^2-1}{n^2})^{2m}t}{(\frac{n^2-1}{n^2})^{2m}t + \varphi((\frac{n^2-1}{n^2})^m u, (\frac{n^2-1}{n^2})^m v, 0, \cdots, 0)} \\ &\geq \lim_{m\to\infty} \frac{(\frac{n^2-1}{n^2})^{2m}t}{(\frac{n^2-1}{n^2})^{2m}t + (\frac{n^2-1}{n^2})^m L^m\varphi(u, v, 0, \cdots, 0)} \end{aligned}$$

for all $u, v \in U(X)$ and all $t > 0$. Since

$$\lim_{m\to\infty} \frac{(\frac{n^2-1}{n^2})^{2m}t}{(\frac{n^2-1}{n^2})^{2m}t + (\frac{n^2-1}{n^2})^m L^m\varphi(u, v, 0, \cdots, 0)} = 1$$

for all $u, v \in U(X)$ and all $t > 0$,

$$d_1([u, v]) = [d_1(u), v] + [u, d_1(v)] + [d_2(u), d_3(v)] + [d_3(u), d_2(v)]$$

for all $u, v \in U(X)$. Since $d_1 : X \to X$ is $\mathbb{C}$-linear and each $x \in X$ is $x = \sum_{i=1}^n \alpha_i u_i$ where $u_i \in U(X)$ and $\alpha_i \in \mathbb{C}$,

$$\begin{aligned} d_1([x, v]) &= d_1(\sum_{i=1}^n [\alpha_i u_i, v]) = \sum_{i=1}^n d_1([\alpha_i u_i, v]) \\ &= \sum_{i=1}^n \alpha_i([d_1(u_i), v] + [u_i, d_1(y)] + [d_2(u_i), d_3(v)] + [d_3(u_i), d_2(v)] \\ &= [d_1(\sum_{i=1}^n \alpha_i u_i), v] + [(\sum_{i=1}^n \alpha_i u_i), d_1(v)] + [d_2(\sum_{i=1}^n \alpha_i u_i), d_3(v)] + [d_3(\sum_{i=1}^n \alpha_i u_i), d_2(v)] \\ &= [d_1(x), v] + [x, d_1(v)] + [d_2(x), d_3(v)] + [d_3(x), d_2(v)] \end{aligned}$$

for all $x \in X$ and $v \in U(X)$. By a similar method, we obtain that

$$d_1[x, y] = [d_1(x), y] + [x, d_1(y)] + [d_2(x), d_3(y)] + [d_3(x), d_2(y)]$$

for all $x, y \in X$. Thus the $\mathbb{C}$-linear mapping $d_1 : X \to X$ is a fuzzy Lie $*$-(d_2, d_3)-double derivation on X, as desired. $\square$

Corollary 4.3. *Let X be a C^*-algebra with norm $\|\cdot\|$, $\delta \geq 0$ and p be a real number with $p > 1$. Let $f_k : X \to X$ be mappings satisfying*

$$N(D_\mu f_k(z, x_1, \cdots, x_n), t) \geq \frac{t}{t + \delta(\|z\|^p + \sum_{i=1}^n \|x_i\|^p)}, \tag{4.3}$$

$$N\left(\Gamma_{f_1,f_2,f_3}((\frac{n^2-1}{n^2})^m u, (\frac{n^2-1}{n^2})^m v), t\right) \geq \frac{t}{t + 2\delta(\frac{n^2-1}{n^2})^{mp}},$$

$$N(f_k((\frac{n^2-1}{n^2})^m u^*) - f_k((\frac{n^2-1}{n^2})^m u)^*, t) \geq \frac{t}{t + \delta(\frac{n^2-1}{n^2})^{mp}}$$

for all $z, x_1, \cdots, x_n \in X$, all $u, v \in U(X)$, all $\mu \in \mathbb{T}^1$, all $t > 0$ and $m = 0, 1, 2, \cdots$. Then there exist unique $\mathbb{C}$-linear mappings $d_k : X \to X$ such that

$$N(f_k(x) - d_k(x), t) \geq \frac{(n^2-1)^{1-p} - n^{2(1-p)}t}{(n^2-1)^{1-p} - n^{2(1-p)}t + 2n\delta(n^2-1)^{-p}\|x\|^p}$$

for all $x \in X$ and all $t > 0$. Moreover, $d_1 : X \to X$ is a fuzzy Lie $*$-(d_2, d_3)-double derivation on X.

Theorem 4.4. *Let $\varphi : X^{n+1} \to [0, \infty)$ be a function such that there exists an $L < 1$ satisfying (2.10) . Let $k \in J$ and $f_k : X \to X$ be mappings satisfying $f_k(0) = 0$, (2.4), (4.1), and (4.2). Then there exist unique $\mathbb{C}$-linear $*$-mappings $f_k : X \to X$ such that*

$$N(f_k(x) - d_k(x), t) \geq \frac{(n^2-1)(1-L)t}{(n^2-1)(1-L)t + n\varphi(x, 0, \cdots, 0, \underbrace{x}_{jth}, 0, \cdots, 0)}$$

for all $x \in X$ and all $t > 0$. Moreover, $d_1 : X \to X$ is a fuzzy Lie $*$-(d_2, d_3)-double derivation on X.

Proof. The proof is similar to the proofs of Theorems 2.5 and 4.2. $\square$

Corollary 4.5. *Let X be a C^*-algebra with norm $\|\cdot\|$, $\delta \geq 0$ and p be a real number with $p < 1$. Let $k \in J$ and $f_k : X \to X$ be mappings satisfying (4.3) and*

$$N\left(\Gamma_{f_1,f_2,f_3}((\frac{n^2}{n^2-1})^m u, (\frac{n^2}{n^2-1})^m v)), t\right) \geq \frac{t}{t + 2\delta(\frac{n^2}{n^2-1})^{mp}}$$

$$N(f_k((\frac{n^2}{n^2-1})^m u^*) - f_k((\frac{n^2}{n^2-1})^m u)^*, t) \geq \frac{t}{t + \delta(\frac{n^2}{n^2-1})^{mp}}$$

for all $u, v \in U(X)$, all $t > 0$ and $m = 0, 1, 2, \cdots$. Then there exist unique $\mathbb{C}$-linear mappings $d_k : X \to X$ such that

$$N(f_k(x) - d_k(x), t) \geq \frac{(n^2-1)^{p-1} - n^{2(p-1)}t}{(n^2-1)^{p-1} - n^{2(p-1)}t + 2\delta n^{2p-1}(n^2-1)^{-1}\|x\|^p}$$

for all $x \in X$ and all $t > 0$. Moreover, $d_1 : X \to X$ is a fuzzy Lie $*$-(d_2, d_3)-double derivation on X.

Acknowledgments

C. Park and D. Y. Shin were supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2012R1A1A2004299). G. Lu was supported by The PhD Start-up Fund of Science Foundation of Liaoning Province, China (No. 20121055), and (NRF-2010-0021792), respectively.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 19, NO.4, 2015, COPYRIGHT 2015 EUDOXUS PRESS, LLC

SOME NEW GENERALIZED RESULTS ON OSTROWSKI TYPE INTEGRAL INEQUALITIES WITH APPLICATION

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Abstract. The aim of this paper is to establish some new inequalities similar to the Ostrowski's inequalities which are more generalized than the inequalities of Dragomir and Cerone. The current article obtains bounds for the deviation of a function from a combination of integral means over the end intervals covering the entire interval. Some new purterbed results are obtained. Application for cumulative distribution function is also discussed.

1. Introduction

In 1938, Ostrowski [13] established an interesting integral inequality associated with differentiable mappings. This Ostrowski inequality has powerful applications in numerical integration, probability and optimization theory, stochastic, statistics, information and integral operator theory. A number of Ostrowski type inequalities have been derived by Cerone [1], [2] and Cheng [3] with applications in Numerical analysis and Probability. Dragomir et.al [5] combined Ostrowski and Grüss inequality to give a new inequality which they named Ostrowski-Grüss type inequality. Milovanović and Pecarić [12] gave the first generalization of Ostrowski's inequality. More recent results concerning the generalizations of Ostrowski inequality are given by Liu [11], Hussain [10] and Qayyum [16]. In this paper, we will extend and generalize the results of Cerone [1] and Dragomir et.al [5]-[8] by using a new kernel.

Let $S(f;a,b)$ be defined by

$$S(f;a,b) := f(x) - M(f;a,b), \tag{1.1}$$

where

$$M(f;a,b) = \frac{1}{b-a}\int_a^b f(x)\,dx \tag{1.2}$$

is the integral mean of f over $[a,b]$. The functional $S(f;a,b)$ represents the deviation of $f(x)$ from its integral mean over $[a,b]$.

Ostrowski [13] proved the following integral inequality:

Theorem 1. *Let* $f:[a,b]\to\mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* (a,b), *whose derivative* $f':(a,b)\to\mathbb{R}$ *is bounded on* (a,b), *i.e.* $\|f'\|_\infty =$

1991 *Mathematics Subject Classification.* 2010 Mathematics Subject Classification : 26D07; 26D10; 26D15.

Key words and phrases. Ostrowski inequality, Čebyŝev-Grüss inequality,Čebyŝev functional.

This paper is in final form and no version of it will be submitted for publication elsewhere.

$\sup_{t\in[a,b]} |f'(t)| < \infty$, *then*

$$|S(f;a,b)| \leq \left[\left(\frac{b-a}{2}\right)^2 + \left(x - \frac{a+b}{2}\right)^2\right] \frac{M}{b-a} \tag{1.3}$$

for all $x \in [a,b]$.

In a series of papers, Dragomir et al [5]-[8] proved (1.3) and some of it's variants for $f' \in L_p[a,b]$ when $p \geq 1$, for Lebesgue norms making use of a peano kernel.

If we assume that $f' \in L_\infty[a,b]$ and $\|f'\|_\infty = \underset{t\in[a,b]}{ess} |f'(t)|$ then M in (1.3) may be replaced by $\|f'\|_\infty$.

Dragomir et al [5]-[8] utilizing an integration by parts argument, obtained

$$|S(f;a,b)| \tag{1.4}$$

$$\leq \begin{cases} \frac{1}{b-a}\left[\left(\frac{b-a}{2}\right)^2 + \left(x-\frac{a+b}{2}\right)^2\right] \|f'\|_\infty, f' \in L_\infty[a,b], \\ \frac{1}{b-a}\left[\frac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1}\right]^{\frac{1}{q}} \|f'\|_p, f' \in L_p[a,b], p>1, \frac{1}{p}+\frac{1}{q}=1, \\ \frac{1}{b-a}\left[\frac{b-a}{2} + \left|x-\frac{a+b}{2}\right|\right] \|f'\|_1, f' \in L_1[a,b], \end{cases}$$

where $f : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$ and the constants $\frac{1}{4}$, $\left[\frac{1}{q+1}\right]^{\frac{1}{q}}$ and $\frac{1}{2}$ are sharp. In [14], Pachpatte established Čebyŝev type inequalities by using Pecarić's extension of the Montgomery identity [17]. Cerone [1], proved the following inequality:

Lemma 1. *Let* $f : [a,b] \to \mathbb{R}$ *be absolutely continuous function. Define*

$$\tau(x;\alpha,\beta) := f(x) - \frac{1}{\alpha+\beta}[\alpha M(f;a,x) + \beta M(f;x,b)], \tag{1.5}$$

where

$$|\tau(x;\alpha,\beta)| \tag{1.6}$$

$$\leq \begin{cases} \frac{1}{2(\alpha+\beta)}[\alpha(x-a)+\beta(b-x)] \|f'\|_\infty, f' \in L_\infty[a,b], \\ \frac{1}{(\alpha+\beta)(q+1)^{\frac{1}{q}}}[\alpha^q(x-a)+\beta^q(b-x)]^{\frac{1}{q}} \|f'\|_p \\ , f' \in L_p[a,b], p>1, \frac{1}{p}+\frac{1}{q}=1, \\ \frac{1}{2}\left(1+\frac{|\alpha-\beta|}{\alpha+\beta}\right) \|f'\|_1, f' \in L_1[a,b], \end{cases}$$

where the usual L_p *norms* $\|k\|_p$ *defined for a function* $k \in L_p[a,b]$ *as follows:*

$$\|k\|_\infty := ess \sup_{t\in[a,b]} |k(t)|$$

and

$$\|k\|_p := \left(\int_a^b |k(t)|^p \, dt\right)^{\frac{1}{p}}, 1 \leq p \leq \infty.$$

With the help of two different kernels (1.7) and (1.9) given below, we extended the version of Cerone [1] and Dragomir's result [5]-[8].

Lemma 2. *Let $P(x,.) : [a,b] \to \mathbb{R}$, the peano type kernel is given by*

$$p(x,t) = \begin{cases} \frac{\alpha}{\alpha+\beta}\frac{1}{x-a}\left[t-\left(a+h\frac{b-a}{2}\right)\right], & a \le t \le x, \\ \frac{\beta}{\alpha+\beta}\frac{1}{b-x}\left[t-\left(b-h\frac{b-a}{2}\right)\right], & x < t \le b, \end{cases} \tag{1.7}$$

Then,

$$\begin{aligned} |\tau(x;\alpha,\beta)| &= \frac{1}{\alpha+\beta}\left[\begin{array}{c} \frac{\alpha}{x-a}\left\{x-\left(a+h\frac{b-a}{2}\right)\right\} \\ -\frac{\beta}{b-x}\left\{x-\left(b-h\frac{b-a}{2}\right)\right\} \end{array}\right] f(x) \\ &\quad + \frac{h}{\alpha+\beta}\left(\frac{b-a}{2}\right)\left(\frac{\alpha}{x-a}f(a)+\frac{\beta}{b-x}f(b)\right) \\ &\quad - \frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\int_a^x f(t)\,dt + \frac{\beta}{b-x}\int_x^b f(t)\,dt\right] \\ &\le \begin{cases} \left(\begin{array}{c} \frac{\alpha}{x-a}\left\{\frac{(x-a)^2}{4}+\left[\left(a+h\frac{b-a}{2}\right)-\frac{a+x}{2}\right]^2\right\} \\ +\frac{\beta}{b-x}\left\{\frac{(b-x)^2}{4}+\left[\left(b-h\frac{b-a}{2}\right)-\frac{x+b}{2}\right]^2\right\} \end{array}\right)\frac{1}{(\alpha+\beta)}\left\|f'\right\|_\infty \\ , \quad f' \in L_\infty[a,b] \\ \left[\begin{array}{c} \frac{\alpha^q}{(x-a)^q}\left\{\left(x-\left(a+h\frac{b-a}{2}\right)\right)^{q+1}-\left(h\frac{a-b}{2}\right)^{q+1}\right\} \\ +\frac{\beta^q}{(b-x)^q}\left\{\left(b-\left(x+h\frac{b-a}{2}\right)\right)^{q+1}-\left(h\frac{a-b}{2}\right)^{q+1}\right\} \end{array}\right]^{\frac{1}{q}} \frac{1}{(q+1)^{\frac{1}{q}}(\alpha+\beta)}\left\|f'\right\|_p, \\ f' \in L_p[a,b], p>1, \frac{1}{p}+\frac{1}{q}=1, \\ \left(\begin{array}{c} (\alpha+\beta)-h\frac{b-a}{2}\left[\frac{\alpha(b-x)+\beta(x-a)}{(x-a)(b-x)}\right] \\ +\left|(\alpha-\beta)+h\frac{b-a}{2}\left[\frac{\beta(x-a)-\alpha(b-x)}{(x-a)(b-x)}\right]\right| \end{array}\right)\frac{\left\|f'\right\|_1}{2(\alpha+\beta)}. \end{cases} \end{aligned} \tag{1.8}$$

Lemma 3. *Denote by $P(x,.) : [a,b] \to \mathbb{R}$ the kernel is given by*

$$P(x,t) := \begin{cases} \frac{\alpha}{2(\alpha+\beta)(x-a)}(t-a)^2, & a \le t \le x, \\ \frac{\beta}{2(\alpha+\beta)(b-x)}(t-b)^2, & x < t \le b, \end{cases} \tag{1.9}$$

Then,

$$
\begin{aligned}
& |\tau(x;\alpha,\beta)| \qquad (1.10)\\
= \; & \frac{1}{2(\alpha+\beta)}\left[\alpha(x-a)-\beta(b-x)\right]f'(x)-f(x)\\
& +\frac{1}{\alpha+\beta}\left[\alpha M(f;a,x)+\beta M(f;x,b)\right],\\
\leq \; & \begin{cases}
\left[\alpha(x-a)^2+\beta(b-x)^2\right]\frac{\|f''\|_\infty}{6(\alpha+\beta)}, f''\in L_\infty[a,b]\\
\frac{1}{(2q+1)^{\frac{1}{q}}}\left[\alpha^q(x-a)^{q+1}+\beta^q(b-x)^{q+1}\right]^{\frac{1}{q}}\frac{\|f''\|_p}{2(\alpha+\beta)},\\
f''\in L_p[a,b], p>1, \frac{1}{p}+\frac{1}{q}=1,\\
\left(\alpha(x-a)+\beta(b-x)+|\alpha(x-a)-\beta(b-x)|\right)\frac{\|f''\|_1}{4(\alpha+\beta)}\\
, f''\in L_1[a,b].
\end{cases}
\end{aligned}
$$

Using a generalized form of (1.9), we constructed a number of new results for twice differentiable functions. These results are given in Lemma 4 and theorem 2 which are more generalized by (1.8)-(1.10). These generalized inequalities will have applications in approximation theory, probability theory and numerical analysis. We will show in our paper an application of the obtained inequalities for cumulative distribution function.

2. Main Results

We will start our main result with this lemma.

Lemma 4. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous mapping. Denote by $P(x,.):[a,b]\to\mathbb{R}$ the kernel $P(x,t,h)$ is given by*

$$
P(x,t)=\begin{cases}
\frac{\alpha}{\alpha+\beta}\frac{1}{x-a}\frac{1}{2}\left[t-\left(a+h\frac{b-a}{2}\right)\right]^2, & a\leq t\leq x,\\
\frac{\beta}{\alpha+\beta}\frac{1}{b-x}\frac{1}{2}\left[t-\left(b-h\frac{b-a}{2}\right)\right]^2, & x<t\leq b,
\end{cases} \tag{2.1}
$$

for all $x\in\left[a+h\frac{b-a}{2}, b-h\frac{b-a}{2}\right]$ and $h\in[0,1]$, where $\alpha,\beta\in\mathbb{R}$ are non negative and not both zero. Before we state and prove our main theorem, we will prove the following identity:

$$
\begin{aligned}
\int_a^b P(x,t)f''(t)dt = \; & \frac{1}{2(\alpha+\beta)}\left[\frac{\alpha}{x-a}\left(x-\left(a+h\frac{b-a}{2}\right)\right)^2\right. \qquad (2.2)\\
& \left.-\frac{\beta}{b-x}\left(x-\left(b-h\frac{b-a}{2}\right)\right)^2\right]f'(x)\\
& -\frac{1}{(\alpha+\beta)}\left[\frac{\alpha}{x-a}\left(x-\left(a+h\frac{b-a}{2}\right)\right)\right.\\
& \left.-\frac{\beta}{b-x}\left(x-\left(b-h\frac{b-a}{2}\right)\right)\right]f(x)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{\alpha+\beta} h \frac{b-a}{2}\left[\frac{\alpha}{x-a} f(a)+\frac{\beta}{b-x} f(b)\right] \\
&+\frac{1}{\alpha+\beta} h^{2} \frac{(b-a)^{2}}{8}\left[\frac{\beta}{b-x} f'(b)-\frac{\alpha}{x-a} f'(a)\right] \\
&+\frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a} \int_{a}^{x} f(t)\, dt+\frac{\beta}{b-x} \int_{x}^{b} f(t)\, dt\right].
\end{aligned}
$$

Proof. From (2.1), we have

$$
\begin{aligned}
\int_{a}^{b} P(x,t) f''(t) dt &= \frac{\alpha}{(\alpha+\beta)} \frac{1}{x-a} \int_{a}^{x} \frac{\left[t-\left(a+h\frac{b-a}{2}\right)\right]^{2}}{2} f''(t) dt \\
&\quad+\frac{\beta}{(\alpha+\beta)} \frac{1}{b-x} \int_{x}^{b} \frac{\left[t-\left(b-h\frac{b-a}{2}\right)\right]^{2}}{2} f''(t) dt.
\end{aligned}
$$

After simplification, we get the required identity (2.2). □

We now give our main theorem.

Theorem 2. *Let* $f:[a,b]\to\mathbb{R}$ *be an absolutely continuous mapping. Define*

$$
\begin{aligned}
\tau(x;\alpha,\beta) \; : &= \frac{1}{2(\alpha+\beta)}\left[\frac{\alpha}{x-a}\left(x-\left(a+h\frac{b-a}{2}\right)\right)^{2}\right. \qquad (2.3)\\
&\left.-\frac{\beta}{b-x}\left(x-\left(b-h\frac{b-a}{2}\right)\right)^{2}\right] f'(x) \\
&-\frac{1}{(\alpha+\beta)}\left[\frac{\alpha}{x-a}\left(x-\left(a+h\frac{b-a}{2}\right)\right)\right. \\
&\left.-\frac{\beta}{b-x}\left(x-\left(b-h\frac{b-a}{2}\right)\right)\right] f(x) \\
&-\frac{1}{\alpha+\beta} h \frac{b-a}{2}\left[\frac{\alpha}{x-a} f(a)+\frac{\beta}{b-x} f(b)\right] \\
&+\frac{1}{\alpha+\beta} h^{2} \frac{(b-a)^{2}}{8}\left[\frac{\beta}{b-x} f'(b)-\frac{\alpha}{x-a} f'(a)\right] \\
&+\frac{1}{\alpha+\beta}\left[\alpha M(f;a,x)+\beta M(f;x,b)\right],
\end{aligned}
$$

where $M(f;a,b)$ is the integral mean defined in (1.2), then

$$
|\tau(x;\alpha,\beta)| \tag{2.4}
$$
$$
\leq \begin{cases}
\left[\begin{array}{c} \frac{\alpha}{x-a}\left\{\left[x-\left(a+h\frac{b-a}{2}\right)\right]^3+\left(h\frac{b-a}{2}\right)^3\right\} \\ -\frac{\beta}{b-x}\left\{\left(h\frac{b-a}{2}\right)^3+\left[x-\left(b-h\frac{b-a}{2}\right)\right]^3\right\}\end{array}\right]\frac{\|f''\|_\infty}{6(\alpha+\beta)}, \quad f''\in L_\infty[a,b], \\[2ex]
\left[\begin{array}{c} \frac{\alpha^q}{(x-a)^q}\left\{\left[x-\left(a+h\frac{b-a}{2}\right)\right]^{2q+1}-\left(h\frac{a-b}{2}\right)^{2q+1}\right\} \\ +\frac{\beta^q}{(b-x)^q}\left\{\left(h\frac{a-b}{2}\right)^{2q+1}-\left[x-\left(b-h\frac{b-a}{2}\right)\right]^{2q+1}\right\}\end{array}\right]^{\frac{1}{q}}\frac{\|f''\|_p}{2(2q+1)^{\frac{1}{q}}(\alpha+\beta)}, \\[2ex]
f''\in L_p[a,b],\, p>1,\, \frac{1}{p}+\frac{1}{q}=1, \\[2ex]
\left\{\begin{array}{c} \alpha(x-a)+\beta(b-x)-h(b-a)(\alpha+\beta) \\ +\frac{h^2(b-a)}{2}\left[\frac{\alpha}{x-a}+\frac{\beta}{b-x}\right] \\ +\left|\begin{array}{c}\beta(b-x)-\alpha(x-a)+h(b-a)(\alpha-\beta) \\ +\frac{h^2(b-a)}{2}\left[\frac{\beta}{b-x}-\frac{\alpha}{x-a}\right]\end{array}\right|\end{array}\right\}\frac{\|f''\|_1}{4(\alpha+\beta)}, \, f''\in L_1[a,b]
\end{cases}
$$

for all $x\in[a,b]$,where $\|k\|$ is the usual Lebesgue norm for $k\in L[a,b]$ with

$$
\|k\|_\infty := ess \sup_{t\in[a,b]} |k(t)| < \infty
$$

and

$$
\|k\|_p := \left(\int_a^b |k(t)|^p\,dt\right)^{\frac{1}{p}}, 1\leq p\leq\infty
$$

Proof. Taking the modulus of (2.2) and using (2.3) and (1.2), we have

$$
|\tau(x;\alpha,\beta)| = \left|\int_a^b P(x,t)f''(t)dt\right| \leq \int_a^b |P(x,t)|\,|f''(t)|\,dt. \tag{2.5}
$$

Therefore, for $f''\in L_\infty[a,b]$ we obtain

$$
|\tau(x;\alpha,\beta)| \leq \|f''\|_\infty \int_a^b |P(x,t)|\,dt.
$$

Now let us observe that

$$
\begin{aligned}
&\int_a^b |P(x,t)|\,dt \\
= \;& \frac{\alpha}{2(\alpha+\beta)(x-a)}\int_a^x \left[t-\left(a+h\frac{b-a}{2}\right)\right]^2 dt \\
&+\frac{\beta}{2(\alpha+\beta)(b-x)}\int_x^b \left[t-\left(b-h\frac{b-a}{2}\right)\right]^2 dt.
\end{aligned}
$$

After simple integration, we get

$$\begin{aligned}&\int_a^b |P(x,t)|\,dt\\ =\;& \frac{1}{6(\alpha+\beta)}\left[\begin{array}{c} \frac{\alpha}{x-a}\left\{\left[x-\left(a+h\frac{b-a}{2}\right)\right]^3+\left(h\frac{b-a}{2}\right)^3\right\} \\ -\frac{\beta}{b-x}\left\{\left(h\frac{b-a}{2}\right)^3-\left[x-\left(b-h\frac{b-a}{2}\right)\right]^3\right\}\end{array}\right].\end{aligned}$$

Hence the first inequality is obtained.

$$\begin{aligned}&|\tau(x;\alpha,\beta)|\\ \leq\;& \left[\begin{array}{c} \frac{\alpha}{x-a}\left\{\left[x-\left(a+h\frac{b-a}{2}\right)\right]^3+\left(h\frac{b-a}{2}\right)^3\right\} \\ -\frac{\beta}{b-x}\left\{\left(h\frac{b-a}{2}\right)^3+\left[x-\left(b-h\frac{b-a}{2}\right)\right]^3\right\}\end{array}\right]\frac{1}{6(\alpha+\beta)}\left\|f''\right\|_\infty.\end{aligned}$$

Further, using Hölder's integral inequality in (2.5) we have for $f'' \in L_p[a,b]$

$$|\tau(x;\alpha,\beta)| \leq \left\|f''\right\|_p \left(\int_a^b |P(x,t)|^q\,dt\right)^{\frac{1}{q}}.$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $p>1$. Now

$$\begin{aligned}&(\alpha+\beta)\left(\int_a^b |P(x,t)|^q\,dt\right)^{\frac{1}{q}}\\ =\;& \left[\begin{array}{c} \frac{\alpha^q}{2^q(x-a)^q}\int_a^x \left[t-\left(a+h\frac{b-a}{2}\right)\right]^{2q}dt \\ +\frac{\beta^q}{2^q(b-x)^q}\int_x^b \left[t-\left(b-h\frac{b-a}{2}\right)\right]^{2q}dt^{\frac{1}{q}}\end{array}\right]^{\frac{1}{q}}\\ =\;& \left[\begin{array}{c} \frac{\alpha^q}{2^q(2q+1)(x-a)^q}\left[t-\left(a+h\frac{b-a}{2}\right)\right]^{2q+1}/_a^x \\ +\frac{\beta^q}{2^q(2q+1)(b-x)^q}\left[t-\left(b-h\frac{b-a}{2}\right)\right]^{2q}/_x^b\end{array}\right]^{\frac{1}{q}}.\end{aligned}$$

Again, after simple integration, we get

$$\begin{aligned}&(\alpha+\beta)\left(\int_a^b |P(x,t)|^q\,dt\right)^{\frac{1}{q}}\\ =\;& \frac{1}{2(2q+1)^{\frac{1}{q}}}\left[\begin{array}{c} \frac{\alpha^q}{(x-a)^q}\left\{\left[x-\left(a+h\frac{b-a}{2}\right)\right]^{2q+1}-\left(h\frac{a-b}{2}\right)^{2q+1}\right\} \\ \\ +\frac{\beta^q}{(b-x)^q}\left\{\left(h\frac{a-b}{2}\right)^{2q+1}-\left[x-\left(b-h\frac{b-a}{2}\right)\right]^{2q+1}\right\}\end{array}\right]^{\frac{1}{q}}.\end{aligned}$$

Hence the second inequality is obtained as below.

$$\begin{aligned} & |\tau(x;\alpha,\beta)| \\ \leq \quad & \frac{1}{2(2q+1)^{\frac{1}{q}}} \left[\begin{array}{c} \frac{\alpha^q}{(x-a)^q} \left\{ \left[x-\left(a+h\frac{b-a}{2}\right)\right]^{2q+1} - \left(h\frac{a-b}{2}\right)^{2q+1} \right\} \\ + \frac{\beta^q}{(b-x)^q} \left\{ \left(h\frac{a-b}{2}\right)^{2q+1} - \left[x-\left(b-h\frac{b-a}{2}\right)\right]^{2q+1} \right\} \end{array} \right]^{\frac{1}{q}} \|f''\|_p . \end{aligned}$$

Finally, for $f'' \in L_1[a,b]$, using (2.1), we have the following inequality from (2.5),

$$|\tau(x;\alpha,\beta)| \leq \sup_{t\in[a,b]} |P(x,t)| \, \|f''\|_1 ,$$

where

$$\begin{aligned} (\alpha+\beta)\sup_{t\in[a,b]}|P(x,t)| \quad = \quad & \frac{1}{4}\left\{ \begin{array}{c} \alpha(x-a)+\beta(b-x)-h(b-a)(\alpha+\beta) \\ +\frac{h^2(b-a)}{2}\left[\frac{\alpha}{x-a}+\frac{\beta}{b-x}\right] \end{array} \right\} \\ + & \begin{array}{c} \frac{1}{4}|\beta(b-x)-\alpha(x-a)+h(b-a)(\alpha-\beta) \\ +\frac{h^2(b-a)}{2}\left[\frac{\beta}{b-x}-\frac{\alpha}{x-a}\right]| \end{array} \end{aligned}$$

This gives us the last inequality as below.

$$|\tau(x;\alpha,\beta)| \leq \left\{ \begin{array}{c} \alpha(x-a)+\beta(b-x)-h(b-a)(\alpha+\beta) \\ +\frac{h^2(b-a)}{2}\left[\frac{\alpha}{x-a}+\frac{\beta}{b-x}\right] \\ +|\beta(b-x)-\alpha(x-a) \\ +h(b-a)(\alpha-\beta)+\frac{h^2(b-a)}{2}\left[\frac{\beta}{b-x}-\frac{\alpha}{x-a}\right]| \end{array} \right\} \frac{\|f''\|_1}{4(\alpha+\beta)}.$$

This completes the proof of theorem. □

Some special cases of Theorem 2

In this section, we will give some useful cases.

Remark 1. *If we put $h=0$ in (2.4), we get (1.10).*

Remark 2. *If we put $h=0$ in (1.8), we get Cerone's result given in (1.6).*

Remark 3. *If we put $h=1$ in (2.4), we get a new result.*

$$|\tau(x;\alpha,\beta)| \tag{2.6}$$

$$\leq \begin{cases} \left[\begin{array}{c} \frac{\alpha}{x-a}\left\{(x-A)^3+\left(\frac{b-a}{2}\right)^3\right\} \\ -\frac{\beta}{b-x}\left\{\left(\frac{b-a}{2}\right)^3+(x-A)^3\right\} \end{array}\right] \frac{1}{6(\alpha+\beta)}\left\|f''\right\|_\infty, \quad f''\in L_\infty[a,b], \\ \\ \left[\begin{array}{c} \frac{1}{(x-a)^q}\left\{(x-A)^{2q+1}-\left(\frac{a-b}{2}\right)^{2q+1}\right\} \\ \\ +\frac{1}{(b-x)^q}\left\{\left(\frac{a-b}{2}\right)^{2q+1}-(x-A)^{2q+1}\right\} \end{array}\right]^{\frac{1}{q}} \frac{1}{(2q+1)^{\frac{1}{q}}}\frac{1}{4}\left\|f''\right\|_p \\ , f''\in L_p[a,b], p>1, \frac{1}{p}+\frac{1}{q}=1, \\ \\ \left\{\begin{array}{c} \alpha(x-a)+\beta(b-x)-(b-a)(\alpha+\beta) \\ +\frac{b-a}{2}\left[\frac{\alpha}{x-a}+\frac{\beta}{b-x}\right] \\ + \quad |\beta(b-x)-\alpha(x-a)+(b-a)(\alpha-\beta) \\ +\frac{b-a}{2}\left[\frac{\beta}{b-x}-\frac{\alpha}{x-a}\right]| \end{array}\right\} \frac{\left\|f''\right\|_1}{4(\alpha+\beta)}, f''\in L_1[a,b], \end{cases}$$

where $A=\frac{a+b}{2}$.

Corollary 1. *If we put $x=A$ in above , we get*

$$|\tau(A;\alpha,\beta)| \tag{2.7}$$

$$\leq \begin{cases} \frac{(b-a)^2}{24}\frac{\alpha-\beta}{\alpha+\beta}\left\|f''\right\|_\infty, \quad f''\in L_\infty[a,b], \\ \\ \left[\frac{\beta^q}{(b-a)^q}\left(\frac{a-b}{2}\right)^{2q+1}-\frac{\alpha^q}{(b-a)^q}\left(\frac{a-b}{2}\right)^{2q+1}\right]^{\frac{1}{q}} \frac{1}{(2q+1)^{\frac{1}{q}}}\frac{1}{(\alpha+\beta)}\left\|f''\right\|_p \\ , f''\in L_p[a,b], p>1, \frac{1}{p}+\frac{1}{q}=1, \\ \\ \left\{\left(1-\frac{a-b}{2}\right)+\left|\frac{\beta-\alpha}{\alpha+\beta}\left(1-\frac{a-b}{2}\right)\right|\right\}\frac{\left\|f''\right\|_1}{4}, f''\in L_1[a,b]. \end{cases}$$

Remark 4. *If we put $h=\frac{1}{2}$ in (2.3) and (2.4) we get the following inequality:*

$$\left|\begin{array}{c} \frac{1}{2(\alpha+\beta)}\left[\frac{\alpha}{x-a}\left(x-\frac{3a+b}{4}\right)^2-\frac{\beta}{b-x}\left(x-\frac{a+3b}{4}\right)^2\right]f'(x) \\ -\frac{1}{(\alpha+\beta)}\left[\frac{\alpha}{x-a}\left(x-\frac{3a+b}{4}\right)-\frac{\beta}{b-x}\left(x-\frac{a+3b}{4}\right)\right]f(x) \\ -\frac{1}{\alpha+\beta}\frac{b-a}{4}\left[\frac{\alpha}{x-a}f(a)+\frac{\beta}{b-x}f(b)\right] \\ +\frac{1}{\alpha+\beta}\frac{(b-a)^2}{32}\left[\frac{\beta}{b-x}f'(b)-\frac{\alpha}{x-a}f'(a)\right] \\ +\frac{1}{\alpha+\beta}\left[\alpha M(f;a,x)+\beta M(f;x,b)\right] \end{array}\right| \tag{2.8}$$

$$\leq\left\{\begin{array}{l} \left[\begin{array}{c}\frac{\alpha}{x-a}\left\{\left[x-\left(a+\frac{b-a}{4}\right)\right]^3+\left(\frac{b-a}{4}\right)^3\right\} \\ -\frac{\beta}{b-x}\left\{\left(\frac{b-a}{4}\right)^3+\left[x-\left(b-\frac{b-a}{4}\right)\right]^3\right\}\end{array}\right]\frac{\|f''\|_\infty}{6(\alpha+\beta)},\quad f''\in L_\infty[a,b], \\ \\ \left[\begin{array}{c}\frac{\alpha^q}{(x-a)^q}\left\{\left[x-\left(a+\frac{b-a}{4}\right)\right]^{2q+1}-\left(\frac{a-b}{4}\right)^{2q+1}\right\} \\ +\frac{\beta^q}{(b-x)^q}\left\{\left(\frac{a-b}{4}\right)^{2q+1}-\left[x-\left(b-\frac{b-a}{4}\right)\right]^{2q+1}\right\}\end{array}\right]^{\frac{1}{q}}\frac{\|f''\|_p}{2(2q+1)^{\frac{1}{q}}(\alpha+\beta)} \\ ,f''\in L_p[a,b],p>1,\frac{1}{p}+\frac{1}{q}=1, \\ \\ \left\{\begin{array}{c}\alpha(x-a)+\beta(b-x)-\frac{1}{2}(b-a)(\alpha+\beta) \\ +\frac{(b-a)}{8}\left[\frac{\alpha}{x-a}+\frac{\beta}{b-x}\right] \\ +\left|\begin{array}{c}\beta(b-x)-\alpha(x-a)+\frac{1}{2}(b-a)(\alpha-\beta) \\ +\frac{(b-a)}{8}\left[\frac{\beta}{b-x}-\frac{\alpha}{x-a}\right]\end{array}\right|\end{array}\right\}\frac{\|f''\|_1}{4(\alpha+\beta)},f''\in L_1[a,b]. \end{array}\right.$$

Corollary 2. *If we put $\alpha=\beta$ and $x=A$ in (2.8) we get another result.*

$$\begin{array}{c}\left|\frac{1}{2}f\left(\frac{a+b}{2}\right)+\frac{1}{4}\left[f(a)+f(b)\right]\right. \\ \left.-\frac{b-a}{32}\left[f'(b)-f'(a)\right]-(b-a)\int_a^b f(t)dt\right|\end{array} \tag{2.9}$$

$$\leq\left\{\begin{array}{l}\frac{(b-a)^2\|f''\|_\infty}{192},\ f''\in L_\infty[a,b], \\ \\ \left[\left\{(b-a)^{2q+1}-(a-b)^{2q+1}\right\}\right]^{\frac{1}{q}}\frac{\|f''\|_p}{16(b-a)2^{2\cdot\frac{1}{q}}(2q+1)^{\frac{1}{q}}} \\ ,f''\in L_p[a,b],p>1,\frac{1}{p}+\frac{1}{q}=1, \\ \\ \frac{(b-a)\|f''\|_1}{16},f''\in L_1[a,b].\end{array}\right.$$

Corollary 3. *If we put $\alpha=\beta$ and $x=\frac{a+3b}{4}$ in (2.8) we get another new result.*

$$\left|\begin{array}{c} \frac{b-a}{12}f'(x)-\left(\frac{1}{3}-\frac{2}{b-a}\right)f(x)-\frac{1}{2}\left(\frac{1}{3}f(a)+f(b)\right) \\ +\frac{b-a}{16}\left(f'(b)-\frac{1}{3}f'(a)\right) \\ +\frac{2}{b-a}\left(\frac{1}{3}\int\limits_a^{\frac{a+3b}{4}}f(t)dt+\int\limits_{\frac{a+3b}{4}}^{b}f(t)dt\right) \end{array}\right| \tag{2.10}$$

$$\leq\begin{cases} \frac{(b-a)^2\left\|f''\right\|_\infty}{96}, \quad f''\in L_\infty[a,b], \\ \left[\frac{1}{3^q}\left\{\left(\frac{b-a}{2}\right)^{2q+1}-\left(\frac{a-b}{4}\right)^{2q+1}\right\}+\left(\frac{a-b}{4}\right)^{2q+1}\right]^{\frac{1}{q}}\frac{\left\|f''\right\|_p}{(b-a)(2q+1)^{\frac{1}{q}}} \\ ,f''\in L_p[a,b],p>1,\frac{1}{p}+\frac{1}{q}=1, \\ \left\{\frac{2}{3}+\left|\frac{b-a}{2}-\frac{1}{3}\right|\right\}\frac{\left\|f''\right\|_1}{8},f''\in L_1[a,b]. \end{cases}$$

Hence, for different values of h, we can obtain a variety of results.

Remark 5. *We can write (2.3) in another way. Since*

$$\begin{aligned} &\alpha M(f;a,x)+\beta M(f;x,b) \\ =\ &\alpha M(f;a,x)+\frac{\beta}{b-x}\left(\int\limits_a^b f(u)\,du-\int\limits_a^x f(u)\,du\right). \end{aligned}$$

or

$$\begin{aligned} &\alpha M(f;a,x)+\beta M(f;x,b) \\ =\ &\alpha M(f;a,x)-\frac{\beta}{b-x}\int\limits_a^x f(u)\,du+\frac{\beta}{b-x}\int\limits_a^b f(u)\,du \\ =\ &(\alpha+\beta-\beta\sigma(x))\,M(f;a,x)+\beta\sigma(x)\,M(f;a,b), \end{aligned}$$

where

$$\frac{b-a}{b-x}=\sigma(x). \tag{2.11}$$

Thus, from (2.3),

$$\begin{aligned}
& \tau(x;\alpha,\beta) \\
= \quad & \tfrac{1}{2(\alpha+\beta)}\left[\tfrac{\alpha}{x-a}\left(x-\left(a+h\tfrac{b-a}{2}\right)\right)^2 \right. \\
& \left. -\tfrac{\beta}{b-x}\left(x-\left(b-h\tfrac{b-a}{2}\right)\right)^2\right] f'(x) \\
& -\tfrac{1}{(\alpha+\beta)}\left[\tfrac{\alpha}{x-a}\left(x-\left(a+h\tfrac{b-a}{2}\right)\right)\right. \\
& \left. -\tfrac{\beta}{b-x}\left(x-\left(b-h\tfrac{b-a}{2}\right)\right)\right] f(x) \\
& -\frac{1}{\alpha+\beta}h\frac{b-a}{2}\left[\frac{\alpha}{x-a}f(a)+\frac{\beta}{b-x}f(b)\right] \\
& +\frac{1}{\alpha+\beta}h^2\frac{(b-a)^2}{8}\left[\frac{\beta}{b-x}f'(b)-\frac{\alpha}{x-a}f'(a)\right] \\
& +\left[\left(1-\frac{\beta}{\alpha+\beta}\sigma(x)\right)M(f;a,x)+\frac{\beta}{\alpha+\beta}\sigma(x)M(f;a,b)\right].
\end{aligned} \tag{2.12}$$

so that for fixed $[a,b]$, $M(f;a,b)$ *is also fixed.*

Corollary 4. *If (2.3) and (2.4) is evaluated at* $x=\frac{a+b}{2}$ *and* $\alpha=\beta$ *then*

$$\begin{aligned}
& \left|(h-1)f\left(\tfrac{a+b}{2}\right)-\tfrac{h}{2}(f(a)+f(b))\right. \\
& \left. +h^2\tfrac{b-a}{8}\left(f'(b)-f'(a)\right)+\tfrac{1}{b-a}\int_a^b f(t)\,dt\right|
\end{aligned} \tag{2.13}$$

$$\leq \begin{cases}
(1-h)^3\frac{(b-a)^2}{24}\|f''\|_\infty, \quad f''\in L_\infty[a,b], \\
\left[\begin{array}{c}
\frac{2^q}{(b-a)^q}\left\{\left(\frac{b-a}{2}(1-h)\right)^{2q+1}-\left(h\frac{a-b}{2}\right)^{2q+1}\right\} \\
+\frac{2^q}{(b-a)^q}\left\{\left(h\frac{a-b}{2}\right)^{2q+1}-\left(\frac{b-a}{2}(1-h)\right)^{2q+1}\right\}
\end{array}\right]^{\frac{1}{q}} \frac{\|f''\|_p}{4(2q+1)^{\frac{1}{q}}} \\
, f''\in L_p[a,b], p>1, \frac{1}{p}+\frac{1}{q}=1, \\
\left[(b-a)(1-2h)+2h^2\right]\frac{\|f''\|_1}{8}, f''\in L_1[a,b].
\end{cases}$$

3. Perturbed Results

In 1882, Čebyŝev [4] gave the following inequality.

$$|T(f,g)|\leq\frac{1}{12}(b-a)^2\|f'\|_\infty\|g'\|_\infty \tag{3.1}$$

where $f,g:[a,b]\to\mathbb{R}$ are absolutely continuous functions, which has bounded first derivatives such that

$$\begin{aligned}
T(f,g) \quad &= \quad \frac{1}{b-a}\int_a^b f(x)g(x)\,dx \\
& \quad -\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right) \\
&= \quad M(f,g;a,b)-M(f;a,b)M(g;a,b),
\end{aligned} \tag{3.2}$$

and $\|.\|_\infty$ denotes the norm in $L_\infty[a,b]$ defined as $\|p\|_\infty = ess \sup_{t\in[a,b]} |P(t)|$.

In 1935, Grüss [9] proved the following inequality:

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx\frac{1}{b-a}\int_a^b g(x)\,dx\right| \qquad (3.3)\\
\leq\ &\frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma),
\end{aligned}$$

provided that f and g are two integrable functions on $[a,b]$ and satisfy the condition:

$$\varphi \leq f(x) \leq \Phi \ \text{ and } \gamma \leq g(x) \leq \Gamma, \text{ for all } x\in[a,b]. \qquad (3.4)$$

The constant $\frac{1}{4}$ is best possible. The perturbed version of the results of Theorem 2 can be obtained by using Grüss type results involving the Čebyŝev functional.

$$T(f,g) = M(f,g;a,b) - M(f;a,b)\,M(g;a,b),$$

where M is the integral mean and is defined in (1.2).

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous mapping and α, β are non-negative real numbers, then*

$$\begin{aligned}
&\left|\tau(x;\alpha,\beta) - \frac{1}{(\alpha+\beta)}\left[\begin{array}{c} \frac{\alpha}{x-a}\left\{\left[x-\left(a+h\frac{b-a}{2}\right)\right]^3+\left(h\frac{b-a}{2}\right)^3\right\} \\ -\frac{\beta}{b-x}\left\{\left(h\frac{b-a}{2}\right)^3+\left[x-\left(b-h\frac{b-a}{2}\right)\right]^3\right\}\end{array}\right]\frac{\kappa}{6}\right| \qquad (3.5)\\
\leq\ &(b-a)\,N(x)\left[\frac{1}{b-a}\|f''\|_2^2-\kappa^2\right]^{\frac{1}{2}}\\
\leq\ &(b-a)(\Gamma-\gamma)\,\lambda,
\end{aligned}$$

where, $\tau(x;\alpha,\beta)$ is as given by (2.3) and $\lambda=\Phi-\varphi$. Let

$$\kappa = \frac{f'(b)-f'(a)}{b-a} \qquad (3.6)$$

then

$$\begin{aligned}
N^2(x) &= \frac{1}{20(\alpha+\beta)^2}\left\{\begin{array}{c} \frac{\alpha^2}{(x-a)^2}\left[\left(x-\left(a+h\frac{b-a}{2}\right)\right)^5+\left(h\frac{b-a}{2}\right)^5\right] \\ +\frac{\beta^2}{(b-x)^2}\left[\left(h\frac{b-a}{2}\right)^5-\left(x-\left(b-h\frac{b-a}{2}\right)\right)^5\right]\end{array}\right\} \qquad (3.7)\\
&\quad-\left(\frac{1}{6(b-a)(\alpha+\beta)}\left[\begin{array}{c} \frac{\alpha}{x-a}\left\{\left[x-\left(a+h\frac{b-a}{2}\right)\right]^3+\left(h\frac{b-a}{2}\right)^3\right\} \\ -\frac{\beta}{b-x}\left\{\left(h\frac{b-a}{2}\right)^3+\left[x-\left(b-h\frac{b-a}{2}\right)\right]^3\right\}\end{array}\right]\right)^2
\end{aligned}$$

Proof. Associating $f(t)$ with $P(x,t)$ and $g(t)$ with $f''(t)$, then from (2.1) and (3.2), we obtain

$$T(P(x,.),f''(.);a,b) = M(P(x,.),f''(.);a,b) - M(P(x,.);a,b)\,M(f''(.);a,b)$$

Now using identity (2.2),

$$(b-a)\,T(P(x,.),f''(.);a,b) = \tau(x;\alpha,\beta) - (b-a)\,M(P(x,.);a,b)\,\kappa \qquad (3.8)$$

where κ is the secant slope of $f\,'$ over $[a,b]$, as given in (3.6). Now, from (2.2) and (3.2),

$$\begin{aligned}
& (b-a)\,M\left(P\left(x,.\right);a,b\right) \qquad (3.9)\\
=& \int_a^b P(x,t)dt\\
=& \frac{\alpha}{2\left(\alpha+\beta\right)\left(x-a\right)}\int_a^x\left[t-\left(a+h\frac{b-a}{2}\right)\right]^2 dt\\
&+\frac{\beta}{2\left(\alpha+\beta\right)\left(b-x\right)}\int_x^b\left[t-\left(b-h\frac{b-a}{2}\right)\right]^2 dt\\
=& \frac{1}{6\left(\alpha+\beta\right)}\left[\begin{array}{c} \frac{\alpha}{x-a}\left\{\left[x-\left(a+h\frac{b-a}{2}\right)\right]^3+\left(h\frac{b-a}{2}\right)^3\right\} \\ -\frac{\beta}{b-x}\left\{\left(h\frac{b-a}{2}\right)^3+\left[x-\left(b-h\frac{b-a}{2}\right)\right]^3\right\}\end{array}\right]
\end{aligned}$$

Now combining (3.9) with (3.7) the left hand side of (3.5) is obtained.

Let $f,g:[a,b]\to\mathbb{R}$ and $fg:[a,b]\to\mathbb{R}$ be integrable on $[a,b]$, then [1]

$$\begin{aligned}
\left|T\left(f,g\right)\right| &\leq T^{\frac{1}{2}}\left(f,f\right)T^{\frac{1}{2}}\left(g,g\right) &&\left(f,g\in L_2\left[a,b\right]\right) \qquad (3.10)\\
&\leq \frac{\left(\Gamma-\gamma\right)}{2}T^{\frac{1}{2}}\left(f,f\right) &&\left(\gamma\leq g\left(x\right)\leq\Gamma,\ t\in\left[a,b\right]\right)\\
&\leq \frac{1}{4}\left(\Phi-\varphi\right)\left(\Gamma-\gamma\right) &&\left(\varphi\leq f\left(x\right)\leq\Phi,\ t\in\left[a,b\right]\right).
\end{aligned}$$

Also, note that

$$\begin{aligned}
0 &\leq T^{\frac{1}{2}}\left(f\,''(.),f\,''(.)\right) \qquad (3.11)\\
&= \left[M\left(f\,''(.)^2;a,b\right)-M^2\left(f\,''(.);a,b\right)\right]^{\frac{1}{2}}\\
&= \left[\frac{1}{b-a}\int_a^b\left\|f\,''\left(t\right)\right\|^2dt-\left(\frac{\int_a^b f\,''\left(t\right)dt}{b-a}\right)^2\right]^{\frac{1}{2}}\\
&= \left[\frac{1}{b-a}\left\|f\,''\right\|_2^2-\kappa^2\right]^{\frac{1}{2}}\\
&\leq \frac{\left(\Gamma-\gamma\right)}{2}
\end{aligned}$$

where $\gamma\leq f\,''\left(t\right)\leq\Gamma, t\in\left[a,b\right]$. Now, for the bounds on (3.8), we have to determine $T^{\frac{1}{2}}\left(p\left(x,.\right),p\left(x,.\right)\right)$ and $\varphi\leq p\left(x,.\right)\leq\Phi$ from (3.10) and (3.11).

Now from (2.1), the definition of $P(x,t,h)$, we have

$$T\left(P\left(x,.\right),P\left(x,.\right)\right)=M\left(P^2\left(x,.\right);a,b\right)-M^2\left(P\left(x,.\right);a,b\right). \qquad (3.12)$$

From (3.10) we obtain

$$\begin{aligned} & M\left(P\left(x,.\right);a,b\right) \\ = & \frac{1}{6\left(\alpha+\beta\right)}\left[\begin{array}{c} \frac{\alpha}{x-a}\left\{\left[x-\left(a+h\frac{b-a}{2}\right)\right]^{3}+\left(h\frac{b-a}{2}\right)^{3}\right\} \\ -\frac{\beta}{b-x}\left\{\left(h\frac{b-a}{2}\right)^{3}+\left[x-\left(b-h\frac{b-a}{2}\right)\right]^{3}\right\} \end{array}\right] \end{aligned}$$

and

$$\begin{aligned} & (b-a)\,M\left(P^{2}\left(x,.\right);a,b\right) \\ = & \left(\frac{\alpha}{\alpha+\beta}\right)^{2}\frac{1}{4\left(x-a\right)^{2}}\int_{a}^{x}\left[t-\left(a+h\frac{b-a}{2}\right)\right]^{4}dt \\ & +\left(\frac{\beta}{\alpha+\beta}\right)^{2}\frac{1}{4\left(b-x\right)^{2}}\int_{x}^{b}\left[t-\left(b-h\frac{b-a}{2}\right)\right]^{4}dt \\ = & \left(\frac{\alpha}{\alpha+\beta}\right)^{2}\frac{1}{20\left(x-a\right)^{2}}\left[\left(x-\left(a+h\frac{b-a}{2}\right)\right)^{5}+\left(h\frac{b-a}{2}\right)^{5}\right] \\ & +\left(\frac{\beta}{\alpha+\beta}\right)^{2}\frac{1}{20\left(b-x\right)^{2}}\left[\left(h\frac{b-a}{2}\right)^{5}-\left(x-\left(b-h\frac{b-a}{2}\right)\right)^{5}\right] \\ = & \frac{1}{20\left(\alpha+\beta\right)^{2}}\left\{\begin{array}{c} \frac{\alpha^{2}}{(x-a)^{2}}\left[\left(x-\left(a+h\frac{b-a}{2}\right)\right)^{5}+\left(h\frac{b-a}{2}\right)^{5}\right] \\ +\frac{\beta^{2}}{(b-x)^{2}}\left[\left(h\frac{b-a}{2}\right)^{5}-\left(x-\left(b-h\frac{b-a}{2}\right)\right)^{5}\right] \end{array}\right\} \end{aligned}$$

Thus, substituting the above results into (3.12) gives

$$0\leq N\left(x\right)=T^{\frac{1}{2}}\left(P\left(x,.\right),P\left(x,.\right)\right)$$

which is given explicitly by (3.7). Combining (3.8), (3.12) and (3.11) give from the first inequality in (3.10), the first inequality in (3.5). Now utilizing the inequality in (3.11) produces the second result in (3.5). Further, it may be noticed from the definition of $P\left(x,t\right)$ in (2.1) that for α, $\beta\geq0$, give

$$\begin{aligned} \Phi & = \sup_{t\in[a,b]}P\left(x,t\right) \\ & = \frac{1}{4\left(\alpha+\beta\right)}\left\{\begin{array}{c} \alpha\left(x-a\right)+\beta\left(b-x\right)-h\left(b-a\right)\left(\alpha+\beta\right) \\ +\frac{h^{2}(b-a)}{2}\left[\frac{\alpha}{x-a}+\frac{\beta}{b-x}\right] \\ +\left|\begin{array}{c} \beta\left(b-x\right)-\alpha\left(x-a\right)+h\left(b-a\right)\left(\alpha-\beta\right) \\ +\frac{h^{2}(b-a)}{2}\left[\frac{\beta}{b-x}-\frac{\alpha}{x-a}\right] \end{array}\right| \end{array}\right\} \end{aligned}$$

$$\begin{aligned} \varphi & = \inf_{t\in[a,b]}P\left(x,t\right) \\ & = \frac{h^{2}\left(b-a\right)^{2}}{8\left(\alpha+\beta\right)}\left\{\frac{\alpha}{x-a}+\frac{\beta}{b-x}-\left|\frac{\alpha}{x-a}-\frac{\beta}{b-x}\right|\right\} \end{aligned}$$

where $\Phi-\varphi=\lambda$. □

4. An Application to the Cumulative Distribution Function

Let $X \in [a,b]$ be a random variable with the cumulative distributive function

$$F(x) = P_r(X \leq x) = \int_a^x f(u)\,du,$$

where f is the probability density function. In particular,

$$\int_a^b f(u)\,du = 1.$$

The following theorem holds.

Theorem 4. *Let X and F be as above, then*

$$\begin{array}{c} \left|\frac{1}{2}\left[\alpha(b-x)\left(x-\left(a+h\frac{b-a}{2}\right)\right)^2 - \beta(x-a)\left(x-\left(b-h\frac{b-a}{2}\right)\right)^2\right] f'(x)\right. \\ -\left[\alpha(b-x)\left(x-\left(a+h\frac{b-a}{2}\right)\right) - \beta(x-a)\left(x-\left(b-h\frac{b-a}{2}\right)\right)\right] f(x) \\ -h\frac{b-a}{2}\left[\alpha(b-x)f(a)+\beta(x-a)f(b)\right] \\ +h^2\frac{(b-a)^2}{8}\left[\beta(x-a)f'(b)-\alpha(b-x)f'(a)\right] \\ \left.+\left[\alpha(b-x)-\beta(x-a)\right]F(x)+\beta(x-a)\right| \end{array} \tag{4.1}$$

$$\leq \begin{cases} \frac{\|f''\|_\infty}{6}\begin{bmatrix} \alpha(b-x)\left\{\left[x-\left(a+h\frac{b-a}{2}\right)\right]^3+\left(h\frac{b-a}{2}\right)^3\right\} \\ -\beta(x-a)\left\{\left(h\frac{b-a}{2}\right)^3+\left[x-\left(b-h\frac{b-a}{2}\right)\right]^3\right\}\end{bmatrix}, f''\in L_\infty[a,b], \\ \frac{(b-x)(x-a)\|f''\|_p}{2(2q+1)^{\frac{1}{q}}}\begin{bmatrix} \frac{\alpha^q}{(x-a)^q}\left\{\left[x-\left(a+h\frac{b-a}{2}\right)\right]^{2q+1}-\left(h\frac{a-b}{2}\right)^{2q+1}\right\} \\ +\frac{\beta^q}{(b-x)^q}\left\{\left(h\frac{a-b}{2}\right)^{2q+1}-\left[x-\left(b-h\frac{b-a}{2}\right)\right]^{2q+1}\right\}\end{bmatrix}^{\frac{1}{q}} \\ ,, f''\in L_p[a,b], p>1, \frac{1}{p}+\frac{1}{q}=1, \\ \frac{(b-x)(x-a)\|f''\|_1}{4}\left\{\begin{array}{c} \alpha(x-a)+\beta(b-x)-h(b-a)(\alpha+\beta) \\ +\frac{h^2(b-a)}{2}\left[\frac{\alpha}{x-a}+\frac{\beta}{b-x}\right] \\ +\left|\begin{array}{c}\beta(b-x)-\alpha(x-a)+h(b-a)(\alpha-\beta) \\ +\frac{h^2(b-a)}{2}\left[\frac{\beta}{b-x}-\frac{\alpha}{x-a}\right]\end{array}\right|\end{array}\right\} \\ , f''\in L_1[a,b]. \end{cases}.$$

Proof. From (2.3) , and by using the definition of Probability Density Function, we have

$$\begin{aligned}
\tau(x;\alpha,\beta) \quad : \quad &= \frac{1}{2(\alpha+\beta)}\left[\frac{\alpha}{x-a}\left(x-\left(a+h\frac{b-a}{2}\right)\right)^2-\frac{\beta}{b-x}\left(x-\left(b-h\frac{b-a}{2}\right)\right)^2\right]f'(x)\\
&-\frac{1}{(\alpha+\beta)}\left[\frac{\alpha}{x-a}\left(x-\left(a+h\frac{b-a}{2}\right)\right)-\frac{\beta}{b-x}\left(x-\left(b-h\frac{b-a}{2}\right)\right)\right]f(x)\\
&-\frac{1}{\alpha+\beta}h\frac{b-a}{2}\left[\frac{\alpha}{x-a}f(a)+\frac{\beta}{b-x}f(b)\right]\\
&+\frac{1}{\alpha+\beta}h^2\frac{(b-a)^2}{8}\left[\frac{\beta}{b-x}f'(b)-\frac{\alpha}{x-a}f'(a)\right]\\
&+\frac{1}{\alpha+\beta}\left[\alpha M(f;a,x)+\beta M(f;x,b)\right],\\
&= \frac{1}{2(\alpha+\beta)}\left[\frac{\alpha}{x-a}\left(x-\left(a+h\frac{b-a}{2}\right)\right)^2-\frac{\beta}{b-x}\left(x-\left(b-h\frac{b-a}{2}\right)\right)^2\right]f'(x)\\
&-\frac{1}{(\alpha+\beta)}\left[\frac{\alpha}{x-a}\left(x-\left(a+h\frac{b-a}{2}\right)\right)-\frac{\beta}{b-x}\left(x-\left(b-h\frac{b-a}{2}\right)\right)\right]f(x)\\
&-\frac{1}{\alpha+\beta}h\frac{b-a}{2}\left[\frac{\alpha}{x-a}f(a)+\frac{\beta}{b-x}f(b)\right]\\
&+\frac{1}{\alpha+\beta}h^2\frac{(b-a)^2}{8}\left[\frac{\beta}{b-x}f'(b)-\frac{\alpha}{x-a}f'(a)\right]\\
&+\frac{1}{\alpha+\beta}\left\{\left[\frac{\alpha(b-x)-\beta(x-a)}{(x-a)(b-x)}\right]F(x)+\frac{\beta}{(b-x)}\right\}
\end{aligned}$$

or

$$\begin{aligned}
&(\alpha+\beta)(x-a)(b-x)\tau(x;\alpha,\beta) \qquad (4.2)\\
= \;& \frac{1}{2}\left[\begin{array}{c}\alpha(b-x)\left(x-\left(a+h\frac{b-a}{2}\right)\right)^2\\ -\beta(x-a)\left(x-\left(b-h\frac{b-a}{2}\right)\right)^2\end{array}\right]f'(x)\\
&-\left[\begin{array}{c}\alpha(b-x)\left(x-\left(a+h\frac{b-a}{2}\right)\right)\\ -\beta(x-a)\left(x-\left(b-h\frac{b-a}{2}\right)\right)\end{array}\right]f(x)\\
&-h\frac{b-a}{2}\left[\alpha(b-x)f(a)+\beta(x-a)f(b)\right]\\
&+h^2\frac{(b-a)^2}{8}\left[\beta(x-a)f'(b)-\alpha(b-x)f'(a)\right]\\
&+\left[\alpha(b-x)-\beta(x-a)\right]F(x)+\beta(x-a)
\end{aligned}$$

Now using (2.4) and (4.2), we get our required result (4.1). □

Putting $\alpha=\beta=\frac{1}{2}$ in Theorem 5 gives the following result.

Corollary 5. *Let X be a random variable, $F(x)$ cumulative distributive function and f is a probability density function. Then*

$$\begin{array}{c} |\frac{1}{4}\left[(b-x)\left(x-\left(a+h\frac{b-a}{2}\right)\right)^2-(x-a)\left(x-\left(b-h\frac{b-a}{2}\right)\right)^2\right]f'(x) \\ -\frac{1}{2}\left[(b-x)\left(x-\left(a+h\frac{b-a}{2}\right)\right)-(x-a)\left(x-\left(b-h\frac{b-a}{2}\right)\right)\right]f(x) \\ -h\frac{b-a}{4}\left[(b-x)f(a)+(x-a)f(b)\right] \\ +h^2\frac{(b-a)^2}{16}\left[(x-a)f'(b)-(b-x)f'(a)\right] \\ +\frac{1}{2}\left[(b-x)-(x-a)\right]F(x)+\frac{1}{2}(x-a)| \end{array} \tag{4.3}$$

$$\leq\left\{\begin{array}{l} \left[\begin{array}{c} (b-x)\left\{\left[x-\left(a+h\frac{b-a}{2}\right)\right]^3+\left(h\frac{b-a}{2}\right)^3\right\} \\ -(x-a)\left\{\left(h\frac{b-a}{2}\right)^3+\left[x-\left(b-h\frac{b-a}{2}\right)\right]^3\right\} \end{array}\right]\frac{\left\|f''\right\|_\infty}{12}, f''\in L_\infty[a,b], \\ \\ \left[\begin{array}{l} \frac{1}{(x-a)^q}\left\{\left[x-\left(a+h\frac{b-a}{2}\right)\right]^{2q+1}-\left(h\frac{a-b}{2}\right)^{2q+1}\right\} \\ \\ +\frac{1}{(b-x)^q}\left\{\left(h\frac{a-b}{2}\right)^{2q+1}-\left[x-\left(b-h\frac{b-a}{2}\right)\right]^{2q+1}\right\} \end{array}\right]^{\frac{1}{q}}\frac{(b-x)(x-a)}{4(2q+1)^{\frac{1}{q}}}\left\|f''\right\|_p \\ , f''\in L_p[a,b], p>1, \frac{1}{p}+\frac{1}{q}=1, \\ \\ \left\{\begin{array}{l} \frac{1}{2}(b-a)-h(b-a)+\frac{h^2(b-a)}{4}\left[\frac{1}{x-a}+\frac{1}{b-x}\right] \\ \quad+\left|\frac{1}{2}(a+b-2x)+\frac{h^2(b-a)}{4}\left[\frac{1}{b-x}-\frac{1}{x-a}\right]\right| \end{array}\right\}\frac{(b-x)(x-a)\left\|f''\right\|_1}{4} \\ , f''\in L_1[a,b]. \end{array}\right.$$

Remark 6. *The above result allow the approximation of $F(x)$ in terms of $f(x)$. The approximation of*

$$R(x)=1-F(x)$$

could also be obtained by a simple substitution. $R(x)$ is of importance in reliability theory where $f(x)$ is the probability density function of failure.

Remark 7. *We put $\beta=0$ in (4.1), assuming that $\alpha\neq 0$ to obtain*

$$\left|\alpha(b-x)\left\{\begin{array}{c} \frac{1}{2}\left[\left(x-\left(a+h\frac{b-a}{2}\right)\right)^2\right]f'(x) \\ -\left[\left(x-\left(a+h\frac{b-a}{2}\right)\right)\right]f(x) \\ -h\frac{b-a}{2}f(a)-h^2\frac{(b-a)^2}{8}\left[f'(a)\right]+F(x) \end{array}\right\}\right| \tag{4.4}$$

$$\leq\left\{\begin{array}{l} \left[\alpha(b-x)\left\{\left[x-\left(a+h\frac{b-a}{2}\right)\right]^3+\left(h\frac{b-a}{2}\right)^3\right\}\right]\frac{\left\|f''\right\|_\infty}{6}, f''\in L_\infty[a,b], \\ \\ \left[\frac{\alpha^q}{(x-a)^q}\left\{\left[x-\left(a+h\frac{b-a}{2}\right)\right]^{2q+1}-\left(h\frac{a-b}{2}\right)^{2q+1}\right\}\right]^{\frac{1}{q}}\frac{(b-x)(x-a)\left\|f''\right\|_p}{2(2q+1)^{\frac{1}{q}}} \\ , f''\in L_p[a,b], p>1, \frac{1}{p}+\frac{1}{q}=1, \\ \\ \left(\begin{array}{c} \alpha(x-a)-h\alpha(b-a)+\frac{h^2(b-a)}{2}\left(\frac{\alpha}{x-a}\right) \\ +\left|-\alpha(x-a)+h\alpha(b-a)-\frac{h^2(b-a)}{2}\left(\frac{\alpha}{x-a}\right)\right| \end{array}\right)\frac{(b-x)(x-a)\left\|f''\right\|_1}{4} \\ , f''\in L_1[a,b]. \end{array}\right.$$

We may replace f by F in any of the equations (4.1),(4.3)and (4.4) so that the bounds are in terms of $\|f''\|_p$, $p \geq 1$. Further we note that

$$\int_a^b F(u)\,du = uF(u)|_a^b - \int_a^b xf(x)\,dx = b - E(X).$$

Competing interests:
The authors declare that they have no competing interests.
Authors' contributions:
All authors have contributed equally and significantly in writing this article.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 19, NO.4, 2015, COPYRIGHT 2015 EUDOXUS PRESS, LLC

A Novel Multistep Generalized Differential Transform Method for Solving Fractional-order Lü Chaotic and Hyperchaotic Systems

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Abstract. This paper investigates the approximate numerical solutions of the fractional-order Lü chaotic and hyperchaotic systems based on a multistep generalized differential transform method (MGDTM). This method has the advantage of giving an analytical form of the solution within each time interval which is not possible using purely numerical techniques. In addition, this paper presents a comparative study between a new scheme and the classical Runge-Kutta method to demonstrate the applicability of the MGDTM. Furthermore, numerical results are presented graphically and reveal that the proposed scheme is an effective, simple and convenient method for solving nonlinear fractional-order chaotic systems with less computational and iteration steps.

keywords: Chaos, Fractional calculus, Lü system, Differential transform method.
AMS Subject Classification: 34C28; 34A08; 74H15

1. Introduction

Recently, mathematical modeling of chaotic systems becomes a challenging and interesting for contemporary scientists due to its complex, extremely sensitive, unpredictable dynamical behaviors and potential applications in many scientific and engineering fields such as chaos control [1], electronic circuits [2], secure communication [3], information processing [4], power converters [5], electrical engineering [6], biological and chemical systems [7,8], and so on. Therefore, a wide variety of research activities in the field of chaotic systems with complex topological structures as well as in related areas have been proposed by many pioneer researchers, for more details see [9-13] and references therein. In particular, there are some detailed investigations and studies of the Lü system which is transition system that bridges the gap between the Lorenz and Chen systems [10].

The characterization of fractional-order dynamical systems has become a powerful tool to describe memory and hereditary properties of various complex materials, processes and models. Whereas many investigations are devoted to generate new fundamentals which exist

in the case of fractional-order and disappear when it is reduced to integer. However, along with the development of research on chaos, it was found that many fractional-order differential systems behave chaotically, for instance, fractional-order Chen system [14], fractional-order Lü system [15], fractional-order Lorenz system [16], fractional-order Chua's system [17], fractional-order Liu system [18], fractional-order Rössler system [19,20], and so forth.

In this paper, the MGDTM is implemented to give approximate numerical solutions for the fractional-order Lü chaotic system, in consideration that the hyperchaotic is chaotic system with two positive Lyapunov exponents. These systems are found to be chaotic in a wide parameter range and have many interesting complex dynamical behaviors. In this point, the chaos and hyperchaos exist in the fractional-order Lü system of order as low as 2.4 and 3.28, respectively. In contrast, the proposed method is just a simple modification of the generalized differential transform method (GDTM), in which it is treated as an algorithm in a sequence of small intervals for finding accurate approximate solutions to the corresponding systems. Moreover, it provides an accurate numerical solutions over a longer time frame compared to the standard GDTM, and has many advantages over the classical methods.

The remainder of this paper is organized as follows. In Section 2, we present basic facts and notations related to the fractional calculus and MGDTM. In Section 3, the MGDTM is applied to the fractional-order Lü chaotic and hyperchaotic systems. In Section 4, Examples and corresponding numerical simulations are shown graphically to illustrate the feasibility and effectiveness of the proposed method. Finally, the conclusions are drawn in Section 5.

2. The multistep generalized differential transform method (MGDTM)

In this section, we present some basic definitions and properties of the fractional calculus theory and MGDTM which will be used in the remainder of this paper.

Definition .1 A function $f(x)\,(x>0)$ is said to be in the space $C_\alpha\,(\alpha\in\mathbb{R})$ if it can be written as $f(x)=x^p f_1(x)$ for some $p>\alpha$, where $f_1(x)$ is continuous in $[0,\infty)$, and it is said to be in the space C_α^m if $f^{(m)}\in C_\alpha,\ m\in\mathbb{N}$.

Definition .2 The Riemann–Liouville integral operator of order α with $a\geq 0$ is defined as

$$\begin{aligned}(J_a^\alpha f)(x) &= \tfrac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\,dt,\ x>a,\\ (J_a^0 f)(x) &= f(x).\end{aligned} \tag{1}$$

Now, for $f\in C_\alpha$, $\alpha,\beta>0$, $a\geq 0$, $c\in\mathbb{R}$, $\gamma>-1$, one can get

$$\begin{aligned}\left(J_a^\alpha J_a^\beta f\right)(x) &= \left(J_a^\beta J_a^\alpha f\right)(x)=\left(J_a^{\alpha+\beta} f\right)(x),\\ J_a^\alpha\, x^\gamma &= \tfrac{x^{\gamma+\alpha}}{\Gamma(\alpha)} B_{\frac{x-a}{x}}(\alpha,\gamma+1),\end{aligned} \tag{2}$$

where $B_\tau(\alpha,\gamma+1)$ is the incomplete beta function which is defined as

$$\begin{aligned}B_\tau(\alpha,\gamma+1) &= \int_0^\tau t^{\alpha-1}(1-t)^\gamma dt,\\ J_a^\alpha e^{cx} &= e^{ac}(x-a)^\alpha \sum_{k\,=\,0}^{\infty} \tfrac{[c(x-a)]^k}{\Gamma(\alpha+k+1)}.\end{aligned} \tag{3}$$

The Riemann–Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Thus, we shall introduce a modified fractional differential operator D_a^α proposed by Caputo in his work on the theory of viscoelasticity. For more details about the fractional calculus theory (see [21-23]).

Definition .3 The Caputo fractional derivative of $f(x)$ of order $\alpha > 0$ with $a \geq 0$ is defined as

$$(D_a^\alpha f)(x) = \left(J_a^{m-\alpha} f^{(m)}\right)(x) = \tfrac{1}{\Gamma(m-\alpha)} \int_a^x \frac{f^{(m)}(t)}{(x-t)^{\alpha+1-m}} dt, \tag{4}$$

for $m-1 < \alpha \leq m$, $m \in \mathbb{N}$, $x \geq a$, $f \in C_{-1}^m$.

Now, for $m-1 < \alpha \leq m$, $f(x) \in C_\alpha^m$ and $\alpha \geq -1$, one can get

$$\left(J_a^\alpha D_a^\alpha f\right)(x) = J^m D^m f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(a) \frac{(x-a)^k}{k!}. \tag{5}$$

In order to describe the MGDTM, we consider the following initial value problem (IVP) for systems of fractional differential equations

$$D_*^{\alpha_i} y_i(t) = f_1(t, y_1, y_2, \ldots, y_n), \; i = 1, 2, ..., n, \tag{6}$$

subject to the initial conditions

$$y_i(t_0) = c_i, \; i = 1, \, 2, \, \ldots \, , \, n \tag{7}$$

where $D_*^{\alpha_i}$ is the Caputo fractional derivative of order α_i, and $0 < \alpha_i \leqslant 1$, for $i = 1, \, 2, \, \ldots, \, n$.

On the other hand, by using the GDTM, the K^{th}-order approximate solution of IVPs (6)-(7) can be expressed by the finite series

$$y_i(t) = \sum_{i=0}^{K} Y_i(k)(t-t_0)^{k\,\alpha_i}, \; t \in [0, T], \tag{8}$$

where $Y_i(k)$ satisfied the recurrence relation

$$\frac{\Gamma((k+1)\alpha_i + 1)}{\Gamma(k\alpha_i + 1)} Y_i(k+1) = F_i(k, Y_1, Y_2, \ldots, Y_n),$$

$Y_i(0) = c_i$ and $F_i(k, Y_1, Y_2, \ldots, Y_n)$ is the differential transform of function $f_i(t, y_1, y_2, \ldots, y_n)$ for $i = 1, \, 2, \, \ldots \, , \, n$.

Prior to applying the MGDTM [24,25], we have to divide the solution interval $[t_0, T]$ into M subintervals $[t_{m-1}, t_m]$, $m = 1, 2, \ldots, M$, of equal step size $h = (T - t_0)/M$ with $t_m = t_0 + m\,h$. Afterward, we apply the GDTM to IVP (6) over $[t_0, t_1]$ in order to obtain the approximate solution $y_{i,1}(t), i = 1, 2, ..., n$, using the initial condition (7). For $m \geq 2$ and at each subinterval $[t_{m-1}, t_m]$, we apply the GDTM to IVP (6) to obtain the approximate solutions $y_{i,m}(t), t \in [t_{m-1}, t_m], i = 1, 2, ..., n$, using the initial condition $y_{i,m}(t_{m-1}) = y_{i,m-1}(t_{m-1})$. As a consequence, the MGDTM assumes the solutions as follows

$$y_i(t) = \begin{cases} y_{i,1}(t), & t \in [t_0, t_1], \\ y_{i,2}(t), & t \in [t_1, t_2], \\ \vdots & \vdots \\ y_{i,M}(t), & t \in [t_{M-1}, t_M]. \end{cases} \tag{9}$$

Further, this new scheme of our proposed method is simple for computational performance for all values of h, whereas the obtained solution converges for wide time regions.

3. Solving the fractional-order Lü chaotic systems using the MGDTM

In this section, in order to illustrate the performance and efficiency of the MGDTM, we applied the method for solving the fractional-order Lü chaotic and hyperchaotic systems.

3.1.The fractional-order Lü chaotic system

For fractional-order Lü system which is the lowest-order chaotic system among all the chaotic systems [26], the integer-order derivatives are replaced by the fractional-order derivatives as follows:

$$\begin{aligned} D^{\alpha_1}x(t) &= a(y-x), \\ D^{\alpha_2}y(t) &= -xz+cy, \\ D^{\alpha_3}z(t) &= xy-bz, \end{aligned} \tag{10}$$

where (a, b, c) are the system parameters, (x, y, z) are the state variables, and α_i, $i = 1, 2, 3$, are parameters describing the order of the fractional time-derivatives in the Caputo sense.

Applying the MGDT algorithm to (10) yields

$$\begin{cases} X(k+1) = a\Gamma_{\alpha_1}(Y(\kappa) - X(\kappa)), \\ Y(k+1) = \Gamma_{\alpha_2}(\sum\limits_{l=0}^{k} X(l)Z(k-l) + cY(\kappa)), \\ Z(k+1) = \Gamma_{\alpha_3}(\sum\limits_{l=0}^{k} X(l)Y(k-l) - bZ(k)), \end{cases} \tag{11}$$

where $\Gamma_{\alpha_i} = \frac{\Gamma(\alpha_i\, k+1)}{\Gamma(\alpha_i(\, k+1)+1)}$, $i = 1, 2, 3$, $X(k)$, $Y(k)$ and $Z(k)$ are the differential transformation of $x(t)$, $y(t)$ and $z(t)$, respectively. The differential transform of the initial conditions are given by $X(0) = c_1$, $Y(0) = c_2$ and $Z(0) = c_3$. In view of the differential inverse transform, the differential transform series solution for the system (10) can be obtained as

$$\begin{cases} x(t) = \sum_{n=0}^{N} X(n)t^{\alpha_1 n}, \\ y(t) = \sum_{n=0}^{N} Y(n)t^{\alpha_2 n}, \\ z(t) = \sum_{n=0}^{N} Z(n)t^{\alpha_3 n}. \end{cases} \tag{12}$$

According to the MSGDT method, the series solution for system (10) is suggested by

$$x(t)=\begin{cases}\sum\limits_{n=0}^{K} X_1(n)t^{\alpha_1 n}, & t\in[0,t_1],\\ \sum\limits_{n=0}^{K} X_2(n)(t-t_1)^{\alpha_1 n}, & t\in[t_1,t_2],\\ \vdots & \\ \sum\limits_{n=0}^{K} X_M(n)(t-t_{M-1})^{\alpha_1 n}, & t\in[t_{M-1},t_M],\end{cases} \tag{13}$$

$$y(t)=\begin{cases}\sum\limits_{n=0}^{K} Y_1(n)t^{\alpha_2 n}, & t\in[0,t_1],\\ \sum\limits_{n=0}^{K} Y_2(n)(t-t_1)^{\alpha_2 n}, & t\in[t_1,t_2],\\ \vdots & \\ \sum\limits_{n=0}^{K} Y_M(n)(t-t_{M-1})^{\alpha_2 n}, & t\in[t_{M-1},t_M],\end{cases} \tag{14}$$

$$z(t)=\begin{cases}\sum\limits_{n=0}^{K} Z_1(n)t^{\alpha_3 n}, & t\in[0,t_1],\\ \sum\limits_{n=0}^{K} Z_2(n)(t-t_1)^{\alpha_3 n}, & t\in[t_1,t_2],\\ \vdots & \\ \sum\limits_{n=0}^{K} Z_M(n)(t-t_{M-1})^{\alpha_3 n}, & t\in[t_{M-1},t_M],\end{cases} \tag{15}$$

where $X_i(n)$, $Y_i(n)$ and $Z_i(n)$, for $i=1,\ 2,\ \ldots,M$, satisfy the following recurrence relations

$$\begin{cases} X_i(k+1)=a\Gamma_{\alpha_1}(Y_i(\kappa)-X_i(\kappa)),\\ Y_i(k+1)=-\Gamma_{\alpha_2}(\sum\limits_{l=0}^{k} X_i(l)Z_i(k-l)+Y_i(k)),\\ Z_i(k+1)=\Gamma_{\alpha_3}(\sum\limits_{l=0}^{k} X_i(l)Y_i(k-l)-bZ_i(k)),\end{cases} \tag{16}$$

such that $X_i(0)=x_i(t_{i-1})=x_{i-1}(t_{i-1}), Y_i(0)=y_i(t_{i-1})=y_{i-1}(t_{i-1})$ and $Z_i(0)=z_i(t_{i-1})=z_{i-1}(t_{i-1})$. Finally, starting with $X_0(0)=c_1, Y_0(0)=c_2, Z_0(0)=c_3$ and using the recurrence relation given in (16), then the multi-step solution can be obtained as in (13)-(15).

3.2.The fractional-order Lü hyperchaotic system

Consider the fractional-order Lü hyperchaotic system [27]. In this system, the integer-order derivatives are replaced by the fractional-order derivatives as follows:

$$\begin{aligned} D^{\alpha_1}x(t) &= a(y-x)+u,\\ D^{\alpha_2}y(t) &= -xz+cy,\\ D^{\alpha_3}z(t) &= xy-bz,\\ D^{\alpha_4}u(t) &= xz+du,\end{aligned} \tag{17}$$

where (a, b, c, d) are the system parameters, (x, y, z, u) are the state variables and $\alpha_i, i = 1, 2, 3, 4$, are parameters describing the order of fractional time-derivatives in Caputo sense.

In the same manner, applying the MGDT algorithm to system (17) yields

$$\begin{cases} X(k+1) = \Gamma_{\alpha_1}(a(Y(\kappa) - X(\kappa)) + U(\kappa)), \\ Y(k+1) = -\Gamma_{\alpha_2}(\sum\limits_{l=0}^{k} X(l)Z(k-l) + cY(\kappa))), \\ Z(k+1) = \Gamma_{\alpha_3}(\sum\limits_{l=0}^{k} X(l)Y(k-l) - bZ(k)), \\ U(k+1) = \Gamma_{\alpha_4}(\sum\limits_{l=0}^{k} X(l)Z(k-l) - dU(\kappa)), \end{cases} \tag{18}$$

where $\Gamma_{\alpha_i} = \frac{\Gamma(\alpha_i k+1)}{\Gamma(\alpha_i(k+1)+1)}, i = 1, 2, 3, 4$, $X(k)$, $Y(k)$, $Z(k)$ and $U(k)$ are the differential transformation of $x(t)$, $y(t)$, $z(t)$ and $u(t)$, respectively. The differential transform of the initial conditions are given by $X(0) = c_1$, $Y(0) = c_2$, $Z(0) = c_3$ and $U(0) = c_4$. In view of the differential inverse transform, the differential transform series solution for the system (17) can be obtained as

$$\begin{cases} x(t) = \sum\limits_{n=0}^{N} X(n)t^{\alpha_1 n}, \\ y(t) = \sum\limits_{n=0}^{N} Y(n)t^{\alpha_2 n}, \\ z(t) = \sum\limits_{n=0}^{N} Z(n)t^{\alpha_3 n}, \\ u(t) = \sum\limits_{n=0}^{N} U(n)t^{\alpha_4 n} \end{cases} \tag{19}$$

According to the MSGDT method, the series solution for the system (17) is suggested by

$$x(t) = \begin{cases} \sum\limits_{n=0}^{K} X_1(n)t^{\alpha_1 n}, & t \in [0, t_1], \\ \sum\limits_{n=0}^{K} X_2(n)(t-t_1)^{\alpha_1 n}, & t \in [t_1, t_2], \\ \vdots & \\ \sum\limits_{n=0}^{K} X_M(n)(t-t_{M-1})^{\alpha_1 n}, & t \in [t_{M-1}, t_M], \end{cases} \tag{20}$$

$$y(t) = \begin{cases} \sum\limits_{n=0}^{K} Y_1(n)t^{\alpha_2 n}, & t \in [0, t_1], \\ \sum\limits_{n=0}^{K} Y_2(n)(t-t_1)^{\alpha_2 n}, & t \in [t_1, t_2], \\ \vdots & \\ \sum\limits_{n=0}^{K} Y_M(n)(t-t_{M-1})^{\alpha_2 n}, & t \in [t_{M-1}, t_M], \end{cases} \tag{21}$$

$$z(t) = \begin{cases} \sum_{n=0}^{K} Z_1(n) t^{\alpha_3 n}, & t \in [0, t_1], \\ \sum_{n=0}^{K} Z_2(n)(t - t_1)^{\alpha_3 n}, & t \in [t_1, t_2], \\ \vdots & \\ \sum_{n=0}^{K} Z_M(n)(t - t_{M-1})^{\alpha_3 n}, & t \in [t_{M-1}, t_M], \end{cases} \tag{22}$$

$$u(t) = \begin{cases} \sum_{n=0}^{K} U_1(n) t^{\alpha_4 n}, & t \in [0, t_1], \\ \sum_{n=0}^{K} U_2(n)(t - t_1)^{\alpha_4 n}, & t \in [t_1, t_2], \\ \vdots & \\ \sum_{n=0}^{K} U_M(n)(t - t_{M-1})^{\alpha_4 n}, & t \in [t_{M-1}, t_M], \end{cases} \tag{23}$$

where $X_i(n)$, $Y_i(n)$, $Z_i(n)$ and $U_i(n)$, for $i = 1,\ 2,\ \ldots, M$, satisfy the following recurrence relations

$$\begin{cases} X_i(k+1) = \Gamma_{\alpha_1}(a(Y_i(\kappa) - X_i(\kappa)) + U_i(k)), \\ Y_i(k+1) = -\Gamma_{\alpha_2}(\sum_{l=0}^{k} X_i(l) Z_i(k-l) + Y_i(k)), \\ Z_i(k+1) = \Gamma_{\alpha_3}(\sum_{l=0}^{k} X_i(l) Y_i(k-l) - b Z_i(k)), \\ U_i(k+1) = \Gamma_{\alpha_4}(\sum_{l=0}^{k} X_i(l) Z_i(k-l) + d U_i(k))), \end{cases} \tag{24}$$

such that $X_i(0) = x_i(t_{i-1}) = x_{i-1}(t_{i-1})$, $Y_i(0) = y_i(t_{i-1}) = y_{i-1}(t_{i-1})$, $Z_i(0) = z_i(t_{i-1}) = z_{i-1}(t_{i-1})$ and $U_i(0) = u_i(t_{i-1}) = u_{i-1}(t_{i-1})$.

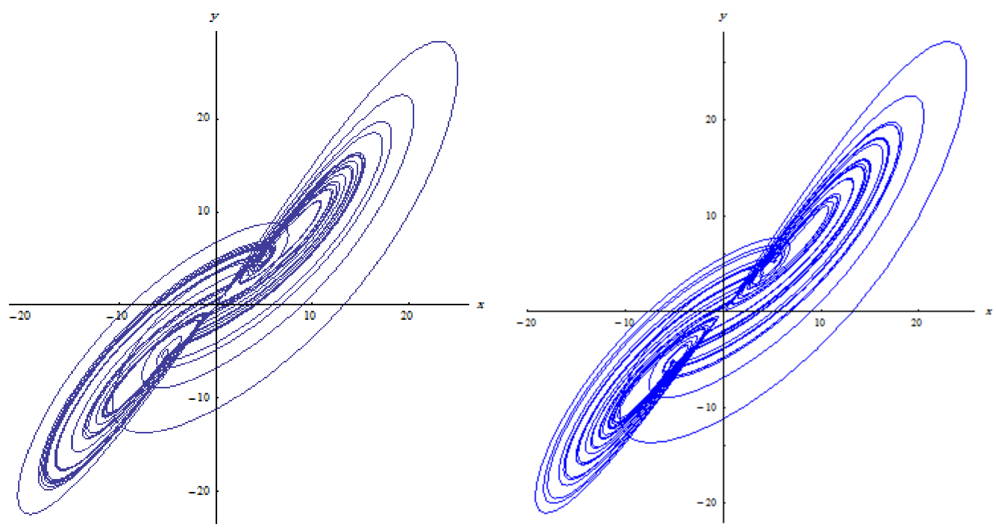
Finally, starting with $X_0(0) = c_1$, $Y_0(0) = c_2$, $Z_0(0) = c_3$ and $U_0(0) = c_4$ and using the recurrence relation that given in (24), then the multi-step solution can be obtained as in (20)-(23).

4. MGDTM simulation results

In this section, some numerical simulations are proposed to show the accuracy of the present method. The method provides immediate and visible symbolic terms of analytic solutions as well as numerical approximate solutions to both Lü chaotic and hyperchaotic systems. Results obtained by the MGDTM are compared with the fourth-order Runge-Kutta (RK4) method and are found to be in good agreement. All computations are performed by Mathematica 0.7. However, the time range studied in this work is $[0, 25]$ and the step size $\Delta t = 0.025$. In this regard, we take the initial condition for Lü chaotic system such as $x(0) = 1$, $y(0) = 1$ and $z(0) = 1$, with parameters $a = 36$, $b = 3$ and $c = 20$ and the initial condition for Lü hyperchaotic system such as $x(0) = 5$, $y(0) = 8$, $z(0) = -1$ and $u(0) = -3$.

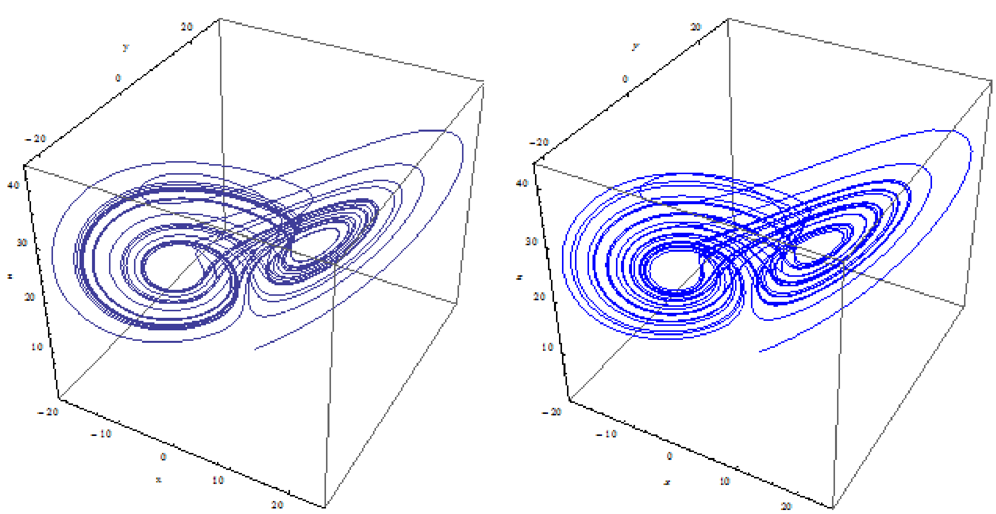
Figures 1 and 2 show the phase portrait for the classical Lü chaotic system, when $\alpha_1 = \alpha_2 = \alpha_3 = 1$, using the MGDTM against RK4 methods. While that Figure 3 shows the phase portrait for the classical Lü hyperchaotic system, when $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$. Accordingly, from the graphical results of these figures, it can be seen the results obtained using the MGDTM match the results of the RK4 method very well, which implies that the present method can predict the behavior of these variables accurately for the region under consideration. In contrast, Figure 4 shows the phase portrait for the fractional Lü

chaotic system using the MGDTM. Obviously, the approximate solutions in Figure 4 depend continuously on the time-fractional derivative α_i, $i = 1, 2, 3$, where the effective dimension $\sum$ of (10) is defined as the sum of orders $\alpha_1 + \alpha_2 + \alpha_3 = \sum$. In the meanwhile, we found that the chaos exists in the fractional-order Lü chaotic system with order as low as 2.4. Further, Figures 5 and 6 show that the hyperchaos exists in the fractional-order Lü hyperchaotic system using the MGDTM. This system has a hyperchaotic attractor when $d = 1.3$ with order as low as 3.28.



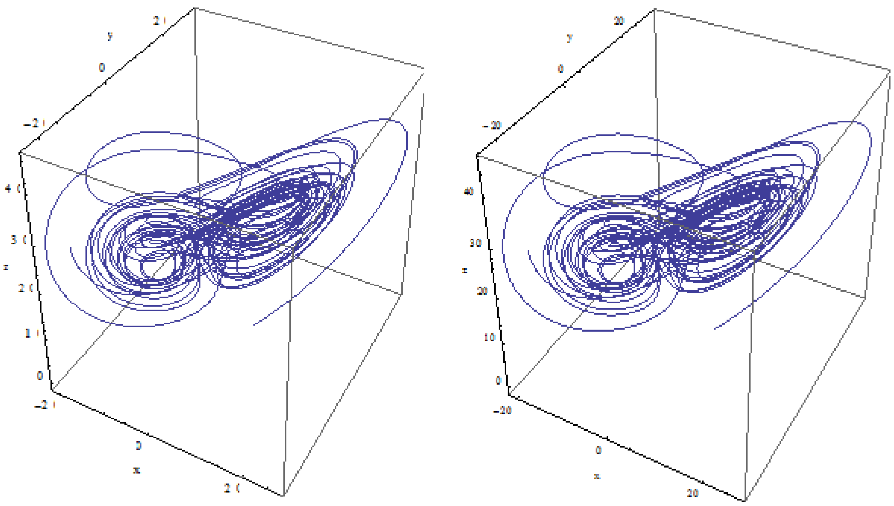
(a) The MSGDT method (b) The RK4 method

Figure 1: Phase plot of chaotic attractor in the $x - y$ space, with $\alpha_1 = \alpha_2 = \alpha_3 = 1$.



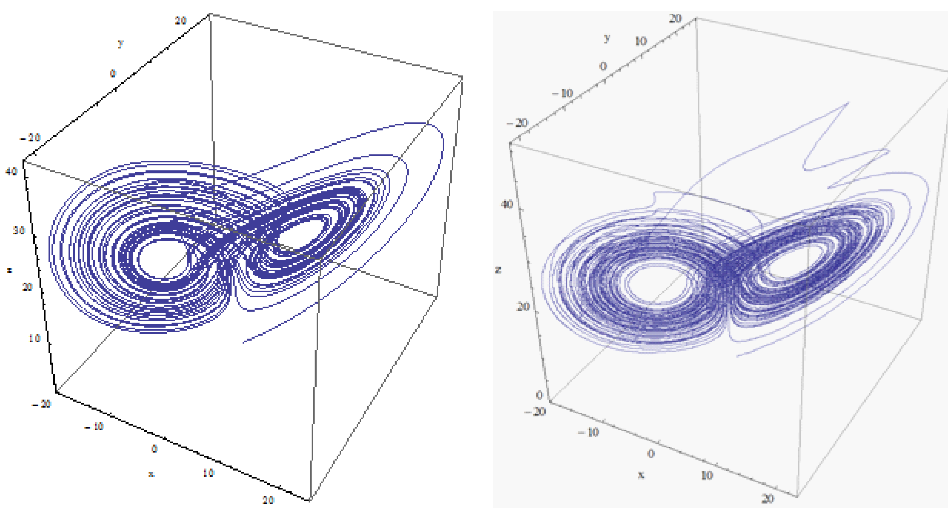
(a) The MSGDT method (b) The RK4 method

Figure 2: Phase plot of chaotic attractor in the $x - y - z$ space, with $\alpha_1 = \alpha_2 = \alpha_3 = 1$.



(a) The MSGDT method (b) The RK4 method

Figure 3: Phase plot of hyperchaotic attractor in the $x-y-z$ space, with $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$.



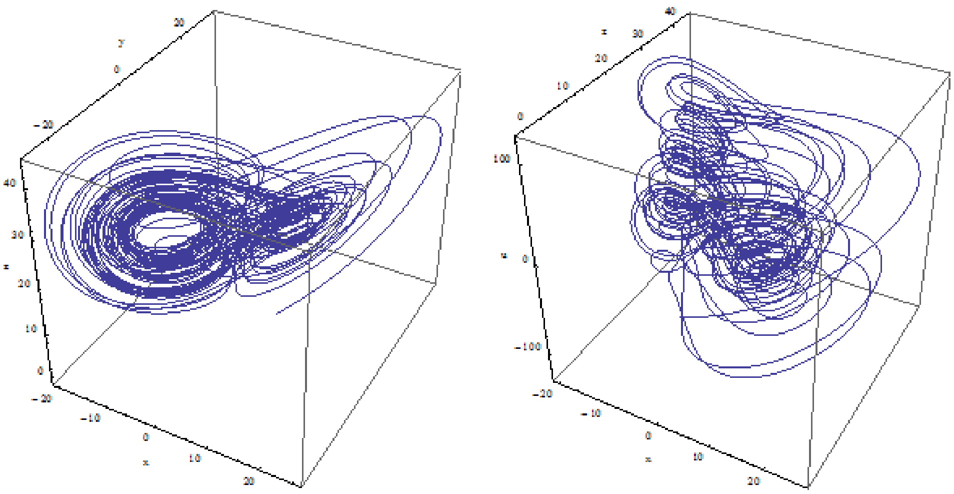
(a) (b)

Figure 4: Phase plot of chaotic attractor in the $x-y-z$ space, (a) $\alpha_1 = \alpha_2 = \alpha_3 = 0.9$, (b) $\alpha_1 = 0.79, \alpha_2 = 0.81, \alpha_3 = 0.8$.

5. Conclusions

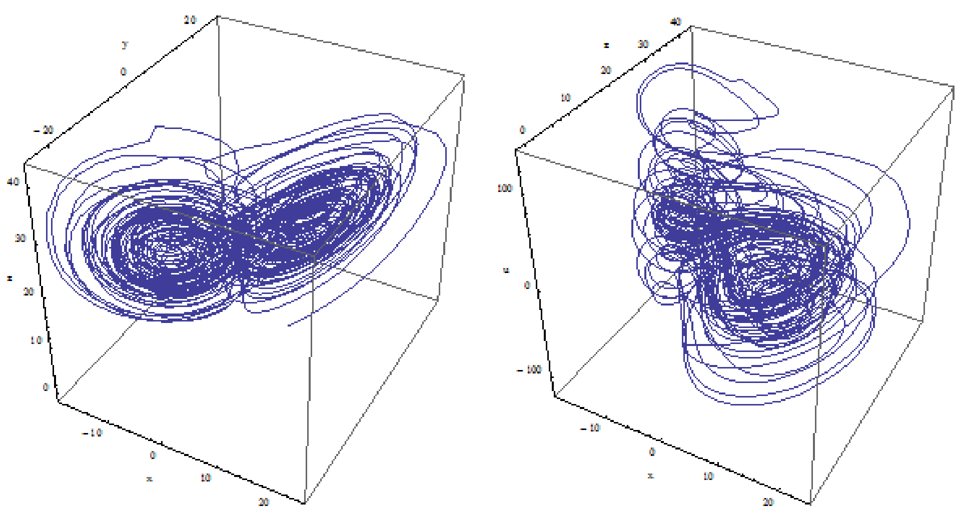
In this paper, a multi-step generalized differential transform method has been successfully applied to find the numerical solutions of the fractional-order Lü chaotic and hyperchaotic systems. This method has the advantage of giving an analytical form of the solution within each time interval which is not possible using purely numerical techniques like the fourth-order Runge-Kutta method (RK4). We conclude that the present method is a highly accurate method in solving a broad array of dynamical problems in fractional calculus due to its consistency used in a longer time frame.

The reliability of the method and the reduction in the size of computational domain give this method a wider applicability. Many of the results obtained in this paper can be extended to significantly more general classes of linear and nonlinear differential equations of fractional order.



(a) in the $x-y-z$ space (b) in the $x-z-u$ space

Figure 5: Phase plot of hyperchaotic attractor, with $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0.9$.



(a) in the $x-y-z$ space (b) in the $x-z-u$ space

Figure 6: Phase plot of hyperchaotic attractor, with $\alpha_1 = 0.80, \alpha_2 = 0.83, \alpha_3 = 0.81, \alpha_4 = 0.84$.

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NEW INEQUALITIES OF HERMITE-HADAMARD AND FEJÉR TYPE VIA PREINVEXITY

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Abstract. Several new weighted inequalities connected with Hermite-Hadamard and Fejér type inequalities are established for functions whose derivatives in absolute value are preinvex. The results presented in this paper provide extensions of those given in earlier works.

1. Introduction

Many inequalities have been established for convex functions but the most famous is the Hermite-Hadamard inequality, due to its rich geometrical significance and applications, which is stated as (see [23]):

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both the inequalities hold in reversed direction if f is concave.

In [7], Fejér gave a weighted generalization of (1.1) as follows:

$$f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx \leq \int_a^b f(x)w(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b w(x)dx, \tag{1.2}$$

where $f : [a, b] \to \mathbb{R}$ be a convex function and $f : [a, b] \to \mathbb{R}$ is nonnegative, integrable and symmetric about $x = \frac{a+b}{2}$.

For several results which generalize, improve and extend the inequalities (1.1) and (1.2) we refer the interested reader [5, 6, 8], [10]-[13], [23, 24], [26]-[31].

In recent years, lot of efforts have been made by many mathematicians to generalize the classical convexity. These studies include among others the work of Hanson in [9], Ben-Israel and Mond [4], Pini [21], M.A.Noor [18, 19], Yang and Li [33] and Weir [32]. Mond [5], Weir [32] and Noor [18, 19], have studied the basic properties of the preinvex functions and their role in optimization, variational inequalities and equilibrium problems. Hanson in [9], introduced invex functions as a significant generalization of convex functions. Ben-Israel and Mond [4], gave the concept of preinvex function which is special case of invexity. Pini [21], introduced the concept of prequasiinvex functions as a generalization of invex functions.

Let us recall some known results concerning invexity and preinvexity.

Date: Today.

2000 *Mathematics Subject Classification.* 26D15, 26D20, 26D07.

Key words and phrases. Hermite-Hadamard's inequality,Fejér's inequality, invex set, preinvex function, Hölder's integral inequality.

This paper is in final form and no version of it will be submitted for publication elsewhere.

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A set $K \subseteq \mathbb{R}^n$ is said to be invex if there exists a function $\eta : K \times K \to \mathbb{R}$ such that

$$x + t\eta(y, x) \in K, \ \forall x, y \in K, t \in [0, 1].$$

The invex set K is also called a η-connected set.

Definition 1. [32] *The function f on the invex set K is said to be preinvex with respect to η, if*

$$f(u + t\eta(v, u)) \le (1 - t) f(u) + tf(v), \forall u, v \in K, t \in [0, 1].$$

The function f is said to be preconcave if and only if $-f$ is preinvex.

It is to be noted that every convex function is preinvex with respect to the map $\eta(x, y) = x - y$ but the converse is not true see for instance [32].

In the recent paper, Noor [16] has obtained the following Hermite-Hadamard inequalities for the preinvex functions:

Theorem 1. [16] *Let $f : [a, a + \eta(b, a)] \to (0, \infty)$ be a preinvex function on the interval of the real numbers K° (the interior of K) and a, $b \in K\circ$ with $a < a + \eta(b, a)$. Then the following inequality holds:*

$$f\left(\frac{2a + \eta(b, a)}{2}\right) \le \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(x)\, dx \le \frac{f(a) + f(b)}{2}. \tag{1.3}$$

For several new results on inequalities for preinvex functions we refer the interested reader to [3, 15, 20, 25] and the references therein.

In the present paper we give new inequalities of Hermite-Hadamard and Fejér type for functions whose derivatives are preinvex. Our results generalize those results presented in a very recent papers of M. Z. Sarikaya [25, 28].

2. Main Results

The following Lemma is essential in establishing our main results in this section:

Lemma 1. *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and a, $b \in K$ with $a < a + \eta(b, a)$. Suppose $f : K \to \mathbb{R}$ is a differentiable mapping on K such that $f' \in L([a, a + \eta(b, a)])$. If $w : [a, a + \eta(b, a)] \to [0, \infty)$ is an integrable mapping, then the following equality holds:*

$$\begin{aligned} \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(x)\, w(x) dx - \frac{1}{\eta(b, a)} f\left(a + \frac{1}{2}\eta(b, a)\right) \int_a^{a+\eta(b,a)} w(x) dx \\ = \eta(b, a) \int_0^1 k(t) f'(a + t\eta(b, a))\, dt, \end{aligned} \tag{2.1}$$

where

$$k(t) = \begin{cases} \int_0^t w(a + s\eta(b, a))\, ds, & t \in \left[0, \frac{1}{2}\right) \\ -\int_t^1 w(a + s\eta(b, a))\, ds, & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

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Proof. We observe that

$$\begin{aligned} I &= \int_0^1 k(t) f'(a + t\eta(b,a))\,dt \\ &= \int_0^{\frac{1}{2}} \left(\int_0^t w(a + s\eta(b,a))\,ds \right) f'(a + t\eta(b,a))\,dt \\ &\quad + \int_{\frac{1}{2}}^1 \left(-\int_t^1 w(a + s\eta(b,a))\,ds \right) f'(a + t\eta(b,a))\,dt = I_1 + I_2. \end{aligned} \tag{2.2}$$

By integration by parts, we get

$$\begin{aligned} I_1 &= \left(\int_0^t w(a + s\eta(b,a))\,ds \right) \frac{f(a + t\eta(b,a))}{\eta(b,a)} \Bigg|_0^{\frac{1}{2}} \\ &\quad - \frac{1}{\eta(b,a)} \int_0^{\frac{1}{2}} w(a + t\eta(b,a)) f(a + t\eta(b,a))\,dt \\ &= \frac{f\left(a + \frac{1}{2}\eta(b,a)\right)}{\eta(b,a)} \int_0^{\frac{1}{2}} w(a + t\eta(b,a))\,dt \\ &\quad - \frac{1}{\eta(b,a)} \int_0^{\frac{1}{2}} w(a + t\eta(b,a)) f(a + t\eta(b,a))\,dt. \end{aligned} \tag{2.3}$$

Similarly, we also have

$$\begin{aligned} I_2 &= \left(-\int_t^1 w(a + s\eta(b,a))\,ds \right) \frac{f(a + t\eta(b,a))}{\eta(b,a)} \Bigg|_{\frac{1}{2}}^1 \\ &\quad - \frac{1}{\eta(b,a)} \int_{\frac{1}{2}}^1 w(a + t\eta(b,a)) f(a + t\eta(b,a))\,dt \\ &= \frac{f\left(a + \frac{1}{2}\eta(b,a)\right)}{\eta(b,a)} \int_{\frac{1}{2}}^1 w(a + t\eta(b,a))\,dt \\ &\quad - \frac{1}{\eta(b,a)} \int_{\frac{1}{2}}^1 w(a + t\eta(b,a)) f(a + t\eta(b,a))\,dt. \end{aligned} \tag{2.4}$$

From (2.3) and (2.4), we get

$$\begin{aligned} I &= \frac{f\left(a + \frac{1}{2}\eta(b,a)\right)}{\eta(b,a)} \int_0^1 w(a + t\eta(b,a))\,dt \\ &\quad - \frac{1}{\eta(b,a)} \int_0^1 w(a + t\eta(b,a)) f(a + t\eta(b,a))\,dt. \end{aligned}$$

Using the change of variable $x = a + t\eta(b,a)$ for $t \in [0,1]$ and multiplying both sides by $\eta(b,a)$, we get (2.1). This completes the proof of the lemma. □

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Remark 1. *If we take* $w(x) = 1$, $x \in [a, a + t\eta(b,a)]$ *in Lemma 1, then (2.1) reduces to*

$$\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx + f\left(a + \frac{1}{2}\eta(b,a)\right) = \eta(b,a) \int_0^1 k(t) f'(a + t\eta(b,a))\,dt, \tag{2.5}$$

where

$$k(t) = \begin{cases} t, & t \in \left[0, \frac{1}{2}\right) \\ t-1, & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Which is one of the results from [25].

Remark 2. *If* $\eta(b,a) = b - a$ *in Lemma 1, then (2.1) becomes Lemma 2.1 from* [28, page 379].

Now using Lemma 1, we prove our results:

Theorem 2. *Let* $K \subseteq \mathbb{R}$ *be an open invex subset with respect to* $\eta : K \times K \to \mathbb{R}$ *and* $a, b \in K$ *with* $a < a + \eta(b,a)$. *Suppose* $f : K \to \mathbb{R}$ *is a differentiable mapping on* K *such that* $f' \in L([a, a + \eta(b,a)])$ *and* $w : [a, a + \eta(b,a)] \to [0, \infty)$ *is an integrable mapping and symmetric to* $a + \frac{1}{2}\eta(b,a)$. *If* $\left|f'\right|$ *is preinvex on* K, *then we have the following inequality:*

$$\left|\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x) w(x)dx - \frac{1}{\eta(b,a)} f\left(a + \frac{1}{2}\eta(b,a)\right) \int_a^{a+\eta(b,a)} w(x)dx\right| \leq \left(\frac{1}{\eta(b,a)} \int_a^{a+\frac{1}{2}\eta(b,a)} [\eta(b,a) - 2(x-a)] w(x)dx\right) \left(\frac{\left|f'(a)\right| + \left|f'(b)\right|}{2}\right). \tag{2.6}$$

Proof. From Lemma 1 and the preinvexity of $\left|f'\right|$ on K, we have

$$\begin{aligned} &\left|\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x) w(x)dx - \frac{1}{\eta(b,a)} f\left(a + \frac{1}{2}\eta(b,a)\right) \int_a^{a+\eta(b,a)} w(x)dx\right| \\ &\leq \eta(b,a) \int_0^{\frac{1}{2}} \left(\int_0^t w(a + s\eta(b,a))\,ds\right) \left[(1-t)\left|f'(a)\right| + t\left|f'(b)\right|\right] dt \\ &\quad + \eta(b,a) \int_{\frac{1}{2}}^1 \left(\int_t^1 w(a + s\eta(b,a))\,ds\right) \left[(1-t)\left|f'(a)\right| + t\left|f'(b)\right|\right] dt \end{aligned} \tag{2.7}$$

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By the change of the order of integration, we have

$$\begin{aligned}(2.8)\quad &\int_0^{\frac{1}{2}}\int_0^t w(a+s\eta(b,a))\left[(1-t)\left|f'(a)\right|+t\left|f'(b)\right|\right]dsdt\\ &=\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}} w(a+s\eta(b,a))\left[(1-t)\left|f'(a)\right|+t\left|f'(b)\right|\right]dtds\\ &=\int_0^{\frac{1}{2}} w(a+s\eta(b,a))\left[\left(\frac{(1-s)^2}{2}-\frac{1}{8}\right)\left|f'(a)\right|+\left(\frac{1}{8}-\frac{s^2}{2}\right)\left|f'(b)\right|\right]ds.\end{aligned}$$

Using the change of variable $x=a+s\eta(b,a)$ for $s\in[0,1]$, we have from (2.8) that

$$\begin{aligned}(2.9)\quad &\int_0^{\frac{1}{2}}\int_0^t w(a+s\eta(b,a))\left[(1-t)\left|f'(a)\right|+t\left|f'(b)\right|\right]dsdt\\ &=\frac{1}{\eta(b,a)}\left|f'(a)\right|\int_a^{a+\frac{1}{2}\eta(b,a)}\left(\frac{1}{2}\left(1-\frac{x-a}{\eta(b,a)}\right)^2-\frac{1}{8}\right)w(x)\,dx\\ &\quad+\frac{1}{\eta(b,a)}\left|f'(b)\right|\int_a^{a+\frac{1}{2}\eta(b,a)}\left(\frac{1}{8}-\frac{1}{2}\left(\frac{x-a}{\eta(b,a)}\right)^2\right)w(x)\,dx.\end{aligned}$$

Similarly by change of order of integration and using the fact that w is symmetric to $a+\frac{1}{2}\eta(b,a)$, we obtain

$$\begin{aligned}(2.10)\quad &\int_{\frac{1}{2}}^{1}\int_t^1 w(a+s\eta(b,a))\left[(1-t)\left|f'(a)\right|+t\left|f'(b)\right|\right]dsdt\\ &=\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{s} w(a+(1-s)\eta(b,a))\left[(1-t)\left|f'(a)\right|+t\left|f'(b)\right|\right]dtds\\ &=\frac{1}{\eta(b,a)}\left|f'(a)\right|\int_{\frac{1}{2}}^{1}\left(\frac{1}{8}-\frac{1}{2}(1-s)^2\right)w(a+(1-s)\eta(b,a))\,ds\\ &\quad+\frac{1}{\eta(b,a)}\left|f'(b)\right|\int_{\frac{1}{2}}^{1}\left(\frac{s^2}{2}-\frac{1}{8}\right)w(a+(1-s)\eta(b,a))\,ds\end{aligned}$$

By the change of variable $x=a+(1-s)\eta(b,a)$, we get form (2.10) that

$$\begin{aligned}(2.11)\quad &\int_{\frac{1}{2}}^{1}\int_t^1 w(a+s\eta(b,a))\left[(1-t)\left|f'(a)\right|+t\left|f'(b)\right|\right]dsdt\\ &=\frac{1}{\eta(b,a)}\left|f'(a)\right|\int_a^{a+\frac{1}{2}\eta(b,a)}\left(\frac{1}{8}-\frac{1}{2}\left(\frac{x-a}{\eta(b,a)}\right)^2\right)w(x)\,dx\\ &\quad+\frac{1}{\eta(b,a)}\left|f'(b)\right|\int_a^{a+\frac{1}{2}\eta(b,a)}\left(\frac{1}{2}\left(1-\frac{x-a}{\eta(b,a)}\right)^2-\frac{1}{8}\right)w(x)\,dx.\end{aligned}$$

Substituting (2.9) and (2.11) in (2.7) and simplifying, we get the inequality (2.6). This completes the proof of the theorem. □

Corollary 1. *If we take $w(x)=1$, for $x\in[a,a+\eta(b,a)]$ in Theorem 2, we get*

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx-f\left(a+\frac{1}{2}\eta(b,a)\right)\right|\le\frac{\eta(b,a)}{8}\left[\left|f'(a)\right|+\left|f'(b)\right|\right]. \tag{2.12}$$

Which is Theorem 5 from [25].

Remark 3. *If $\left|f'\right|$ is convex on $[a,b]$, then $\eta(b,a)=b-a$. Hence from Theorem 2, and using the symmetricity of w about $\frac{a+b}{2}$, we get Theorem 2.3 from* [28, page 380].

Theorem 3. *Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:K\times K\to\mathbb{R}$ and $a,b\in K$ with $a<a+\eta(b,a)$. Suppose $f:K\to\mathbb{R}$ is a differentiable mapping on K such that $f'\in L([a,a+\eta(b,a)])$ and $w:[a,a+\eta(b,a)]\to[0,\infty)$ is an integrable mapping and symmetric to $a+\frac{1}{2}\eta(b,a)$. If $\left|f'\right|^q$, $q>1$, is preinvex on K, then we have the following inequality:*

$$\begin{aligned}&\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,w(x)dx-\frac{1}{\eta(b,a)}f\left(a+\frac{1}{2}\eta(b,a)\right)\int_a^{a+\eta(b,a)}w(x)dx\right|\\&\le\eta(b,a)\left(\frac{1}{(\eta(b,a))^2}\int_a^{a+\frac{1}{2}\eta(b,a)}\left[\frac{\eta(b,a)}{2}-(x-a)\right]w^p(x)\,dx\right)^{\frac{1}{p}}\\&\quad\times\left[\left(\frac{2\left|f'(a)\right|^q+\left|f'(b)\right|^q}{24}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+2\left|f'(b)\right|^q}{24}\right)^{\frac{1}{q}}\right],\end{aligned} \tag{2.13}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 1 and change of order of integration, we get

$$\begin{aligned}&\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,w(x)dx-\frac{1}{\eta(b,a)}f\left(a+\frac{1}{2}\eta(b,a)\right)\int_a^{a+\eta(b,a)}w(x)dx\right|\\&\le\eta(b,a)\int_0^{\frac{1}{2}}\left(\int_0^t w(a+s\eta(b,a))\,ds\right)\left|f'(a+t\eta(b,a))\right|dt\\&+\eta(b,a)\int_{\frac{1}{2}}^1\left(\int_t^1 w(a+s\eta(b,a))\,ds\right)\left|f'(a+t\eta(b,a))\right|dt\\&=\eta(b,a)\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}}w(a+s\eta(b,a))\left|f'(a+t\eta(b,a))\right|dtds\\&\quad+\eta(b,a)\int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^s w(a+s\eta(b,a))\left|f'(a+t\eta(b,a))\right|dtds.\end{aligned} \tag{2.14}$$

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By the Hölder's inequality, we have

$$\eta(b,a)\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}} w(a+s\eta(b,a))\left|f'(a+t\eta(b,a))\right| dtds \tag{2.15}$$
$$\leq \eta(b,a)\left(\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}} w^p(a+s\eta(b,a))\,dtds\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}}\left|f'(a+t\eta(b,a))\right|^q dtds\right)^{\frac{1}{q}}.$$

Since $\left|f'\right|^q$ for $q>1$, is preinvex on K, for every $a,b\in K$ and $t\in[0,1]$ we have

$$\left|f'(a+t\eta(b,a))\right|^q \leq (1-t)\left|f'(a)\right|^q + t\left|f'(b)\right|^q$$

hence by solving elementary integrals and using the substitution $x=a+s\eta(b,a)$, $s\in[0,1]$, we have from (2.15) that

$$\eta(b,a)\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}} w(a+s\eta(b,a))\left|f'(a+t\eta(b,a))\right| dtds \tag{2.16}$$
$$\leq \eta(b,a)\left(\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}} w^p(a+s\eta(b,a))\,dtds\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}}\left[(1-t)\left|f'(a)\right|^q+t\left|f'(b)\right|^q\right]dtds\right)^{\frac{1}{q}}$$
$$=\eta(b,a)\left(\frac{1}{(\eta(b,a))^2}\int_a^{a+\frac{1}{2}\eta(b,a)}\left[\frac{\eta(b,a)}{2}-(x-a)\right]w^p(x)\,dx\right)^{\frac{1}{p}}$$
$$\times\left(\frac{2\left|f'(a)\right|^q+\left|f'(b)\right|^q}{24}\right)^{\frac{1}{q}}.$$

Analogously, using the symmetricity of w about $a+\frac{1}{2}\eta(b,a)$, we also have

$$\eta(b,a)\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{s} w(a+s\eta(b,a))\left|f'(a+t\eta(b,a))\right| dtds \tag{2.17}$$
$$\leq \eta(b,a)\left(\frac{1}{(\eta(b,a))^2}\int_a^{a+\frac{1}{2}\eta(b,a)}\left[\frac{\eta(b,a)}{2}-(x-a)\right]w^p(x)\,dx\right)^{\frac{1}{p}}$$
$$\times\left(\frac{\left|f'(a)\right|^q+2\left|f'(b)\right|^q}{24}\right)^{\frac{1}{q}}$$

Using (2.16) and (2.17) in (2.14), we get the required inequality. This completes the proof of the theorem. □

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Corollary 2. *If the conditions of Theorem 3 are satisfied and if* $w(x) = 1$, $x \in [a, a+\eta(b,a)]$, *then the following inequality holds:*

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx + f\left(a+\frac{1}{2}\eta(b,a)\right)\right| \leq \eta(b,a)\left(\frac{1}{8}\right)^{\frac{1}{p}} \times\left[\left(\frac{2\left|f'(a)\right|^q+\left|f'(b)\right|^q}{24}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+2\left|f'(b)\right|^q}{24}\right)^{\frac{1}{q}}\right], \tag{2.18}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Corollary 3. [28, Theorem 2.5, page 381] *Suppose* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *is a differentiable mapping on* I°, $a, b \in I$ *with* $a < b$. *Let* $w : [a,b] \to [0,\infty)$ *be an integrable mapping and symmetric to* $\frac{a+b}{2}$ *such that* $f' \in L([a,b])$. *If* $\left|f'\right|^q$, $q > 1$, *is convex on* $[a,b]$, *then we have the following inequality:*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,w(x)dx - \frac{1}{b-a}f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx\right| \leq (b-a)\left(\frac{1}{(b-a)^2}\int_{\frac{a+b}{2}}^b\left(x-\frac{a+b}{2}\right)w^p(x)\,dx\right)^{\frac{1}{p}} \times\left[\left(\frac{2\left|f'(a)\right|^q+\left|f'(b)\right|^q}{24}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+2\left|f'(b)\right|^q}{24}\right)^{\frac{1}{q}}\right], \tag{2.19}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. It follows from Theorem 3 by taking $\eta(b,a) = b-a$ and using the symmetry of w about $\frac{a+b}{2}$. □

For our next results we need the following Lemma:

Lemma 2. *Let* $K \subseteq \mathbb{R}$ *be an open invex subset with respect to* $\eta : K \times K \to \mathbb{R}$ *and* $a, b \in K$ *with* $a < a+\eta(b,a)$. *Suppose* $f : K \to \mathbb{R}$ *is a differentiable mapping on* K *such that* $f' \in L([a, a+\eta(b,a)])$. *If* $w : [a, a+\eta(b,a)] \to [0,\infty)$ *is an integrable mapping, then the following inequality holds:*

$$-\frac{f(a)+f(a+\eta(b,a))}{2\eta(b,a)}\int_a^{a+\eta(b,a)} w(x)dx + \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,w(x)dx = \frac{\eta(b,a)}{2}\int_0^1 p(t)f'(a+t\eta(b,a))\,dt, \tag{2.20}$$

where

$$p(t) = \int_t^1 w(a+s\eta(b,a))\,ds - \int_0^t w(a+s\eta(b,a))\,ds,\ t\in[0,1].$$

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Proof. It suffices to note that

$$\begin{aligned} J &= \int_0^1 p(t) f'(a+t\eta(b,a))\,dt \\ &= -\int_0^1 \left(\int_0^t w(a+s\eta(b,a))\,ds\right) f'(a+t\eta(b,a))\,dt \\ &\quad + \int_0^1 \left(\int_t^1 w(a+s\eta(b,a))\,ds\right) f'(a+t\eta(b,a))\,dt = J_1 + J_2 \end{aligned} \tag{2.21}$$

By integration by parts, we get

$$\begin{aligned} J_1 &= -\left.\frac{\left(\int_0^t w(a+s\eta(b,a))\,ds\right) f(a+t\eta(b,a))}{\eta(b,a)}\right|_0^1 \\ &\quad + \frac{1}{\eta(b,a)}\int_0^1 w(a+t\eta(b,a)) f(a+t\eta(b,a))\,dt \\ &= -\frac{f(a+\eta(b,a))}{\eta(b,a)}\int_0^1 w(a+t\eta(b,a))\,dt \\ &\quad + \frac{1}{\eta(b,a)}\int_0^1 w(a+t\eta(b,a)) f(a+t\eta(b,a))\,dt. \end{aligned} \tag{2.22}$$

Similarly. we also have

$$J_2 = -\frac{f(a)}{\eta(b,a)}\int_0^1 w(a+t\eta(b,a))\,dt + \frac{1}{\eta(b,a)}\int_0^1 w(a+t\eta(b,a)) f(a+t\eta(b,a))\,dt. \tag{2.23}$$

Using (2.22) and (2.23) in (2.21), we obtain

$$\begin{aligned} J &= -\frac{f(a)+f(a+\eta(b,a))}{\eta(b,a)}\int_0^1 w(a+t\eta(b,a))\,dt \\ &\quad + \frac{2}{\eta(b,a)}\int_0^1 w(a+t\eta(b,a)) f(a+t\eta(b,a))\,dt \end{aligned} \tag{2.24}$$

By the change of variable $x = a + t\eta(b,a)$ for $t \in [0,1]$ and by multiplying both sides if (2.6) by $\frac{\eta(b,a)}{2}$, we get (2.20). This completes the proof of the lemma. □

Remark 4. *If we take $w(x) = 1$, $x \in [a, a+\eta(b,a)]$, then we get*

$$\begin{aligned} &-\frac{f(a)+f(a+\eta(b,a)}{2} + \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx \\ &\qquad = \frac{\eta(b,a)}{2}\int_0^1 (1-2t) f'(a+t\eta(b,a))\,dt, \end{aligned} \tag{2.25}$$

which is Lemma 2.1 from [3, Page 3].

Theorem 4. *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $a, b \in K$ with $a < a + \eta(b,a)$. Suppose $f : K \to \mathbb{R}$ is a differentiable mapping on K such that $f' \in L([a, a+\eta(b,a)])$ and $w : [a, a+\eta(b,a)] \to [0,\infty)$ is an*

integrable mapping and symmetric to $a+\frac{1}{2}\eta(b,a)$. *If* $\left|f'\right|^q$, $q>1$, *is preinvex on* K, *then we have the following inequality:*

$$\left|\frac{f(a)+f(a+\eta(b,a)}{2\eta(b,a)}\int_a^{a+\eta(b,a)}w(x)dx-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,w(x)dx\right| \tag{2.26}$$
$$\leq \frac{1}{2}\left(\int_0^1 g^p(t)dt\right)^{\frac{1}{p}}\left(\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}},$$

where

$$g(t)=\left|\int_{a+t\eta(b,a)}^{a+(1-t)\eta(b,a)}w(x)\,dx\right|, t\in[0,1] \text{ and } \frac{1}{p}+\frac{1}{q}=1.$$

Proof. From Lemma 2, we get

$$\left|\frac{f(a)+f(a+\eta(b,a)}{2\eta(b,a)}\int_a^{a+\eta(b,a)}w(x)dx-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,w(x)dx\right| \tag{2.27}$$
$$\leq \frac{\eta(b,a)}{2}\int_0^1\left|\int_t^1 w(a+s\eta(b,a))\,ds-\int_0^t w(a+s\eta(b,a))\,ds\right|\left|f'(a+t\eta(b,a))\right|dt.$$

Since w is symmetric to $a+\frac{1}{2}\eta(b,a)$, we can write

$$\begin{aligned}
&\int_t^1 w(a+s\eta(b,a))\,ds-\int_0^t w(a+s\eta(b,a))\,ds \\
&=\int_t^1 w(a+s\eta(b,a))\,ds-\int_0^t w(a+(1-s)\eta(b,a))\,ds \\
&=\frac{1}{\eta(b,a)}\int_{a+t\eta(b,a)}^{a+\eta(b,a)}w(x)\,dx+\frac{1}{\eta(b,a)}\int_{a+\eta(b,a)}^{a+(1-t)\eta(b,a)}w(x)\,dx \\
&=\begin{cases}\frac{1}{\eta(b,a)}\int_{a+t\eta(b,a)}^{a+(1-t)\eta(b,a)}w(x)\,dx, & t\in\left[0,\frac{1}{2}\right] \\ -\frac{1}{\eta(b,a)}\int_{a+(1-t)\eta(b,a)}^{a+t\eta(b,a)}w(x)\,dx, & t\in\left[\frac{1}{2},1\right].\end{cases}
\end{aligned} \tag{2.28}$$

Using (2.28) in (2.27) we obtain

$$\left|\frac{f(a)+f(a+\eta(b,a)}{2\eta(b,a)}\int_a^{a+\eta(b,a)}w(x)dx-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,w(x)dx\right| \tag{2.29}$$
$$\leq \frac{1}{2}\int_0^1 g(x)\left|f'(a+t\eta(b,a))\right|dt,$$

where

$$g(t)=\left|\int_{a+t\eta(b,a)}^{a+(1-t)\eta(b,a)}w(x)\,dx\right|, t\in[0,1].$$

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By Hölder's inequality, it follows from (2.29) that

$$\left|\frac{f(a)+f(a+\eta(b,a)}{2\eta(b,a)}\int_a^{a+\eta(b,a)} w(x)dx - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)w(x)dx\right| \le \frac{1}{2}\left(\int_0^1 g^p(t)dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}. \tag{2.30}$$

Since $\left|f'(a+t\eta(b,a))\right|^q$ is preinvex on K, for every $a, b\in K$ and $t\in[0,1]$, we have

$$\left|f'(a+t\eta(b,a))\right|^q \le (1-t)\left|f'(a)\right|^q + t\left|f'(b)\right|^q$$

and hence from (2.30), we get that

$$\begin{aligned}\left|\frac{f(a)+f(a+\eta(b,a)}{2\eta(b,a)}\int_a^{a+\eta(b,a)} w(x)dx - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)w(x)dx\right| &\le \frac{1}{2}\left(\int_0^1 g^p(t)dt\right)^{\frac{1}{p}}\left(\int_0^1 \left[(1-t)\left|f'(a)\right|^q + t\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}} \\ &= \frac{1}{2}\left(\int_0^1 g^p(t)dt\right)^{\frac{1}{p}}\left(\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}},\end{aligned} \tag{2.31}$$

which completes the proof of the theorem. □

Corollary 4. *If we take $\eta(b,a)=b-a$ in Theorem 4, then we have the inequality:*

$$\left|\frac{f(a)+f(b)}{2(b-a)}\int_a^b w(x)dx - \frac{1}{b-a}\int_a^b f(x)w(x)dx\right| \le \frac{1}{2}\left(\int_0^1 g^p(t)dt\right)^{\frac{1}{p}}\left(\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}, \tag{2.32}$$

where

$$g(t)=\left|\int_{tb+(1-t)a}^{ta+(1-t)b} w(x)\,dx\right|,\ t\in[0,1]\ \text{ and } \frac{1}{p}+\frac{1}{q}=1.$$

Which is Theorem 2.8 from [28, page 383].

Corollary 5. [3, Theorem 2.2, page 4] *Under the assumptions of Theorem 4, if we take $w(x)=1$, $x\in[a,a+\eta(b,a)]$. Then*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \le \frac{\eta(b,a)}{2(p+1)^{\frac{1}{p}}}\left(\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}, \tag{2.33}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

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Proof. It follows from the fact that

$$\int_0^1 g^p(t)dt = \int_0^1 \left(\left| \int_{a+t\eta(b,a)}^{a+(1-t)\eta(b,a)} dx \right|^p \right) dt$$
$$= (\eta(b,a))^p \int_0^1 |1-2t|^p \, dt = \frac{(\eta(b,a))^p}{p+1}.$$

□

Corollary 6. [5] *If the conditions of Theorem 4 are fulfilled and if* $w(x) = 1$, $x \in [a,b]$ *and* $\eta(b,a) = b-a$, *then we have the inequality:*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \leq \frac{b-a}{2(p+1)^{\frac{1}{p}}} \left(\frac{\left|f'(a)\right|^q + \left|f'(b)\right|^q}{2} \right)^{\frac{1}{q}}, \tag{2.34}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. It follows from Corollary 5. □

3. Applications to Special Means

In what follows we give certain generalizations of some notions for a positive valued function of a positive variable.

Definition 2. [31]*A function* $M : \mathbb{R}_+^2 \to \mathbb{R}_+$, *is called a Mean function if it has the following properties:*

1. *Homogeneity:* $M(ax, ay) = aM(x,y)$, *for all* $a > 0$,
2. *Symmetry :* $M(x,y) = M(y,x)$,
3. *Reflexivity :* $M(x,x) = x$,
4. *Monotonicity: If* $x \leq x'$ *and* $y \leq y'$, *then* $M(x,y) \leq M(x',y')$,
5. *Internality:* $\min\{x,y\} \leq M(x,y) \leq \max\{x,y\}$.

We consider some means for arbitrary positive real numbers α, β (see for instance [31]).

1. The arithmetic mean:
$$A := A(\alpha,\beta) = \frac{\alpha+\beta}{2}$$
2. The geometric mean:
$$G := G(\alpha,\beta) = \sqrt{\alpha\beta}$$
3. The harmonic mean:
$$H := H(\alpha,\beta) = \frac{2}{\frac{1}{\alpha}+\frac{1}{\beta}}$$
4. The power mean:
$$P_r := P_r(\alpha,\beta) = \left(\frac{\alpha^r+\beta^r}{2}\right)^{\frac{1}{r}}, \; r \geq 1$$

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(5) The identric mean:

$$I := I(\alpha,\beta) = \begin{cases} \frac{1}{e}\left(\frac{\beta^{\beta}}{\alpha^{\alpha}}\right), & \alpha \neq \beta \\ \alpha, & \alpha = \beta \end{cases}$$

(6) The logarithmic mean:

$$L := L(\alpha,\beta) = \frac{\alpha-\beta}{\ln|\alpha| - \ln|\beta|}, \ \ |\alpha| \neq |\beta|$$

(7) The generalized log-mean:

$$L_p := L_p(\alpha,\beta) = \left[\frac{\beta^{p+1}-\alpha^{p+1}}{(p+1)(\beta-\alpha)}\right], \ \alpha \neq \beta, \ p \in \mathbb{R}\backslash\{-1,0\}.$$

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$, with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality $H \leq G \leq L \leq I \leq A$.

Now, let a and b be positive real numbers such that $a < b$. Consider the function $M := M(a,b) : [a, a+\eta(b,a)] \times [a, a+\eta(b,a)] \to \mathbb{R}^+$, which is one of the above mentioned means, therefore one can obtain variant inequalities for these means as follows:

Setting $\eta(b,a) = M(b,a)$ in (2.12), (2.18) and (2.33), one can obtain the following interesting inequalities involving means:

$$\left|\frac{1}{M(b,a)}\int_a^{a+M(b,a)} f(x)\,dx - f\left(a+\frac{1}{2}M(b,a)\right)\right| \leq \frac{M(b,a)}{8}\left[\left|f'(a)\right| + \left|f'(b)\right|\right], \tag{3.1}$$

$$\left|\frac{1}{M(b,a)}\int_a^{a+M(b,a)} f(x)\,dx + f\left(a+\frac{1}{2}M(b,a)\right)\right| \leq M(b,a)\left(\frac{1}{8}\right)^{\frac{1}{p}} \times \left[\left(\frac{2\left|f'(a)\right|^q + \left|f'(b)\right|^q}{24}\right)^{\frac{1}{q}} + \left(\frac{\left|f'(a)\right|^q + 2\left|f'(b)\right|^q}{24}\right)^{\frac{1}{q}}\right] \tag{3.2}$$

and

$$\left|\frac{f(a)+f(a+M(b,a))}{2} - \frac{1}{M(b,a)}\int_a^{a+M(b,a)} f(x)\,dx\right| \leq \frac{M(b,a)}{2(p+1)^{\frac{1}{p}}}\left(\frac{\left|f'(a)\right|^q + \left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}. \tag{3.3}$$

Letting $M = A, G, H, P_r, I, L, L_p$ in (3.1), (3.2) and (3.3), we can get the required inequalities, and the details are left to the interested reader.

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APPROXIMATE QUADRATIC FORMS IN PARANORMED SPACES

JAE-HYEONG BAE AND WON-GIL PARK*

Abstract. In this paper, we investigate approximate quadratic forms related to the functional equations

$$f(x+y,\, z+w)+f(x-y,\, z-w)=2f(x,\, z)+2f(y,\, w)$$

and

$$f(x+y,\, z-w)+f(x-y,\, z+w)=2f(x,\, z)+2f(y,\, w)$$

in paranormed spaces.

1. Introduction

The concept of statistical convergence for sequences of real numbers was introduced by Fast [3] and Steinhaus [15] independently, and since then several generalizations and applications of this notion have been investigated by various authors (see [4, 7, 9, 10, 14]). This notion was defined in normed spaces by Kolk [8].

We recall some basic facts concerning Fréchet spaces (see [17]).

Definition. Let X be a vector space. A *paranorm* on X is a function $P : X \to \mathbb{R}$ such that for all $x, y \in X$

(i) $P(0) = 0$;

(ii) $P(-x) = P(x)$;

(iii) $P(x + y) \le P(x) + P(y)$ (triangle inequality);

(iv) If $\{t_n\}$ is a sequence of scalars with $t_n \to t$ and $\{x_n\} \subset X$ with $P(x_n - x) \to 0$, then $P(t_n x_n - tx) \to 0$ (continuity of scalar multiplication).

The pair (X, P) is called a *paranormed space* if P is a paranorm on X. Note that

$$P(nx) \le nP(x)$$

for all $n \in \mathbb{N}$ and all $x \in (X, P)$. The paranorm P on X is called *total* if, in addition, P satisfies (v) $P(x) = 0$ implies $x = 0$. A *Fréchet space* is a total and complete paranormed

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2000 *Mathematics Subject Classification.* Primary 39B52, 39B82.

Key words and phrases. Approximate quadratic form, Paranormed space.

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space. Note that each seminorm P on X is a paranorm, but the converse need not be true.

In 1940, S. M. Ulam proposed the general Ulam stability problem (see [16]):

Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot, \cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$ then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

In 1941, this problem was solved by D. H. Hyers [6] in the case of Banach space. Thereafter, we call that type the Hyers-Ulam stability. In 1978, Th. M. Rassias [13] extended the Hyers-Ulam stability by considering variables. It also has been generalized to the function case by P. Găvruta [5]. In recent, C. Park [11] obtained some stability results in paranormed spaces.

Let X and Y be vector spaces. A mapping $g : X \to Y$ is called a quadratic mapping [1] if g satisfies the quadratic functional equation $g(x + y) + g(x - y) = 2g(x) + 2g(y)$.

The authors [2, 12] considered the following functional equations:

$$f(x + y, z + w) + f(x - y, z - w) = 2f(x, z) + 2f(y, w) \tag{1}$$

and

$$f(x + y, z - w) + f(x - y, z + w) = 2f(x, z) + 2f(y, w). \tag{2}$$

It is easy to show that the functions $f(x, y) = ax^2 + bxy + cy^2$ and $f(x, y) = ax^2 + by^2$ satisfy the functional equations (1) and (2), respectively.

From now on, assume that (X, P) is a Fréchet space and $(Y, \|\cdot\|)$ is a Banach space. The authors [2, 12] solved the solutions of (1) and (2).

In this paper, we investigate approximate quadratic forms related to the equations (1) and (2) in paranormed spaces.

2. Approximate quadratic forms related to (1)

Theorem 2.1. *Let r, θ be positive real numbers with $r > 2$, and let $f : Y \times Y \to X$ be a mapping satisfying $f(0, 0) = 0$ such that*

$$P\big(f(x+y, z+w)+f(x-y, z-w)-2f(x, z)-2f(y, w)\big) \le \theta(\|x\|^r+\|y\|^r+\|z\|^r+\|w\|^r) \tag{3}$$

for all $x, y, z, w \in Y$. Then there exists a unique mapping $F : Y \times Y \to X$ satisfying (1) such that

$$P\big(f(x, y) - F(x, y)\big) \le \frac{2\theta}{2^r - 4}(\|x\|^r + \|y\|^r) \tag{4}$$

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for all $x, y \in Y$.

Proof. Letting $x = y$ and $z = w$ in (3), $P\big(f(2x, 2z) - 4f(x, z)\big) \le 2\theta(\|x\|^r + \|z\|^r)$ for all $x, z \in Y$. Substituting x and z by $\frac{x}{2^{j+1}}$ and $\frac{z}{2^{j+1}}$ in the above inequality, respectively, we have

$$P\Big(f\Big(\frac{x}{2^j}, \frac{z}{2^j}\Big) - 4f\Big(\frac{x}{2^{j+1}}, \frac{z}{2^{j+1}}\Big)\Big) \le \frac{2\theta}{2^r}\frac{1}{2^{rj}}(\|x\|^r + \|z\|^r)$$

for all $j \in \mathbb{N}$ and all $x, z \in Y$. For given integers $l, m (0 \le l < m)$,

$$\begin{aligned} P\Big(4^l f\Big(\frac{x}{2^l}, \frac{z}{2^l}\Big) - 4^m f\Big(\frac{x}{2^m}, \frac{z}{2^m}\Big)\Big) &\le \sum_{j=l}^{m-1} P\Big(4^j f\Big(\frac{x}{2^j}, \frac{z}{2^j}\Big) - 4^{j+1} f\Big(\frac{x}{2^{j+1}}, \frac{z}{2^{j+1}}\Big)\Big) \\ &\le \frac{2\theta}{2^r}\sum_{j=l}^{m-1}\frac{4^j}{2^{rj}}(\|x\|^r + \|z\|^r) \qquad (5) \end{aligned}$$

for all $x, z \in Y$. By (5), the sequence $\{4^j f(\frac{x}{2^j}, \frac{z}{2^j})\}$ is a Cauchy sequence in X for all $x, z \in Y$. Since X is complete, the sequence $\{4^j f(\frac{x}{2^j}, \frac{z}{2^j})\}$ converges for all $x, z \in Y$. Define $F : Y \times Y \to X$ by $F(x, z) := \lim_{j\to\infty} 4^j f\big(\frac{x}{2^j}, \frac{z}{2^j}\big)$ for all $x, z \in Y$. By (3), we see that

$$\begin{aligned} &P\big(F(x+y, z+w) + F(x-y, z-w) - 2F(x, z) - 2F(y, w)\big) \\ &= \lim_{j\to\infty} P\bigg(4^j\Big[f\Big(\frac{x+y}{2^j}, \frac{z+w}{2^j}\Big) + f\Big(\frac{x-y}{2^j}, \frac{z-w}{2^j}\Big)\Big] - 2\cdot 4^j\Big[f\Big(\frac{x}{2^j}, \frac{z}{2^j}\Big) - f\Big(\frac{y}{2^j}, \frac{w}{2^j}\Big)\Big]\bigg) \\ &\le \lim_{j\to\infty} 4^j P\bigg(\Big[f\Big(\frac{x+y}{2^j}, \frac{z+w}{2^j}\Big) + f\Big(\frac{x-y}{2^j}, \frac{z-w}{2^j}\Big)\Big] - 2\Big[f\Big(\frac{x}{2^j}, \frac{z}{2^j}\Big) - f\Big(\frac{y}{2^j}, \frac{w}{2^j}\Big)\Big]\bigg) \\ &\le \lim_{j\to\infty}\frac{4^j\theta}{2^{jr}}(\|x\|^r + \|y\|^r + \|z\|^r + \|w\|^r) = 0 \end{aligned}$$

for all $x, y, z, w \in Y$. Thus F satisfies (1). Setting $l = 0$ and taking $m \to \infty$ in (5), one can obtain the inequality (4).

Let $F' : Y \times Y \to X$ be another mapping satisfying (1) and (4). By [2], there exist four symmetric bi-additive mappings $S, T, S', T' : Y \times Y \to X$ and two bi-additive mappings $B, B' : Y \times Y \to X$ such that $F(x, y) = S(x, x) + B(x, y) + T(y, y)$ and $F'(x, y) =$

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$S'(x,x)+B'(x,y)+T'(y,y)$ for all $x,y\in Y$. Since $r>2$, we obtain that

$$\begin{aligned}
&P(F(x,y)-F'(x,y))\\
&=P\bigg(4^n\Big[S\Big(\frac{x}{2^n},\frac{x}{2^n}\Big)+B\Big(\frac{x}{2^n},\frac{y}{2^n}\Big)+T\Big(\frac{y}{2^n},\frac{y}{2^n}\Big)-S'\Big(\frac{x}{2^n},\frac{x}{2^n}\Big)-B'\Big(\frac{x}{2^n},\frac{y}{2^n}\Big)-T'\Big(\frac{y}{2^n},\frac{y}{2^n}\Big)\Big]\bigg)\\
&\le 4^nP\bigg(S\Big(\frac{x}{2^n},\frac{x}{2^n}\Big)+B\Big(\frac{x}{2^n},\frac{y}{2^n}\Big)+T\Big(\frac{y}{2^n},\frac{y}{2^n}\Big)-S'\Big(\frac{x}{2^n},\frac{x}{2^n}\Big)-B'\Big(\frac{x}{2^n},\frac{y}{2^n}\Big)-T'\Big(\frac{y}{2^n},\frac{y}{2^n}\Big)\bigg)\\
&\le 4^n\bigg[P\bigg(F\Big(\frac{x}{2^n},\frac{y}{2^n}\Big)-f\Big(\frac{x}{2^n},\frac{y}{2^n}\Big)\bigg)+P\bigg(f\Big(\frac{x}{2^n},\frac{y}{2^n}\Big)-G\Big(\frac{x}{2^n},\frac{y}{2^n}\Big)\bigg)\bigg]\\
&\le 4^n\frac{2\theta}{2^r-4}\frac{2}{2^{nr}}(\|x\|^r+\|y\|^r)\to 0 \text{ as } n\to\infty
\end{aligned}$$

for all $x,y\in Y$. Hence the mapping F is a unique mapping satisfying (1) and (4), as desired. □

Theorem 2.2. *Let r be a positive real number with $r<2$, and let $f: X\times X\to Y$ be a mapping satisfying $f(0,0)=0$ such that*

$$\big\|f(x+y,z+w)+f(x-y,z-w)-2f(x,z)-2f(y,w)\big\|\le P(x)^r+P(y)^r+P(z)^r+P(w)^r \tag{6}$$

for all x, y, z, $w\in X$. Then there exists a unique mapping $F: X\times X\to Y$ satisfying (1) *such that*

$$\big\|f(x,y)-F(x,y)\big\|\le\frac{2}{4-2^r}[P(x)^r+P(y)^r] \tag{7}$$

for all $x,y\in X$.

Proof. Letting $x=y$ and $z=w$ and dividing by 4 in (6), we have

$$\Big\|f(x,z)-\frac{1}{4}f(2x,2z)\Big\|\le\frac{1}{2}\big[P(x)^r+P(z)^r\big]$$

for all $x,z\in X$. Substituting x and z by 2^jx and 2^jz, respectively, we obtain that

$$\Big\|f\big(2^jx,2^jz\big)-\frac{1}{4}f\big(2^{j+1}x,2^{j+1}z\big)\Big\|\le 2^{rj-1}\big[P(x)^r+P(z)^r\big]$$

for all $j \in \mathbb{N}$ and all $x, z \in X$. For given integers $l, m (0 \le l < m)$, we obtain

$$\begin{aligned} \left\| \frac{1}{4^l} f\big(2^l x, 2^l z\big) - \frac{1}{4^m} f\big(2^m x, 2^m z\big) \right\| &\le \sum_{j=l}^{m-1} \left\| \frac{1}{4^j} f\big(2^j x, 2^j z\big) - \frac{1}{4^{j+1}} f\big(2^{j+1} x, 2^{j+1} z\big) \right\| \\ &\le \sum_{j=l}^{m-1} \frac{2^{rj-1}}{4^j} \big[P(x)^r + P(z)^r\big] \end{aligned} \tag{8}$$

for all $x, z \in X$. By (8), the sequence $\left\{\frac{1}{4^j} f\big(2^j x, 2^j z\big)\right\}$ is a Cauchy sequence in Y for all $x, z \in X$. Since Y is complete, the sequence $\left\{\frac{1}{4^j} f\big(2^j x, 2^j z\big)\right\}$ converges for all $x, z \in X$. Define $F : X \times X \to Y$ by $F(x, z) := \lim_{j\to\infty} \frac{1}{4^j} f\big(2^j x, 2^j z\big)$ for all $x, z \in X$. By (6), we see that

$$\begin{aligned} &\big\|F(x+y, z+w) + F(x-y, z-w) - 2F(x,\, z) - F(y,\, w)\big\| \\ &= \lim_{j\to\infty} \left\| \frac{1}{4^j} \big[f\big(2^j(x+y), 2^j(z+w)\big) + f\big(2^j(x-y), 2^j(z-w)\big)\big] - \frac{2}{4^j} \big[f\big(2^j x, 2^j z\big) - f\big(2^j y, 2^j w\big)\big] \right\| \\ &\le \lim_{j\to\infty} \frac{1}{4^j} \left\| \big[f\big(2^j(x+y), 2^j(z+w)\big) + f\big(2^j(x-y), 2^j(z-w)\big)\big] - 2\big[f\big(2^j x, 2^j z\big) - f\big(2^j y, 2^j w\big)\big] \right\| \\ &\le \lim_{j\to\infty} \left(\frac{2^r}{4}\right)^j \big[P(x)^r + P(y)^r + P(z)^r + P(w)^r\big] = 0 \end{aligned}$$

for all $x, y, z, w \in X$. Thus F satisfies (1). Setting $l = 0$ and taking $m \to \infty$ in (8), one can obtain the inequality (7).

Let $F' : X \times X \to Y$ be another mapping satisfying (1) and (7). By [2], there exist four symmetric bi-additive mappings $S, T, S', T' : X \times X \to Y$ and two bi-additive mappings $B, B' : X \times X \to Y$ such that $F(x, y) = S(x, x) + B(x, y) + T(y, y)$ and $F'(x, y) = S'(x, x) + B'(x, y) + T'(y, y)$ for all $x, y \in X$. Since $0 < r < 2$, we obtain that

$$\begin{aligned} &\|F(x, y) - F'(x, y)\| \\ &= \left\| \frac{1}{4^n} [S(2^n x, 2^n x) + B(2^n x, 2^n y) + T(2^n y, 2^n y) - S'(2^n x, 2^n x) - B'(2^n x, 2^n y) - T'(2^n y, 2^n y)] \right\| \\ &= \frac{1}{4^n} \|S(2^n x, 2^n x) + B(2^n x, 2^n y) + T(2^n y, 2^n y) - S'(2^n x, 2^n x) - B'(2^n x, 2^n y) - T'(2^n y, 2^n y)\| \\ &\le \frac{1}{4^n} \left[\left\| F\Big(\frac{x}{2^n}, \frac{y}{2^n}\Big) - f\Big(\frac{x}{2^n}, \frac{y}{2^n}\Big) \right\| + \left\| f\Big(\frac{x}{2^n}, \frac{y}{2^n}\Big) - G\Big(\frac{x}{2^n}, \frac{y}{2^n}\Big) \right\| \right] \\ &\le \frac{1}{4^n} \frac{2}{4 - 2^r} \frac{2}{2^{nr}} \big[P(x)^r + P(y)^r\big] \to 0 \text{ as } n \to \infty \end{aligned}$$

for all $x, y \in X$. Hence the mapping F is a unique mapping satisfying (1) and (7), as desired. □

3. Approximate quadratic forms related to (2)

Theorem 3.1. *Let r, θ be positive real numbers with $r > 2$, and let $f : Y \times Y \to X$ be a mapping satisfying $f(0,0) = 0$ such that*

$$(9)\quad P\big(f(x+y, z-w)+f(x-y, z+w)-2f(x,\, z)-2f(y,\, w)\big) \le \theta(\|x\|^r+\|y\|^r+\|z\|^r+\|w\|^r)$$

for all $x,\, y,\, z,\, w \in Y$. Then there exists a unique mapping $F : Y \times Y \to X$ satisfying (2) *such that*

$$P\Big(4f\Big(\frac{x}{2}, \frac{y}{2}\Big) - F(x,y)\Big) \le \frac{24\theta}{2^r(2^r-4)}(\|x\|^r + \|y\|^r) \tag{10}$$

for all $x, y \in Y$.

Proof. Letting $y = x$ and $w = -z$ in (9), we get

$$P\big(f(2x, 2z) - 2f(x, z) - 2f(x, -z)\big) \le 2\theta(\|x\|^r + \|z\|^r) \tag{11}$$

for all $x, z \in Y$. Putting $x = 0$ in the above inequality, we have

$$P\big(f(0, 2z) - 2f(0, z) - 2f(0, -z)\big) \le 2\theta\|z\|^r$$

for all $z \in Y$. Replacing z by $-z$ in the above inequality, we obtain

$$P\big(f(0, -2z) - 2f(0, z) - 2f(0, -z)\big) \le 2\theta\|z\|^r$$

for all $z \in Y$. By the above two inequalities, we see that

$$P\big(f(0, 2z) - f(0, -2z)\big) \le 4\theta\|z\|^r \tag{12}$$

for all $z \in Y$. Taking $y = x$ and $w = z$ in (9), we get

$$P\big(f(2x, 0) + f(0, 2z) - 4f(x, z)\big) \le 2\theta(\|x\|^r + \|z\|^r)$$

for all $x, z \in Y$. Replacing z by $-z$ in the above inequality, we have

$$P\big(f(2x, 0) + f(0, -2z) - 4f(x, -z)\big) \le 2\theta(\|x\|^r + \|z\|^r)$$

for all $x, z \in Y$. By the above two inequalities, we obtain

$$P\big(4f(x, z) - 4f(x, -z) - f(0, 2z) + f(0, -2z)\big) \le 4\theta(\|x\|^r + \|z\|^r)$$

for all $x, z \in Y$. By (11) and the above inequality, we see that

$$P\big(8f(x, z) - 2f(2x, 2z) - f(0, 2z) + f(0, -2z)\big) \le 8\theta(\|x\|^r + \|z\|^r)$$

for all $x, z \in Y$. By (12) and the above inequality, we get

$$P\big(8f(x,z) - 2f(2x,2z)\big) \leq 12\theta(\|x\|^r + \|z\|^r)$$

for all $x, z \in Y$. Replacing x and z by $\frac{x}{2}$ and $\frac{z}{2}$ in the above inequality, respectively, we have

$$P\Big(8f\Big(\frac{x}{2}, \frac{z}{2}\Big) - 2f(x,z)\Big) \leq \frac{12\theta}{2^r}(\|x\|^r + \|z\|^r)$$

for all $x, z \in Y$. Substituting x and z by $\frac{x}{2^j}$ and $\frac{z}{2^j}$, respectively, we obtain

$$P\Big(2f\Big(\frac{x}{2^j}, \frac{z}{2^j}\Big) - 8f\Big(\frac{x}{2^{j+1}}, \frac{z}{2^{j+1}}\Big)\Big) \leq \frac{12\theta}{2^r}\frac{1}{2^{rj}}(\|x\|^r + \|z\|^r)$$

for all $j \in \mathbb{N}$ and all $x, z \in Y$. Thus we see that

$$\begin{aligned}
P\Big(4^j f\Big(\frac{x}{2^j}, \frac{z}{2^j}\Big) - 4^{j+1} f\Big(\frac{x}{2^{j+1}}, \frac{z}{2^{j+1}}\Big)\Big) &= P\Big(\frac{4^j}{2} \cdot 2f\Big(\frac{x}{2^j}, \frac{z}{2^j}\Big) - \frac{4^j}{2} \cdot 8f\Big(\frac{x}{2^{j+1}}, \frac{z}{2^{j+1}}\Big)\Big) \\
&\leq \frac{4^j}{2} P\Big(2f\Big(\frac{x}{2^j}, \frac{z}{2^j}\Big) - 8f\Big(\frac{x}{2^{j+1}}, \frac{z}{2^{j+1}}\Big)\Big) \leq \frac{4^j}{2} \cdot \frac{12\theta}{2^r}\frac{1}{2^{rj}}(\|x\|^r + \|z\|^r) \\
&= \frac{6\theta}{2^r}\Big(\frac{4}{2^r}\Big)^j(\|x\|^r + \|z\|^r)
\end{aligned}$$

for all $j \in \mathbb{N}$ and all $x, z \in Y$. For given integers l, m $(1 \leq l < m)$, we have

$$\begin{aligned}
P\Big(4^l f\Big(\frac{x}{2^l}, \frac{z}{2^l}\Big) - 4^m f\Big(\frac{x}{2^m}, \frac{z}{2^m}\Big)\Big) &\leq \sum_{j=l}^{m-1} P\Big(4^j f\Big(\frac{x}{2^j}, \frac{z}{2^j}\Big) - 4^{j+1} f\Big(\frac{x}{2^{j+1}}, \frac{z}{2^{j+1}}\Big)\Big) \\
&\leq \frac{6\theta}{2^r}\sum_{j=l}^{m-1}\Big(\frac{4}{2^r}\Big)^j(\|x\|^r + \|z\|^r) \qquad (13)
\end{aligned}$$

for all $x, z \in Y$. By (13), the sequence $\{4^j f(\frac{x}{2^j}, \frac{z}{2^j})\}$ is a Cauchy sequence in X for all $x, z \in Y$. Since X is complete, the sequence $\{4^j f(\frac{x}{2^j}, \frac{z}{2^j})\}$ converges for all $x, z \in Y$. Define $F : Y \times Y \to X$ by $F(x,z) := \lim_{j\to\infty} 4^j f\big(\frac{x}{2^j}, \frac{z}{2^j}\big)$ for all $x, z \in Y$. By (9), we see that

$$\begin{aligned}
&P\big(F(x+y, z-w) + F(x-y, z+w) - 2F(x,z) - 2F(y,w)\big) \\
&= \lim_{j\to\infty} P\bigg(4^j\Big[f\Big(\frac{x+y}{2^j}, \frac{z-w}{2^j}\Big) + f\Big(\frac{x-y}{2^j}, \frac{z+w}{2^j}\Big)\Big] - 2\cdot 4^j\Big[f\Big(\frac{x}{2^j}, \frac{z}{2^j}\Big) - f\Big(\frac{y}{2^j}, \frac{w}{2^j}\Big)\Big]\bigg) \\
&\leq \lim_{j\to\infty} 4^j P\bigg(\Big[f\Big(\frac{x+y}{2^j}, \frac{z-w}{2^j}\Big) + f\Big(\frac{x-y}{2^j}, \frac{z+w}{2^j}\Big)\Big] - 2\Big[f\Big(\frac{x}{2^j}, \frac{z}{2^j}\Big) - f\Big(\frac{y}{2^j}, \frac{w}{2^j}\Big)\Big]\bigg) \\
&\leq \lim_{j\to\infty} \frac{4^j\theta}{2^{jr}}(\|x\|^r + \|y\|^r + \|z\|^r + \|w\|^r) = 0
\end{aligned}$$

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for all $x, y, z, w \in Y$. Thus F is a mapping satisfying (2). Setting $l = 1$ and taking $m \to \infty$ in (13), one can obtain the inequality (10).

Let $F' : Y \times Y \to X$ be another mapping satisfying (2) and (10). By [12], there exist four symmetric bi-additive mappings $S, T, S', T' : Y \times Y \to X$ such that $F(x, y) = S(x, x) + T(y, y)$ and $F'(x, y) = S'(x, x) + T'(y, y)$ for all $x, y \in Y$. Since $r > 2$, we obtain that

$$\begin{aligned}
&P(F(x, y) - F'(x, y)) \\
&\quad = P\Big(4^n\Big[S\Big(\frac{x}{2^n}, \frac{x}{2^n}\Big) + T\Big(\frac{y}{2^n}, \frac{y}{2^n}\Big) - S'\Big(\frac{x}{2^n}, \frac{x}{2^n}\Big) - T'\Big(\frac{y}{2^n}, \frac{y}{2^n}\Big)\Big]\Big) \\
&\quad \le 4^n P\Big(S\Big(\frac{x}{2^n}, \frac{x}{2^n}\Big) + T\Big(\frac{y}{2^n}, \frac{y}{2^n}\Big) - S'\Big(\frac{x}{2^n}, \frac{x}{2^n}\Big) - T'\Big(\frac{y}{2^n}, \frac{y}{2^n}\Big)\Big) \\
&\quad \le 4^n\Big[P\Big(F\Big(\frac{x}{2^n}, \frac{y}{2^n}\Big) - f\Big(\frac{x}{2^n}, \frac{y}{2^n}\Big)\Big) + P\Big(f\Big(\frac{x}{2^n}, \frac{y}{2^n}\Big) - G\Big(\frac{x}{2^n}, \frac{y}{2^n}\Big)\Big)\Big] \\
&\quad \le 4^n \frac{2\theta}{2^r - 4}\frac{2}{2^{nr}}(\|x\|^r + \|y\|^r) \to 0 \text{ as } n \to \infty
\end{aligned}$$

for all $x, y \in Y$. Hence the mapping F is a unique mapping satisfying (2) and (10), as desired. □

Theorem 3.2. *Let r be a positive real number with $r < 2$, and let $f : X \times X \to Y$ be a mapping satisfying $f(0, 0) = 0$ such that*

(14)
$$\big\|f(x+y, z-w) + f(x-y, z+w) - 2f(x,\, z) - 2f(y,\, w)\big\| \le P(x)^r + P(y)^r + P(z)^r + P(w)^r$$

for all x, y, z, $w \in X$. Then there exists a unique mapping $F : X \times X \to Y$ such that

$$\big\|f(x, y) - F(x, y)\big\| \le \frac{6}{4 - 2^r}[P(x)^r + P(y)^r] \tag{15}$$

for all $x, y \in X$.

Proof. Letting $y = x$ and $w = -z$ in (14), we get

$$\Big\|\frac{1}{2}f(2x, 2z) - f(x, z) - f(x, -z)\Big\| \le P(x)^r + P(z)^r \tag{16}$$

for all $x, z \in X$. Putting $x = 0$ in the above inequality, we have

$$\Big\|\frac{1}{2}f(0, 2z) - f(0, z) - f(0, -z)\Big\| \le P(z)^r$$

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for all $z \in X$. Replacing z by $-z$ in the above inequality, we obtain

$$\left\|\frac{1}{2}f(0,-2z) - f(0,z) - f(0,-z)\right\| \leq P(z)^r$$

for all $z \in X$. By the above two inequalities, we see that

$$\left\|\frac{1}{4}f(0,2z) - \frac{1}{4}f(0,-2z)\right\| \leq P(z)^r \tag{17}$$

for all $z \in X$. Taking $y = x$ and $w = z$ in (9), we get

$$\left\|\frac{1}{4}f(2x,0) + \frac{1}{4}f(0,2z) - f(x,z)\right\| \leq \frac{1}{2}[P(x)^r + P(z)^r]$$

for all $x, z \in X$. Replacing z by $-z$ in the above inequality, we have

$$\left\|\frac{1}{4}f(2x,0) + \frac{1}{4}f(0,-2z) - f(x,-z)\right\| \leq \frac{1}{2}[P(x)^r + P(z)^r]$$

for all $x, z \in X$. By the above two inequalities, we obtain

$$\left\|f(x,z) - f(x,-z) - \frac{1}{4}f(0,2z) + \frac{1}{4}f(0,-2z)\right\| \leq P(x)^r + P(z)^r$$

for all $x, z \in X$. By (16) and the above inequality, we see that

$$\left\|f(x,z) - \frac{1}{4}f(2x,2z) - \frac{1}{8}f(0,2z) + \frac{1}{8}f(0,-2z)\right\| \leq P(x)^r + P(z)^r$$

for all $x, z \in X$. By (17) and the above inequality, we get

$$\left\|f(x,z) - \frac{1}{4}f(2x,2z)\right\| \leq \frac{3}{2}[P(x)^r + P(z)^r]$$

for all $x, z \in X$. Substituting x and z by $2^j x$ and $2^j z$, respectively, we have

$$\left\|f(2^j x, 2^j z) - \frac{1}{4}f(2^{j+1}x, 2^{j+1}z)\right\| \leq \frac{3}{2} \cdot 2^{jr}[P(x)^r + P(z)^r]$$

for all nonnegative integers j and all $x, z \in X$. Thus we obtain

$$\left\|\frac{1}{4^j}f(2^j x, 2^j z) - \frac{1}{4^{j+1}}f(2^{j+1}x, 2^{j+1}z)\right\| \leq \frac{3}{2} \cdot 2^{(r-2)j}[P(x)^r + P(z)^r]$$

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for all nonnegative integers j and all $x, z \in X$. For given integers l, m $(0 \le l < m)$, we see that

$$\begin{aligned}\Big\|\frac{1}{4^l}f(2^l x, 2^l z) - \frac{1}{4^m}f(2^m x, 2^m z)\Big\| &\le \sum_{j=l}^{m-1}\Big\|\frac{1}{4^j}f(2^j x, 2^j z) - \frac{1}{4^{j+1}}f(2^{j+1}x, 2^{j+1}z)\Big\| \\ &\le \frac{3}{2}\sum_{j=l}^{m-1} 2^{(r-2)j}[P(x)^r + P(z)^r] \qquad (18)\end{aligned}$$

for all $x, z \in X$. By (18), the sequence $\{\frac{1}{4^j}f(2^j x, 2^j z)\}$ is a Cauchy sequence in Y for all $x, z \in X$. Since Y is complete, the sequence $\{\frac{1}{4^j}f(2^j x, 2^j z)\}$ converges for all $x, z \in X$. Define $F : X \times X \to Y$ by $F(x, z) := \lim_{j\to\infty} \frac{1}{4^j}f(2^j x, 2^j z)$ for all $x, z \in X$. By (14), we see that

$$\begin{aligned}&\big\|F(x+y, z-w) + F(x-y, z+w) - 2F(x, z) - 2F(y, w)\big\| \\ &= \lim_{j\to\infty}\Big\|\frac{1}{4^j}[f(2^j(x+y), 2^j(z-w)) + f(2^j(x-y), 2^j(z+w))] - \frac{2}{4^j}[f(2^j x, 2^j z) - f(2^j y, 2^j w)]\Big\| \\ &\le \lim_{j\to\infty}\frac{1}{4^j}\|f(2^j(x+y), 2^j(z-w)) + f(2^j(x-y), 2^j(z+w)) - 2[f(2^j x, 2^j z) - f(2^j y, 2^j w)]\| \\ &\le \lim_{j\to\infty} 2^{(r-2)j}[P(x)^r + P(y)^r + P(z)^r + P(w)^r] = 0\end{aligned}$$

for all $x, y, z, w \in X$. Thus F is a mapping satisfying (2). Setting $l = 0$ and taking $m \to \infty$ in (18), one can obtain the inequality (15).

Let $F' : X \times X \to Y$ be another mapping satisfying (2) and (15). By [12], there exist four symmetric bi-additive mappings $S, T, S', T' : X \times X \to Y$ such that $F(x, y) = S(x, x) + T(y, y)$ and $F'(x, y) = S'(x, x) + T'(y, y)$ for all $x, y \in X$. Since $0 < r < 2$, we obtain that

$$\begin{aligned}\|F(x, y) - F'(x, y)\| &= \Big\|\frac{1}{4^n}[S(2^n x, 2^n x) + T(2^n y, 2^n y) - S'(2^n x, 2^n x) - T'(2^n y, 2^n y)]\Big\| \\ &\le \frac{1}{4^n}\big\|S(2^n x, 2^n x) + T(2^n y, 2^n y) - S'(2^n x, 2^n x) - T'(2^n y, 2^n y)\big\| \\ &\le \frac{1}{4^n}\big[\big\|F(2^n x, 2^n y) - f(2^n x, 2^n y)\big\| + \big\|f(2^n x, 2^n y) - G(2^n x, 2^n y)\big\|\big] \\ &\le \frac{12}{4 - 2^r}\cdot\Big(\frac{1}{2}\Big)^{n(2-r)}[P(x)^r + P(y)^r] \to 0 \text{ as } n \to \infty\end{aligned}$$

for all $x, y \in X$. Hence the mapping F is a unique mapping satisfying (2) and (15), as desired. □

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Acknowledgment

This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(grant number 2013013682)

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Common Fixed Point Theorems for the R-weakly Commuting Mappings in S-metric Spaces

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Abstract. In this paper, we prove common fixed point theorems for two mappings under the condition of R-weakly commuting in complete S-metric spaces. A lot of fixed point theorems on ordinary metric spaces are special cases of our main result.

Keywords: S-metric space, contractive mapping, R-weakly commuting self-mappings, common fixed points.

2010 AMS Subject Classification: Premary 54H25, Secondary 47H10.

1 Introduction and preliminaries

Recently, Mustafa and Sims [12] introduced a new structure of generalized metric spaces which are called G-metric spaces as generalization of metric space (X, d), to develop and introduce a new fixed point theory for various mappings in this new structure. Some authors [3, 13, 28] have proved some fixed point theorems in these spaces. Fixed point problems of contractive mappings in partically ordered metric spaces have been studied in a number of works (see [2, 4, 7, 10, 11, 23, 24]). Metric spaces are very important in mathematics and applied sciences. So, some authors have tried to give generalizations of metric spaces in several ways. For example, Gähler [6] and Dhage [5] introduced the concepts of 2-matric spaces and D-metric spaces, respectively, but some authors pointed out that these attempts are not valid (see [9], [18-22]).

Recently, Sedghi et al. introduced D^*-metric which is a probable modification of the definition of D-metric introduced by Dhage [5] and proved some basic properties in D^*-metric spaces(see [25-27]).

[0]This work was supported by the Basic Science Research Program through the National Research Foundation Grant funded by Ministry of Education of the republic of Korea(2013R1A1A2054617).

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In this paper, we introduce the concept of S-metric spaces and give some properties of them. And we prove common fixed point theorem for a self-mapping on complete S-metric spaces.

In [30], Sedghi et al. have introduced the notion of an S-metric space as follows.

Definition 1.1. [30] Let X be a nonempty set. An *S-metric* on X is a function $S : X^3 \to [0, \infty)$ that satisfies the following conditions:

(1) $S(x, y, z) = 0$ if and only if $x = y = z$, for all $x, y, z \in X$.

(2) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$, for all $x, y, z, a \in X$.

The pair (X, S) is said to be an *S-metric space*.

Example 1.2. [30] Let R be the real line. Then

$$S(x, y, z) = |x - z| + |y - z|$$

for all $x, y, z \in R$ is an S-metric on R. This S-metric on R is called the *usual S-metric* on R.

From the definition, we know that the S-metric space is a generalization of a G-metric space [17] and a D^*-metric space [26]. That is, every G-metric is a D^*-metric, and every D^*-metric is an S-metric, but each converse is not true.

Example 1.3. [30] For $X = R$, let we define

$$d(x, y, z) = |x + y - 2x| + |x + z - 2y| + |y + z - 2x|.$$

Then d is a D^*-metric, but it is not a G-metric.

Example 1.4. [30] For $X = R^n$, let we define

$$S(x, y, z) = ||y + z - 2x|| + ||y - z||.$$

Then S is an S-metric, but it is not a D^*-metric.

For the fixed point problem in generalized metric spaces, many results have been proved (see [1, 8, 14-16, 29, 31]). In [30], the authors have proved some properties of S-metric spaces. Also, they have been proved some fixed point theorems for a self-map in an S-metric space.

Definition 1.5. [30] Let (X, S) be an S-metric space. For $r > 0$ and $x \in X$, we define the *open ball* $B_S(x, r)$ and the *closed ball* $B_S[x, r]$ with center x and radius r as follows.

$$B_S(x, r) = \{y \in X : S(y, y, x) < r\},$$
$$B_S[x, r] = \{y \in X : S(y, y, x) \leq r\}.$$

The *topology induced by the S-metric* is the topology generated by the base of all open balls in X.

Definition 1.6. [30] Let (X, S) be an S-metric space.

(1) A sequence $\{x_n\} \subset X$ is *S-convergent* to $x \in X$ if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. That is, for all $\varepsilon > 0$, there exists $n_0 \in N$ such that for all $n \geq n_0$, $S(x_n, x_n, x) < \varepsilon$. We write $x_n \to x$ for brevity.

(2) A sequence $\{x_n\} \subset X$ is a *Cauchy sequence* if $S(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$. That is, for all $\varepsilon > 0$, there exists $n_0 \in N$ such that for all $n, m \geq n_0$, $S(x_n, x_n, x_m) < \varepsilon$.

(3) The S-metric space (X, S) is said to be *complete* if every Cauchy sequence in X is S-convergent.

Lemma 1.7. [30] Let (X, S) be an S-metric space, Then, we have

$$S(x, x, y) = S(y, y, x)$$

for all $x, y \in X$.

Lemma 1.8. [30] Let (X, S) be an S-metric space. If $x_n \to x$ and $y_n \to y$ then $S(x_n, x_n, y_n) \to S(x, x, y)$.

2 Common fixed point results

In this section we give some fixed point results on S-metric spaces.

Definition 2.1. Let (X, S_1) and (Y, S_2) be two S-metric spaces and $T : X \to Y$ be a map. T is called sequentially convergent if $\{x_n\}$ is S_1-convergent in X provided $\{Tx_n\}$ is S_2-convergent in Y.

Definition 2.2. Let (X, S) be an S-metric space and let f and g be maps from X into itself. The maps f and g are said to be weakly commuting if

$$S(fgx, fgx, gfx) \leq S(fx, fx, gx)$$

for each $x \in X$.

Definition 2.3. Let (X, S) be an S-metric space and let f and g be maps from X into itself. The maps f and g are said to be R-weakly commuting if there exists a positive real number R such that

$$S(fgx, fgx, gfx) \leq RS(fx, fx, gx)$$

for each $x \in X$.

Weak commutativity implies R-weak commutativity in an $S-$metric space. However, R-weak commutativity implies weak commutativity only when $R \geq 1$.

Example 2.4. Let $X = R^2$. Let S be the $S-$metric on X^3 defined as follows: $S(x, y, z) = ||x - y|| + ||y - z||$. Then (X, S) is an $S-$metric space. Define $f(x, y) = (x^2, \sin y)$ and $g(x, y) = (2x - 1, \sin y)$. Then, we have

$$\begin{aligned}
& S(fg(x,y), fg(x,y), gf(x,y)) \\
= \quad & S(((2x-1)^2, \sin(\sin y)), ((2x-1)^2, \sin(\sin y)), (2x^2-1, \sin(\sin y))) \\
= \quad & \sqrt{(2(x-1)^2)^2} = 2(x-1)^2 \\
= \quad & 2S(f(x,y), f(x,y), g(x,y)) \\
\leq \quad & RS(f(x,y), f(x,y), g(x,y)).
\end{aligned}$$

Therefore, for $R \geq 2$, f and g are R-weakly commuting. But f and g are not weakly commuting.

Now we introduce and prove the main theorem in this paper.

Theorem 2.5. Let (X, S_1) be an S-metric space, (Y, S_2) be a complete S-metric space and let f and g be R-weakly commuting self-mappings of X satisfying the following conditions:

(a) $f(X) \subseteq g(X)$;

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(b) f or g is continuous;

(c) $S_2(Ffx, Ffx, Ffy) \le kS_2(Fgx, Fgx, Fgy))$, where $0 < k < 1$ and $F : X \to Y$ is one-to-one, continuous and sequentially convergent.

Then there exists a unique common fixed point $z \in X$ of f and g.

proof. Let x_0 be an arbitrary point in X. By (a), there exists a point x_1 in X such that $fx_0 = gx_1$. Continuing in this process, we can choose x_{n+1} such that $fx_n = gx_{n+1}$. Set $y_n = Ffx_n$. Then

$$\begin{aligned}
S_2(y_n, y_n, y_{n+1}) &= S_2(Ffx_n, Ffx_n, Ffx_{n+1}) \\
&\le kS_2(Fgx_n, Fgx_n, Fgx_{n+1})) \\
&= kS_2(Ffx_{n-1}, Ffx_{n-1}, Ffx_n)) \\
&= kS_2(y_{n-1}, y_{n-1}, y_n) \\
&\le k^2 S_2(y_{n-2}, y_{n-2}, y_{n-1}) \\
&\vdots \\
&\le k^n S_2(y_0, y_0, y_1).
\end{aligned}$$

Thus for all $n < m$, we have

$$\begin{aligned}
S_2(y_n, y_n, y_m) &\le 2S_2(y_n, y_n, y_{n+1}) + S_2(y_m, y_m, y_{n+1}) \\
&= 2S_2(y_n, y_n, y_{n+1}) + S_2(y_{n+1}, y_{n+1}, y_m) \\
&\vdots \\
&\le 2[k^n + \cdots + k^{m-1}]S_2(y_0, y_0, y_1) \\
&\le \frac{2k^n}{1-k} S_2(y_0, y_0, y_1).
\end{aligned}$$

Taking the limit as $n, m \to \infty$, we get $S_2(y_n, y_n, y_m) \to 0$. This means that $\{y_n\}$ is a Cauchy sequence. Since (Y, S_2) is complete, the sequence $\{y_n\}$ converges to some $y \in Y$. Since F is sequentially convergent, $\{fx_n\}$ converges to some $z \in X$ and also from the continuity of F, $\{Ffx_n\}$ converges to Fz. Note that $\{y_n\}$ converges to y, then $y_n = Ffx_n = Fgx_{n+1} \to Fz = y$. Also $\{gx_n\}$ converges to z in X. Let us suppose that the mapping f is continuous. Then $\lim_{n\to\infty} ffx_n = fz$ and $\lim_{n\to\infty} fgx_n = fz$. Further, since f and g are R-weakly commuting, we have

$$S_1(fgx_n, fgx_n, gfx_n) \le RS_1(fx_n, fx_n, gx_n).$$

Taking the limit as $n \to \infty$ in the above inequality, we have

$$\begin{aligned}
S_1(fz, fz, \lim_{n\to\infty} gfx_n) &= \lim_{n\to\infty} S_1(fgx_n, fgx_n, gfx_n) \\
&\le R \lim_{n\to\infty} S_1(fx_n, fx_n, gx_n) \\
&= RS_1(z, z, z) \\
&= 0.
\end{aligned}$$

Hence, we get $\lim_n gfx_n = fz$. We now prove that $z = fz$. By (c)

$$\begin{aligned}
S_2(Ffz, Ffz, Fz) &= \lim_{n\to\infty} S_2(Fffx_n, Fffx_n, Ffx_n) \\
&\le k \lim_{n\to\infty} S_2(Fgfx_n, Fgfx_n, Fgx_n) \\
&= kS_2(Ffz, Ffz, Fz)).
\end{aligned}$$

By the above inequality, we get $Ffz = Fz$. Since F is one-to-one, it follows that $fz = z$. Since $f(X) \subseteq g(X)$, we can find $z_1 \in X$ such that $z = fz = gz_1$. Now,

$$S_2(Fffx_n, Fffx_n, Ffz_1) \leq kS_2(Fgfx_n, Fgfx_n, Fgz_1)).$$

Taking the limit as $n \to \infty$, we get

$$\begin{aligned} S_2(Ffz, Ffz, Ffz_1) &= \lim_{n\to\infty} S_2(Fffx_n, Fffx_n, Ffz_1) \\ &\leq k \lim_{n\to\infty} S_2(Fgfx_n, Fgfx_n, Fgz_1) \\ &= kS_2(Ffz, Ffz, Fgz_1)) \\ &= 0, \end{aligned}$$

which implies that $Ffz = Ffz_1$, i.e., $z = fz = fz_1 = gz_1$. Also, we have

$$S_1(fz, fz, gz) = S_1(fgz_1, fgz_1, gfz_1) \leq RS_1(fz_1, fz_1, gz_1) = 0,$$

which implies that $fz = gz$. Thus z is a common fixed point of f and g.

Now, in order to prove uniqueness, let $z' \neq z$ be another common fixed point of f and g. Then

$$\begin{aligned} S_2(Fz, Fz, Fz') &= S_2(Ffz, Ffz, Ffz') \\ &\leq kS_2(Fgz, Fgz, Fgz') \\ &= kS_2(Fz, Fz, Fz') \\ &< S_2(Fz, Fz, Fz'), \end{aligned}$$

which is a contradiction. Therefore, $Fz = Fz'$, i.e., $z = z'$ is a unique common fixed point of f and g. This completes the proof.

Now we give an example to support our Theorem 2.5.

Example 2.6. Let $X = [1, \infty)$, $Y = R^2$ and $S_1(x, y, z) = \max\{|x - y|, |y - z|\}$, $S_2(x, y, z) = ||x-y||+||y-z||$. Then (X, S_1) is an S-metric space. Define $f(x) = 2x-1$ and $g(x) = x^2$ on X. It is evident that $f(X) \subseteq g(X)$, f is continuous.

$$S_1(fgx, fgx, gfx) = 2|x - 1|^2 = 2S_1(fx, fx, gx)$$

for all $x \in X$, it is easy to see that f and g are R-weakly commuting for $R \geq 2$. If define $F : X \to Y$ by $F(x) = (\frac{x}{2}, \frac{x-1}{2})$, then F is one-to-one, continuous and sequentially convergent. Also, we have

$$\begin{aligned} S_2(Ffx, Ffy, Ffz) &= S_2((\frac{2x-1}{2}, \frac{2x-2}{2}), (\frac{2y-1}{2}, \frac{2y-2}{2}), (\frac{2z-1}{2}, \frac{2z-2}{2})) \\ &= \sqrt{(x-y)^2 + (x-y)^2} + \sqrt{(y-z)^2 + (y-z)^2} \\ &= \sqrt{2}(|x-y| + |y-z|) \end{aligned}$$

and

$$\begin{aligned} S_2(Fgx, Fgy, Fgz) &= S_2((\frac{2x^2-1}{2}, \frac{2x^2-2}{2}), (\frac{2y^2-1}{2}, \frac{2y^2-2}{2}), (\frac{2z^2-1}{2}, \frac{2z^2-2}{2})) \\ &= \sqrt{(x^2-y^2)^2 + (x^2-y^2)^2} + \sqrt{(y^2-z^2)^2 + (y^2-z^2)^2} \\ &= \sqrt{2}(|x-y||x+y| + |y-z||y+z|) \\ &\geq 2\sqrt{2}(|x-y| + |y-z|) \\ &= 2S_2(Ffx, Ffy, Ffz). \end{aligned}$$

Therefore

$$S_2(Ffx, Ffy, Ffz) \leq kS_2(Fgx, Fgy, Fgz),$$

for $k = \frac{1}{2}$. Thus all the conditions of Theorem 2.5 are satisfied and 1 is a common fixed point of f and g.

Corollary 2.7. Let (X, S) be a complete S-metric space and let f and g be R-weakly commuting self-mappings of X satisfying the following conditions:

(a) $f(X) \subseteq g(X)$;

(b) f or g is continuous;

(c) $S(Ffx, Ffx, Ffy) \leq kS(Fgx, Fgx, Fgy)$, where $0 < k < 1$ and $F : X \to X$ is one-to-one, continuous and sequentially convergent.

Then f and g have a unique common fixed point $z \in X$. Moreover, if $Ff = fF$ and $Fg = gF$ then F, f and g have a unique common fixed point $z \in X$.

Proof. By Theorem 2.5, f and g have a unique common fixed point $z \in X$. Now we show that $Fz = z$.

$$\begin{aligned} S(Fz, Fz, FFz) &= S(Ffz, Ffz, FFfz) \\ &= S(Ffz, Ffz, FfFz) \\ &\leq kS(Fgz, Fgz, FgFz) \\ &= kS(Fz, Fz, FFz), \end{aligned}$$

it follows that $FFz = Fz$, hence $Fz = z$ from the injectivity of F.

Corollary 2.8. Let (X, S_1) be an S-metric space, (Y, S_2) be a complete S-metric space and let f be a self-mapping of X satisfying the following conditions:

(a) f is continuous;

(b) $S(Ffx, Ffx, Ffy) \leq kS(Ffx, Ffx, Ffy)$, where $0 < k < 1$ and $F : X \to Y$ is one-to-one, continuous and sequentially convergent.

Then f has a unique fixed point $z \in X$.

Corollary 2.9. Let (X, S) be a complete S-metric space and let f and g be R-weakly commuting self-mappings of X satisfying the following conditions:

(a) $f(X) \subseteq g(X)$;

(b) f or g is continuous;

(c) $S(fx, fx, fy) \leq kS(gx, gx, gy))$, where $0 < k < 1$.

Then f and g have a unique fixed point $z \in X$.

Proof. If set $F = I$ identity map then by Corollary 2.7 follows that f and g have a unique common fixed point $z \in X$.

Corollary 2.10. Let (X, S) be a complete S-metric space, F, f and g be self-mappings of X and let Ff and Fg be R-weakly commuting satisfying the following conditions:

(a) $Ff(X) \subseteq Fg(X)$;

(b) Ff or Fg is continuous;

(c) $S(Ffx, Ffx, Ffy) \leq kS(Fgx, Fgx, Fgy))$, where $0 < k < 1$.

If $Ff = fF$ and $Fg = gF$, then F, f and g have a unique common fixed point $z \in X$.

Proof. By Corollary 2.9, Ff and Fg have a unique common fixed point $z \in X$. That is $Ffz = Fgz = z$. Now we show that $fz = z$.

$$\begin{aligned} S(Fz, Fz, z) &= S(FFfz, FFfz, Ffz) \\ &= S(FfFz, FfFz, Ffz) \\ &\leq kS(FgFz, FgFz, Fgz) \\ &= kS(FFgz, FFgz, Fgz) \\ &= kS(Fz, Fz, z), \end{aligned}$$

it follows that $Fz = z$, hence $z = Ffz = fFz = fz$. Similarly we can show that $gz = z$. This completes the proof.

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On a solution of system of three fractional difference equations

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ABSTRACT

In this paper, we investigate the form of the solution of the following systems of difference equations

$$x_{n+1}=\frac{x_{n-2}}{\pm1+x_{n-2}z_{n-1}y_n},\ y_{n+1}=\frac{y_{n-2}}{\pm1+y_{n-2}x_{n-1}z_n},\ z_{n+1}=\frac{z_{n-2}}{\pm1+z_{n-2}y_{n-1}x_n},$$

with initial conditions are nonzero real numbers.

Keywords: difference equations, recursive sequences, periodic solutions, system of difference equations, stability.

Mathematics Subject Classification: 39A10.

1. INTRODUCTION

Difference equations appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations having applications in biology, ecology, economy, physics, and so on. Although difference equations are very simple in form, it is extremely difficult to understand thoroughly the behaviors of their solution see [1]–[10] and the references cited therein. Recently a great effort has been made in studying the qualitative analysis of rational difference equations and rational difference system see [11-15].

There are many papers related to the difference equations system, for example, The periodicity of the positive solutions of the rational difference equations systems

$$\begin{aligned} x_{n+1} &= \frac{1}{z_n},\ \ y_{n+1}=\frac{y_n}{x_{n-1}y_{n-1}},\ \ z_{n+1}=\frac{1}{x_{n-1}}. \\ x_{n+1} &= \frac{1}{z_n},\ \ y_{n+1}=\frac{x_n}{x_{n-1}},\ \ z_{n+1}=\frac{1}{x_{n-1}}. \end{aligned}$$

has been obtained by Cinar et al. in [4], [5].

Elabbasy et al. [12] has obtained the solution of particular cases of the following general system of difference equations

$$x_{n+1}=\frac{a_1+a_2y_n}{a_3z_n+a_4x_{n-1}z_n},\ y_{n+1}=\frac{b_1z_{n-1}+b_2z_n}{b_3x_ny_n+b_4x_ny_{n-1}},\ z_{n+1}=\frac{c_1z_{n-1}+c_2z_n}{c_3x_{n-1}y_{n-1}+c_4x_{n-1}y_n+c_5x_ny_n}.$$

Kurbanli [26]-[28] investigated the behavior of the solutions of the difference equation systems

$$\begin{aligned} x_{n+1} &= \frac{x_{n-1}}{x_{n-1}y_n-1},\ y_{n+1}=\frac{y_{n-1}}{y_{n-1}x_n-1},\ \ z_{n+1}=\frac{1}{z_ny_n}, \\ x_{n+1} &= \frac{x_{n-1}}{x_{n-1}y_n-1},\ y_{n+1}=\frac{y_{n-1}}{y_{n-1}x_n-1},\ \ z_{n+1}=\frac{z_{n-1}}{z_{n-1}y_n-1}, \\ x_{n+1} &= \frac{x_{n-1}}{x_{n-1}y_n-1},\ y_{n+1}=\frac{y_{n-1}}{y_{n-1}x_n-1},\ z_{n+1}=\frac{x_n}{z_{n-1}y_n}. \end{aligned}$$

In [32] Yalçınkaya investigated the sufficient condition for the global asymptotic stability of the following system of difference equations

$$z_{n+1}=\frac{t_nz_{n-1}+a}{t_n+z_{n-1}},\ t_{n+1}=\frac{z_nt_{n-1}+a}{z_n+t_{n-1}}.$$

In [34] Yalcinkaya et al. investigated the solutions of the system of difference equations

$$x_{n+1}^{(1)}=\frac{x_n^{(2)}}{x_n^{(2)}-1},\ x_{n+1}^{(2)}=\frac{x_n^{(3)}}{x_n^{(3)}-1},...,x_{n+1}^{(k)}=\frac{x_n^{(1)}}{x_n^{(1)}-1}.$$

Similar to difference equations and nonlinear systems of rational difference equations were investigated see [16]-[36].

In this paper we deal with the solution and the periodic nature of the following third order systems of a rational difference equations

$$x_{n+1}=\frac{x_{n-2}}{\pm 1+x_{n-2}z_{n-1}y_n},\ y_{n+1}=\frac{y_{n-2}}{\pm 1+y_{n-2}x_{n-1}z_n},\ z_{n+1}=\frac{z_{n-2}}{\pm 1+z_{n-2}y_{n-1}x_n},$$

with initial conditions are nonzero real numbers.

2. THE FIRST SYSTEM: $X_{N+1}=\frac{X_{N-2}}{-1+X_{N-2}Z_{N-1}Y_N}$, $Y_{N+1}=\frac{Y_{N-2}}{1+Y_{N-2}X_{N-1}Z_N}$, $Z_{N+1}=\frac{Z_{N-2}}{1+Z_{N-2}Y_{N-1}X_N}$

In this section, we get the solutions of the system of the difference equations

$$x_{n+1}=\frac{x_{n-2}}{-1+x_{n-2}z_{n-1}y_n},\ y_{n+1}=\frac{y_{n-2}}{1+y_{n-2}x_{n-1}z_n},\ z_{n+1}=\frac{z_{n-2}}{1+z_{n-2}y_{n-1}x_n},\tag{1}$$

where $n\in\mathbb{N}_0$ and the initial conditions are arbitrary nonzero real numbers with $x_{-2}z_{-1}y_0\neq 1,\neq\frac{1}{2},\neq\frac{1}{3}$, $y_{-2}x_{-1}z_0\neq\pm 1,\neq\frac{1}{2}$, $z_{-2}y_{-1}x_0\neq\pm 1,-\frac{1}{2}$.

Theorem 2.1. *Suppose that $\{x_n,y_n,z_n\}$ are solutions of system (1). Then the solution of system (1) are given by the following formula for $n=0,1,2,...$,*

$$\begin{aligned}
x_{6n-2}&=\frac{(-1)^n x_{-2}(-1+3x_{-2}z_{-1}y_0)^n}{(-1+x_{-2}z_{-1}y_0)^n(-1+2x_{-2}z_{-1}y_0)^n},\ x_{6n-1}=\frac{x_{-1}(1+y_{-2}x_{-1}z_0)^n(1-2y_{-2}x_{-1}z_0)^n}{(1-y_{-2}x_{-1}z_0)^n},\\
x_{6n}&=\frac{x_0(1+2z_{-2}y_{-1}x_0)^n(1-z_{-2}y_{-1}x_0)^n}{(1+z_{-2}y_{-1}x_0)^n},\ x_{6n+1}=\frac{(-1)^n x_{-2}(-1+3x_{-2}z_{-1}y_0)^n}{(-1+x_{-2}z_{-1}y_0)^{n+1}(-1+2x_{-2}z_{-1}y_0)^n},\\
x_{6n+2}&=\frac{-x_{-1}(1+y_{-2}x_{-1}z_0)^{n+1}(1-2y_{-2}x_{-1}z_0)^n}{(1-y_{-2}x_{-1}z_0)^n},\ x_{6n+3}=\frac{-x_0(1+2z_{-2}y_{-1}x_0)^{n+1}(1-z_{-2}y_{-1}x_0)^n}{(1+z_{-2}y_{-1}x_0)^{n+1}},\\
y_{6n-2}&=\frac{y_{-2}(1-y_{-2}x_{-1}z_0)^n}{(1+y_{-2}x_{-1}z_0)^n(1-2y_{-2}x_{-1}z_0)^n},\ y_{6n-1}=\frac{y_{-1}(1+z_{-2}y_{-1}x_0)^n}{(1-z_{-2}y_{-1}x_0)^n(1+2z_{-2}y_{-1}x_0)^n},\\
y_{6n}&=\frac{(-1)^n y_0(-1+x_{-2}z_{-1}y_0)^n(-1+2x_{-2}z_{-1}y_0)^n}{(-1+3x_{-2}z_{-1}y_0)^n},\ y_{6n+1}=\frac{y_{-2}(1-y_{-2}x_{-1}z_0)^n}{(1+y_{-2}x_{-1}z_0)^{n+1}(1-2y_{-2}x_{-1}z_0)^n},\\
y_{6n+2}&=\frac{y_{-1}(1+z_{-2}y_{-1}x_0)^{n+1}}{(1-z_{-2}y_{-1}x_0)^n(1+2z_{-2}y_{-1}x_0)^{n+1}},\ y_{6n+3}=\frac{(-1)^n y_0(-1+x_{-2}z_{-1}y_0)^n(-1+2x_{-2}z_{-1}y_0)^{n+1}}{(-1+3x_{-2}z_{-1}y_0)^{n+1}},
\end{aligned}$$

$$\begin{aligned}
z_{6n-2}&=z_{-2},\quad z_{6n-1}=z_{-1},\quad z_{6n}=z_0,\quad z_{6n+1}=\frac{z_{-2}}{1+z_{-2}y_{-1}x_0},\\
z_{6n+2}&=\frac{z_{-1}(-1+x_{-2}z_{-1}y_0)}{(-1+2x_{-2}z_{-1}y_0)},\quad z_{6n+3}=\frac{z_0}{1-y_{-2}x_{-1}z_0}.
\end{aligned}$$

Proof: For $n=0$ the result holds. Now suppose that $n>1$ and that our assumption holds for $n-1$. that is,

$$\begin{aligned}
x_{6n-8}&=\frac{(-1)^{n-1}x_{-2}(-1+3x_{-2}z_{-1}y_0)^{n-1}}{(-1+x_{-2}z_{-1}y_0)^{n-1}(-1+2x_{-2}z_{-1}y_0)^{n-1}},\quad x_{6n-7}=\frac{x_{-1}(1+y_{-2}x_{-1}z_0)^{n-1}(1-2y_{-2}x_{-1}z_0)^{n-1}}{(1-y_{-2}x_{-1}z_0)^{n-1}},\\
x_{6n-6}&=\frac{x_0(1+2z_{-2}y_{-1}x_0)^{n-1}(1-z_{-2}y_{-1}x_0)^{n-1}}{(1+z_{-2}y_{-1}x_0)^{n-1}},\quad x_{6n-5}=\frac{(-1)^{n-1}x_{-2}(-1+3x_{-2}z_{-1}y_0)^{n-1}}{(-1+x_{-2}z_{-1}y_0)^n(-1+2x_{-2}z_{-1}y_0)^{n-1}},\\
x_{6n-4}&=\frac{-x_{-1}(1+y_{-2}x_{-1}z_0)^n(1-2y_{-2}x_{-1}z_0)^{n-1}}{(1-y_{-2}x_{-1}z_0)^{n-1}},\quad x_{6n-3}=\frac{-x_0(1+2z_{-2}y_{-1}x_0)^n(1-z_{-2}y_{-1}x_0)^{n-1}}{(1+z_{-2}y_{-1}x_0)^n},
\end{aligned}$$

$$\begin{aligned}
y_{6n-8}&=\frac{y_{-2}(1-y_{-2}x_{-1}z_0)^{n-1}}{(1+y_{-2}x_{-1}z_0)^{n-1}(1-2y_{-2}x_{-1}z_0)^{n-1}},\quad y_{6n-7}=\frac{y_{-1}(1+z_{-2}y_{-1}x_0)^{n-1}}{(1-z_{-2}y_{-1}x_0)^{n-1}(1+2z_{-2}y_{-1}x_0)^{n-1}},\\
y_{6n-6}&=\frac{(-1)^{n-1}y_0(-1+x_{-2}z_{-1}y_0)^{n-1}(-1+2x_{-2}z_{-1}y_0)^{n-1}}{(-1+3x_{-2}z_{-1}y_0)^{n-1}},\quad y_{6n-5}=\frac{y_{-2}(1-y_{-2}x_{-1}z_0)^{n-1}}{(1+y_{-2}x_{-1}z_0)^n(1-2y_{-2}x_{-1}z_0)^{n-1}},\\
y_{6n-4}&=\frac{y_{-1}(1+z_{-2}y_{-1}x_0)^n}{(1-z_{-2}y_{-1}x_0)^{n-1}(1+2z_{-2}y_{-1}x_0)^n},\quad y_{6n-3}=\frac{(-1)^{n-1}y_0(-1+x_{-2}z_{-1}y_0)^{n-1}(-1+2x_{-2}z_{-1}y_0)^n}{(-1+3x_{-2}z_{-1}y_0)^n},
\end{aligned}$$

and

$$\begin{aligned}
z_{6n-8} &= z_{-2}, \quad z_{6n-7} = z_{-1}, \quad z_{6n-6} = z_0, \quad z_{6n-5} = \tfrac{z_{-2}}{1+z_{-2}y_{-1}x_0}, \\
z_{6n-4} &= \tfrac{z_{-1}(-1+x_{-2}z_{-1}y_0)}{(-1+2x_{-2}z_{-1}y_0)}, \quad z_{6n-3} = \tfrac{z_0}{1-y_{-2}x_{-1}z_0}.
\end{aligned}$$

It follows from Eq.(1) that

$$\begin{aligned}
x_{6n-2} &= \tfrac{x_{6n-5}}{-1+x_{6n-5}z_{6n-4}y_{6n-3}} \\
&= \frac{\frac{(-1)^{n-1}x_{-2}(-1+3x_{-2}z_{-1}y_0)^{n-1}}{(-1+x_{-2}z_{-1}y_0)^n(-1+2x_{-2}z_{-1}y_0)^{n-1}}}{\left(-1+\frac{(-1)^{n-1}x_{-2}(-1+3x_{-2}z_{-1}y_0)^{n-1}}{(-1+x_{-2}z_{-1}y_0)^n(-1+2x_{-2}z_{-1}y_0)^{n-1}}\frac{z_{-1}(-1+x_{-2}z_{-1}y_0)}{(-1+2x_{-2}z_{-1}y_0)}\frac{(-1)^{n-1}y_0(-1+x_{-2}z_{-1}y_0)^{n-1}(-1+2x_{-2}z_{-1}y_0)^n}{(-1+3x_{-2}z_{-1}y_0)^n}\right)} \\
&= \frac{(-1)^{n-1}x_{-2}(-1+3x_{-2}z_{-1}y_0)^{n-1}}{(-1+x_{-2}z_{-1}y_0)^n(-1+2x_{-2}z_{-1}y_0)^{n-1}}\frac{1}{\left(-1+\frac{x_{-2}z_{-1}y_0}{(-1+3x_{-2}z_{-1}y_0)}\right)} \\
&= \frac{(-1)^{n-1}x_{-2}(-1+3x_{-2}z_{-1}y_0)^{n-1}}{(-1+x_{-2}z_{-1}y_0)^n(-1+2x_{-2}z_{-1}y_0)^{n-1}}\frac{1}{\left(-1+\frac{x_{-2}z_{-1}y_0}{(-1+3x_{-2}z_{-1}y_0)}\right)}\left(\frac{-1+3x_{-2}z_{-1}y_0}{-1+3x_{-2}z_{-1}y_0}\right) \\
&= \frac{(-1)^{n-1}x_{-2}(-1+3x_{-2}z_{-1}y_0)^{n-1}}{(-1+x_{-2}z_{-1}y_0)^n(-1+2x_{-2}z_{-1}y_0)^{n-1}}\frac{(-1+3x_{-2}z_{-1}y_0)}{(1-2x_{-2}z_{-1}y_0)} \\
&= \frac{(-1)^n x_{-2}(-1+3x_{-2}z_{-1}y_0)^n}{(-1+x_{-2}z_{-1}y_0)^n(-1+2x_{-2}z_{-1}y_0)^n} = \frac{(-1)^n x_{-2}(-1+3x_{-2}z_{-1}y_0)^n}{(-1+x_{-2}z_{-1}y_0)^n(-1+2x_{-2}z_{-1}y_0)^n}.
\end{aligned}$$

Also, we see from Eq.(1) that

$$\begin{aligned}
y_{6n-2} &= \tfrac{y_{6n-5}}{1+y_{6n-5}x_{6n-4}z_{6n-3}} \\
&= \frac{\frac{y_{-2}(1-y_{-2}x_{-1}z_0)^{n-1}}{(1+y_{-2}x_{-1}z_0)^n(1-2y_{-2}x_{-1}z_0)^{n-1}}}{\left(1+\frac{y_{-2}(1-y_{-2}x_{-1}z_0)^{n-1}}{(1+y_{-2}x_{-1}z_0)^n(1-2y_{-2}x_{-1}z_0)^{n-1}}\frac{-x_{-1}(1+y_{-2}x_{-1}z_0)^n(1-2y_{-2}x_{-1}z_0)^{n-1}}{(1-y_{-2}x_{-1}z_0)^{n-1}}\frac{z_0}{1-y_{-2}x_{-1}z_0}\right)} \\
&= \frac{y_{-2}(1-y_{-2}x_{-1}z_0)^{n-1}}{(1+y_{-2}x_{-1}z_0)^n(1-2y_{-2}x_{-1}z_0)^{n-1}}\frac{1}{\left(1-\frac{y_{-2}x_{-1}z_0}{1-y_{-2}x_{-1}z_0}\right)}\left(\frac{1-y_{-2}x_{-1}z_0}{1-y_{-2}x_{-1}z_0}\right) \\
&= \frac{y_{-2}(1-y_{-2}x_{-1}z_0)^{n-1}}{(1+y_{-2}x_{-1}z_0)^n(1-2y_{-2}x_{-1}z_0)^{n-1}}\frac{1-y_{-2}x_{-1}z_0}{(1-y_{-2}x_{-1}z_0-y_{-2}x_{-1}z_0)}.
\end{aligned}$$

Then

$$y_{6n-2} = \frac{y_{-2}(1-y_{-2}x_{-1}z_0)^n}{(1+y_{-2}x_{-1}z_0)^n(1-2y_{-2}x_{-1}z_0)^n}.$$

Finally from Eq.(1), we see that

$$\begin{aligned}
z_{6n-2} &= \frac{z_{6n-5}}{1+z_{6n-5}y_{6n-4}x_{6n-3}} \\
&= \frac{\frac{z_{-2}}{1+z_{-2}y_{-1}x_0}}{\left(1+\frac{z_{-2}}{1+z_{-2}y_{-1}x_0}\frac{y_{-1}(1+z_{-2}y_{-1}x_0)^n}{(1-z_{-2}y_{-1}x_0)^{n-1}(1+2z_{-2}y_{-1}x_0)^n}\frac{-x_0(1+2z_{-2}y_{-1}x_0)^n(1-z_{-2}y_{-1}x_0)^{n-1}}{(1+z_{-2}y_{-1}x_0)^n}\right)} \\
&= \frac{z_{-2}}{1+z_{-2}y_{-1}x_0}\frac{1}{\left(1-\frac{z_{-2}y_{-1}x_0}{1+z_{-2}y_{-1}x_0}\right)} = \frac{z_{-2}}{(1+z_{-2}y_{-1}x_0-z_{-2}y_{-1}x_0)} = z_{-2}.
\end{aligned}$$

Similarly we can prove the other relations. This completes the proof.

Lemma 1. Let $\{y_n, z_n\}$ be a positive solution of system (1), then every solution $\{y_n, z_n\}$ of system (1) is bounded and $\{y_n\}$ converges to zero.

Proof: It follows from Eq.(1) that

$$y_{n+1} = \tfrac{y_{n-2}}{1+z_n x_{n-1} y_{n-2}} \leq y_{n-2}, \quad z_{n+1} = \tfrac{z_{n-2}}{1+x_n y_{n-1} z_{n-2}} \leq z_{n-2}.$$

Then the subsequences $\{y_{3n-2}\}_{n=0}^{\infty}$, $\{y_{3n-1}\}_{n=0}^{\infty}$, $\{y_{3n}\}_{n=0}^{\infty}$,are decreasing and so are bounded from above by $M = \max\{y_{-2}, y_{-1}, y_0\}$ and $\{z_{3n-2}\}_{n=0}^{\infty}$, $\{z_{3n-1}\}_{n=0}^{\infty}$, $\{z_{3n}\}_{n=0}^{\infty}$,are decreasing and also, bounded from above by $M = \max\{z_{-2}, z_{-1}, z_0\}$.

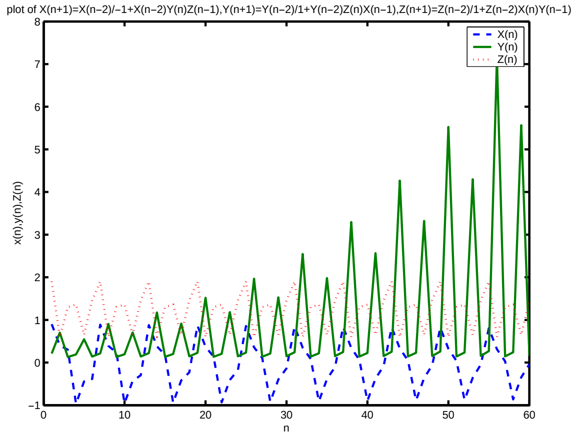

Figure 1.

Lemma 2. It is easy to see that $\{z_n\}$ periodic with period six.

Lemma 3. If $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0, z_{-2}, z_{-1}$ and z_0 arbitrary real numbers and let $\{x_n, y_n, z_n\}$ are solutions of system (1) then the following statements are true:-

(i) If $x_{-2} = 0,\ \ y_0 \neq 0,\ \ z_{-1} \neq 0$, then we have $x_{6n-2} = x_{6n+1} = 0$ *and* $y_{6n} = y_{6n+3} = y_0,\ z_{6n+2} = z_{-1}$.

(ii) If $x_{-1} = 0,\ \ y_{-2} \neq 0,\ \ z_0 \neq 0$, then we have $x_{6n-1} = x_{6n+2} = 0$ *and* $y_{6n-2} = y_{6n+1} = y_{-2},\ z_{6n+3} = z_0$.

(iii) If $x_0 = 0,\ \ y_{-1} \neq 0,\ \ z_{-2} \neq 0$, then we have $x_{6n} = x_{6n+3} = 0$ *and* $y_{6n-1} = y_{6n+2} = y_{-1},\ z_{6n+1} = z_{-2}$.

(iv) If $y_{-2} = 0,\ \ x_{-1} \neq 0,\ \ z_0 \neq 0$, then we have $y_{6n-2} = y_{6n+1} = 0$ *and* $x_{6n-1} = x_{-1},\ x_{6n+2} = -x_{-1},\ z_{6n+3} = z_0$.

(v) If $y_{-1} = 0,\ \ x_0 \neq 0,\ \ z_{-2} \neq 0$, then we have $y_{6n-1} = y_{6n+2} = 0$ *and* $x_{6n} = x_0,\ x_{6n+3} = -x_0,\ z_{6n+1} = z_{-2}$.

(vi) If $y_0 = 0,\ \ x_{-2} \neq 0,\ \ z_{-1} \neq 0$, then we have $y_{6n} = y_{6n+3} = 0$ *and* $x_{6n-2} = x_{6n+1} = x_{-2},\ z_{6n+2} = z_{-1}$.

(vii) If $z_{-2} = 0,\ \ x_0 \neq 0,\ y_{-1} \neq 0$, then we have $z_{6n-2} = z_{6n+1} = 0$ *and* $x_{6n} = x_0,\ x_{6n+3} = -x_0,\ y_{6n-1} = y_{6n+2} = y_{-1}$.

(viii) If $z_{-1} = 0,\ \ x_{-2} \neq 0,\ \ y_0 \neq 0$, then we have $z_{6n-1} = z_{6n+2} = 0$ *and* $x_{6n-2} = x_{6n+1} = x_{-2},\ y_{6n} = y_{6n+3} = y_0$.

(ix) If $z_0 = 0,\ x_{-1} \neq 0,\ y_{-2} \neq 0$, then we have $z_{6n} = z_{6n+3} = 0$ *and* $x_{6n-1} = x_{-1},\ x_{6n+2} = -x_{-1},\ y_{6n-2} = y_{6n+1} = y_{-2}$.

Proof: The proof follows from the form of the solutions of system (1).

Example 1. We consider interesting numerical example for the difference system (1) with the initial conditions $x_{-2} = 0.9,\ x_{-1} = 0.4,\ x_0 = 0.3,\ y_{-2} = 0.21,\ y_{-1} = 0.7,\ y_0 = 0.13,\ z_{-2} = 1.9,\ z_{-1} = 0.6$ *and* $z_0 = 1.3$. (See Fig. 1).

Example 2. See Figure (2) when we put the initial conditions $x_{-2} = 0.9,\ x_{-1} = 0.4,\ x_0 = -0.3,\ y_{-2} = -0.21,\ y_{-1} = 0.7,\ y_0 = 0,\ z_{-2} = -1.9,\ z_{-1} = 0.6$ *and* $z_0 = 1.3$ for the difference system (1).

3. THE SECOND SYSTEM:

$$X_{N+1} = \frac{X_{N-2}}{-1+X_{N-2}Z_{N-1}Y_N},\ Y_{N+1} = \frac{Y_{N-2}}{-1+Y_{N-2}X_{N-1}Z_N},\ Z_{N+1} = \frac{Z_{N-2}}{1+Z_{N-2}Y_{N-1}X_N}$$

In this section, we study the solutions of the following system of the difference equations

$$x_{n+1} = \frac{x_{n-2}}{-1 + x_{n-2}z_{n-1}y_n},\ y_{n+1} = \frac{y_{n-2}}{-1 + y_{n-2}x_{n-1}z_n},\ z_{n+1} = \frac{z_{n-2}}{1 + z_{n-2}y_{n-1}x_n}, \tag{2}$$

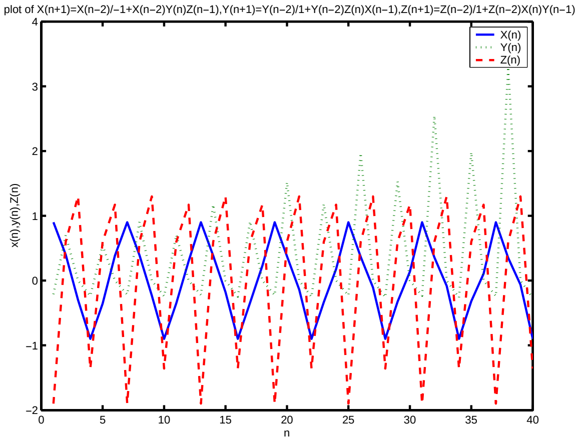

Figure 2.

where $n \in \mathbb{N}_0$ and the initial conditions are arbitrary non zero real numbers such that $y_{-2}x_{-1}y_0 \neq 1$, $x_{-2}y_{-1}x_0 \neq 1$.

THEOREM 3.1. *Suppose that $\{x_n, y_n, z_n\}$ are solutions of system (2). Then the solution of system (2) are given by*

$$\begin{aligned} x_{3n-2} &= \frac{(-1)^{n+1}x_{-2}}{(-1+nx_{-2}z_{-1}y_0)}, \quad x_{3n-1} = \frac{(-1)^{n+1}x_{-1}(-1+y_{-2}x_{-1}z_0)}{(1+(n-1)y_{-2}x_{-1}z_0)}, \quad x_{3n} = \frac{(-1)^n x_0}{(1+nz_{-2}y_{-1}x_0)}, \\ y_{3n-2} &= \frac{(-1)^{n+1}y_{-2}(1+(n-1)y_{-2}x_{-1}z_0)}{(-1+y_{-2}x_{-1}z_0)}, \quad y_{3n-1} = (-1)^n y_{-1}(1+nz_{-2}y_{-1}x_0), \\ y_{3n} &= \frac{(-1)^n y_0(1-(n+1)x_{-2}z_{-1}y_0)}{(1-x_{-2}z_{-1}y_0)}, \end{aligned}$$

$$z_{3n-2} = \frac{z_{-2}}{(1+nz_{-2}y_{-1}x_0)}, \quad z_{3n-1} = \frac{z_{-1}(-1+x_{-2}z_{-1}y_0)}{(-1+(n+1)x_{-2}z_{-1}y_0)}, \quad z_{3n} = \frac{z_0}{(1+ny_{-2}x_{-1}z_0)}.$$

Proof: For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n-1$. that is,

$$\begin{aligned} x_{3n-5} &= \frac{(-1)^n x_{-2}}{(-1+(n-1)x_{-2}z_{-1}y_0)}, \quad x_{3n-4} = \frac{(-1)^n x_{-1}(-1+y_{-2}x_{-1}z_0)}{(1+(n-2)y_{-2}x_{-1}z_0)}, \quad x_{3n-3} = \frac{(-1)^{n-1}x_0}{(1+(n-1)z_{-2}y_{-1}x_0)}, \\ y_{3n-5} &= \frac{(-1)^n y_{-2}(1+(n-2)y_{-2}x_{-1}z_0)}{(-1+y_{-2}x_{-1}z_0)}, \quad y_{3n-4} = (-1)^{n-1}y_{-1}(1+(n-1)\,z_{-2}y_{-1}x_0), \\ y_{3n-3} &= \frac{(-1)^{n-1}y_0(1-nx_{-2}z_{-1}y_0)}{(1-x_{-2}z_{-1}y_0)}, \end{aligned}$$

$$z_{3n-5} = \frac{z_{-2}}{(1+(n-1)z_{-2}y_{-1}x_0)}, \quad z_{3n-4} = \frac{z_{-1}(-1+x_{-2}z_{-1}y_0)}{(-1+nx_{-2}z_{-1}y_0)}, \quad z_{3n-3} = \frac{z_0}{(1+(n-1)y_{-2}x_{-1}z_0)}.$$

Now from Eq.(2) it follows that

$$\begin{aligned} x_{3n-2} &= \frac{x_{3n-5}}{-1+x_{3n-5}z_{3n-4}y_{3n-3}} \\ &= \frac{\frac{(-1)^n x_{-2}}{(-1+(n-1)x_{-2}z_{-1}y_0)}}{\left(-1+\frac{(-1)^n x_{-2}}{(-1+(n-1)x_{-2}z_{-1}y_0)}\frac{z_{-1}(-1+x_{-2}z_{-1}y_0)}{(-1+nx_{-2}z_{-1}y_0)}\frac{(-1)^{n-1}y_0(1-nx_{-2}z_{-1}y_0)}{(1-x_{-2}z_{-1}y_0)}\right)} \\ &= \frac{(-1)^n x_{-2}}{(-1+(n-1)x_{-2}z_{-1}y_0)\left(-1-\frac{x_{-2}z_{-1}y_0}{(-1+(n-1)x_{-2}z_{-1}y_0)}\right)} = \frac{(-1)^n x_{-2}}{(1-(n-1)x_{-2}z_{-1}y_0-x_{-2}z_{-1}y_0)} = \frac{(-1)^{n+1}x_{-2}}{(-1+nx_{-2}z_{-1}y_0)}, \end{aligned}$$

$$
\begin{aligned}
y_{3n-2} &= \frac{y_{3n-5}}{-1+y_{3n-5}x_{3n-4}z_{3n-3}} = \frac{\frac{(-1)^n y_{-2}(1+(n-2)y_{-2}x_{-1}z_0)}{(-1+y_{-2}x_{-1}z_0)}}{\left(-1+\frac{(-1)^n y_{-2}(1+(n-2)y_{-2}x_{-1}z_0)}{(-1+y_{-2}x_{-1}z_0)}\frac{(-1)^n x_{-1}(-1+y_{-2}x_{-1}z_0)}{(1+(n-2)y_{-2}x_{-1}z_0)}\frac{z_0}{(1+(n-1)y_{-2}x_{-1}z_0)}\right)} \\
&= \frac{(-1)^n y_{-2}(1+(n-2)y_{-2}x_{-1}z_0)}{(-1+y_{-2}x_{-1}z_0)\left(-1+\frac{y_{-2}x_{-1}z_0}{(1+(n-1)y_{-2}x_{-1}z_0)}\right)} \\
&= \frac{(-1)^n y_{-2}(1+(n-2)y_{-2}x_{-1}z_0)(1+(n-1)y_{-2}x_{-1}z_0)}{(-1+y_{-2}x_{-1}z_0)(-(1+(n-1)y_{-2}x_{-1}z_0+y_{-2}x_{-1}z_0)} \\
&= \frac{(-1)^{n+1} y_{-2}(1+(n-1)y_{-2}x_{-1}z_0)}{(-1+y_{-2}x_{-1}z_0)},
\end{aligned}
$$

and so,

$$
\begin{aligned}
z_{3n-2} &= \frac{z_{3n-5}}{1+z_{3n-5}y_{3n-4}x_{3n-3}} \\
&= \frac{\frac{z_{-2}}{(1+(n-1)z_{-2}y_{-1}x_0)}}{\left(1+\frac{z_{-2}}{(1+(n-1)z_{-2}y_{-1}x_0)}(-1)^{n-1}y_{-1}(1+(n-1)z_{-2}y_{-1}x_0)\frac{(-1)^{n-1}x_0}{(1+(n-1)z_{-2}y_{-1}x_0)}\right)}
\end{aligned}
$$

$$
= \frac{z_{-2}}{(1+(n-1)z_{-2}y_{-1}x_0)\left(1+\frac{z_{-2}y_{-1}x_0}{(1+(n-1)z_{-2}y_{-1}x_0)}\right)} = \frac{z_{-2}}{(1+(n-1)z_{-2}y_{-1}x_0+z_{-2}y_{-1}x_0)} = \frac{z_{-2}}{(1+nz_{-2}y_{-1}x_0)}.
$$

Also, we can prove the other relation. The proof is complete.

Lemma 4. Let $\{z_n\}$ be a positive solution of system (2), then every the sequence $\{z_n\}$ is bounded and converges to zero.

Lemma 5. If $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0, z_{-2}, z_{-1}$ and z_0 arbitrary real numbers and let $\{x_n, y_n, z_n\}$ are solutions of system (2) then the following statements are true:-

(i) If $x_{-2} = 0,\ \ y_0 \neq 0,\ \ z_{-1} \neq 0$, then we have $x_{3n-2} = 0$ *and* $y_{3n} = (-1)^n y_0,\ z_{3n-1} = z_{-1}$.

(ii) If $x_{-1} = 0,\ \ y_{-2} \neq 0,\ \ z_0 \neq 0$, then we have $x_{3n-1} = 0$ *and* $y_{3n-2} = (-1)^n y_{-2},\ z_{3n} = z_0$.

(iii) If $x_0 = 0,\ \ y_{-1} \neq 0,\ \ z_{-2} \neq 0$, then we have $x_{3n} = 0$ *and* $y_{3n-1} = (-1)^n y_{-1},\ z_{3n-2} = z_{-2}$.

(iv) If $y_{-2} = 0,\ \ x_{-1} \neq 0,\ \ z_0 \neq 0$, then we have $y_{3n-2} = 0$ *and* $x_{3n-1} = (-1)^n x_{-1},\ z_{3n} = z_0$.

(v) If $y_{-1} = 0,\ \ x_0 \neq 0,\ \ z_{-2} \neq 0$, then we have $y_{3n-1} = 0$ *and* $x_{3n} = (-1)^n x_0,\ z_{3n-2} = z_{-2}$.

(vi) If $y_0 = 0,\ \ x_{-2} \neq 0,\ \ z_{-1} \neq 0$, then we have $y_{3n} = 0$ *and* $x_{3n-2} = (-1)^n x_{-2},\ z_{3n-1} = z_{-1}$.

(vii) If $z_{-2} = 0,\ \ x_0 \neq 0,\ \ y_{-1} \neq 0$, then we have $z_{3n-2} = 0$ *and* $x_{3n} = (-1)^n x_0,\ y_{3n-1} = (-1)^n y_{-1}$.

(viii) If $z_{-1} = 0,\ \ x_{-2} \neq 0,\ \ y_0 \neq 0$, then we have $z_{3n-1} = 0$ *and* $x_{3n-2} = (-1)^n x_{-2},\ y_{3n} = (-1)^n y_0$.

(ix) If $z_0 = 0,\ \ x_{-1} \neq 0,\ \ y_{-2} \neq 0$, then we have $z_{3n} = 0$ *and* $x_{3n-1} = (-1)^n x_{-1},\ y_{3n-2} = (-1)^n y_{-2}$.

Example 3. Figure (3) shows the behavior of the solution of the difference system (2) with the initial conditions $x_{-2} = 0.9,\ x_{-1} = 0.4,\ x_0 = 0.3,\ y_{-2} = 0.21,\ y_{-1} = 0.7,\ y_0 = 0,\ z_{-2} = 1.9,\ z_{-1} = 0.6$ *and* $z_0 = 1.3$.

4. THE THIRD SYSTEM: $X_{N+1} = \frac{X_{N-2}}{1+X_{N-2}Z_{N-1}Y_N}$, $Y_{N+1} = \frac{Y_{N-2}}{1+Y_{N-2}X_{N-1}Z_N}$, $Z_{N+1} = \frac{Z_{N-2}}{-1+Z_{N-2}Y_{N-1}X_N}$

In this section, we obtain the form of the solutions of the system of three difference equations

$$
x_{n+1} = \frac{x_{n-2}}{1+x_{n-2}z_{n-1}y_n},\ y_{n+1} = \frac{y_{n-2}}{1+y_{n-2}x_{n-1}z_n},\ z_{n+1} = \frac{z_{n-2}}{-1+z_{n-2}y_{n-1}x_n}, \tag{3}
$$

where $n \in \mathbb{N}_0$ and the initial conditions are arbitrary nonzero real numbers such that $x_{-2}z_{-1}y_0 \neq \pm 1, \neq \frac{1}{2}$, $z_{-2}y_{-1}x_0 \neq 1, \neq, \frac{1}{2} \neq \frac{1}{3}$ and $y_{-2}x_{-1}z_0 \neq \pm 1, \neq \frac{-1}{2}$.

The following theorem is devoted to the expression of the form of the solutions of system (3).

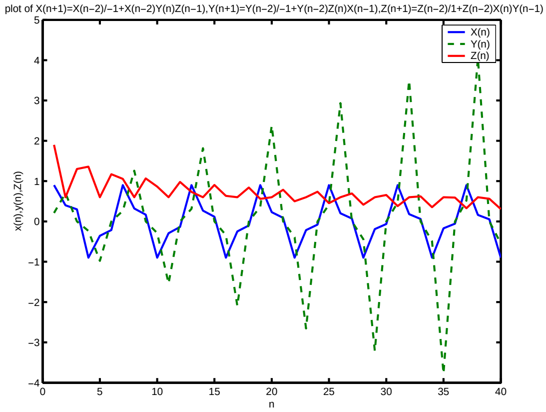

Figure 3.

THEOREM 4.1. *If $\{x_n, y_n, z_n\}$ are solutions of difference equation system (3). Then every solution of system (3) are takes the following form for $n = 0, 1, 2, ...,$*

$$\begin{aligned} x_{6n-2} &= \frac{x_{-2}(-1+x_{-2}z_{-1}y_0)^n}{(1+x_{-2}z_{-1}y_0)^n(-1+2x_{-2}z_{-1}y_0)^n}, \quad x_{6n-1} = \frac{(-1)^n x_{-1}(1+y_{-2}x_{-1}z_0)^n}{(-1+y_{-2}x_{-1}z_0)^n(1+2y_{-2}x_{-1}z_0)^n}, \\ x_{6n} &= \frac{(-1)^n x_0(-1+z_{-2}y_{-1}x_0)^n(-1+2z_{-2}y_{-1}x_0)^n}{(-1+3z_{-2}y_{-1}x_0)^n}, \quad x_{6n+1} = \frac{x_{-2}(-1+x_{-2}z_{-1}y_0)^n}{(1+x_{-2}z_{-1}y_0)^{n+1}(-1+2x_{-2}z_{-1}y_0)^n}, \\ x_{6n+2} &= \frac{(-1)^n x_{-1}(1+y_{-2}x_{-1}z_0)^{n+1}}{(-1+y_{-2}x_{-1}z_0)^n(1+2y_{-2}x_{-1}z_0)^{n+1}}, \quad x_{6n+3} = \frac{(-1)^n x_0(-1+2z_{-2}y_{-1}x_0)^{n+1}(-1+z_{-2}y_{-1}x_0)^n}{(-1+3z_{-2}y_{-1}x_0)^{n+1}}, \end{aligned}$$

$$\begin{aligned} y_{6n-2} &= y_{-2}, \quad y_{6n-1} = y_{-1}, \quad y_{6n} = y_0, \quad y_{6n+1} = \frac{y_{-2}}{(1+y_{-2}x_{-1}z_0)}, \\ y_{6n+2} &= \frac{y_{-1}(-1+z_{-2}y_{-1}x_0)}{(-1+2z_{-2}y_{-1}x_0)}, \quad y_{6n+3} = \frac{y_0}{(1-x_{-2}z_{-1}y_0)}, \end{aligned}$$

and

$$\begin{aligned} z_{6n-2} &= \frac{(-1)^n z_{-2}(-1+3z_{-2}y_{-1}x_0)^n}{(-1+z_{-2}y_{-1}x_0)^n(-1+2z_{-2}y_{-1}x_0)^n}, \quad z_{6n-1} = \frac{z_{-1}(1+x_{-2}z_{-1}y_0)^n(-1+2x_{-2}z_{-1}y_0)^n}{(-1+x_{-2}z_{-1}y_0)^n}, \\ z_{6n} &= \frac{(-1)^n z_0(-1+y_{-2}x_{-1}z_0)^n(1+2y_{-2}x_{-1}z_0)^n}{(1+y_{-2}x_{-1}z_0)^n}, \quad z_{6n+1} = \frac{(-1)^n z_{-2}(-1+3z_{-2}y_{-1}x_0)^n}{(-1+z_{-2}y_{-1}x_0)^{n+1}(-1+2z_{-2}y_{-1}x_0)^n}, \end{aligned}$$

$$z_{6n+2} = \frac{-z_{-1}(1+x_{-2}z_{-1}y_0)^{n+1}(-1+2x_{-2}z_{-1}y_0)^n}{(-1+x_{-2}z_{-1}y_0)^n}, \quad z_{6n+3} = \frac{(-1)^{n+1}z_0(-1+y_{-2}x_{-1}z_0)^n(1+2y_{-2}x_{-1}z_0)^{n+1}}{(1+y_{-2}x_{-1}z_0)^{n+1}}.$$

Proof: As the proof of Theorem 2.1 and so will be omitted.

Lemma 6. Let $\{x_n, y_n, z_n\}$ be a positive solution of system (3), then the sequences $\{x_n, y_n\}$ are bounded and $\{x_n\}$ converges to zero.

Lemma 7. It is easy to see that $\{y_n\}$ periodic with period six.

Lemma 8. If $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0, z_{-2}, z_{-1}$ and z_0 arbitrary real numbers and let $\{x_n, y_n, z_n\}$ are solutions of system (3) then the following statements are true:-

(i) If $x_{-2} = 0, \quad y_0 \neq 0, \quad z_{-1} \neq 0$, then $x_{6n-2} = x_{6n+1} = 0$ *and* $y_{6n+3} = y_0, \; z_{6n-1} = z_{-1}, \; z_{6n+2} = -z_{-1}$.

(ii) If $x_{-1} = 0, \quad y_{-2} \neq 0, \quad z_0 \neq 0$, then $x_{6n-1} = x_{6n+2} = 0$ *and* $y_{6n+1} = y_{-2}, \; z_{6n} = z_0, \; z_{6n+3} = -z_0$.

(iii) If $x_0 = 0, \quad y_{-1} \neq 0, \quad z_{-2} \neq 0$, then $x_{6n} = x_{6n+3} = 0$ *and* $y_{6n+2} = y_{-1}, \; z_{6n-2} = z_{-2}, \; z_{6n+1} = -z_{-2}$.

(iv) If $y_{-2} = 0, \quad x_{-1} \neq 0, \quad z_0 \neq 0$, then $y_{6n-2} = y_{6n+1} = 0$ *and* $x_{6n-1} = x_{6n+2} = x_{-1}, \; z_{6n} = z_0, \; z_{6n+3} = -z_0$.

(v) If $y_{-1} = 0, \quad x_0 \neq 0, \quad z_{-2} \neq 0$, then $y_{6n-1} = y_{6n+2} = 0$ *and* $x_{6n} = x_{6n+3} = x_0, \; z_{6n-2} = z_{-2}, \; z_{6n+1} = -z_{-2}$.

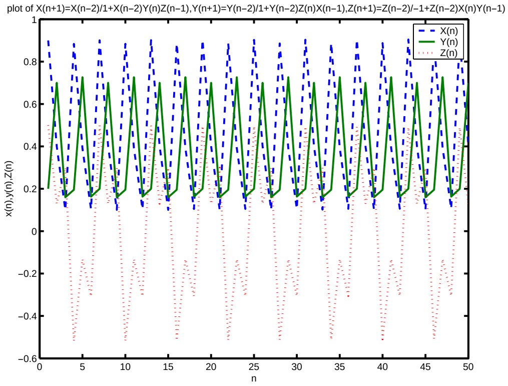

Figure 4.

(vi) If $y_0 = 0,\quad x_{-2} \neq 0,\quad z_{-1} \neq 0$, then $y_{6n} = y_{6n+3} = 0$ *and* $x_{6n-2} = x_{6n+1} = x_{-2},\ z_{6n-1} = z_{-1},\ z_{6n+2} = -z_{-1}$.

(vii) If $z_{-2} = 0,\quad x_0 \neq 0,\ y_{-1} \neq 0$, then $z_{6n-2} = z_{6n+1} = 0$ *and* $x_{6n} = x_{6n+3} = x_0,\ y_{6n+2} = y_{-1}$.

(viii) If $z_{-1} = 0,\quad x_{-2} \neq 0,\quad y_0 \neq 0$, then $z_{6n-1} = z_{6n+2} = 0$ *and* $x_{6n-2} = x_{6n+1} = x_{-2},\ y_{6n+3} = y_0$.

(ix) If $z_0 = 0,\ x_{-1} \neq 0,\ y_{-2} \neq 0$, then $z_{6n} = z_{6n+3} = 0$ *and* $x_{6n-1} = x_{6n+2} = x_{-1},\ y_{6n+1} = y_{-2}$.

Example 4. See Figure (4) for an example for the system (3) with the initial values $x_{-2} = 0.9,\ x_{-1} = 0.4,\ x_0 = 0.1,\ y_{-2} = 0.2,\ y_{-1} = 0.7,\ y_0 = 0.16,\ z_{-2} = 0.5,\ z_{-1} = 0.13$ *and* $z_0 = 0.3$.

5. THE FOURTH SYSTEM: $X_{N+1} = \frac{X_{N-2}}{1+X_{N-2}Z_{N-1}Y_N},\ Y_{N+1} = \frac{Y_{N-2}}{-1+Y_{N-2}X_{N-1}Z_N},\ Z_{N+1} = \frac{Z_{N-2}}{-1+Z_{N-2}Y_{N-1}X_N}$

In this section, we investigate the solutions of the system of three difference equations

$$x_{n+1} = \frac{x_{n-2}}{1 + x_{n-2}z_{n-1}y_n},\ y_{n+1} = \frac{y_{n-2}}{-1 + y_{n-2}x_{n-1}z_n},\ z_{n+1} = \frac{z_{n-2}}{-1 + z_{n-2}y_{n-1}x_n}, \tag{4}$$

where $n = 0, 1, 2, ...$, and the initial conditions are arbitrary non zero real numbers with $z_{-2}y_{-1}x_0 \neq 1$ and $y_{-2}x_{-1}z_0 \neq 1$.

Theorem 5.1. *Assume that $\{x_n, y_n, z_n\}$ are solutions of system (4). Then for $n = 0, 1, 2, ...,$*

$$\begin{aligned} x_{3n-2} &= \tfrac{x_{-2}}{1+nx_{-2}z_{-1}y_0}, \quad x_{3n-1} = \tfrac{x_{-1}(-1+y_{-2}x_{-1}z_0)}{(-1+(n+1)y_{-2}x_{-1}z_0)}, \quad x_{3n} = \tfrac{x_0}{1+nz_{-2}y_{-1}x_0}, \\ y_{3n-2} &= \tfrac{(-1)^{n+1}y_{-2}}{-1+ny_{-2}x_{-1}z_0}, \quad y_{3n-1} = \tfrac{(-1)^{n+1}y_{-1}(-1+z_{-2}y_{-1}x_0)}{(1+(n-1)z_{-2}y_{-1}x_0)}, \\ y_{3n} &= \tfrac{(-1)^n y_0}{1+nx_{-2}z_{-1}y_0}, \end{aligned}$$

and

$$\begin{aligned} z_{3n-2} &= \tfrac{(-1)^{n+1}z_{-2}(1+(n-1)z_{-2}y_{-1}x_0)}{(-1+z_{-2}y_{-1}x_0)}, \quad z_{3n-1} = (-1)^n z_{-1}(1 + nx_{-2}z_{-1}y_0), \\ z_{3n} &= \tfrac{(-1)^n z_0(-1+(n+1)y_{-2}x_{-1}z_0)}{(-1+y_{-2}x_{-1}z_0)}. \end{aligned}$$

Proof: As the proof of Theorem 2 and so will be omitted.

Lemma 9. Let $\{x_n, y_n, z_n\}$ be a positive solution of system (4), then the sequences $\{x_n\}$ are bounded and converges to zero.

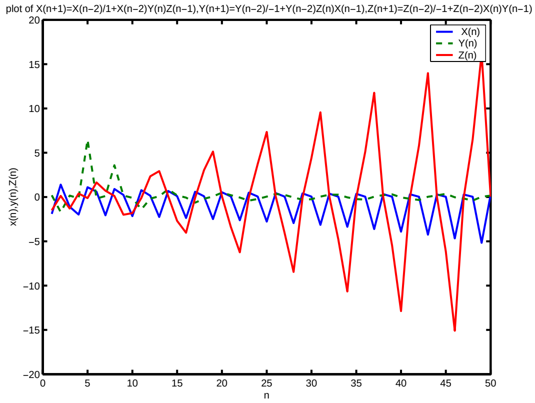

Figure 5.

Example 5. We assume the initial conditions $x_{-2} = -1.9,\ x_{-1} = 1.4,\ x_0 = -1.1, y_{-2} = 0.2,\ y_{-1} = -1.7,\ y_0 = 0.16,\ z_{-2} = -1.5,\ z_{-1} = 0.13$ *and* $z_0 = -1.3$, for the difference system (4), see Fig. 5.

Acknowledgements

This article was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. The authors, therefore, acknowledge with thanks DSR technical and financial support.

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Generalized Equi-Statistical Convergence of Positive Linear Operators

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Abstract

The concepts of λ-equi-statistical convergence, λ-statistical pointwise convergence and λ-statistical uniform convergence for sequences of functions were introduced recently by Srivastava, Mursaleen and Khan [Math. Comput. Modelling, 55 (2012) 2040–2051.]. In this paper, we apply the notion of λ-equi-statistical convergence to prove a Korovkin type approximation theorem by using test functions $1, \frac{x}{1+x}, (\frac{x}{1+x})^2$ and apply our result for the Bleimann, Butzer and Hahn [4] operators. We also study the rate of λ-equi-statistical convergence of a sequence of positive linear operators.

Keywords and phrases: Statistical convergence; λ-Statistical convergence; equi-statistical convergence; λ-equi-statistical convergence; positive linear operators; Korovkin type approximation theorem.

AMS subject classification (2000): 41A10, 41A25, 41A36, 40A30, 33C45.

1. Introduction and Preliminaries

The following concept of statistical convergence for sequences of real numbers was introduced by Fast [7].

Let $K \subseteq \mathbb{N}$ and $K_n = \{j \in K : j \leq n\}$, where $\mathbb{N}$ is the set of natural numbers . Then the *natural density* of K is defined by

$$\delta(K) := \lim_{n\to\infty} \frac{|K_n|}{n}$$

if the limit exists, where $|K_n|$ denotes the *cardinality* of the set K_n.

A sequence $x = (x_j)$ of real numbers is said to be *statistically convergent* to the number L if, for every $\epsilon > 0$, the set

$$\{j : j \in \mathbb{N} \quad \text{and} \quad |x_j - L| \geq \epsilon\}$$

has natural density zero, that is, if, for each $\epsilon > 0$, we have

$$\lim_n \frac{1}{n}\big|\{j : j \leq n \quad \text{and} \quad |x_j - L| \geq \epsilon\}\big| = 0.$$

The concept of λ-statistical convergence was studied in [14], [1] and [12] and further applied for deriving approximation theorems in [3], [5], [6], [13] and [15].

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that

$$\lambda_{n+1} \leq \lambda_n + 1 \qquad \text{and} \qquad \lambda_1 = 1.$$

Also let

$$I_n = [n - \lambda_n + 1, n] \qquad \text{and} \qquad K \subseteq \mathbb{N}.$$

Then the λ-*density* of K is defined by

$$\delta_\lambda(K) = \lim_{n\to\infty} \frac{1}{\lambda_n} \big|\{j : n - \lambda_n + 1 \leq j \leq n \quad \text{and} \quad j \in K\}\big|.$$

Clearly, in the special case when $\lambda_n = n$, the λ-density reduces to the above-defined natural density.

The number sequence $x = (x_j)$ is said to be λ-*statistically convergent* to the number L if, for each $\epsilon > 0$,

$$\delta_\lambda(K_\epsilon) = 0,$$

where

$$K_\epsilon = \{j : j \in I_n \quad \text{and} \quad |x_j - L| > \epsilon\},$$

that is, if, for each $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \big|\{j : j \in I_n \quad \text{and} \quad |x_j - L| > \epsilon\}\big| = 0.$$

In this case, we write

$$\mathrm{st}_\lambda\text{-} \lim_{n\to\infty} x_n = L$$

and we denote the set of all λ-statistically convergent sequences by S_λ.

The concept of equi-statistical convergence was introduced by Balcerzak *et al.* [2] and was subsequently applied for deriving approximation theorems in [8], [9] and [10]. In [16], the concepts of λ-equi-statistical convergence, λ-statistical pointwise convergence and λ-statistical uniform convergence for a sequence of real-valued functions were introduced. Further the notion of λ-equi-statistical convergence was used to prove a Korovkin type approximation theorem. In this paper, we prove such type of theorem by using the test functions 1, $\frac{x}{1+x}$ and $(\frac{x}{1+x})^2$.

Let f and f_n $(n \in \mathbb{N})$ be real-valued functions defined on a subset X of the set $\mathbb{N}$ of positive integers.

Definition 1.1. A sequence (f_n) of real-valued functions is said to be λ-*equi-statistically convergent* to f on X if, for every $\epsilon > 0$, the sequence $\big(S_n(\epsilon, x)\big)_{n\in\mathbb{N}}$ of real-valued functions converges uniformly to the zero function on X, that is, if, for every $\epsilon > 0$, we have

$$\lim_{n\to\infty} \|S_n(\epsilon, x)\|_{\mathcal{C}(X)} = 0,$$

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where

$$S_n(\epsilon, x) := \frac{1}{\lambda_n}\big|\{k : k \in I_n \quad \text{and} \quad |f_k(x) - f(x)| \geq \epsilon\}\big| = 0$$

and $\mathcal{C}(X)$ denotes the space of all continuous functions on X. In this case, we write

$$f_n \rightsquigarrow f \;\; (\lambda\text{-equi-stat}).$$

Definition 1.2. A sequence (f_n) is said to be λ-*statistically pointwise convergent* to f on X if, for every $\epsilon > 0$ and for each $x \in X$, we have

$$\lim_{n\to\infty} \frac{1}{\lambda_n}\big|\{k : k \in I_n \quad \text{and} \quad | f_k(x) - f(x) |\geq \epsilon\}\big| = 0.$$

In this case, we write

$$f_n \longrightarrow f \;\; (\lambda\text{-stat}).$$

Definition 1.3. A sequence (f_n) is said to be λ-*statistically uniform convergent* to f on X if (for every $\epsilon > 0$), we have

$$\lim_{n\to\infty} \frac{1}{\lambda_n}\big|\{k : k \in I_n \quad \text{and} \quad \|f_k - f\|_{\mathcal{C}(X)} \geq \epsilon\}\big| = 0.$$

In this case, we write

$$f_n \rightrightarrows f \;\; (\lambda\text{-stat})$$

Definition 1.4 (see [10]). A sequence (f_n) of real-valued functions is said to be *equi-statistically convergent* to f on X if, for every $\epsilon > 0$, the sequence $\big(P_{n,\epsilon}(x)\big)_{n\in\mathbb{N}}$ of real-valued functions converges uniformly to the zero function on X, that is, if (for every $\epsilon > 0$) we have

$$\lim_{n\to\infty} \|P_{n,\epsilon}(x)\|_{\mathcal{C}(X)} = 0,$$

where

$$P_{n,\epsilon}(x) = \frac{1}{n}\big|\{k : k \leqq n \quad \text{and} \quad |f_k(x) - f(x)| \geq \epsilon\}\big| = 0.$$

In this case, we write

$$f_n \rightsquigarrow f \;\; (\text{equi-stat}).$$

The following implications of the above definitions and concepts are trivial.

Lemma 1.1. *Each of the following implications holds true*:

$$f_n \rightrightarrows f \;\; (\lambda\text{-stat}) \Longrightarrow f_n \rightsquigarrow f \;\; (\lambda\text{-equi-stat}) \Longrightarrow f_n \to f \;\; (\lambda\text{-stat}).$$

Furthermore, in general, the reverse implications do not hold true.

Example 1.1. Let $\lambda = (\lambda_n)$ be a sequence as described above and consider the sequence of continuous functions $f_r : [0, 1] \to \mathbb{R}$ $(r \in \mathbb{N})$, defined as follows:

$$f_r(x) = \begin{cases} -\dfrac{(r+1)^2\left(x - \dfrac{1}{r+1}\right)\left(x + \dfrac{1}{r+1}\right)}{x^2+1} & \left(x \in \left[0, \frac{1}{r+1}\right]\right) \\[2ex] 0 & (\text{otherwise}). \end{cases}$$

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Then, for every $\epsilon > 0$, we have

$$\frac{1}{\lambda_n}\big|\{r : r \in I_n \quad \text{and} \quad |f_r(x)| \geq \epsilon\}\big|$$
$$\leq \frac{1}{\lambda_n} \to 0 \qquad (n \to \infty)$$

uniformly in x. This implies that $f_r \rightsquigarrow 0$ (λ-equi-stat). But, since $\sup_{x\in[0,1]} |f_r(x)| = 1 \quad (r \in \mathbb{N})$, we conclude that the following condition: $f_r \rightrightarrows 0$ (λ-stat) does not hold true.

Example 1.2. Let $\lambda_n \doteq [\sqrt{n}\,]$ and consider the sequence of continuous functions

$$f_r : [0,1] \to \mathbb{R} \qquad \big(f_r(x) = x^r \ \ (r \in \mathbb{N})\big).$$

If f is the pointwise limit of f_r (in the ordinary sense), then

$$f_r \to f \ \ (\lambda\text{-stat}),$$

but the condition:

$$f_r \rightsquigarrow f \ \ (\text{equi-stat})$$

does not hold true. Let us take $\epsilon = \frac{1}{2}$. Then, for all $n \in \mathbb{N}$, there exists $r > N$ such that

$$m \in [n - [\lambda_n] + 1, n] \qquad \text{and} \qquad x \in \left(\sqrt[n]{\frac{1}{2}}, 1\right),$$

so that

$$|f_m(x)| = |x^m| \geq \left|\left(\sqrt[n]{\frac{1}{2}}\right)^m\right| \geq \left|\left(\sqrt[n]{\frac{1}{2}}\right)^n\right| = \frac{1}{2}.$$

2. Main Result

Let $\mathcal{C}[a,b]$ be the linear space of all real-valued continuous functions f on $[a,b]$ and let T be a linear operator which maps $\mathcal{C}[a,b]$ into itself. We say that T is *positive* if, for every non-negative $f \in \mathcal{C}[a,b]$, we have

$$T(f,x) \geq 0 \qquad (x \in [a,b]).$$

We know that $\mathcal{C}[a,b]$ is a Banach space with the norm given by

$$\|f\|_{\mathcal{C}[a,b]} := \sup_{x\in[a,b]} |f(x)| \qquad (f \in \mathcal{C}[a,b]).$$

The classical Korovkin approximation theorem states as follows [11].

Let $\{T_n\}$ be a sequence of positive linear operators from $\mathcal{C}[a,b]$ into $\mathcal{C}[a,b]$. Then

$$\lim_n \|T_n(f,x) - f(x)\|_{\mathcal{C}[a,b]} = 0 \qquad (f \in \mathcal{C}[a,b])$$
$$\iff \quad \lim_n \|T_n(f_i,x) - e_i(x)\|_{\mathcal{C}[a,b]} = 0 \qquad (i = 0,1,2),$$

where

$$e_i(x) = x^i \qquad (i = 0, 1, 2).$$

In [16], the following result was proved which is an extension of the result of Karakuş *et al.* [10].

Theorem 2.1. *Let X be a compact subset of the set $\mathbb{R}$ of real numbers. Also let $\{L_n\}$ be a sequence of positive linear operators from $\mathcal{C}(X)$ into itself. Then, for all $f \in \mathcal{C}(X)$, $L_n(f) \rightsquigarrow f(\lambda\text{-equi-stat})$ on X if and only if $L_n(e_i) \rightsquigarrow e_i(\lambda\text{-equi-stat})$ on X.*

In this paper, we prove such type of theorem by using the test functions 1, $\frac{x}{1+x}$ and $(\frac{x}{1+x})^2$.

Let $K = [0, \infty)$ and $C_B(K)$ denote the space of all bounded and continuous real valued functions on K equipped with norm

$$\|f\|_{C_B(K)} := \sup_{x \in K} |f(x)|, \; f \in C_B(K).$$

Let $H_\omega(K)$ denote the space of all real valued functions f on K such that

$$| f(s) - f(x) | \leq \omega(f; |\frac{s}{1+s} - \frac{x}{1+x} |),$$

where ω is the modulus of continuity, i.e.

$$\omega(f; \delta) = \sup_{s,x \in K} \{|f(s) - f(x)| : |s - x | \leq \delta\} \; (\delta > 0).$$

It is to be noted that any function $f \in H_\omega(K)$ is bounded and continuous on K, and a necessary and sufficient condition for $f \in H_\omega(K)$ is that

$$\lim_{\delta \to 0} \omega(f; \delta) = 0.$$

We prove the following theorem.

Theorem 2.2. *Let $\{L_n\}$ be a sequence of positive linear operators from $H_\omega(K)$ into $C_B(K)$. Then, for all $f \in H_\omega(K)$,*

$$L_n(f) \rightsquigarrow f(\lambda\text{-equi-stat}) \tag{2.1}$$

if and only if

$$L_n(f_i) \rightsquigarrow g_i(\lambda\text{-equi-stat}) \; (i = 0, 1, 2), \qquad (2.2).$$

with

$$g_0(x) = 1, \;\; g_1(x) = \frac{x}{1+x} \quad \text{and} \quad g_2(x) = (\frac{x}{1+x})^2.$$

Proof. Since each of the functions f_i belongs to $H_\omega(K)$, conditions (3.2) follow immediately. Let $g \in H_\omega(K)$ and $x \in K$ be fixed. Then for $\varepsilon > 0$ there exist $\delta > 0$ such that $| f(s) - f(x) | < \varepsilon$ holds for all $s \in K$ satisfying $|\frac{s}{1+s} - \frac{x}{1+x}| < \delta$. Let

$$K(\delta) := \{s \in K : |\frac{s}{1+s} - \frac{x}{1+x} | < \delta\}.$$

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Hence

$$| f(s) - f(x) |=| f(s) - f(x) |_{\chi_{K(\delta)}(s)} + | f(s) - f(x) |_{\chi_{K\setminus K(\delta)}(s)}$$
$$\leq \varepsilon + 2N_{\chi_{K\setminus K(\delta)}(s)}, \tag{2.3}$$

where χ_D denotes the characteristic function of the set D and $N = \|f\|_{C_B(K)}$. Further we get

$$\chi_{K\setminus K(\delta)}(s) \leq \frac{1}{\delta^2}(\frac{s}{1+s} - \frac{x}{1+x})^2. \tag{2.4}$$

Combining (2.3) and (2.4), we get

$$| f(s) - f(x) |\leq \varepsilon + \frac{2N}{\delta^2}(\frac{s}{1+s} - \frac{x}{1+x})^2. \tag{2.5}$$

After using the linearity and positivity of operators $\{L_n\}$, we get

$$| L_n(f;x) - f(x) |\leq \varepsilon + M\{ \mid L_n(g_0;x) - g_0(x) \mid + \mid L_n(g_1;x) - g_1(x) \mid$$
$$+ \mid L_n(g_2;x) - g_2(x) \mid + \mid L_n(g_3;x) - g_3(x) \mid \}, \tag{2.6}$$

which implies that

$$|L_n(f;x) - f(x)| \leq \epsilon + B\sum_{i=o}^{2} |L_n(g_i;x) - g_i(x)|, \tag{2.7}$$

where $M := \varepsilon + N + \frac{4N}{\delta^2}$. Now for a given $r > 0$, choose $\epsilon > 0$ such that $\epsilon < r$. Then, for each $i = 0, 1, 2$, set $\psi_n(x, r) := |\{k \in I_n : |L_k(f;x) - f(x)| \geq r\}|$ and $\psi_{i,n}(x, r) := |\{k \in I_n : |L_k(g_i;x) - g_i(x)| \geq \frac{r-\epsilon}{3K}\}|$ for $(i = 0, 1, 2)$, it follows from (2.7) that $\psi_n(x, r) \leq \sum_{i=o}^{2} \psi_{i,n}(x, r)$. Hence

$$\frac{\|\psi_n(.,r)\|_{C_B(K)}}{\lambda_n} \leq \sum_{i=o}^{2}\left(\frac{\|\psi_{i,n}(.,r)\|_{C_B(K)}}{\lambda_n}\right). \tag{2.8}$$

Now using the hypothesis (2.2) and the Definition 1.1, the right hand side of (2.8) tends to zero as $n \to \infty$. Therefore, we have

$$\lim_{n\to\infty} \frac{\|\psi_n(.,r)\|_{C_B(K)}}{\lambda_n} = 0 \text{ for every } r > 0,$$

i.e. (2.1) holds.

This completes the proof of the theorem.

3. An Application

We shall apply our result for the following Bleimann, Butzer and Hahn [4] operators: Let

$$B_n(f;x) := \frac{1}{(1+x)^n}\sum_{k=0}^{n} f\left(\frac{j}{n-k+1}\right)\binom{n}{k}x^k,$$

be a sequence of positive linear operators from $H_\omega(K)$ into $C_B(K)$, $K=[0,\infty)$ and $n\in\mathbb{N}$. Then, for all $f\in H_\omega(K)$,

$$B_n(f)\rightsquigarrow f(\lambda\text{-equi-stat}).$$

Since

$$(1+x)^n=\sum_{k=0}^{n}\binom{n}{k}x^k \ ,$$

it is easy to see that

$$\begin{aligned} B_n(f_0;x) &= \frac{1}{(1+x)^n}\sum_{k=0}^{n}\binom{n}{k}x^k \\ &= 1=g_0(x). \end{aligned}$$

Also by simple calculation, we obtain

$$B_n(f_1;x)=\frac{n}{n+1}\left(\frac{x}{1+x}\right)\to\frac{x}{1+x}=g_1(x),$$

and

$$\begin{aligned} B_n(f_2;x) &= \frac{n(n-1)}{(n+1)^2}\left(\frac{x}{1+x}\right)^2+\frac{n}{(n+1)^2}\left(\frac{x}{1+x}\right) \\ &\to (\frac{x}{1+x})^2=g_2(x). \end{aligned}$$

Therefore

$$B_n(f_i;x)\to g_i(x)\ \ (n\to\infty)\ (i=0,1,2),$$

and cosequently, we have

$$B_n(f_i)\rightsquigarrow g_i(\lambda\text{-equi-stat})\ (i=0,1,2).$$

Hence by Theorem 2.2, we have

$$B_n(f)\rightsquigarrow f(\lambda\text{-equi-stat}).$$

4. Rate of $\lambda-$ equistatistical convergence

In this section we study the rate of λ-equi-statistical convergence of a sequence of positive linear operators as given in [16].

Definition 4.1. Let (a_n) be a positive non-increasing sequence. A sequence (f_n) is equi-statistically convergent to a function f with the rate $o(a_n)$ if for every $\epsilon>0$,

$$\lim_{n\to\infty}\frac{\Lambda_n(x,\epsilon)}{a_n}=0$$

uniformly with respect to $x \in K$ or equivalently, for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{\|\Lambda_n(.,\epsilon)\|_{C_B(X)}}{a_n} = 0,$$

where

$$\Lambda_n(x,\epsilon) := \frac{1}{\lambda_n} \mid \{k \in I_n :\mid f_k(x) - f(x) \mid\geq \varepsilon\} \mid= 0.$$

In this case, it is denoted by $f_n - f = o(a_n))$ (λ-equi-stat) on K.

We have the following basic lemma.

Lemma 4.1. *Let (f_n) and (g_n) be sequences of function belonging to $\mathcal{C}(X)$. Assume that $f_n - f = o(a_n)$ (λ-equi-stat) on X and $g_n - g = o(b_n)$ (λ-equi-stat) on X . Let $c_n = \max\{a_n, b_n\}$. Then the following statement holds:*

(i) $(f_n + g_n) - (f + g) = o(c_n)$ *(λ-equi-stat) on X ,*

(ii) $(f_n - f)(g_n - g) = o(a_n b_n)$ *(λ-equi-stat) on X,*

(iii) $\mu(f_n - f) = o(a_n)$ *(λ-equi-stat) on X for any real number μ,*

(iv) $\sqrt{|f_n - f|} = o(a_n)$ *(λ-equi-stat) on X.*

We recall that the modulus of continuity of a function $f \in H_\omega(K)$ is defined by

$$\omega(f;\delta) = \sup_{s,x\in K} \{|f(s) - f(x)| : |s - x \mid\leq \delta\} \ (\delta > 0).$$

Now we prove the following result.

Theorem 4.2. *Let X be a compact subset of the real numbers, and $\{L_n\}$ be a sequence of positive linear operators from $H_\omega(K)$ into $C_B(K)$. Assume that the following conditions hold:*
(a) $L_n(g_0; x) - g_0 = o(a_n)$ *(λ-equi-stat) on K,*
(b) $\omega(f, \delta_n) = o(b_n)$ *(λ-equi-stat) on K , where* $\delta_n(x) = \sqrt{L_n(\phi^2; x)}$
with $\phi(x) = \left(\frac{s}{1+s} - \frac{x}{1+x}\right)$*. Then for all $f \in H_\omega(K)$, we have*

$$L_n(f) - f = o(c_n) \ (\lambda\text{-}equi\text{-}stat) \ on \ K,$$

where $c_n = \max\{a_n, b_n\}$.

Proof. Let $f \in H_\omega(K)$ and $x \in K$. Then it is well known that,

$$|L_n(f;x) - f(x)| \leq M|L_n(g_0;x) - g_0(x)| + (L_n(g_0;x) + \sqrt{L_n(g_0;x)})\omega(f,\delta_n),$$

where $M = \|f\|_{H_\omega(K)}$. This yields that

$$\begin{aligned} |L_n(f;x) - f(x)| &\leq M|(L_n(g_0;x) - g_0(x)| + 2\omega(f,\delta_n) \\ &\quad +\omega(f,\delta_n)|(L_n(g_0;x) - g_0(x)| + \omega(f,\delta_n)\sqrt{|(L_n(g_0;x) - g_0(x)|}. \end{aligned}$$

Now using the conditions (a), (b) and Lemma 4.1 in the above inequality, we get $L_n(f) - f = o(c_n)$ (λ-equi-stat) on K.
This completes the proff of the theorem.

Acknowledgement
This Project was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, under grant No. (197/130/1434). The authors, therefore, acknowledge with thanks DSR technical and financial support.

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Volume 19, Number 5 November 2015
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

An international publication of Eudoxus Press, LLC
(twelve times annually)

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Journal of Computational Analysis and Applications

A quartely international publication of Eudoxus Press, LLC, of TN.

Editor in Chief: George Anastassiou

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ON THE INTERVAL-VALUED PSEUDO-LAPLACE TRANSFORM BY MEANS OF THE INTERVAL-VALUED PSEUDO-INTEGRAL

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Abstract. Pap and Ralevic (1998) introduced the pseudo-Laplace transform and proved the analog of exchange formula. Jang (2013) defined the interval-valued generalized fuzzy integral by using an interval- representable pseudo-operations and Jang (2014) defined the interval- valued $\bar{g}$-integral.

In this paper, by using the concept of the interval-valued pseudo-integral, we define interval-valued pseudo-Laplace transform. Furthermore, we investigate pseudo-exchange formula and inverse of pseudo-Laplace transform.

1. Introduction

Many researchers [3,4,15,16,17,18,19,21,22] have studied the theory of the pseudo-integral, for examples, fuzzy integrals, generalized fuzzy integral, and generated pseudo-integral, etc. Aubin [1], Aumann [2], Grbic et al. [5], Grabisch [6], Guo and Zhang [7], Jang [7-14], and Wechselberger [20] have been researching various integrals of measurable multi-valued functions which are used for representing uncertain functions.

Pap and Ralevic [17] introduced the pseudo-Laplace transform and proved the analogue of exchange formula. They also discussed inverse of pesudo-Laplace transform.

The purpose of this paper is to define the interval-valued pseudo-Laplace transform and investigate some characterizations of them. Furthermore, we also investigate pseudo-exchange formula and inverse of pseudo-Laplace transform.

The paper is organized in five sections. In section 2, we list definitions and some properties of the pseudo-integral and the pseudo-Laplace transform by means of the corresponding pseudo-interval. We also discuss the special forms of the pseudo-Laplace transform. In section 3, we define the interval-valued pseudo-Laplace transform and investigate some characterizations of them. Furthermore, we discuss the special forms of the interval-valued pseudo-Laplace transform. In section 4, we prove pseudo-exchange formula of the special forms of interval-valued pseudo-Laplace transform and discuss inverse of them. In section 5, we give a brief summary results and some conclusions.

1991 *Mathematics Subject Classification.* 28E10, 28E20, 03E72, 26E50 11B68.

Key words and phrases. σ-$\bigoplus$-decomposable measure, pseudo-integral, pseudo-Laplace transform, pseudo-exchange formular, interval-representable pseudo-operations, interval-valued pseudo-Laplace transform .

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2. Definitions and Preliminaries

In this section, we introduce a pseudo-addition and a pseudo-multiplication. Let $[a,b]$ be a closed (in some cases can be considered semiclosed) subinterval of $\bar{\mathbb{R}} = [-\infty, \infty]$. The full order on $[a,b]$ will be denoted by $\preceq$.

Definition 2.1. ([3,4,5,12-19,21,22]) (1) The pseudo-addition $\oplus$ is an operation $\oplus : [a,b] \times [a,b] \longrightarrow [a,b]$ which is commutative, non-decreasing with respect to $\preceq$, that is, $x \preceq y$ implies $x \oplus z \preceq y \oplus z$ for all $z \in [a,b]$, associative, and with a zero (natural) element denoted by $\mathbf{0}$, that is, for each $x \in [a,b]$, $\mathbf{0} \oplus x = x$ holds (usually $\mathbf{0}$ is either a or b).

(2) The pseudo-multiplication $\odot$ is an operation $\odot : [a,b] \times [a,b] \longrightarrow [a,b]$ which is commutative, positively non-decreasing with respect to $\preceq$, that is, $x \preceq y$ implies $x \odot z \preceq y \odot z$ for all $z \in [a,b]+$, associative, and there exists a unit element $\mathbf{1} \in [a,b]$, that is, for each $x \in [a,b], \mathbf{1} \odot x = x$, where $[a,b]_+ = \{x | x \in [a,b], \mathbf{0} \preceq x\}$.

Definition 2.2. ([16,17,21,22]) (1) Let X be a set and $\mathcal{M}$ be a σ -algebra of subsets of X. An elementary measurable function e is a mapping $e : X \longrightarrow [a,b]$ if it has the following representation :

$$e(x) = \oplus_{i=1}^{\infty} a_i \odot \chi_{A_i}(x), \tag{1}$$

where $a_i \in [a,b], A_i \in \mathcal{M}$ for $i = 1, 2, \cdots$, $x \in X$, and pseudo-characteristic function χ_A of A is defines by

$$\chi_A(x) = \begin{cases} \mathbf{1} & \text{for } x \in A \\ \mathbf{0} & \text{for } x \notin A. \end{cases}$$

(2) The pseudo-integral of an elementary function e with respect to the $\sigma-\oplus$-decomposable measure μ is

$$\int^{\oplus} e \odot d\mu = \oplus_{i=1}^{\infty} a_i \odot \mu(A_i) \tag{2}$$

where $A_i \in \mathcal{M}$ disjoint if $\oplus$ is not idempotent. (3) The pseudo-integral of a measurable function $f : X \longrightarrow [a,b]$ is defined by

$$\int^{\oplus} f \odot d\mu = \lim_{n\to\infty} \int^{\oplus} e_n \odot d\mu \tag{3}$$

where $\{e_n\}$ is the sequence of elementary functions with $\lim_{n\to\infty} e_n = f$.

(4) A measurable function f is said to be integrable if $\int^{\oplus} f \odot d\mu$ is finite.

Let $(X, +)$ be a commutative group for $X \subset \mathbb{R}$ and $([a,b], \oplus, \odot)$ be a semiring. We also consider the pseudo-Laplace transform of a function.

Definition 2.3. ([17]) (1) The semiring $B(X, [a,b])$ is the set of all bounded (with respect to the order in $[a,b]$) functions.

(2) The space $L_1(X)$ is the set of Lebesgue integrable functions which satisfy the condition $\int_X |f(x)| dx < +\infty$.

(3) The pseudo-character of the group $(X, +), X \subset \mathbb{R}$ is a map $\xi : X \to [a,b]$, of the group $(X, +)$ into semiring $([a,b], \oplus, \odot)$, with the property

$$\xi(x+y) = \xi(x) \odot \xi(y),\ x, y \in X. \tag{4}$$

(4) The pseudo-Laplace transform $\mathfrak{L}^{\oplus}(f)$ of a function $f \in B(X, [a, b])$ is defined by

$$(\mathfrak{L}^{\oplus} f)(\xi)(z) = \int_{X \cap [0,\infty)}^{\oplus} \xi(x, -z) \odot dm_f \tag{5}$$

where ξ is the continuous pseudo-character for $z \in \mathbb{R}$ for which the right side is meaningful $m_f(A) = \inf_{x \in A} f(x)$, and $dm_f = f \odot m_f$.

Remark 2.4. ([17]) We have the special cases the following forms of the pseudo-Laplace transform for $X = \mathbb{R}$.

(1) If $\oplus = \max$, $\odot = +$, and $\xi(x, -z) = -xz$, then we have $(\mathfrak{L}_1^{\oplus} f)(z) = \sup_{x \geq 0}(-xz + f(x))$

(2) If $\oplus = \max$, $\odot = \cdot$, and $\xi(x, -z) = e^{-xz}$, then we have $(\mathfrak{L}_2^{\oplus} f)(z) = \sup_{x \geq 0}(e^{-xz} f(x))$

(3) If $x \oplus y = g^{-1}(g(x) + g(y))$, $x \odot y = g^{-1}(g(x)g(y))$, and $\xi(x, -z) = e^{-xz}$, then we have $(\mathfrak{L}_3^{\oplus} f)(z) = g^{-1}(\int_0^{\infty} e^{-xz} g(f(x))dx)$
where $g : [-\infty, \infty] \to [-\infty, \infty], g(0) = 0$ is an odd, strictly increasing, continuous function, $f \in B(X, [a, b])$, and $g|f| \in L_1(X)$.

We note that $\mathfrak{L}_3^{\oplus}(f)(z) = g^{-1}(\mathfrak{L}(g \circ f)(z))$, where $(\mathfrak{L} f)(z) = \int_0^{\infty} e^{-zx} f(x)dx$, and that $(f * h)(x) = \int f(x - t)h(t)dt$ for all $x \in X$. In [17], they proved the following pseudo-exchange formula and inverse of the pseudo-Laplace transform.

Theorem 2.1. ([17]) *Let $f_1, f_2 \in B(X, [a, b])$. Then we have*

$$\mathfrak{L}_k^{\oplus}(f_1 *_k f_2) = \mathfrak{L}^{\oplus}(f_1) \odot \mathfrak{L}^{\oplus}(f_2), \ \textit{for } k = 1, 2, 3 \tag{6}$$

where $(f_1 *_1 f_2)(x) = \sup_{0 \leq t \leq x}[f_1(x - t) + f_2(t)]$, $(f_1 *_2 f_2)(x) = \sup_{0 \leq y \leq x}(f_1(y) f_2(x - y))$, *and* $(f_1 *_3 f_2)(x) = g^{-1}((g \circ f_1 * g \circ f_2)(x))$.

Theorem 2.2. ([17]) *If $f \in B(X, [a, b])$ and $\mathfrak{L}_k^{\oplus}(f) = F_k$ for $k = 1, 2, 3$, and there exists $(\mathfrak{L}_k^{\oplus})^{-1}(F_k)$, then we have*

(1) $(\mathfrak{L}_1^{\oplus})^{-1}(F_1)(x) = \inf_{z \geq 0}(xz + F(z))$,

(2) $(\mathfrak{L}_2^{\oplus})^{-1}(F_2)(x) = \inf_{z \geq 0}(e^{xz} F(z))$, *and*

(3) $(\mathfrak{L}_3^{\oplus})^{-1}(F_3)(x) = g^{-1}(\mathfrak{L}^{-1}(g \circ F)(x))$,

where $f \in B(X, I([a, b])), g|f| \in L_1(X), \mathfrak{L}(f) = \int_0^{\infty} e^{-zx} f(x)dx = F(z)$ *and* $\mathfrak{L}^{-1}(F) = f(x)$.

3. The interval-valued pseudo-Laplace transforms

In this section, we consider a standard interval-valued addition and a standard interval-valued pseudo-multiplication (see [12,13]). Let $I([a, b])$ be the set of all bounded closed intervals in $[a, b]$ as follows :

$$I([a, b]) = \{\bar{u} = [u_l, u_r] \mid u_l, u_r \in [a, b] \text{ and } u_l \preceq u_r\}. \tag{7}$$

We note that the full order on $I([a, b])$ will be denoted by $\preceq_i$. For any $a \in [a, b]$, we define $a = [a, a]$. Obviously, $a \in I([a, b])$.

Definition 3.1. ([12]) (1) An operation $\bigoplus : I([a, b])^2 \longrightarrow I([a, b])$ is called a standard interval-valued pseudo-addition if there exist pseudo-additions $\oplus_l$ and $\oplus_r$ such that $x \oplus_l y \preceq x \oplus_r y$ for all $x, y \in [a, b]$ and such that for each $\bar{u} = [u_l, u_r]$, $\bar{v} = [v_l, v_r] \in I([a, b])$,

$$\bar{u} \bigoplus \bar{v} = [u_l \oplus_l v_l, u_r \oplus_r v_r]. \tag{8}$$

Then $\oplus_l$ and $\oplus_r$ are called represents of $\bigoplus$.

(2) An operation $\bigodot : I([a,b])^2 \longrightarrow I([a,b])$ is called a standard interval-valued pseudo-multiplication if there exist pseudo-multiplications $\odot_l$ and $\odot_r$ such that $x \odot_l y \leq x \odot_r y$ for all $x, y \in [a,b]$ and such that for each $\bar{u} = [u_l, u_r]$, $\bar{v} = [v_l, v_r] \in I([a,b])$,

$$\bar{u} \bigodot \bar{v} = [u_l \odot_l v_l, u_r \odot_r v_r]. \tag{9}$$

Then $\odot_l$ and $\odot_r$ are called represents of $\bigodot$.

Definition 3.2. ([12]) If $\bar{u} = [u_l, u_r], \bar{v} = [v_l, v_r] \in I([a,b])$ and $k \in [a,b]$, then we define

(1) $\bar{u} \bigoplus \bar{v} = [u_l \oplus_l v_l, u_r \oplus_r v_r]$,

(2) $k \bigodot \bar{u} = [k \odot_l v_l, k \odot_r v_r]$,

(3)$\bar{u} \bigodot \bar{v} = [u_l \odot_l v_l, u_r \odot_r v_r]$,

(4) $\bar{u} \vee \bar{v} = [u_l \vee v_l, u_r \vee v_r]$,

(5) $\bar{u} \wedge \bar{v} = [u_l \wedge v_l, u_r \wedge v_r]$,

(6) $\bar{u} \preceq_i \bar{v}$ if and only if $u_l \preceq_l v_l$ and $u_r \preceq_r v_r$,

(7) $\bar{u} \prec_i \bar{v}$ if and only if $\bar{u} \preceq_r \bar{v}$ and $\bar{u} \neq \bar{v}$, and

Definition 3.3. ([11]) If $K \subset \mathbb{R}$ and $\bar{u}_\alpha = [u_{\alpha l}, v_{\alpha r}]$ for all $\alpha \in K$, then we define

(1) $\sup_{\alpha \in K} \bar{u}_\alpha = [\sup_{\alpha \in K} u_{\alpha_l}, \sup_{\alpha \in K} u_{\alpha_r}]$, and

(2) $\inf_{\alpha \in K} \bar{u}_\alpha = [\inf_{\alpha \in K} u_{\alpha_l}, \inf_{\alpha \in K} u_{\alpha_r}]$.

Definition 3.4. An interval-valued mapping $\bar{\mu} = [\mu_l, \mu_r] : \mathcal{M} \longrightarrow I([a,b])$ is called the interval-valued $\sigma - \bigoplus$-decomposable measure $\bar{\mu} = [\mu_l, \mu_r]$ if μ_l and μ_r are $\sigma - \oplus$- decomposable measures.

Let $(X, +)$ be a commutative group for $X \subset \mathbb{R}$ and $(I([a,b]), \bigoplus, \bigodot)$ be a semiring. In [8-14], we see that if d_H is the Hausdorff metric on $I([a,b])$, that is,

$$d_H(A, B) = \max\{\sup_{x \in A} \inf_{y \in B} |x - y|, \inf_{x \in A} \sup_{x \in B} |x - y|\} \tag{10}$$

then we have

$$d_H - \lim_{n \to \infty} [u_n, v_n] = [u, v] \text{ if and only if}$$
$$\lim_{n \to \infty} u_n = u \text{ and } \lim_{n \to \infty} v_n = v. \tag{11}$$

Definition 3.5. (1) The semiring $IB(X, I([a,b]))$ is the set of all bounded (with respect to the pseudo-order $\preceq_i$ in $I([a,b])$) interval-valued functions.

(2) The space $IL_1(X)$ be the set of all Lebesgue integrable interval-valued functions $\bar{f} = [f_l, f_r]$ which satisfy the conditions $\int_X |f_l(x)| dx < \infty$ and $\int_X |f_r(x)| dx < \infty$.

(3) The interval-valued pseudo-character of the group $(X, +), X \subset \mathbb{R}$ is an interval-valued map $\bar{\xi} = [\xi_l, \xi_r] : X \to I([a,b])$, of the group $(X, +)$ into semiring $(I([a,b]), \bigoplus, \bigodot)$, with the property

$$\bar{\xi}(x + y) = \bar{\xi}(x) \bigodot \bar{\xi}(y), \text{ for all } x, y \in X. \tag{12}$$

Definition 3.6. (1) Let X be a set and $\mathcal{M}$ be a σ-algebra of subset of X. A measurable interval-valued elementary function $\bar{e} = [e_l, e_r]$is an interval-valued mapping $\bar{e} : X \longrightarrow I([a,b])$ if it has the following representation :

$$\bar{e}(x) = \bigoplus_{i=1}^{\infty} \bar{a_i} \bigodot \chi_{A_i}(x) \tag{13}$$

where $\bar{a_i} = [a_{il}, a_{ir}] \in I([a,b])$, and χ_{A_i} is a pseudo-characteristic function of A_i for all $i = 1, 2, \cdots$.

(2) The interval-valued pseudo-integral of an interval-valued elementary function $\bar{e} = [e_l, e_r]$ with respect to the $\sigma - \bigoplus$-decomposable measure $\bar{\mu}$ is

$$\int^{\bigoplus} \bar{e} \bigodot d\bar{\mu} = \bigoplus_{i=1}^{\infty} \bar{a_i} \bigodot \bar{\mu}(A_i). \tag{14}$$

where $\bar{\mu} = [\mu_l, \mu_r], \mu_l, \mu_r$ are $\sigma - \bigoplus$-decomposable measures, and $A_i \in \mathcal{M}$ disjoint if $\oplus_l$ and $\oplus_r$ are not idempotent.

(3) The interval-valued pseudo-integral of a measurable interval-valued function $\bar{f} = [f_l, f_r] : [a,b] \longrightarrow I([a,b])$ is defined by

$$\int^{\bigoplus} \bar{f} \bigodot d\bar{\mu} = d_H - \lim_{n\to\infty} \int^{\bigoplus} \bar{e_n} \bigodot d\bar{\mu} \tag{15}$$

where $\{\bar{e_n}\}$ is the sequence of interval-valued elementary functions with $d_H - \lim_{n\to\infty} \bar{e_n} = \bar{f}$.

(4) An interval-valued measurable function $\bar{f}$ is said to be integrable if $\int^{\bigoplus} \bar{f} \bigodot d\bar{\mu} \in I([a,b])$.

By a standard interval-valued addition and a standard interval-valued multiplication, we define the interval-valued pseudo-Laplace transforms as follows.

Definition 3.7. The interval-valued pseudo-Laplace transform $\mathfrak{L}_i^{\bigoplus}(\bar{f})$ of an interval-valued function $\bar{f} = [f_1, f_r] \in B(X, I([a,b]))$ is defined by

$$(\mathfrak{L}_i^{\bigoplus} \bar{f})(\bar{\xi})(z) = \int_{X\cap[0,\infty)}^{\bigoplus} \bar{\xi}(x, -z) \bigodot dm_{\bar{f}} \tag{16}$$

where $\bar{\xi} = [\xi_l, \xi_r]$ is the d_H-continuous interval-valued pseudo-character for $z \in \mathbb{R}$ for which the right hand side is meaningful and $m_{\bar{f}}(A) = \inf_{x\in A} \bar{f}(x)$, and $dm_{\bar{f}} = \bar{f} \bigodot m_{\bar{f}}$.

We obtain some characterizations of the standard interval-valued pseudo-characters and the interval-valued pseudo-Laplace transforms.

Theorem 3.1. *If $\oplus_l$ and $\oplus_r$ are represents of a standard interval-valued pseudo-addition $\bigoplus$, and $\odot_l$ and $\odot_r$ are represents of a standard interval-valued pseudo-multiplication $\bigodot$ and if $\bar{\xi} = [\xi_l, \xi_r]$ is the interval-valued pseudo-character, then ξ_l and ξ_r are pseudo-characters.*

Proof. Since $\bigodot$ is a standard interval-valued pseudo-multiplication, for each $x, y \in X$,

$$\begin{aligned} [\xi_l(x+y), \xi_r(x+y)] &= \bar{\xi}(x+y) = \bar{\xi}(x) \bigodot \bar{\xi}(y) \\ &= [\xi_l(x), \xi_r(x)] \bigodot [\xi_l(y), \xi_r(y)] \\ &= [\xi_l(x) \odot_l \xi_l(y), \xi_r(x) \odot_r \xi_r(y)]. \end{aligned} \tag{17}$$

Thus, we have $\xi_l(x+y) = \xi_l(x) \odot_l \xi_l(y)$ and $\xi_r(x+y) = \xi_r(x) \odot_r \xi_r(y)$. That is, ξ_l and ξ_r are pseudo-characters of $\odot_l$ and $\odot_r$, respectively. □

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Theorem 3.2. *If $\bar{f} = [f_l, f_r] \in B(X, I([a,b]))$, $m_{\bar{f}}(A) = \inf_{x\in A} \bar{f}(x)$ for all $A \in \mathcal{M}, m_{\bar{f}} = [m_{f_l}, m_{f_r}]$, and $dm_{\bar{f}} = \bar{f} \bigodot m_{\bar{f}}$, then we have*
(1) $m_{\bar{f}}(A) = [mf_l(A), m_{f_r}(A)]$ *for all* $A \in \mathcal{M}$
(2) $dm_{\bar{f}} = [dm_{f_l}, dm_{f_r}]$ *where* $dm_{f_l} = f_l \odot_l m_{f_l}$ *and* $dm_{f_r} = f_r \odot_r m_{f_r}$.

Proof. (1) By Definition 3.3 (2),

$$\begin{aligned} m_{\bar{f}} &= \inf_{x\in A} \bar{f}(x) = \inf_{x\in A} [f_l(x), f_r(x)] \\ &= [\inf_{x\in A} f_l(x), \inf_{x\in A} f_r(x)] = [m_{f_l}(A), m_{f_r}(A)]. \end{aligned} \tag{18}$$

(2) Since $\bigodot$ is a standard interval-valued pseudo-multiplication,

$$\begin{aligned} dm_{\bar{f}} &= \bar{f} \bigodot m_{\bar{f}} = [f_l, f_r] \bigodot [m_{f_l}, m_{f_r}] \\ &= [f_l \odot_l m_{f_l}, f_r \odot_r m_{f_r}] = [dm_{f_l}, dm_{f_r}]. \end{aligned} \tag{19}$$

□

We also easily obtain the following theorem without their proof.

Theorem 3.3. (1) *Let $\oplus_l$ and $\oplus_r$ are represents of a standard interval-valued pseudo-addition $\bigoplus$, and $\odot_l$ and $\odot_r$ are represents of a standard interval-valued pseudo-multiplication $\bigodot$. Then we have an interval-valued set function $\bar{\mu} = [\mu_l, \mu_r]$ is an interval-valued $\sigma - \bigoplus$-decomposable measure if and only if μ_l and μ_r are $\sigma - \oplus_l$-decomposable measure and $\sigma - \oplus_r$-decomposable measure, respectively.*
(2) *An interval-valued function $\bar{f} = [f_l, f_r]$ is integrable if and only if f_l and f_r are integrable.*

Theorem 3.4. (1) *If $\bar{e}(x) = \bigoplus_{i=1}^{\infty} \bar{a}_i \bigodot \chi_{A_i}(x)$ is an interval-valued elementary function as in Definition 3.6 (1), then we have*

$$\bar{e}(x) = [\oplus_{l}{}_{i=1}^{\infty} a_{il} \odot_l \chi_{A_i}(x), \oplus_{r}{}_{i=1}^{\infty} a_{ir} \odot_r \chi_A(x)]. \tag{20}$$

(2) *If $\bar{e} = [e_l, e_r]$ is an interval-valued elementary function with respect to the interval-valued $\sigma - \bigoplus$-decomposable measure $\bar{\mu} = [\mu_l, \mu_r]$ is*

$$\int^{\bigoplus} \bar{e} \bigodot d\bar{\mu} = [\int^{\oplus_l} e_l \odot_l d\mu_l, \int^{\oplus_r} e_r \odot_r d\mu_r]. \tag{21}$$

(3) *An interval-valued character $\bar{\xi} : X \longrightarrow I([a,b])$ is d_H-continuous if and only if ξ_l and ξ_r are continuous.*
(4) *If $\bar{f} = [f_l, f_r]$ is an integrable, interval- valued function, then we have*

$$\int^{\bigoplus} \bar{f} \bigodot d\bar{\mu} = [\int^{\oplus_l} f_l \odot_l d\mu_l, \int^{\oplus_r} f_r \odot_r d\mu_r] \tag{22}$$

Proof. (1) Since $\bigoplus$ is a standard interval-valued pseudo-addition and $\bigodot$ is a standard interval-valued pseudo-multiplication,

$$\begin{aligned} \bar{e}(x) &= \bigoplus_{i=1}^{\infty} \bar{a}_i \bigodot \chi_{A_i}(x) \\ &= [\oplus_{l}{}_{i=1}^{\infty} a_{il}, \oplus_{r}{}_{i=1}^{\infty} a_{ir}] \bigodot \chi_{A_i}(x) \\ &= [\oplus_{l}{}_{i=1}^{\infty} a_{il} \odot_l \chi_{A_i}(x), \oplus_{r}{}_{i=1}^{\infty} a_{ir} \odot_r \chi_{A_i}(x)]. \end{aligned} \tag{23}$$

(2) Since $\bigoplus$ is a standard interval-valued pseudo-addition and $\bigodot$ is a standard interval-valued pseudo-multiplication,

$$\begin{aligned}
\int^{\bigoplus} \bar{e} \bigodot d\bar{\mu} &= \bigoplus_{i=1}^{\infty} \bar{a_i} \bigodot \bar{\mu}(A_i) \\
&= [\oplus_l{}_{i=1}^{\infty} a_{il}, \oplus_r{}_{i=1}^{\infty} a_{ir}] \bigodot [\mu_l(A), \mu_r(A)] \\
&= [\oplus_l{}_{i=1}^{\infty} a_{il} \odot_l \mu_l(A), \oplus_r{}_{i=1}^{\infty} a_{ir} \odot_r \mu_r(A)] \\
&= [\int^{\oplus_l} e_l \odot_l d\mu_l, \int^{\oplus_r} e_r \odot_r d\mu_r].
\end{aligned} \tag{24}$$

(3) By (12), the proof is trivial.

(4) Since $\bigoplus$ is a standard interval-valued pseudo-addition and $\bigodot$ is a standard interval-valued pseudo-multiplication, by (11)

$$\begin{aligned}
\int^{\bigoplus} \bar{f} \bigodot d\bar{\mu} &= d_H - \lim_{n\to\infty} \int^{\bigoplus} \bar{e_n} \bigodot d\bar{\mu} \\
&= d_H - \lim_{n\to\infty} [\int^{\oplus_l} e_{nl} \odot_l d\mu_l, \int^{\oplus_r} e_{nr} \odot_r d\mu_r] \\
&= [\lim_{n\to\infty} \int^{\oplus_l} e_{nl} \odot_l d\mu_l, \lim_{n\to\infty} \int^{\oplus_r} e_{nr} \odot_r d\mu_r] \\
&= [\int^{\oplus_l} f_l \odot_l d\mu_l, \int^{\oplus_r} f_r \odot_r d\mu_r].
\end{aligned} \tag{25}$$

□

Theorem 3.5. *If $\oplus_l$ and $\oplus_r$ are represents of a standard interval-valued pseudo-addition $\bigoplus$, and $\odot_l$ and $\odot_r$ are represents of a standard interval-valued pseudo-multiplication $\bigodot$, and $\bar{\xi} = [\xi_l, \xi_r]$ is the d_H-continuous interval-valued pseudo-character, and $\bar{f} = [f_l, f_r] \in B(X, I([a, b]))$, and $\mathfrak{L}_i^{\bigoplus} \bar{f}$ is the interval-valued pseudo-Laplace transform, then we have*

$$(\mathfrak{L}_i^{\bigoplus} \bar{f})(\bar{\xi})(z) = [(\mathfrak{L}^{\oplus_l} f_l)(\xi_l)(z), (\mathfrak{L}^{\oplus_r} f_r)(\xi_r)(z)] \tag{26}$$

for all $z \in \mathbb{R}$.

Proof. Since $\oplus_l$ and $\oplus_r$ are represents of a standard interval-valued pseudo-addition $\bigoplus$ and $\odot_l$ and $\odot_r$ are represents of a standard interval-valued pseudo-multiplication $\bigodot$, by Theorem 3.1, Theorem 3.2 (2), and Theorem 3.3 (2), and Theorem 3.4 (4),

$$\begin{aligned}
(\mathfrak{L}_i^{\bigoplus} \bar{f})(\xi)(z) &= \int_{X\cap[0,\infty)}^{\bigoplus} \bar{\xi}(x, -z) \bigodot dm_{\bar{f}} \\
&= \int_{X\cap[0,\infty)}^{\bigoplus} [\xi_l(x, -z), \xi_r(x, -z)] \bigodot [dm_{f_l}, dm_{f_r}] \\
&= [\int_{X\cap[0,\infty)}^{\oplus_l} \xi_l(x, -z) \odot_l dm_{f_l}, \int_{X\cap[0,\infty)}^{\oplus_r} \xi_r(x, -z) \odot_r dm_{f_r}] \\
&= [\mathfrak{L}^{\oplus_l}(f_l)(\xi_l)(z), \mathfrak{L}^{\oplus_r}(f_r)(\xi_r)(z)].
\end{aligned} \tag{27}$$

□

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4. The interval-valued pseudo-exchange formula and the inverse of interval-valued pseudo-Laplace transform

In this section, we study the interval-valued pseudo-exchange formula and the inverse of the interval-valued pseudo-Laplace transform for $X = \mathbb{R}$.

Definition 4.1. (1) A standard interval-valued composition $\circ_s$ of $\bar{f} = [f_l, f_r]$ and $\bar{g} = [g_l, g_r]$ in $B(X, I([a, b]))$ is defined by

$$\bar{f} \circ_s \bar{h} = [f_l \circ h_l, f_r \circ h_r], \tag{28}$$

where $\circ$ is the composition of f and h, that is

$$(f \circ h)(x) = f(h(x)), \ \forall \in X. \tag{29}$$

(2) Let $\bar{g} = [g_l, g_r] \in B(X, I([a, b]))$ be an odd, strictly increasing d_H-continuous function. If $\bar{g} = [g_l, g_r] \in B(X, I([a, b]))$, then the inverse function $\bar{g}^{-1}$ of $\bar{g}$ with respect to a standard interval-valued composition $\circ_s$ is defined by $\bar{g}^{-1} = [g_l^{-1}, g_r^{-1}]$.

Theorem 4.1. (1) *If $\oplus_{1_l} = \oplus_{1_r} = \max$ are represents of a standard interval-valued pseudo-addition $\bigoplus_1$ and $\odot_{1_l} = \odot_{1_r} = +$ are represents of a standard interval-valued pseudo-multiplication $\bigodot_1$ and $\bar{\xi} = [\xi_l, \xi_r]$ with $\xi_l(x, -z) = \xi_r(x, -z) = -xz$ then we have*

$$(\mathfrak{L}_{i_1}^{\bigoplus_1} \bar{f})(z) = \sup_{x \geq 0}(-xz \bigodot{}_1 \bar{f}(z)) \tag{30}$$

where $\bar{f} \in B(X, I([a, b]))$.

(2) *If $\oplus_{2_l} = \oplus_{2_r} = \max$ are the represents of a standard interval-valued pseudo-addition $\bigoplus_2$, and $\odot_{2_l} = \odot_{2_r} = \cdot$ are the represents of a standard interval-valued pseudo- multiplication $\bigodot_2$ and $\bar{\xi} = [\xi_l, \xi_r]$ with $\xi_l(x, -z) = \xi_r(x, -z) = e^{-xz}$, then we have*

$$(\mathfrak{L}_{i_2}^{\bigoplus_2} \bar{f})(z) = \sup_{x \geq 0} e^{xz} \bigodot{}_2 \bar{f}(z) \tag{31}$$

where $\bar{f} \in B(X, I([a, b]))$.

(3) *Let $g_l, g_r : [-\infty, \infty] \longrightarrow [-\infty, \infty], g(0) = 0$ be an odd, strictly increasing, continuous function. If $x \oplus_l y = g_l^{-1}(g_l(x) + g_l(y)), x \oplus_r y = g_r^{-1}(g_r(x) + g_r(y))$ are the represents of a standard interval-valued pseudo-addition $\bigoplus_3$, and $x \odot_3 y = g_l^{-1}(g_l(x) g_l(y)), x \odot_3 y = g_r^{-1}(g_r(x) g_r(y))$ are the represents of a standard interval-valued pseudo-multiplication $\bigodot$, and $\bar{\xi} = [\xi_l, \xi_r]$ with $\xi_l(x, -z) = \xi_r(x, -z) = e^{-xz}$, then we have*

$$(\mathfrak{L}_{i_3}^{\bigoplus_3} \bar{f})(z) = \bar{g}^{-1} \mathfrak{L}_i(\bar{g} \circ_s \bar{f})(z) \tag{32}$$

where $(\mathfrak{L}_i \bar{f})(z) = [\mathfrak{L}(f_l)(z), \mathfrak{L}(f_r)(z)]$.

Proof. (1) By Theorem 3.5, and Definition 3.3 (1), we have

$$\begin{aligned} (\mathfrak{L}_{i_1}^{\bigoplus_1} \bar{f})(z) &= [\mathfrak{L}_1^{\oplus_{1l}}(f_l)(z), \mathfrak{L}_1^{\oplus_{1r}}(f_r)(z)] \\ &= [\sup_{x \geq 0}(-xz + f_l(z)), \sup_{x \geq 0}(-xz + f_r(z))] \\ &= \sup_{x \geq 0}[-xz \oplus_{1l} f_l(z), -xz \oplus_{1r} f_r(z)] \\ &= \sup_{x \geq 0}[-xz \bigodot{}_1 \bar{f}(z)]. \end{aligned} \tag{33}$$

(2) By Theorem 3.5 and Definition 3.3(1), we have

$$\begin{aligned}(\mathfrak{L}_{i_2}^{\bigoplus_2}\bar{f})(z) &= [\mathfrak{L}_2^{\oplus_{2l}}(f_l)(z), \mathfrak{L}_2^{\oplus_{2r}}(f_r)(z)] \\ &= [\sup_{x\geq 0}(e^{-xz}+f_l(z)), \sup_{x\geq 0}(e^{-xz}+f_r(z))] \\ &= \sup_{x\geq 0}[e^{-xz}\oplus_{2l}f_l(z), e^{-xz}\oplus_{2r}f_r(z)] \\ &= \sup_{x\geq 0}[e^{-xz}\bigodot_2\bar{f}(z)]. \end{aligned} \tag{34}$$

(3) By Theorem 3.5 and Definition 4.1, we have

$$\begin{aligned}(\mathfrak{L}_{i_3}^{\bigoplus_3}\bar{f})(z) &= [\mathfrak{L}_3^{\oplus_{3l}}(f_l)(z), \mathfrak{L}_3^{\oplus_{3r}}(f_r)(z)] \\ &= [g_l^{-1}\mathfrak{L}(g_l\circ f_l)(z), g_r^{-1}\mathfrak{L}(g_r\circ f_r)(z)] \\ &= \bar{g}^{-1}[\mathfrak{L}(g_l\circ f_l)(z), \mathfrak{L}(g_r\circ f_r)(z)] \\ &= \bar{g}^{-1}\mathfrak{L}_i(\bar{g}\circ_s\bar{f})(z). \end{aligned} \tag{35}$$

□

Now, we obtain the following pseudo-exchange formula and inverse of interval-valued pseudo-Laplace transform.

Theorem 4.2. *Let $\bigoplus_k$, and $\bigodot_k$ be as in Theorem 4.1, for $k=1,2,3$, and $\bar{f}_1,\bar{f}_2\in B(X,I([a,b]))$. Then we have*

$$\mathfrak{L}_{i_k}^{\bigoplus_k}(\bar{f}_1 *_k \bar{f}_2) = \mathfrak{L}_{i_k}^{\bigoplus_k}(\bar{f}_1)\bigodot \mathfrak{L}_k^{\bigoplus_k}(\bar{f}_2) \tag{36}$$

for $k=1,2,3$, where $(\bar{f}_1 *_1 \bar{f}_2)(x)=\sup_{0\leq t\leq x}[\bar{f}_1(x-t)\bigodot_1\bar{f}_2(t)]$, $(\bar{f}_1 *_2 \bar{f}_2)(x)=\sup_{0\leq t\leq x}[\bar{f}_1(x-t)\bigodot_2\bar{f}_2(t)]$, and $(\bar{f}_1 *_3 \bar{f}_2)(x)=\bar{g}^{-1}((\bar{g}\circ_s\bar{f}_1 *_I \bar{g}\circ_s\bar{f}_2)(x))$ with $\bar{f} *_I \bar{h}=[f_l * h_l, f_r * h_r]$.

Proof. (1) When $k=1$; we see that

$$\begin{aligned}(\bar{f}_1 *_1 \bar{f}_2)(x) &= \sup_{0\leq t\leq x}[\bar{f}_1(x-t)\bigodot_1\bar{f}_2(t)] \\ &= \sup_{0\leq t\leq x}[f_{1_l}(x-t)+f_{2_l}(t), f_{1_r}(x-t)+f_{2_r}(t)] \\ &= [\sup_{0\leq t\leq x}(f_{1_l}(x-t)+f_{2_l}(t)), \sup_{0\leq t\leq x}(f_{1_r}(x-t)+f_{2_r}(t))] \\ &= [f_{1_l} *_1 f_{2_l}, f_{1_r} *_1 f_{2_r}]. \end{aligned} \tag{37}$$

By Theorem 4.1(1), (37), and Theorem 2.1, we have

$$\begin{aligned}\mathfrak{L}_{i_1}^{\bigoplus_1}(\bar{f}_1 *_1 \bar{f}_2) &= [\mathfrak{L}_1^{\oplus_l}(f_{1_l} *_1 f_{2_l}), \mathfrak{L}_1^{\oplus_r}(f_{1_r} *_1 f_{2_r})] \\ &= [\mathfrak{L}_1^{\oplus_{1_l}}(f_{1_l})\odot_{1_l}\mathfrak{L}_1^{\oplus_{1_l}}(f_{2_l}), \mathfrak{L}_1^{\oplus_{1_r}}(f_{1_r})\odot_{1_r}\mathfrak{L}_1^{\oplus_{1_r}}(f_{2_r})] \\ &= [\mathfrak{L}_1^{\oplus_{1_l}}(f_{1_l}), \mathfrak{L}_1^{\oplus_{1_r}}(f_{1_r})]\bigodot_1[\mathfrak{L}_1^{\oplus_{1_l}}f_{2_l}, \mathfrak{L}_1^{\oplus_{1_r}}(f_{2_r})] \\ &= \mathfrak{L}_{i_1}^{\bigoplus_1}(\bar{f}_1)\bigodot_1\mathfrak{L}_{i_2}^{\bigoplus_1}(\bar{f}_2). \end{aligned} \tag{38}$$

(2) When $k=2$; by the some method in the proof of (1), we can obtain the result.

(3) When $k=3$; we see that

$$\begin{aligned}(\bar{f}_1 *_3 \bar{f}_2)(x) &= \bar{g}^{-1}(\bar{g}\circ_s\bar{f}_1 *_I \bar{g}\circ_s\bar{f}_2)(x) \\ &= \bar{g}^{-1}([g_l\circ f_{1_l}, g_r\circ f_{1_r}] *_I [g_l\circ f_{2_l}, g_r\circ f_{2_l}]) \\ &= \bar{g}^{-1}[(g_l\circ f_{1_l}) * (g_r\circ f_{2_l}), (g_r\circ f_{1_r}) * (g_r\circ f_{2_r})] \\ &= [\bar{g}^{-1}((g_l\circ f_{1_l}) * (g_r\circ f_{2_l})), g_r^{-1}((g_r\circ f_{1_r}) * (g_r\circ f_{2_r}))] \end{aligned}$$

$$= [f_{1_l} *_3 f_{2_l}, f_{1_r} *_3 f_{2_r}]. \tag{39}$$

By Theorem 4.1(3), (39), and Theorem 2.1,

$$\begin{aligned}
\mathfrak{L}_{i_3}^{\bigoplus_3}(\bar{f}_1 *_3 \bar{f}_2) &= [\mathfrak{L}_3^{\oplus_{3_l}}(f_{1_l} *_3 f_{2_l}), \mathfrak{L}_3^{\oplus_{3_r}}(f_{1_r} *_3 f_{2_r})] \\
&= [\mathfrak{L}_3^{\oplus_{3_l}}(f_{1_l}) \odot_l \mathfrak{L}_3^{\oplus_{3_l}}(f_{2_l}), \mathfrak{L}_3^{\oplus_{3_r}}(f_{1_r}) \odot_r \mathfrak{L}_3^{\oplus_{3_r}}(f_{2_r})] \\
&= [\mathfrak{L}_3^{\oplus_{3_l}}(f_{1_l}), \mathfrak{L}_3^{\oplus_{3_r}}(f_{1_r})] \bigodot_3 [\mathfrak{L}_3^{\oplus_{3_l}}(f_{2_l}), \mathfrak{L}_3^{\oplus_{3_r}}(f_{2_r})] \\
&= \mathfrak{L}_{i_3}^{\bigoplus_3}(\bar{f}_1) \bigodot_3 \mathfrak{L}_{i_3}^{\bigoplus_3}(\bar{f}_2).
\end{aligned} \tag{40}$$

□

Finally, we will study the inverse of a standard interval-valued pseudo-Laplace transform.

Theorem 4.3. *If $\bar{f} \in B(X, I([a,b]))$ and $\mathfrak{L}_{i_k}^{\bigoplus_k}(\bar{f}) = \bar{F}_k$ for $k = 1, 2, 3$ as in Theorem 4.1, and there exist $(\mathfrak{L}_{i_k}^{\bigoplus_k})^{-1}(\bar{F}_k) = [\mathfrak{L}_k^{\oplus_{k_l}}{}^{-1}(F_{k_l}), \mathfrak{L}_k^{\oplus_{k_r}}{}^{-1}(F_{k_r})]$, then we have*

(1) $(\mathfrak{L}_{i_1}^{\bigoplus_1})^{-1}(\bar{F}_1)(x) = \inf_{z \geq 0}(xz \bigodot_1 \bar{F}_1(z))$,
(2) $(\mathfrak{L}_{i_2}^{\bigoplus_2})^{-1}(\bar{F}_2)(x) = \inf_{z \geq 0}(e^{xz} \bigodot_2 \bar{F}_2(z))$, *and*
(3) $(\mathfrak{L}_{i_3}^{\bigoplus_3})^{-1}(\bar{F}_3)(x) = \bar{g}^{-1}(\mathfrak{L}_i^{-1}(\bar{g} \circ_s \bar{F}_3)(x))$.

Proof. (1) Since $\mathfrak{L}_{i_1}^{\bigoplus_1}(\bar{f}) = \bar{F}_1$, by Theorem 3.5,

$$\begin{aligned}
[F_{1_l}, F_{1_r}] &= \bar{F}_1 = \mathfrak{L}_{i_1}^{\bigoplus_1}(\bar{f}) \\
&= [\mathfrak{L}^{\oplus_{1_l}}(f_l), \mathfrak{L}^{\oplus_{1_r}}(f_r)].
\end{aligned} \tag{41}$$

Thus, we have $F_{1_l} = \mathfrak{L}_1^{\oplus_{1_l}}(f_l)$ and $F_{1_r} = \mathfrak{L}_1^{\oplus_{1_r}}(f_r)$. By Theorem 2.2 (1), we have

$$(\mathfrak{L}_1^{\oplus_{1_l}})^{-1}(F_{1_l})(x) = \inf_{x \geq 0}(xz + F_{1_l}(z)) \tag{42}$$

and

$$(\mathfrak{L}_1^{\oplus_{1_r}})^{-1})(F_{1_r})(x) = \inf_{z \geq 0}(xz + F_{1_r}(z)) \tag{43}$$

By (42) and (43), Definition 3.3 (2),

$$\begin{aligned}
(\mathfrak{L}_{i_1}^{\bigoplus_1})^{-1}(\bar{F}_1) &= [\mathfrak{L}_1^{\oplus_{1_l}}{}^{-1}(F_{1_l}), \mathfrak{L}_2^{\oplus_{1_r}}{}^{-1}(F_{1_r})] \\
&= [\inf_{z \geq 0}(xz + F_{1_l}(z)), \inf_{z \geq 0}(xz + F_{1_r}(z)]) \\
&= [\inf_{z \geq 0}(xz \odot_{1_l} F_{1_l}(z)), \inf_{z \geq 0}(xz \odot_{1_l} F_{1_r}(z))] \\
&= \inf_{z \geq 0}[xz \odot_{1_l} F_{1_l}(z), xz \odot_{1_r} F_{1_r}(z)] \\
&= \inf_{z \geq 0}[xz \bigodot_1 \bar{F}_1(z)].
\end{aligned} \tag{44}$$

(2) By the same method in the proof of (1), we can obtain the result.

(3) Since $\mathfrak{L}_{i_3}^{\bigoplus_3}(\bar{f}) = \bar{F}_3$, by Theorem 3.5,

$$\begin{aligned}
[F_{3_l}, F_{3_r}] &= \bar{F}_3 = \mathfrak{L}_{i_3}^{\bigoplus_3}(\bar{f}) \\
&= [\mathfrak{L}^{\oplus_{3_l}}(f_l), \mathfrak{L}^{\oplus_{3_r}}(f_r)]
\end{aligned} \tag{45}$$

Thus, we have $F_{3_l} = \mathfrak{L}_1^{\oplus_{3_l}}(f_l)$ and $F_{3_r} = \mathfrak{L}_3^{\oplus_{3_r}}(f_r)$. By Theorem 2.2 (3), we have

$$(\mathfrak{L}_3^{\oplus_{3_l}})^{-1}(F_{3_l})(x) = g_l^{-1}(\mathfrak{L}^{-1}(g_l \circ F_{3_l})(x)) \tag{46}$$

and

$$(\mathfrak{L}_3^{\oplus_{3_r}})^{-1}(F_{3_r})(x) = g_r^{-1}(\mathfrak{L}^{-1}(g_r \circ F_{3_r})(x)) \tag{47}$$

By (46) and (47), Definition 4.1 (2),

$$\begin{aligned}
{\mathfrak{L}_3^{\bigoplus_3}}^{-1}(\bar{F}_3) &= [(\mathfrak{L}_3^{\oplus_{3_l}})^{-1}(F_{3_l}), (\mathfrak{L}_3^{\oplus_{3_r}})^{-1}(F_{3_r})] \\
&= [g_l^{-1}(\mathfrak{L}^{-1}(g_l \circ F_{3_l}))(x), g_r^{-1}(\mathfrak{L}^{-1}(g_r \circ F_{3_r}))(x)] \\
&= \bar{g}^{-1}[g^{-1}(g_l \circ F_{3_l})(x), \mathfrak{L}^{-1}(g_r \circ F_{3_r})(x)] \\
&= \bar{g}^{-1}\mathfrak{L}_i^{-1}([g_l \circ F_{3_l}, g_r \circ F_{3_r}])(x) \\
&= \bar{g}^{-1}\mathfrak{L}_i^{-1}(\bar{g} \circ_s \bar{F}_3)(x). \qquad (48)
\end{aligned}$$

□

Remark 4.2. We find the maximum of the uncertain utility function

$$\bar{U}(x_1, x_2, \cdots, x_n) = \bar{f}_1(x_1) \bigodot_1 \bar{f}_2(x_2) \bigodot_1 \cdots \bigodot_1 \bar{f}_n(x_n)$$

on the domain $R = \{(x_1, x_2, \cdots, x_n) | x_1 + x_2 + \cdots + x_n = x,\ x_i \geq 0,\ i = 1, 2, \cdots, n\}$. Such problems occurs in the mathematical economy and operation research. Let $\bar{f}(x) = \max_R[\bar{f}_1(x_1) \bigodot_1 \bar{f}_2(x_2) \bigodot_1 \cdots \bigodot_1 \bar{f}_n(x_n)]$. Applying the standard interval-valued pseudo-Laplace transfor (for the case $\bigoplus_1$ and $\bigodot_1$)

$$\bar{F}(z) = (\mathfrak{L}_1^{\bigoplus_1} \bar{f})(z) = \max_{x \geq 0}(-xz \bigodot_1 \bar{f}(x)), \tag{49}$$

we obtain by interval-valued pseudo-exchange formula

$$\mathfrak{L}_1^{\bigoplus_1}(\bar{f}) = \bigodot_{1_{i=1}}^{n} \mathfrak{L}^{\bigoplus_1}(\bar{f}_i). \tag{50}$$

Applying the inverse of a standard pseudo-Laplace transform

$$(\mathfrak{L}_1^{\bigoplus_1})^{-1}(\bar{F})(x) = \min_{z \geq 0}(xz \bigodot_1 \bar{F}_1(z)), \tag{51}$$

we obtain the solution

$$f(x) = \min_{z \geq 0}[xz \bigodot_1 \bigodot_{1_{i=1}}^{n} \mathfrak{L}_1^{\bigoplus_1}(f_i)(z)]. \tag{52}$$

5. Conclusions

This study was to define the interval-valued pseudo-Laplace transform by means of the interval-valued pseudo-integral with respect to a $\sigma-\bigoplus$-decomposable measure (See Definitions 3.6 and 3.7). By using the method of the standard interval-valued pseudo-operations in [12-14], we also investigated some characterizations of the standard interval-valued pseudo-characters, the interval-valued pseudo-integrals, and the interval-valued pseudo-Laplace transform (See Theorems 3.1, 3.2, 3.3, 3.4, and 3.5.). In particular, we obtained the pseudo-exchange formula for the interval-valued pseudo-Laplace transform and inverse of interval-valued pseudo-Laplace transforms (See Theorems 4.2 and 4.3) and an application (See Remark 4.4).

In the future, we will focus the pseudo-exchange formula for the interval-valued pseudo-Laplace transform by means of the interval-valued generalized Choquet integrals and the inverse of interval-valued pseudo-Laplace transforms.

Acknowledgement: This paper was supported by Konkuk University in 2014.

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OPERATORS IDEALS OF GENERALIZED MODULAR SPACES OF CESÁRO TYPE DEFINED BY WEIGHTED MEANS

NECİP ŞİMŞEK*, VATAN KARAKAYA, AND HARUN POLAT

ABSTRACT. In this work, we investigate the ideal of all bounded linear operators between any arbitrary Banach spaces whose sequence of approximation numbers belong to the generalized modular spaces of Cesáro type defined by weighted means. Also, we show that the completeness of obtained operator ideals.

1. Introduction

At the turn of this century, there has been a considerable interest to study the behavior of the operator ideals using s-numbers and specially approximation numbers. As has been shown in the famous book of Gochberg and Kreĭn [11], the s-numbers are an important tool in the spectral theory of Hilbert space. We consider s-numbers sequences of operators on Banach spaces in the sense of [4]. He presented the axiomatic theory of s-numbers of operators in Banach spaces. Also, in his work it has been turned out that s-numbers are very powerful tools for estimating eigenvalues of operators in Banach spaces.

Many useful operator ideals have been defined by using sequence of s-numbers. The concept of an operator ideal on the class of Banach spaces and the theory of s-numbers of linear bounded operators among Banach spaces was introduced and first studied by [4]. Due to important applications in spectral theory, geometry of Banach spaces, theory of eigenvalue distributions etc., the theory of operator ideals occupies a special importance in functional analysis. Many useful operator ideals have been defined by using sequence of s-numbers.

In [14], Faried and Bakery use the de la Valée-Poussin means for definition of the operator ideals. By using similar concept, we investigate the ideals of all bounded linear operators generated by the approximation numbers and generalized weighted means defined by [6] and generalized by [16].

2. Preliminary and Notations

By $\mathbb{N}$ and $\mathbb{R}$ it will be signed the set of all natural numbers and the real numbers, respectively and also by w, it is denoted the space of all real valued sequences. The set $B(X,Y)$ of all operators from X into Y becomes a Banach space with respect to the so-called operator norm

$$\|T\| = \sup\{\|Tx\| : \|x\| \leq 1\}.$$

The class of all operators between arbitrary Banach spaces is denoted by B. Also if $T \in B(X,Y)$, then we denote null space and range $N(T)$ and $M(T)$ respectively.

2000 *Mathematics Subject Classification.* Primary 46B20, 46B45; Secondary 47B06.

Key words and phrases. Approximation numbers; operator ideal; generalized weighted mean.

1

Let $X_0 = X \diagup N(T)$ and $Y_0 = \overline{M(T)}$. A map which transform every operator $T \in B(X,Y)$ to a unique sequence $(s_n(T))_{n=0}^{\infty}$ is called $s-function$, and the number $s_n(T)$ is called the n^{th} $s-number$ of T if the following conditions are satisfied: s_1) $||T|| = s_1(T) \geq s_2(T) \geq ... \geq 0$ for all $T \in B(X,Y)$ s_2) $s_n(T_1+T_2) \leq s_n(T_1)+||T_2||$ for all $T_1, T_2 \in B(X,Y)$ s_3) $s_n(RST) \leq ||R||s_n(S)||T||$ for all $T \in B(X_0, X)$, $S \in B(X,Y)$ and $R \in B(Y, Y_0)$ s_4) $s_n(\lambda T) \leq |\lambda| s_n(T)$ for all $T \in B(X,Y)$, $\lambda \in \mathbb{R}$ s_5) $rank(T) \leq n$ if $s_n(T) = 0$ for all $T \in B(X,Y)$ s_6) $s_r(I_n) = \begin{cases} 0 & ; \text{ for } r \geq n \\ 1 & ; \text{ for } r < n \end{cases}$ where I_n is the identity map of an n-dimensional normed space. As important examples of s-numbers, we consider approximation numbers $\alpha_n(T)$

$$\alpha_n(T) = \inf\{||T-A|| : A \in B(X,Y) \text{ and } rank(A) \leq n\}, \tag{2.1}$$

We can show that all these s-numbers satisfy the following condition. This is improvement of condition (s_2) and called additive property:

$$s_{n+m-1}(T_1+T_2) \leq s_n(T_1) + s_m(T_2), \text{for all} T_1, T_2 \in B(X,Y) \text{ and } n = 1, 2, ...$$

Many other famous s-numbers are modifications of them. All these numbers express some finite dimensional approximations for the operator T. Geometrically they may be viewed as different widths of the ball $T(B_X)$ as a subset of Y. For details on s-number sequences, we refer [4, 7, 5, 2]. An *operator ideal* U is a subclass of B such that the components

$$U(X,Y) = U \cap B(X,Y)$$

satisfy the following conditions: O_1) $I_K \in U$, where K denotes the 1-dimensional Banach space, where $U \subset B$, (equally; $F(X,Y) \subseteq U(X,Y)$, where $F(X,Y)$ is the space of all operators of finite rank from the Banach space X into the Banach space Y), O_2) If $T_1, T_2 \in U(X,Y)$, then $\lambda_1 T_1 + \lambda_2 T_2 \in U(X,Y)$ for any scalars λ_1 and λ_2, O_3) If $V \in B(X_0, X)$, $T \in U(X,Y)$ and $R \in B(Y, Y_0)$ then $RTV \in U(X_0, Y_0)$ (see [3, 1, 15]). The generalized modular spaces of Cesáro type defined by weighted means is defined by Şimşek and Karakaya in [16] as follows:

$$\ell_\rho(u,v;p) = \{x \in w : \rho(\lambda x) < \infty, \text{ for some } \lambda > 0\},$$

where

$$\rho(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} u_k v_j |x_j| \right)^{p_k}.$$

It can be seen that $\rho : \ell_\rho(u,v;p) \to [0,\infty]$ is a modular on $\ell_\rho(u,v;p)$. If $p = (p_k)$ for all $k \in \mathbb{N}$, we can write for $\ell_\rho(u,v;p)$ and it's norm respectively;

$$\ell_\rho(u,v;p) = \left\{ x \in w : \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} u_k v_j |x_j| \right)^{p_k} < \infty, \quad \text{for some } \lambda > 0 \right\},$$

$$\|x\| = \inf\left\{\lambda > 0 : \rho\left(\frac{x}{\lambda}\right) \leq 1\right\}, \qquad \forall x \in \ell_\rho(u,v;p).$$

With norm given above, space $\ell_\rho(u,v;p)$ is a Banach space as shown in [16] .One can find detailes about the space $\ell(u,v;p)$ in list given in ([9], [6] [6]). By combining special case of u_k and v_j, we get the following spaces: Firstly, if $(u_k) = \left(\frac{1}{k+1}\right)$ and $v_j = 1$ for all $j, k \in \mathbb{N}$, then the space $\ell_\rho(u,v;p)$ reduces to the space $ces(p)$ (see [12]).

$(u_k) = \left(\frac{1}{Q_k}\right)$ and $(v_j) = (q_j)$ and $Q_k = \sum\limits_{j=0}^{k} q_j$ for all $j, k \in \mathbb{N}$, then the space $\ell_\rho(u, v; p)$ reduces to the space $N_\rho(q)$ (see [8]). In addition, some related papers on the sequence space can be found in [10, 13, 17, 18].

Also we denote $e_i = (0, 0, ..., 0, 1, 0, ...)$ where 1 seats in ith place of e_i for all $i \in \mathbb{N}$.

For any bounded sequence of positive real numbers $p = (p_k)$, we have the following well known inequality: $|a_k + b_k|^{p_k} \leq 2^{h-1}\left(|a_k|^{p_k} + |b_k|^{p_k}\right)$, where $h = \sup p_k$ and $p_k > 1$ for all $k \in \mathbb{N}$

Throughout this paper, the sequence (p_k) is a bounded sequence of positive real numbers with

(a_1) (p_k) is an increasing sequence of positive real numbers,
(a_2) $\underset{k\to\infty}{lim} \sup p_k < \infty$,
(a_3) $\underset{k\to\infty}{lim} \inf p_k > 1$.

Definition 1. [14] *A class E of linear sequence spaces is called a special space of sequences (sss) having the following conditions:*

(i) *E is a linear space and $e_n \in E$, for each $n \in \mathbb{N}$,*

(ii) *If $x \in w$, $y \in E$ and $|x_n| \leq |y_n|$ for all $n \in \mathbb{N}$, than $x \in E$, i.e. E is solid,*

(iii) *If $(x_n)_{n=0}^{\infty} \in E$, then $\left(x_{\left[\frac{n}{2}\right]}\right)_{n=0}^{\infty} = (x_0, x_0, x_1, x_1, x_2, x_2, ..., x_n, x_n, ...) \in E$, where $\left[\frac{n}{2}\right]$ denotes the integral part of $\frac{n}{2}$.*

Example 1. *a) ℓ_p is a special space of sequences for $0 < p < \infty$.*
b) ces_p is a special space of sequences for $1 < p < \infty$.

Definition 2. $U_E^{app} := \{U_E^{app}(X, Y) :$ *X, Y are Banach spaces$\}$ where* $U_E^{app}(X, Y) = \{T \in L(X, Y) : (\alpha_n(T))_{n=0}^{\infty} \in E\}$.

Definition 3. *A class of special sequences(sss) E_ρ is called a pre-modular special space of sequences if there exists a function $\rho : E \longrightarrow [0, \infty)$, satisfies the following conditions:*

(i) *$\rho(x) \geq 0$, $\forall x \in E_\rho$ and $\rho(x) = 0 \iff x = \theta$, where θ is the zero element of E,*

(ii) *there exists a constant $\ell \geq 1$ such that $\rho(\lambda x) \leq \ell|\lambda|\rho(x)$ for all values of $\forall x \in E$ and for any scalar λ,*

(iii) *for some numbers $k \geq 1$, we have the inequality $\rho(x+y) \leq k(\rho(x) + \rho(y))$ for all $x, y \in E$,*

(iv) *if $|x_n| \leq |y_n|$ for all $n \in \mathbb{N}$ then $\rho((x_n)) \leq \rho((y_n))$,*

(v) *for some numbers $k_0 \geq 1$ we have the inequality $\rho((x_n)) \leq \rho\left(\left(x_{\left[\frac{n}{2}\right]}\right)\right) \leq k_0\rho((x_n))$,*

(vi) *for each $x = (x(i))_{i=0}^{\infty} \in E$ there exists $s \in \mathbb{N}$ such that $\rho(x(i))_{i=s}^{\infty} < \infty$. This means that the set of all finite set is $\rho-$dense in E,*

(vii) *for any $\lambda > 0$ there exists a constant $\zeta > 0$ such that $\rho(\lambda, 0, 0, ...) \geq \zeta\lambda\rho(1, 0, 0, ...)$.*

It is clear that ρ is continuous at θ. The function ρ defines a metrizable topology in E endowed with this topology is denoted by E_ρ.

For the arbitrary linear space E, it was provided in the following theorem:

Theorem 1. [14] *U_E^{app} is an operator ideal if E is a special space of sequences (sss).*

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Now we give here the sufficient conditions on the weighted means such that the class of all bounded linear operators between any arbitrary Banach spaces with n-th approximation numbers of the bounded linear operators in the generalized weighted means form an operator ideal.

3. Main Results

For the proof of the main theorems, we need the following theorems which includes the notion of special space of sequences. It helps to construction of operators ideals.

Theorem 2. *$U^{app}_{\ell_\rho(u,v;p)}$ is an operator ideal if the conditions $(a_1),(a_2),(a_3)$ are satisfied.*

Proof. To prove the theorem, we have to satisfy the three conditions of Definition 1. mentioned above.

(1) linearity : Let $x,y\in\ell_\rho(u,v;p)$. Since,

$$\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}u_kv_j|x_j+y_j|\right)^{p_k}\le 2^{h-1}\left(\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}u_kv_j|x_j|\right)^{p_k}+\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}u_kv_j|y_j|\right)^{p_k}\right)$$

where $h=\sup p_k$, then $(x+y)\in\ell_\rho(u,v;p)$.

Let $\lambda\in\mathbb{R}, x\in\ell_\rho(u,v;p)$, then

$$\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}u_kv_j|\lambda x_j|\right)^{p_k}\le\sup_k|\lambda|^{p_k}\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}u_kv_j|x_j|\right)^{p_k}<\infty.$$

Hence we can get $\lambda x\in\ell_\rho(u,v;p)$.

To show that $e_m\in\ell_\rho(u,v;p)$ for each $m\in\mathbb{N}$, by using condition (a_3), we have

$$\rho(e_m)=\sum_{k=m}^{\infty}\left(\sum_{j=0}^{k}u_kv_j|e_m(j)|\right)^{p_k}=\sum_{k=m}^{\infty}(u_kv_m)^{p_k}<\infty.$$

Hence $e_m\in\ell_\rho(u,v;p)$.

(2) solidity: Let $|x_k|\le|y_k|$ for each $k\in\mathbb{N}$, then $\sum\limits_{k=0}^{\infty}\left(\sum\limits_{j=0}^{k}u_kv_j|x_j|\right)^{p_k}\le\sum\limits_{k=0}^{\infty}\left(\sum\limits_{j=0}^{k}u_kv_j|y_j|\right)^{p_k}$, since $y\in\ell_\rho(u,v;p)$. Thus $x\in\ell_\rho(u,v;p)$.

(3) $x=(x_k)\in\ell_\rho(u,v;p)$, then we have

$$\begin{aligned}\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}u_kv_j|x_{[\frac{j}{2}]}|\right)^{p_k}&=\sum_{k=0}^{\infty}\left(\sum_{j=0}^{2k}u_{2k}v_j|x_{[\frac{j}{2}]}|\right)^{p_k}+\sum_{k=0}^{\infty}\left(\sum_{j=0}^{2k+1}u_{2k+1}v_j|x_{[\frac{j}{2}]}|\right)^{p_{2k+1}}\\&\le\sum_{k=0}^{\infty}\left(\left(\sum_{j=0}^{k}u_{2k}\left[(v_{2j}+v_{2j+1})|x_j|\right]\right)+(u_{2k}v_{2k}|x_k|)\right)^{p_{2k}}\end{aligned}$$

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$$
\begin{aligned}
& +\sum_{k=0}^{\infty}\left(u_{2k+1}\sum_{j=0}^{k}\left(v_{2j}+v_{2j+1}\right)|x_j|\right)^{p_{2k+1}} \\
\leq\ & 2^{h-1}\left[\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}u_{2k}\left[\left(v_{2j}+v_{2j+1}\right)|x_j|\right]\right)^{p_{2k}}+\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}u_{2k}v_{2j}|x_j|\right)^{p_{2k}}\right] \\
& +\sum_{k=0}^{\infty}\left(u_{2k+1}\sum_{j=0}^{k}\left(v_{2j}+v_{2j+1}\right)|x_j|\right)^{p_{2k+1}} \\
\leq\ & \left(2^{2h-2}+2^{h-1}\right)\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}u_{2k}v_{2j}|x_j|\right)^{p_{2k}}+2^{2h-2}\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}u_{2k}v_{2j+1}|x_j|\right)^{p_{2k}} \\
& +2^{h-1}\sum_{k=0}^{\infty}\left(u_{2k+1}\sum_{j=0}^{k}v_{2j}|x_j|\right)^{p_{2k+1}}+2^{h-1}\sum_{k=0}^{\infty}\left(u_{2k+1}\sum_{j=0}^{k}v_{2j+1}|x_j|\right)^{p_{2k+1}} \\
\leq\ & \left(2^{2h-1}+32^{h-1}\right)\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}u_kv_j|x_j|\right)^{p_k}<\infty
\end{aligned}
$$

where $u_k \to 0(k\to\infty)$. Hence $\left(x_{\left[\frac{n}{2}\right]}\right)_{n=0}^{\infty} \in \ell_\rho(u,v;p)$. Consequently, from Theorem 1. it follows that $U^{app}_{\ell_\rho(u,v;p)}$ is an operator ideal. ∎

Corollary 1. *(a)* $U^{app}_{N_\rho(q)}$ *is an operator ideal for* $1<p<\infty$.
(b) $U^{app}_{ces(p)}$ *is an operator ideal for* $1<p<\infty$.

Theorem 3. *The linear space* $F(X,Y)$ *is dense in* $U^{app}_{\ell_\rho(u,v;p)}$ *i.e.* $\overline{F(X,Y)} = U^{app}_{\ell_\rho(u,v;p)}$.

Proof. We first verify that every finite mapping $T\in\overline{F(X,Y)}$ belongs to $U^{app}_{\ell_\rho(u,v;p)}(X,Y)$ i.e. $\overline{F(X,Y)}\subset U^{app}_{\ell_\rho(u,v;p)}(X,Y)$. Since $e_m\in\ell_\rho(u,v;p)$ for each $m\in\mathbb{N}$ and $\ell_\rho(u,v;p)$ is a linear space, then for every finite mapping $T\in\overline{F(X,Y)}$, the sequence $(\alpha_k(T))_{n=0}^{\infty}$ contains only finitely many numbers different from zero. Therefore $\overline{F(X,Y)}\subset U^{app}_{\ell_\rho(u,v;p)}(X,Y)$.

Now we have to show that $U^{app}_{\ell_\rho(u,v;p)}(X,Y)\subseteq\overline{F(X,Y)}$. Let $T\in U^{app}_{\ell_\rho(u,v;p)}(X,Y)$. Thus we get $(\alpha_k(T))_{k=0}^{\infty}\in\ell_\rho(u,v;p)$, and since $\rho\left((\alpha_k(T))_{k=0}^{\infty}\right)<\infty$, for given $\varepsilon\in(0,1)$ then there exists a natural number $m>0$ such that $\rho\left((\alpha_k(T))_{k=m}^{\infty}\right)<\varepsilon$. Therefore we have, for $p>0$

$$
\sum_{k=m}^{m+p}\left(\sum_{j=0}^{k}u_kv_j\left|\alpha_j(T)\right|\right)^{p_k}<\varepsilon. \tag{3.1}
$$

It means that $A \in F_m(X,Y)$, $rank(A) \leq m$. Besides, by using the fact

$$|\alpha_j(T)| \leq | \ \|T-A\| \, e_m(j) \ | , \quad j = m, m+1, ... \tag{3.2}$$

and also from (2.1), (3.1) we have

$$\sum_{k=m}^{m+p} (u_k v_m | \ \|T-A\| \, e_m(j) \ |)^{p_k} \leq \infty.$$

we take $m_0 \geq m$ and from (3.2), we get

$$\begin{aligned} d(T,A) &= \rho\left(\alpha_k(T-A)\right)_{k=0}^{\infty} = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} u_k v_j \left|\alpha_j(T-A)\right|\right)^{p_k} \\ &\leq \sum_{k=0}^{m_0-1}\left(\sum_{j=0}^{k} u_k v_j | \ \|T-A\| \, e_m(j) \ |\right)^{p_k} + \sum_{k=m_0}^{\infty}\left(\sum_{j=0}^{k} u_k v_j \left|\alpha_j(T-A)\right|\right)^{p_k} \\ &< \varepsilon. \end{aligned}$$

Hence this completes the proof. ∎

Example 2. *ℓ_p is a pre-modular special space of sequences for $0 < p < \infty$, with $\rho(x) = \sum_{n=0}^{\infty} |x_n|^p$.*

Example 3. *ces_p is a pre-modular special space of sequences for $1 < p < \infty$, with $\rho(x) = \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} |x_k|\right)^p$.*

Theorem 4. *$\ell_\rho(u,v;p)$ with $\rho(x) = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} u_k v_j |x_j|\right)^{p_k}$ is a pre-modular special space of sequences.*

Proof. **(i)** Clearly $\rho(x) \geq 0$ and $\rho(x) = 0 \Longleftrightarrow x = \theta$,

(ii) since (p_n) is bounded then there exists a constant $\ell \geq 1$ such that $\rho(\lambda x) \leq \ell|\lambda|\rho(x)$ for all values of $x \in E$ and for any scalar λ,

(iii) for some numbers $k = \max\left(1, 2^{h-1}\right) \geq 1$, we have inequality

$$\rho(x+y) \leq k\left(\rho(x) + \rho(y)\right), \text{ for all } x, y \in \ell_\rho(u,v;p),$$

(iv) let $|x_k| \leq |y_k|$, for all $k \in \mathbb{N}$ then

$$\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} u_k v_j |x_j|\right)^{p_k} \leq \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} u_k v_j |y_j|\right)^{p_k}$$

(v) there exist some numbers $k_0 = 5C2^{h-1} \geq 1$ and by using (iv) we have the inequality $\rho\left((x_n)\right) \leq \rho\left(\left(x_{\left[\frac{n}{2}\right]}\right)\right) \leq k_0 \rho\left((x_n)\right)$,

(vi) it is clear that the set of all finite sequences is $\rho - dense$ in $\ell_\rho(u,v;p)$,

(vii) for any $\lambda > 0$ there exist a constant $0 < \zeta < \lambda^{p_0-1}$ such that $\rho\left(\lambda, 0, 0, ...\right) \geq \zeta\lambda\rho(1, 0, 0, ...)$. ∎

Theorem 5. *Let X be a normed space, Y be a Banach space and the conditions $(a_1), (a_2)$ and (a_3) are satisfied, then $U_{\ell_\rho(u,v;p)}^{app}(X,Y)$ is complete.*

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Proof. Let (T_m) be a Cauchy sequence in $U^{app}_{\ell_\rho(u,v;p)}(X,Y)$. Since $\ell_\rho(u,v;p)$ is pre-modular special space of sequences, then by using Definition 3 (vii) and since $U^{app}_{\ell_\rho(u,v;p)}(X,Y) \subseteq L(X,Y)$, we have

$$\begin{aligned} \rho\left((\alpha_n(T_i-T_j))_{n=0}^{\infty}\right) &\geq \rho\left(\alpha_0(T_i-T_j),0,0,...\right) \\ &= \rho\left(\alpha_0\left\|T_i-T_j\right\|,0,0,...\right) \\ &\geq \zeta\left\|T_i-T_j\right\|\rho(1,0,0,...). \end{aligned}$$

Then (T_m) is also Cauchy sequence in $L(X,Y)$. Since the space $L(X,Y)$ is a Banach space, then there exists $T \in L(X,Y)$ such that $\|T_m - T\| \to 0$ as $m \to \infty$ and since $(\alpha_n(T_m))_{n=0}^{\infty} \in \ell_\rho(u,v;p)$ for all $m \in \mathbb{N}$, ρ is continuous at θ and using Definition 3 (iii), we have

$$\begin{aligned} \rho\left(\alpha_n(T)_{n=0}^{\infty}\right) &= \rho\left(\alpha_n(T-T_m+T_m)\right)_{n=0}^{\infty} \\ &\leq k\rho\left(\alpha_{\left[\frac{n}{2}\right]}(T_m-T)\right)_{n=0}^{\infty} + k\rho\left(\alpha_{\left[\frac{n}{2}\right]}(T_m)\right)_{n=0}^{\infty} \\ &\leq k\rho\left((\left\|T_m-T\right\|)_{n=0}^{\infty}\right) + k\rho\left(\alpha_n(T_m)\right)_{n=0}^{\infty} \\ &< \varepsilon. \end{aligned}$$

Hence $(\alpha_n(T))_{n=0}^{\infty} \in \ell_\rho(u,v;p)$ as such $T \in U^{app}_{\ell_\rho(u,v;p)}(X,Y)$. ∎

Corollary 2. *Let X be a normed space, Y be a Banach space and (p_k) be an increasing sequence of positive real numbers with $\lim_{k\to\infty}\sup(p_k) < \infty$ and $\lim_{k\to\infty}\inf(p_k) > 1$, then $U^{app}_{ces_{(p)}}(X,Y)$ is complete.*

Corollary 3. *Let X be a normed space, Y be a Banach space and (p_k) be an increasing sequence of positive real numbers with $1 < p < \infty$, then $U^{app}_{ces_p}(X,Y)$ is complete.*

Corollary 4. *Let X be a normed space, Y be a Banach space and (p_k) be an increasing sequence of positive real numbers with $1 < p < \infty$, then $U^{app}_{N_\rho(q)}(X,Y)$ is complete.*

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Certain subclasses of multivalent uniformly starlike and convex functions involving a linear operator

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Abstract

In the present paper, we obtain coefficient inequalities, inclusion relationships involving neighborhoods, subordination results, properties involving modified Hadamard products and also study majorization properties of certain subclasses of multivalent analytic functions, which are defined by means of a certain linear operator. Some useful consequences our main results are mentioned and their relevance with some of the earlier results are also pointed out.

2010 *Mathematics Subject Classification*: 30C45
Key words and phrases: $p-$Valent analytic functions, Uniformly starlike and convex functions, Subordination, Majorization, Hadamard product and modified Hadamard product.

1. Introduction and definitions

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad (p \in \mathbb{N}) \tag{1.1}$$

which are analytic and p-valent in the unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. For convenience, we write $\mathcal{A}_1 = \mathcal{A}$.

For functions f and g analytic in $\mathcal{U}$, we say that f is subordinate to g, written as $f \prec g$ or $f(z) \prec g(z)$ $(z \in \mathcal{U})$, if there exists a Schwarz function ω, which(by definition) is analytic in $\mathcal{U}$ with $\omega(0) = 0, |\omega(z)| < 1$ and $f(z) = g(\omega(z)), z \in \mathcal{U}$.

For $f \in \mathcal{A}_p$ defined by (1.1) and g given by

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k} \quad (z \in \mathcal{U}),$$

their convolution (or Hadamard product) is defined as

$$(f \star g)(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^{p+k} = (g \star f)(z) \quad (z \in \mathcal{U}).$$

A function $f \in \mathcal{A}_p$ is said to be β-uniformly p-valent starlike of order α, denoted by $UST(p, \alpha, \beta)$, if it satisfies the condition

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \beta \left|\frac{zf'(z)}{f(z)} - p\right| + \alpha \quad (-p \le \alpha < p, \beta \ge 0; z \in \mathcal{U}). \tag{1.3}$$

Analogously, a function $f \in \mathcal{A}_p$ is said to be β-uniformly p-valent convex of order α, denoted by $UCV(p, \alpha, \beta)$, if it satisfies the condition

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \beta \left|\frac{zf''(z)}{f'(z)} - (p-1)\right| + \alpha \quad (-p \le \alpha < p, \beta \ge 0; z \in \mathcal{U}). \tag{1.4}$$

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We note that $UST(1,\alpha,\beta) = UST(\alpha,\beta), UCV(1,\alpha,\beta) = UCV(\alpha,\beta), UST(p,\alpha,0) = \mathcal{S}_p^*(\alpha)$ and $UCV(p,\alpha,0) = C_p(\alpha)$, where $UST(\alpha,\beta)$ and $UCV(\alpha,\beta)$ are the classes of β-uniformly starlike and β-uniformly convex functions of order $\alpha\,(-1 \le \alpha < 1)$ studied by Ronning [18, 19] (see also [11,12]). The classes $\mathcal{S}_p^*(\alpha), C_p(\alpha)$ are the well-known classes of p-valently starlike and p-valently convex functions of order $\alpha\,(0 \le \alpha < p)$, respectively. Further, $\mathcal{S}_1^*(\alpha) = \mathcal{S}^*(\alpha), C_1(\alpha) = C(\alpha)$ are the familiar classes of starlike and convex functions of order $\alpha\,(0 \le \alpha < 1)$. We also observe that

$$f \in UCV(p,\alpha,\beta) \Leftrightarrow zf'/p \in UST(p,\alpha,\beta) \quad (-p \le \alpha < p).$$

Let ϕ_p be the incomplete beta function defined by

$$\phi_p(a,c;z) = z^p + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} z^{p+k} \quad (z \in \mathcal{U}),$$

where $a \in \mathbb{R}, c \in \mathbb{R}\setminus\mathbb{Z}_0^- = \{0,-1,-2,\dots\}$ and $(x)_k$ denote the Pochhammer symbol (or shifted factorial) defined by

$$(x)_k = \begin{cases} 1, & k = 0 \\ x(x+1)\cdots(x+k-1), & k \in \mathbb{N}. \end{cases}$$

With the aid of the function ϕ_p, we define the linear operator $\mathcal{L}_p(a,c) : \mathcal{A}_p \longrightarrow \mathcal{A}_p$ by

$$\mathcal{L}_p(a,c)f(z) = \phi_p(a,c;z) \star f(z) \quad (z \in \mathcal{U}) \tag{1.4}$$

and if f is given by (1.1), then it follows from (1.4) that

$$\mathcal{L}_p(a,c)f(z) = z^p + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} a_{p+k} z^{p+k} \quad (z \in \mathcal{U}). \tag{1.5}$$

It follows from (1.4) or (1.5) that

$$z\left(\mathcal{L}_p(a,c)f(z)\right)' = a\mathcal{L}_p(a+1,c)f(z) - (a-p)\mathcal{L}_p(a,c)f(z) \quad (f \in \mathcal{A}_p; z \in \mathcal{U}). \tag{1.6}$$

The operator $\mathcal{L}_p(a,c)$ was introduced and studied by Saitoh [22] which yields the operator $\mathcal{L}(a,c)$ introduced by Carlson and Shaffer [5] for $p = 1$. We also note that for $f \in \mathcal{A}_p$,

(i) $\mathcal{L}_p(a,a)f(z) = f(z)$;

(ii) $\mathcal{L}_p(p+1,p)f(z) = zf'(z)/p$;

(iii) $\mathcal{L}_p(m+p,1)f(z) = D^{m+p-1}f(z)$ $(m \in \mathbb{Z}, m > -p)$, the operator studied by Goel and Sohi [9]. In the case $p = 1$, $D^m f$ is the familiar Ruscheweyh derivative [20] of $f \in \mathcal{A}$.

(iv) $\mathcal{L}_p(p+1,m+p)f(z) = \mathcal{I}_{m,p}f(z)$ $(m \in \mathbb{Z}, m > -p)$, the extended Noor integral operator considered by Liu and Noor [13].

(v) $\mathcal{L}_p(p+1,p+1-\lambda)f(z) = \Omega_z^{(\lambda,p)}f(z)$ $(-\infty < \lambda < p+1)$, the extended fractional differintegral operator considered by Patel and Mishra [16].

(vi) $\mathcal{L}_p(\lambda+p,\lambda+p+1)f(z) = \dfrac{\lambda+p}{z^\lambda}\displaystyle\int_0^z t^{\lambda+p-1}f(t)dt = \mathcal{F}_{\lambda,p}(f)(z)$ $(\lambda > -p; z \in \mathcal{U})$, the generalized Bernardi-Libera-Livingston integral operator (cf., e.g., [6]).

With the help of the operator $\mathcal{L}_p(a,c)$, we introduce a subclass of $\mathcal{A}_p$ as follows:

Definition 1. For the fixed parameters A and B with $-1 \le B < A \le 1$, we say that a function $f \in \mathcal{A}_p$ is in the class $\mathcal{S}_{p,j}(a,c,\beta,A,B)$, if it satisfies the following subordination condition:

$$\frac{z\left(\mathcal{L}_p(a,c)f\right)^{(j)}(z)}{\left(\mathcal{L}_p(a,c)f\right)^{(j-1)}(z)} - \beta\left|\frac{z\left(\mathcal{L}_p(a,c)f\right)^{(j)}(z)}{\left(\mathcal{L}_p(a,c)f\right)^{(j-1)}(z)} - (p+1-j)\right| \prec \frac{(p+1-j)(1+Az)}{1+Bz}, \tag{1.7}$$

where $p \in \mathbb{N}, 1 \le j \le p, \beta \ge 0$ and $z \in \mathcal{U}$.

In view of the definition of the subordination, (1.7) is equivalent to

$$\left|\frac{\dfrac{z\left(\mathcal{L}_p(a,c)f\right)^{(j)}(z)}{\left(\mathcal{L}_p(a,c)f\right)^{(j-1)}(z)} - (p+1-j) - \beta\left|\dfrac{z\left(\mathcal{L}_p(a,c)f\right)^{(j)}(z)}{\left(\mathcal{L}_p(a,c)f\right)^{(j-1)}(z)} - (p+1-j)\right|}{(p+1-j)A - B\left\{\dfrac{z\left(\mathcal{L}_p(a,c)f\right)^{(j)}(z)}{\left(\mathcal{L}_p(a,c)f\right)^{(j-1)}(z)} - \beta\left|\dfrac{z\left(\mathcal{L}_p(a,c)f\right)^{(j)}(z)}{\left(\mathcal{L}_p(a,c)f\right)^{(j-1)}(z)} - (p+1-j)\right|\right\}}\right| < 1. \tag{1.8}$$

It is easily seen that if $f \in \mathcal{S}_{p,j}(a,c,\beta,A,B)$ with $1-A-\beta(1-B) \geq 0$, then

$$\left(\mathcal{L}_p(a,c)f\right)^{(j-1)}(z) \in \mathcal{S}_p^*\left(\frac{(p+1-j)(1-A-\beta(1-B))}{(1-\beta)(1-B)}\right). \tag{1.9}$$

for $1 \leq j \leq p$.

Let $\mathcal{T}_p$ denote the subclass of $\mathcal{A}_p$ consisting of functions of the form

$$f(z) = z^p - \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad (a_{p+k} \geq 0; z \in \mathcal{U}) \tag{1.10}$$

and for ease of notation, we write $\mathcal{T}_1 = \mathcal{T}$.

Now, we use the operator $\mathcal{L}_p(a,c)$ to define a new subclass $\mathcal{T}_{p,j}(a,c,\beta,A,B)$ as follows:

Definition 2. For the fixed parameters A and B with $-1 \leq B < A \leq 1\,(-1 \leq B < 0)$, we say that a function $f \in \mathcal{T}_p$ is in the class $\mathcal{T}_{p,j}(a,c,\beta,A,B)$, if it satisfies the following subordination condition:

$$\frac{z\left(\mathcal{L}_p(a,c)f\right)^{(j)}(z)}{\left(\mathcal{L}_p(a,c)f\right)^{(j-1)}(z)} - \beta\left|\frac{z\left(\mathcal{L}_p(a,c)f\right)^{(j)}(z)}{\left(\mathcal{L}_p(a,c)f\right)^{(j-1)}(z)} - (p+1-j)\right| \prec \frac{(p+1-j)(1+Az)}{1+Bz}, \tag{1.11}$$

where $p \in \mathbb{N}, 1 \leq j \leq p, \beta \geq 0$ and $z \in \mathcal{U}$.

We note that the families $\mathcal{S}_{p,j}(a,c,\beta,A,B)$ and $\mathcal{T}_{p,j}(a,c,\beta,A,B)$ are of special interest for they contain many well-known as well as many new classes of univalent and multivalent analytic functions. For instance,

Example 1. $\mathcal{S}_{p,j}\left(a,c,\beta,1-\dfrac{2\alpha}{p+1-j},-1\right) = \mathcal{S}_{p,j}(a,c,\beta,\alpha)$, the class consisting of functions $f \in \mathcal{A}_p$ which satisfy the condition:

$$\mathrm{Re}\left\{\frac{z\left(\mathcal{L}_p(a,c)f\right)^{(j)}(z)}{\left(\mathcal{L}_p(a,c)f\right)^{(j-1)}(z)}\right\} > \beta\left|\frac{z\left(\mathcal{L}_p(a,c)f\right)^{(j)}(z)}{\left(\mathcal{L}_p(a,c)f\right)^{(j-1)}(z)} - (p+1-j)\right| + \alpha \quad (0 \leq \alpha < p+1-j; z \in \mathcal{U}).$$

We note that

$$\mathcal{S}_{p,j}(a,c,\beta,\alpha) \cap \mathcal{T}_p = \mathcal{T}_{p,j}(a,c,\beta,\alpha)\ (0 \leq \alpha < p+1-j).$$

Example 2. (i) $\mathcal{S}_{p,j}\left(a,a,\beta,1-\frac{2\alpha}{p+1-j},-1\right) = UST(p,j,\beta,\alpha)$, the class of functions $f \in \mathcal{A}_p$ which satisfy the condition:

$$\mathrm{Re}\left\{\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}\right\} > \beta\left|\frac{zf^{(j)}(z)}{f^{(j-1)}(z)} - (p+1-j)\right| + \alpha \quad (0 \leq \alpha < p+1-j; z \in \mathcal{U}).$$

and

(ii) $\mathcal{S}_{p,j}\left(p+2-j,p+1-j,\beta,1-\frac{2\alpha}{p+1-j},-1\right) = UCV(p,j,\beta,\alpha)$, the class consisting of functions $f \in \mathcal{A}_p$ which satisfy the condition:

$$\mathrm{Re}\left\{1+\frac{zf^{(j+1)}(z)}{f^{(j)}(z)}\right\} > \beta\left|\frac{zf^{(j+1)}(z)}{f^{(j)}(z)} - (p-j)\right| + \alpha \quad (0 \leq \alpha < p+1-j; z \in \mathcal{U}).$$

We note that

$$UST(p,j,0,\alpha) \cap \mathcal{T}_p = \mathcal{T}^*(p,\alpha,j) \quad \text{and} \quad UCV(p,j,0,\alpha) \cap \mathcal{T}_p = \mathcal{C}(p,\alpha,j)$$

are the classes introduced and studied by Aouf [4].

Example 3. $\mathcal{S}_{p,j}\left(m+p,1,\beta,1-\frac{2\alpha}{p+1-j},-1\right) = \mathcal{S}_{p,j,m}(a,c,\beta,\alpha)$ $(0 \leq \alpha < p+1-j)$ is the class of functions in $\mathcal{A}_p$ satisfying the condition:

$$\mathrm{Re}\left\{\frac{z\left(\mathcal{D}^{m+p-1}f\right)^{(j)}(z)}{\left(\mathcal{D}^{m+p-1}f\right)^{(j-1)}(z)}\right\} > \beta\left|\frac{z\left(\mathcal{D}^{m+p-1}f\right)^{(j)}(z)}{\left(\mathcal{D}^{m+p-1}f\right)^{(j-1)}(z)} - (p+1-j)\right| + \alpha \quad (z \in \mathcal{U}),$$

where $p \in \mathbb{N}, 1 \le j \le p, 0 \le \alpha < p+1-j$ and $m > -p (m \in \mathbb{Z})$.

Example 4. $\mathcal{S}_{p,j}\left(p+1, p+1-\mu, \beta, 1-\frac{2\alpha}{p+1-j}, -1\right) = \mathcal{S}_p(\alpha, \beta, j, \mu)$, the subclass of functions $f \in \mathcal{A}_p$ satisfying the condition:

$$\mathrm{Re}\left\{\frac{z\left(\Omega_z^{(\mu,p)} f\right)^{(j)}(z)}{\left(\Omega_z^{(\mu,p)} f\right)^{(j-1)}(z)}\right\} > \beta\left|\frac{z\left(\Omega_z^{(\mu,p)} f\right)^{(j)}(z)}{\left(\Omega_z^{(\mu,p)} f\right)^{(j-1)}(z)} - p\right| + \alpha \quad (z \in \mathcal{U}),$$

where $p \in \mathbb{N}, 0 \le \alpha < p+1-j, 1 \le j \le p$ and $-\infty < \mu < p+1$. For $j = 1$, this class was studied by Pathak and Sharma [17].

Example 5. $\mathcal{S}_{p,j}\left(p+1, p+1+\lambda, \beta, 1-\frac{2\alpha}{p+1-j}, -1\right) = \mathcal{S}_p^{\lambda}(\alpha, \beta, j)$, the subclass of functions $f \in \mathcal{A}_p$ satisfying the condition:

$$\mathrm{Re}\left\{\frac{z\left(\mathcal{F}_{\lambda,p}(f)\right)^{(j)}(z)}{\left(\mathcal{F}_{\lambda,p}(f)\right)^{(j-1)}(z)}\right\} > \beta\left|\frac{z\left(\mathcal{F}_{\lambda,p}(f)\right)^{(j)}(z)}{\left(\mathcal{F}_{\lambda,p}(f)\right)^{(j-1)}(z)} - p\right| + \alpha \quad (z \in \mathcal{U}),$$

where $p \in \mathbb{N}, 0 \le \alpha < p+1-j, 1 \le j \le p$ and $\lambda > -p$.

Example 6. $\mathcal{S}_{p,1}(a, c, \beta, 1-\dfrac{2\,\alpha}{p}, -1) = \mathcal{S}_p(a, c, \beta, \alpha)$, the class of functions $f \in \mathcal{A}_p$ satisfying

$$\mathrm{Re}\left\{\frac{a\mathcal{L}_p(a+1,c)f(z)}{\mathcal{L}_p(a,c)f(z)} - a + p\right\} > \beta\left|\frac{a\mathcal{L}_p(a+1,c)f(z)}{\mathcal{L}_p(a,c)f(z)} - a\right| + \alpha \ (z \in \mathcal{U}),$$

where $0 \le \alpha < p$. We note that $\mathcal{S}_{1,1}(a, c, \beta, \alpha) = \mathcal{L}(a, c, \alpha, \beta)$ was the class introduced and studied by Frasin [8].

Analogous to the classes defined above (i.e., Example 1 to Example 6), we can also define subclasses of multivalent analytic functions with negative coefficients by taking the intersection of these classes with $\mathcal{T}_p$.

2. Coefficient estimates

Unless otherwise mentioned, we assume throughout the sequel that $p \in \mathbb{N}, a > 0, c > 0$, $\beta \ge 0, 1 \le j \le p, -1 \le B < A \le 1$

Next, we give a sufficient condition for functions belonging to the class $\mathcal{S}_{p,j}(a, c, \beta, A, B)$.

Theorem 1. Let the function f be defined by (1.1). If

$$\sum_{k=1}^{\infty} \frac{\delta(p+k, j-1)\left\{k(1+\beta+\beta|B|) + |(p+1-j)A - (p+k+1-j)B|\right)\}}{\delta(p,j)(A-B)} \frac{(a)_k}{(c)_k}|a_{p+k}| \le 1, \tag{2.1}$$

then $f \in \mathcal{S}_{p,j}(a, c, \beta, A, B)$, where

$$\delta(p,j) = \begin{cases} p(p-1)\dots(p+1-j), & j \in \mathbb{N} \\ 1, & j = 0. \end{cases} \tag{2.2}$$

Proof. In view of (1.8), we need to show that for $z \in \partial\mathcal{U}$,

$$\Big| z\left(\mathcal{L}_p(a,c)f\right)^{(j)}(z) - (p+1-j)\left(\mathcal{L}_p(a,c)f\right)^{(j-1)}(z)$$
$$-\beta\, e^{i\theta}\left|z\left(\mathcal{L}_p(a,c)f\right)^{(j)}(z) - (p+1-j)\left(\mathcal{L}_p(a,c)f\right)^{(j-1)}(z)\right|\Big| - \Big|(p+1-j)A\left(\mathcal{L}_p(a,c)f\right)^{(j-1)}(z)$$
$$-B\,z\left(\mathcal{L}_p(a,c)f\right)^{(j)}(z) + \beta B e^{i\theta}\left|z\left(\mathcal{L}_p(a,c)f\right)^{(j)}(z) - (p+1-j)\left(\mathcal{L}_p(a,c)f\right)^{(j-1)}(z)\right|\Big| < 0.$$

Substituting the series representation of $(\mathcal{L}_p(a,c)f)^{(j-1)}(z)$ and $(\mathcal{L}_p(a,c)f)^{(j)}(z)$ in the above expression, we deduce that for $-\pi < \theta < \pi$.

$$\begin{aligned}
&\Big|\, z\left(\mathcal{L}_p(a,c)f\right)^{(j)}(z) - (p+1-j)\left(\mathcal{L}_p(a,c)f\right)^{(j-1)}(z) \\
&-\beta\, e^{i\theta}\left|z\left(\mathcal{L}_p(a,c)f\right)^{(j)}(z) - (p+1-j)\left(\mathcal{L}_p(a,c)f\right)^{(j-1)}(z)\right|\Big| - \Big|(p+1-j)A\left(\mathcal{L}_p(a,c)f\right)^{(j-1)}(z) \\
&-B\,z\left(\mathcal{L}_p(a,c)f\right)^{(j)}(z) + \beta B e^{i\theta}\left|z\left(\mathcal{L}_p(a,c)f\right)^{(j)}(z) - (p+1-j)\left(\mathcal{L}_p(a,c)f\right)^{(j-1)}(z)\right|\Big| \\
&\le \sum_{k=1}^{\infty} k(1+\beta)\frac{(p+k)!}{(p+k+1-j)!}\frac{(a)_k}{(c)_k}|a_{p+k}||z|^k - \frac{p!}{(p-j)!}(A-B) \\
&+\sum_{k=1}^{\infty}\frac{(p+k)!}{(p+k+1-j)!}\big|(p+j-1)A-(p+k+1-j)B\big|\frac{(a)_k}{(c)_k}|a_{p+k}||z|^k \\
&+\beta|B|\sum_{k=1}^{\infty} k\frac{(p+k)!}{(p+k+1-j)!}\frac{(a)_k}{(c)_k}|a_{p+k}||z|^k \\
&\le \sum_{k=1}^{\infty}\delta(p+k,j-1)\left\{k(1+\beta+\beta B)+\big|(p+j-1)A-(p+k+1-j)B\big|\right\}\frac{(a)_k}{(c)_k}|a_{p+k}| \\
&\quad -\delta(p,j)(A-B) \le 0,
\end{aligned}$$

by applying the hypothesis (2.1). Now, the result follows by using the maximum modulus theorem.

Theorem 2. Let the function f be given by (1.10) and $-1 \le B < 0$. Then $f \in \mathcal{T}_{p,j}(a,c,\beta,A,B)$, if and only if

$$\sum_{k=1}^{\infty}\frac{\delta(p+k,j-1)\left\{k(1+\beta)(1-B)+(p+1-j)(A-B)\right\}}{\delta(p,j)(A-B)}\frac{(a)_k}{(c)_k}a_{p+k} \le 1, \tag{2.3}$$

where $\delta(p,j)$ is defined by (2.2). The result is sharp.

Proof. In view of Theorem 1, we only need to prove the sufficient part of Theorem 2. We assume that f, given by (1.10) belongs to the class $\mathcal{T}_{p,j}(a,c,\beta,A,B)$. Thus, by (1.11), we deduce that

$$\begin{aligned}
&(1-B)\mathrm{Re}\left\{\frac{z\left(\mathcal{L}_p(a,c)f\right)^{(j)}(z)}{\left(\mathcal{L}_p(a,c)f\right)^{(j-1)}(z)}\right\} - (p+1-j)(1-A) \\
&\quad -\beta(1-B)\left|\frac{z\left(\mathcal{L}_p(a,c)f\right)^{(j)}(z)}{\left(\mathcal{L}_p(a,c)f\right)^{(j-1)}(z)} - (p+1-j)\right| > 0 \quad (z\in\mathcal{U}).
\end{aligned} \tag{2.4}$$

Substituting the series expansion of $(\mathcal{L}_p(a,c)f)^{(j)}(z)$ and $(\mathcal{L}_p(a,c)f)^{(j-1)}(z)$ in (2.4), we get

$$\begin{aligned}
&\mathrm{Re}\left\{\frac{\delta(p,j)(A-B)-\sum_{k=1}^{\infty}\delta(p+k,j-1)\left\{k(1-B)+(p+1-j)(A-B)\right\}\frac{(a)_k}{(c)_k}a_{p+k}z^k}{\delta(p,j-1)-\sum_{k=1}^{\infty}\delta(p+k,j-1)(p+1-j)\frac{(a)_k}{(c)_k}a_{p+k}z^k}\right\} \\
&\quad -\left|\frac{\beta(1-B)\sum_{k=1}^{\infty}-k\delta(p+k,j-1)\frac{(a)_k}{(c)_k}a_{p+k}z^k}{\delta(p,j-1)-\sum_{k=1}^{\infty}\delta(p+k,j-1)(p+1-j)\frac{(a)_k}{(c)_k}a_{p+k}z^k}\right| > 0 \quad (z\in\mathcal{U}).
\end{aligned}$$

Using the fact that $\mathrm{Re}(z) \le |z|$ for all $z \in \mathbb{C}$ and upon choosing the values of z on the real axis, the above

inequality reduces to

$$\begin{aligned}&\mathrm{Re}\left\{\frac{\delta(p,j)(A-B)-\sum_{k=1}^{\infty}\delta(p+k,j-1)\left\{k(1-B)+(p+1-j)(A-B)\right\}\dfrac{(a)_k}{(c)_k}a_{p+k}z^k}{\delta(p,j-1)-\sum_{k=1}^{\infty}\delta(p+k,j-1)(p+1-j)\dfrac{(a)_k}{(c)_k}a_{p+k}z^k}\right\}\\&-\mathrm{Re}\left\{e^{i\theta}\frac{\beta(1-B)\sum_{k=1}^{\infty}-k\delta(p+k,j-1)\dfrac{(a)_k}{(c)_k}a_{p+k}z^k}{\delta(p,j-1)-\sum_{k=1}^{\infty}\delta(p+k,j-1)(p+1-j)\dfrac{(a)_k}{(c)_k}a_{p+k}z^k}\right\}>0\ (-\pi<\theta<\pi;z\in\mathbb{R}\cap\mathcal{U}).\end{aligned}\tag{2.5}$$

Since $\mathrm{Re}\left(e^{i\theta}\right)\geq-|e^{i\theta}|=-1$, (2.5) can be rewritten as

$$\begin{aligned}&\mathrm{Re}\left\{\frac{\delta(p,j)(A-B)-\sum_{k=1}^{\infty}\delta(p+k,j-1)\left\{k(1-B)+(p+1-j)(A-B)\right\}\dfrac{(a)_k}{(c)_k}a_{p+k}z^k}{\delta(p,j-1)-\sum_{k=1}^{\infty}\delta(p+k,j-1)(p+1-j)\dfrac{(a)_k}{(c)_k}a_{p+k}z^k}\right\}\\&-\mathrm{Re}\left\{\frac{\beta(1-B)\sum_{k=1}^{\infty}-k\delta(p+k,j-1)\dfrac{(a)_k}{(c)_k}a_{p+k}z^k}{\delta(p,j-1)-\sum_{k=1}^{\infty}\delta(p+k,j-1)(p+1-j)\dfrac{(a)_k}{(c)_k}a_{p+k}z^k}\right\}>0\quad(z\in\mathbb{R}\cap\mathcal{U}).\end{aligned}\tag{2.6}$$

Letting $r\to1^-$ in (2.6), we get the desired inequality.

It is easily verified that the estimate (2.3) is sharp for the function f_0, defined by

$$f_0(z)=z^p-\frac{(p+1-j)(p+2-j)(A-B)c}{(p+1)\left\{(1+\beta)(1-B)+(p+1-j)(A-B)\right\}a}z^{p+1}\ (-1\leq B<0;z\in\mathcal{U}).\tag{2.7}$$

Letting $A=1-\{2\alpha/(p+1-j)\}$ $(0\leq\alpha<P+1-j)$ and $B=-1$ in Theorem 2, we obtain

Corollary 1. Let the function f be given by (1.10). Then $f\in\mathcal{T}_{p,j}(a,c,\beta,\alpha)$, if and only if

$$\sum_{k=1}^{\infty}\frac{\delta(p+k,j-1)\left\{k(1+\beta)+(p+1-j-\alpha)\right\}}{\delta(p,j-1)(p+1-j-\alpha)}\frac{(a)_k}{(c)_k}a_{p+k}\leq1.$$

The result is sharp.

Theorem 3. Let $a\geq c>0$ and $-1\leq B<0$. If $f\in\mathcal{T}_{p,j}(a,c,\beta,A,B)$, then the unit disk is mapped onto a domain that contains the disk

$$\left\{z\in\mathbb{C}:|z|<\frac{(p+1)(1+\beta)(1-B)a+(p+1-j)(A-B)\{(p+1)(a-c)+(j-1)c\}}{(p+1)\{(1+\beta)(1-B)+(p+1-j)(A-B)\}a}\right\}.$$

The result is sharp.

Proof. Let f, defined by (1.10) belongs to the class $f\in\mathcal{T}_{p,j}(a,c,\beta,A,B)$. Then, by virtue of Theorem 2, we get

$$\begin{aligned}&\frac{\delta(p+1,j-1)\left\{(1+\beta)(1-B)+(p+1-j)(A-B)\right\}a}{\delta(p,j)(A-B)c}\sum_{k=1}^{\infty}a_{p+k}\\&\leq\sum_{k=1}^{\infty}\frac{\delta(p+k,j-1)\left\{k(1+\beta)(1-B)+(p+1-j)(A-B)\right\}(a)_k}{\delta(p,j)(A-B)(c)_k}a_{p+k}\leq1,\end{aligned}$$

which evidently yields

$$\begin{aligned}\sum_{k=1}^{\infty}a_{p+k}&\leq\frac{\delta(p,j)(A-B)c}{\delta(p+1,j-1)\left\{(1+\beta)(1-B)+(p+1-j)(A-B)\right\}a}\\&=\frac{(p+1-j)(p+2-j)(A-B)c}{(p+1)\left\{(1+\beta)(1-B)+(p+1-j)(A-B)\right\}a}.\end{aligned}$$

Consequently, for $|z| = r < 1$, we deduce that

$$\begin{aligned}|f(z)| &\geq r^p - \left(\sum_{k=1}^{\infty} a_{p+k}\right) r^{p+1} \\ &\geq r^p - \frac{\delta(p,j)(A-B)c}{\delta(p+1,j-1)\left\{(1+\beta)(1-B)+(p+1-j)(A-B)\right\} a} r^{p+1}.\end{aligned}$$

Hence, by letting $r \to 1^-$ in the above inequality, we get the required result. The result is sharp for the function f_0 given by (2.7). This completes the proof of Theorem 3.

Remarks 1. A special case of Theorem 2 when $j = 1, A = 1 - 2\alpha\,(0 \leq \alpha < p)$ and $B = -1$ was obtained by Frasin [8, Theorem 1]. Further, if in Theorem 2 with $\beta = 0, A = 1 - \{2\alpha/(p+1-j)\}, B = -1$, we set $a = c$ and $a = p+2-j, c = p+1-j$, we shall obtain the corresponding results obtained by Aouf [4] for the classes $\mathcal{T}^*(p,\alpha,j)$ and $\mathcal{C}(p,\alpha,j)$, respectively.

2. If, in Theorem 3 with $\beta = 0, A = 1 - \{2\alpha/(p+1-j)\}\,(0 \leq \alpha < p+1-j)$ and $B = -1$, we set $a = c$ and $a = p+2-j, c = p+1-j$, we obtain the corresponding results obtained by Aouf [4, Corollary 5 and Corollary 6].

Theorem 4. If $a \geq c > 0$ and $-1 \leq B < 0$, then

$$\mathcal{T}_{p,j}(a+1,c,\beta,A,B) \subset \mathcal{T}_{p,j}(a,c,\beta,C,B),$$

where

$$C = B + \frac{a(1+\beta)(1-B)(A-B)}{(a+1)(1+\beta)(1-B)+(p+1-j)(A-B)}.$$

The result is sharp.

Proof. Suppose f, given by (1.10) belongs to the class $\mathcal{T}_{p,j}(a+1,c,\beta,A,B)$. Then by Theorem 2, we obtain

$$\sum_{k=1}^{\infty} \frac{\delta(p+k,j-1)\left\{k(1+\beta)(1-B)+(p+1-j)(A-B)\right\}}{\delta(p,j)(A-B)} \frac{(a+1)_k}{(c)_k} a_{p+k} \leq 1, \tag{2.8}$$

In view of Theorem 2 and (2.8), we need to find the largest value of C so that

$$\frac{\left\{k(1+\beta)(1-B)+(p+1-j)(C-B)\right\} a}{C-B} \leq \frac{\left\{k(1+\beta)(1-B)+(p+1-j)(A-B)\right\}\,(a+k)}{A-B}$$

each positive integer $k \in \mathbb{N}$. Or, equivalently,

$$C \geq B + \frac{a(1+\beta)(1-B)(A-B)}{(a+k)(1+\beta)(1-B)+(p+1-j)(A-B)} \quad (k \in \mathbb{N}).$$

Since the right side of the above inequality is a decreasing function of $k (k \in \mathbb{N})$, putting $k = 1$ in the above inequality, we get the required result.

The result is sharp for the function f_0, defined by

$$f_0(z) = z^p - \frac{(p+1-j)(p+2-j)(A-B)c}{(p+1)\{(1+\beta)(1-B)+(p+1-j)(A-B)\}(a+1)} z^{p+1} \quad (-1 \leq B < 0; z \in \mathcal{U}).$$

Denoting

$$UST^+(p,j,\alpha,\beta) = UST(p,j,\alpha,\beta) \cap \mathcal{T}_p, \quad UCV^+(p,j,\alpha,\beta) = UCV(p,j,\alpha,\beta) \cap \mathcal{T}_p$$

and taking $a = c = p+1-j, A = 1 - \{2\alpha/(p+1-j)\}\ (0 \leq \alpha < p+1-j)$ and $B = -1$ in Theorem 4, we get the following result.

Corollary 2. We have

$$UCV^+(p,j,\alpha,\beta) \subset UST^+(p,j,\rho,\beta),$$

where

$$\rho = \frac{(p+1-j)\{\beta(1+\alpha)+(p+2-j)\}}{(p+2-j)(1+\beta)+(p+1-j-\alpha)}.$$

The result is sharp with the extremal function f_0, given by

$$f_0(z) = z^p - \frac{(p+1-j)(p+1-j-\alpha)}{(p+1)(p+2+\beta-j-\alpha)} z^{p+1} \quad (z \in \mathcal{U}).$$

Theorem 5. If $0 \leq \beta \leq (1-A)/(1-B)$ and the function f, defined by (1.1) belongs to the class $\mathcal{S}_{p,j}(a, c, \beta, A, B)$, then

$$|a_{p+k}| \leq \frac{\delta(p, j-1)(c)_k}{k!\, \delta(p+k, j-1)(a)_k} \prod_{j=1}^{k} \left(j-1) + \frac{2(p+1-j)(A-B)}{(1-\beta)(1-B)} \right) \quad (k \in \mathbb{N}), \tag{2.9}$$

where $\delta(p, j)$ is defined as in (2.2).

Proof. Suppose f, defined by (1.1) belongs to the class $\mathcal{S}_{p,j}(a, c, \beta, A, B)$. Then, by using (1.9), we deduce that

$$\begin{aligned} z\left(\mathcal{L}_p(a,c)f\right)^{(j)}(z) - \frac{(p+1-j)\{1-A-\beta(1-B)\}}{(1-\beta)(1-B)} \left(\mathcal{L}_p(a,c)f\right)^{(j-1)}(z) \\ = \frac{(p+1-j)(A-B)}{(1-\beta)(1-B)} \left(\mathcal{L}_p(a,c)f\right)^{(j-1)}(z)\, q(z) \quad (z \in \mathcal{U}), \end{aligned} \tag{2.10}$$

where $q(z) = 1 + q_1 z + q_2 z^2 + \ldots$ is analytic in $\mathcal{U}$ and $\mathrm{Re}\{q(z)\} > 0$ for $z \in \mathcal{U}$. Substituting the series expansions of $\left(\mathcal{L}_p(a,c)f\right)^{(j-1)}, \left(\mathcal{L}_p(a,c)f\right)^{(j)}, q$ in (2.10) and comparing the coefficient of z^{p+k} on both the sides of the resulting equation, we obtain

$$\begin{aligned} k\,\delta(p+k, j-1)\frac{(a)_k}{(c)_k} a_{p+k} = \frac{(p+1-j)(A-B)}{(1-\beta)(1-B)} \times \\ \times \left\{\delta(p, j-1)q_k + \delta(p+1, j-1)q_{k-1}a_{p+1} + \cdots + \delta(p+k-1, j-1)q_1 a_{p+k-1}\right\}. \end{aligned} \tag{2.11}$$

Using the fact that $|q_k| \leq 2$ for $k \geq 1$ for Carathéodory function q [7] in (2.11), we have

$$\begin{aligned} |a_{p+k}| \leq \frac{2(p+1-j)(A-B)(c)_k}{k\,\delta(p+k, j-1)(1-\beta)(1-B)(a)_k} \times \\ \times \{\delta(p, j-1) + \delta(p+1, j-1)|a_{p+1}| + \cdots + \delta(p+k-1, j-1)|a_{p+k-1}|\} \end{aligned} \tag{2.12}$$

for each $k \in \mathbb{N}$. We shall prove the estimate (2.9) by induction on k. For $k = 1$, (2.12) gives

$$|a_{p+1}| \leq \frac{2\delta(p, j-1)(p+1-j)(A-B)(c)_1}{\delta(p+1, j-1)(1-\beta)(1-B)(a)_1}$$

so that (2.9) is certainly true. For $k = 2$, (2.12) yields

$$\begin{aligned} |a_{p+2}| &\leq \frac{2(p+1-j)(A-B)(c)_2}{2\delta(p+2, j-1)(1-\beta)(1-B)(a)_2} \{\delta(p, j-1) + \delta(p+1, j-1)|a_{p+1}|\} \\ &\leq \frac{\delta(p, j-1)(c)_2}{2!\,\delta(p+2, j-1)(a)_2} \frac{2(p+1-j)(A-B)}{(1-\beta)(1-B)} \left(1 + \frac{2(p+1-j)(A-B)}{(1-\beta)(1-B)}\right). \end{aligned}$$

Thus, (2.9) is true for $k = 2$. We assume that (2.9) holds true for $k = n$. Then for $k = n+1$, we obtain

$$\begin{aligned} |a_{p+n+1}| \leq \frac{\delta(p, j-1)(c)_{n+1}}{(n+1)\delta(p+n+1, j-1)(a)_{n+1})} x_j \times \\ \times \left\{(1+x_j) + \frac{1}{2!}x_j(1+x_j) + \frac{1}{3!}x_j(1+x_j)(2+x_j) + \cdots + \frac{1}{n!}x_j(1+x_j)\cdots(n+x_j)\right\}, \end{aligned} \tag{2.13}$$

where $x_j = 2(p+1-j)(A-B)/\{(1-\beta)(1-B)\}$. Since

$$(1+x_j) + \frac{1}{2!}x_j(1+x_j) + \frac{1}{3!}x_j(1+x_j)(2+x_j) + \cdots + \frac{1}{n!}x_j(1+x_j)\cdots(n+x_j) = \frac{1}{n!}\prod_{j=1}^{n}(j+x_j),$$

the estimate (2.13) implies that (2.9) is true for $k = n+1$. This evidently completes the proof of Theorem 5.

Remark. Putting $j = 1, a = p+1, c = p+1-\mu (0 \leq \mu < 1), A = 1-(2\alpha/p)(0 \leq \alpha < p)$ and $B = -1$ in Theorem 5, we shall obtain the result due to Pathak and Sharma [17, Theorem 2].

3. Mazorization problem for the class $\mathcal{S}_{p,j}(a,c,\beta,A,B)$

Let f and g be analytic functions in the unit disk $\mathcal{U}$. We say that f is majorized by g in $\mathcal{U}$ (see [14]) and write

$$f(z) \ll g(z) \quad (z \in \mathcal{U}),$$

if there exists a function φ, analytic in $\mathcal{U}$ such that $|\varphi(z)| < 1$ and $f(z) = \varphi(z)g(z)$ for all $z \in \mathcal{U}$.

To establish our result, we need the following lemma.

Lemma 1. If $a > 0, 1 \le j \le p, 0 \le \beta < 1$ and $0 \le aB + (p+1-j)(A-B)$, then

$$\left|aB + \frac{(p+1-j)(A-B)}{1-\beta e^{i\theta}}\right| \le aB + \frac{(p+1-j)(A-B)}{1-\beta} \quad (\theta \in \mathbb{R}). \tag{3.1}$$

Proof. Since $|1-\beta e^{i\theta}| \ge (1-\beta)$ for all $\theta \in \mathbb{R}$, we have for $0 \le B < A \le 1$

$$\begin{aligned}\left|aB + \frac{(p+1-j)(A-B)}{1-\beta e^{i\theta}}\right| &\le aB + \left|\frac{(p+1-j)(A-B)}{1-\beta e^{i\theta}}\right| \\ &\le aB + \frac{(p+1-j)(A-B)}{1-\beta} \quad (\theta \in \mathbb{R}).\end{aligned}$$

This proves the inequality (3.1) for $0 \le B < A \le 1$. Next, we assume that $-1 \le B < 0 \le A \le 1$. We note that

$$\begin{aligned}&\{aB(1-\beta) + (p+1-j)(A-B)\}^2 - \left|aB(1-\beta e^{i\theta}) + (p+1-j)(A-B)\right|^2 \\ &= 2\beta(aB)^2(\cos\theta - 1) + 2(p+1-j)a\beta(A-B)(\cos\theta - 1) \\ &= 2a\beta\, B(\cos\theta - 1)\{aB + (p+1-j)(A-B)\} \\ &\ge 0\end{aligned}$$

by using the hypothesis and the fact that $B < 0$ and $\cos\theta \le 1$ for all $\theta \in \mathbb{R}$. Thus, the inequality (3.1) holds for $-1 \le B < 0$ and the proof of Lemma 1 is completed.

Now, we prove

Theorem 6. Let $a > 0, 0 \le \beta < 1, aB + (p+1-j)(A-B) \ge 0$ and $a(1-\beta-B) - (p+1-j)(A-B) \ge 0$. If the functions $f \in \mathcal{A}_p, g \in \mathcal{S}_{p,j}(a,c,\beta,A,B)$ and

$$(\mathcal{L}_p(a,c)f)^{(j-1)}(z) \ll (\mathcal{L}_p(a,c)f)^{(j-1)}(z) \quad (z \in \mathcal{U}),$$

then

$$\left|(\mathcal{L}_p(a+1,c)f)^{(j-1)}(z)\right| \le \left|(\mathcal{L}_p(a+1,c)f)^{(j-1)}(z)\right| \quad (|z| < r_0 = r_0(p,a,j,\beta,A,B)), \tag{3.2}$$

where r_0 is the smallest positive root of the equation:

$$\left(aB + \frac{(p+1-j)(A-B)}{1-\beta}\right)r^3 - (a+2|B|)r^2 - \left((2 + aB + \frac{(p+1-j)(A-B)}{1-\beta}\right)r + a = 0. \tag{3.3}$$

Proof. Since $g \in \mathcal{S}_{p,j}(a,c,\beta,A,B)$, it follows from (1.7) that

$$\frac{z\,(\mathcal{L}_p(a,c)g)^{(j)}(z)}{(\mathcal{L}_p(a,c)g)^{(j-1)}(z)} - \beta e^{i\theta}\left\{\frac{z\,(\mathcal{L}_p(a,c)g)^{(j)}(z)}{(\mathcal{L}_p(a,c)g)^{(j-1)}(z)} - (p+1-j)\right\} \prec \frac{(p+1-j)(1+Az)}{1+Bz} \quad (z \in \mathcal{U})$$

for $-\pi < \theta < \pi$ so that

$$\frac{z\,(\mathcal{L}_p(a,c)g)^{(j)}(z)}{(\mathcal{L}_p(a,c)g)^{(j-1)}(z)} = \frac{(p+1-j)\left\{1 + \dfrac{A-\beta Be^{i\theta}}{1-\beta e^{i\theta}}\omega(z)\right\}}{1+B\omega(z)} \quad (z \in \mathcal{U}), \tag{3.4}$$

where ω is analytic in $\mathcal{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ in $\mathcal{U}$. Also from (1.6), we deduce that

$$z\,(\mathcal{L}_p(a,c)g)^{(j)}(z) = a\,(\mathcal{L}_p(a+1,c)g)^{(j-1)}(z) - (a-p+j-1)\,(\mathcal{L}_p(a,c)g)^{(j-1)}(z) \quad (z \in \mathcal{U}). \tag{3.5}$$

Now, by using (3.5) in (3.4), we get

$$\begin{aligned}&\left|(\mathcal{L}_p(a,c)g)^{(j-1)}(z)\right|\\ &\le \frac{a(1+B|z|)}{a-\left\{aB+\dfrac{(p+1-j)(A-B)}{1-\beta}\right\}|z|}\left|(\mathcal{L}_p(a+1,c)g)^{(j-1)}(z)\right| \quad (z\in\mathcal{U}).\end{aligned} \tag{3.6}$$

From (3.1) and the hypothesis, we have

$$z\left(\mathcal{L}_p(a,c)f\right)^{(j)}(z)=z\varphi(z)\left(\mathcal{L}_p(a,c)g\right)^{(j)}(z)+z\varphi'(z)\left(\mathcal{L}_p(a,c)g\right)^{(j-1)}(z) \quad (z\in\mathcal{U}). \tag{3.7}$$

Using (3.5) in (3.7), we obtain

$$\begin{aligned}&a\left|(\mathcal{L}_p(a+1,c)f)^{(j-1)}(z)\right|\\ &\le a|\varphi(z)|\left|(\mathcal{L}_p(a+1,c)g)^{(j-1)}(z)\right|+|z||\varphi'(z)|\left|(\mathcal{L}_p(a,c)g)^{(j-1)}(z)\right| \quad (z\in\mathcal{U}).\end{aligned} \tag{3.8}$$

Using the following inequality [15]

$$|\varphi'(z)|\le\frac{1-|\varphi(z)|^2}{1-|z|^2} \quad (z\in\mathcal{U})$$

in (3.8) and applying the resulting inequality in (3.6), we deduce that

$$\begin{aligned}\left|(\mathcal{L}_p(a+1,c)f)^{(j-1)}(z)\right|\le&\left\{|\varphi(z)|+\frac{|z|(1+B|z|)(1-|\varphi(z)|^2)}{(1-|z|^2)\left\{a-\left(aB+\dfrac{(p+1-j)(A-B)}{1-\beta}\right)|z|\right\}}\right\}\times\\ &\times\left|(\mathcal{L}_p(a+1,c)g)^{(j-1)}(z)\right| \quad (z\in\mathcal{U}),\end{aligned}$$

which upon setting $|z|=r<1$ and $|\varphi(z)|=x$ $(0\le x\le 1)$ leads us to the inequality

$$\begin{aligned}\left|(\mathcal{L}_p(a+1,c)f)^{(j-1)}(z)\right|\le&\frac{\psi(x)}{(1-r^2)\left\{a-\left(aB+\dfrac{(p+1-j)(A-B)}{1-\beta}\right)r\right\}}\times\\ &\times\left|(\mathcal{L}_p(a+1,c)g)^{(j-1)}(z)\right| \quad (z\in\mathcal{U}),\end{aligned}$$

where

$$\psi(x)=-r(1+|B|r)x^2+(1-r^2)\left\{a-\left(aB+\frac{(p+1-j)(A-B)}{1-\beta}\right)r\right\}x+r(1+|B|r).$$

The function ψ takes its maximum value at $x=1$ with $r_0=r_0(p,a,j,\beta,A,B)$, where $r_0(p,a,j,\beta,A,B)$ is the smallest positive root of the equation (3.2). Further, for $0\le y\le r_0(p,a,j,\beta,A,B)$, the function

$$\Phi(x)=-y(1+|B|y)x^2+(1-y^2)\left\{a-\left(aB+\frac{(p+1-j)(A-B)}{1-\beta}\right)y\right\}x+y(1+|B|y) \tag{3.9}$$

is increasing on the interval $0\le x\le 1$ so that

$$\Phi(x)\le\Phi(1)=(1-y^2)\left\{a-\left(aB+\frac{(p+1-j)(A-B)}{1-\beta}\right)y\right\}$$

Hence, upon setting $x=1$ in (3.9), we conclude that (3.1) holds true for $|z|<r_0(p,a,j,\beta,A,B)$, where $r_0(p,a,j,\beta,A,B)$ is the smallest positive root of the equation (3.2). This completes the proof of Theorem 5.

Letting $A=1-\{2\alpha/(p+1-j)\}, B=-1$ and $a=c=p+1-j$ in Theorem 5, we get the following result which yields the corresponding work of MacGregor [14] for $j=1$ and $\alpha=0$.

Corollary 3. Let $1 \le j \le p, 0 \le \alpha \le (p+1-j)/2$ and $0 \le \beta \le \alpha/(p+1-j)$. If the functions $f \in \mathcal{A}_p$ and $g \in UST(p,\alpha,\beta,j)$ satisfies

$$|f^{(j-1)}(z)| \ll |g^{(j-1)}(z)| \quad (z \in \mathcal{U}),$$

then

$$|f^{(j)}(z)| \le |g^{(j)}(z)| \quad (|z| < \widetilde{r_0}),$$

where $\widetilde{r_0}$ is the smallest positive root of the equation:

$$\left\{\frac{2(p+1-j-\alpha)}{1-\beta} - (p+1-j)\right\} r^2 - 2\left\{1 + \frac{p+1-j-\alpha}{1-\beta}\right\} r + (p+1-j) = 0.$$

4. Inclusion relatioships involving neighborhoods

Following the earlier investigations by Goodman [10], Ruscheweyh [21] and others including Altintas and Owa [2] (see also [1,3]), we define the $\varepsilon-$neighborhoods of a function $f \in \mathcal{A}_p$ given by (1.1) as follows:

$$\mathcal{N}_\varepsilon(f) = \left\{ g \in \mathcal{A}_p : g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k} \text{ and } \sum_{k=1}^{\infty} (p+k)|b_{p+k} - a_{p+k}| \le \varepsilon; \varepsilon > 0 \right\}. \tag{4.1}$$

In particular, for the identity function $e(z) = z^p (z \in \mathcal{U})$, we immediately have

$$\mathcal{N}_\varepsilon(e) = \left\{ g \in \mathcal{A}_p : g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k} \text{ and } \sum_{k=1}^{\infty} (p+k)|b_{p+k}| \le \varepsilon; \varepsilon > 0 \right\}. \tag{4.2}$$

For convenience, we denote

$$\mathcal{N}_\varepsilon^+(f) = \mathcal{N}_\varepsilon(f) \cap \mathcal{T}_p \quad \text{and} \quad \mathcal{N}_\varepsilon^+(e) = \mathcal{N}_\varepsilon(e) \cap \mathcal{T}_p.$$

Theorem 7. If $a \ge c > 0, 1 \le j \le p, -1 \le B < 0$ and

$$\varepsilon = \frac{(p+1)\delta(p,j)(1+\beta)(1-B)(A-B)c}{\delta(p+1,j-1)\{(1+\beta)(1-B)+(p+1-j)(A-B)\}a},$$

then

$$\mathcal{T}_{p,j}(a,c,\beta,A,B) \subset \mathcal{N}_\varepsilon^+(e).$$

Proof. Suppose f, given by (1.10) belongs to the class $\mathcal{T}_{p,j}(a,c,\beta,A,B)$. Then by Theorem 2, we have

$$\frac{\delta(p+1,j-1)\{(1+\beta)(1-B)+(p+1-j)(A-B)\}}{\delta(p,j)(A-B)} \frac{a}{c} \sum_{k=1}^{\infty} a_{p+k} \le 1,$$

so that

$$\sum_{k=1}^{\infty} a_{p+k} \le \frac{\delta(p,j)(A-B)c}{\delta(p+1,j-1)\{(1+\beta)(1-B)+(p+1-j)(A-B)\}a}. \tag{4.3}$$

Making use of Theorem 2 again in conjunction with (4.3), we deduce that

$$\begin{aligned}
&\frac{\delta(p+1,j-1)(1+\beta)(1-B)\,a}{c} \sum_{k=1}^{\infty} (p+k) a_{p+k} \\
&\le \delta(p,j)(A-B) + \{p(1+\beta)(1-B) - (p+1-j)(A-B)\} \frac{\delta(p+1,j-1)\,a}{c} \sum_{k=1}^{\infty} a_{p+k} \\
&\le \delta(p,j)(A-B) + \frac{\delta(p,j)(A-B)\,\{p(1+\beta)(1-B) - (p+1-j)(A-B)\}}{(1+\beta)(1-B)+(p+1-j)(A-B)} \\
&\le \frac{\delta(p,j)(p+1)(1+\beta)(1-B)(A-B)}{(1+\beta)(1-B)+(p+1-j)(A-B)}
\end{aligned}$$

that is,

$$\sum_{k=1}^{\infty}(p+k)a_{p+k} \leq \frac{\delta(p,j)(p+1)(A-B)c}{\delta(p+1,j-1)\{(1+\beta)(1-B)+(p+1-j)(A-B)\}a},$$

which in view of (4.2) establishes the inclusion relation asserted by Theorem 4.

Setting $A = 1-\{2\alpha/(p+1-j)\}$ and $B=-1$ in Theorem 7, we have

Corollary 4. If $a \geq c > 0, 1 \leq j \leq p$ and

$$\varepsilon = \frac{\delta(p,j-1)(p+1)(p+1-j-\alpha)c}{\delta(p+1,j-1)\{(1+\beta)+(p+1-j-\alpha)\}a} \quad (0 \leq \alpha < p+1-j),$$

then

$$\mathcal{T}_{p,j}(a,c,\beta,\alpha) \subset \mathcal{N}_{\varepsilon}^{+}(e).$$

Next, we determine the neighborhood for the class $\mathcal{T}_{p,j}^{(\gamma)}(a,c,\beta,A,B)$ which is defined as follows:

Definition 3. A function $f \in \mathcal{T}_p$ is said to be in the class $\mathcal{T}_{p,j}^{(\gamma)}(a,c,\beta,A,B)$, if there exists a function $g \in \mathcal{T}_{p,j}(a,c,\beta,A,B)$ such that

$$\left|\frac{f(z)}{g(z)}-1\right| < p-\gamma \quad (0 \leq \gamma < p; z \in \mathcal{U}) \tag{4.4}$$

Theorem 8. If $g \in \mathcal{T}_{p,j}(a,c,\beta,A,B)$ and

$$\varepsilon = \frac{(p-\gamma)\left[(p+1)\{(1+\beta)(1-B)+(p+1-j)(A-B)\}a-(p+1-j)(p+2-j)(A-B)c\right]}{\{(1+\beta)(1-B)+(p+1-j)(A-B)\}a}, \tag{4.5}$$

then

$$\mathcal{N}_{\varepsilon}(g) \subset \mathcal{T}_{p,j}^{(\gamma)}(a,c,\beta,A,B).$$

Proof. Suppose that $f \in \mathcal{N}_{\varepsilon}(g)$, where g, defined by

$$g(z) = z^p - \sum_{k=1}^{\infty} b_{p+k}z^{p+k} \quad (z \in \mathcal{U}). \tag{4.6}$$

belongs to the class $\mathcal{T}_{p,j}(a,c,\beta,A,B)$. It follows from (3.1) that

$$\sum_{k=1}^{\infty}(p+k)|a_{p+k}-b_{p+k}| \leq \varepsilon$$

which implies that

$$\sum_{k=1}^{\infty}|a_{p+k}-b_{p+k}| \leq \frac{\varepsilon}{p+1}.$$

Since $g \in \mathcal{T}_{p,j}(a,c,\beta,A,B)$, we deduce from (2.3) that

$$\sum_{k=1}^{\infty} b_{p+k} \leq \frac{(p+1-j)(p+2-j)(A-B)c}{(p+1)\{(1+\beta)(1-B)+(p+1-j)(A-B)\}a}$$

so that

$$\begin{aligned}\left|\frac{f(z)}{g(z)}-1\right| &< \frac{\sum_{k=1}^{\infty}|a_{p+k}-b_{p+k}|}{1-\sum_{k=1}^{\infty}b_{p+k}} \\ &\leq \frac{\varepsilon\{(1+\beta)(1-B)+(p+1-j)(A-B)\}a}{(p+1)\{(1+\beta)(1-B)+(p+1-j)(A-B)\}a-(p+1-j)(p+2-j)(A-B)c} \\ &= p-\gamma,\end{aligned}$$

provided that ε is given by (4.5). Thus, by (4.4), the function $f \in \mathcal{T}_{p,j}^{(\gamma)}(a,c,\beta,A,B)$. This completes the proof of Theorem 5.

5. Subordination results

To prove our main result, we need the following definition and lemma.

Definition 4. A sequence $\{b_{p+k}\}_{k=0}^{\infty}$ of complex numbers is said to be a subordinating factor sequence, if for any analytic convex(univalent) function

$$g(z) = z^p + \sum_{k=1}^{\infty} c_{p+k} z^{p+k} \quad (z \in \mathcal{U}), \tag{5.1}$$

we have

$$\sum_{k=0}^{\infty} b_{p+k} c_{p+k} z^{p+k} \prec g(z) \quad (c_p = 1; z \in \mathcal{U}). \tag{5.2}$$

Wilf [23] gave the following necessary and sufficient condition for a sequence to be a subordinating factor sequence.

Lemma 2. A sequence $\{b_{p+k}\}_{k=0}^{\infty}$ of complex numbers is said to be a subordinating factor sequence, if and only if

$$\operatorname{Re}\left\{1 + \sum_{k=0}^{\infty} b_{p+k} z^{p+k}\right\} > 0 \quad (z \in \mathcal{U}). \tag{5.3}$$

Theorem 9. Suppose f, given by (1.10) belongs to the class $\mathcal{T}_{p,j}(a, c, \beta, A, B)$ and g defined by (4.1) is an analytic convex(univalent) function in $\mathcal{U}$. Then

$$\rho\,(f \star g)(z) \prec g(z) \quad (z \in \mathcal{U}), \tag{5.4}$$

where

$$\rho = \frac{\delta(p+1, j-1)\{(1+\beta)(1-B) + (p+1-j)(A-B)\}a}{2\left[\delta(p+1, j-1)\{(1+\beta)(1-B) + (p+1-j)(A-B)\}a + \delta(p,j)(A-B)c\right]} \tag{5.5}$$

Moreover,

$$\operatorname{Re}\{f(z)\} > (-1)^p \rho \quad (z \in \mathcal{U}) \tag{5.6}$$

and the subordinating result (5.4) is the best possible for the maximum factor ρ given by (5.5).

Proof. We have

$$\rho\,(f \star g)(z) = \sum_{k=0}^{\infty} b_{p+k} z^{p+k} \quad (z \in \mathcal{U}),$$

where

$$b_{p+k} = \begin{cases} \rho, & k = 0 \\ -\rho\, a_{p+k}, & k = 1. \end{cases}$$

To prove the subordination (5.4), we need to show that $\{b_{p+k}\}_{k=0}^{\infty}$ is a subordinating factor sequence which in view of Lemma 2 is true, if

$$\operatorname{Re}\left\{1 + 2\sum_{k=0}^{\infty} b_{p+k} z^{p+k}\right\} > 0 \quad (z \in \mathcal{U}).$$

Since

$$\frac{\delta(p+k, j-1)\{k(1+\beta)(1-B) + (p+1-j)(A-B)\}(a)_k}{(c)_k}$$

is an increasing function of $k(k \in \mathbb{N})$, we get

$$\begin{aligned}
&\mathrm{Re}\left\{1+\sum_{k=0}^{\infty} b_{p+k} z^{p+k}\right\} \\
&= 1+\mathrm{Re}\left\{2\rho\, z^p - \frac{\sum_{k=1}^{\infty}\left[\delta(p+1,j-1)\{(1+\beta)(1-B)+(p+1-j)(A-B)\}a\right] a_{p+k} z^{p+k}}{\left[\delta(p+1,j-1)\{(1+\beta)(1-B)+(p+1-j)(A-B)\}a+\delta(p,j)(A-B)c\right]}\right\} \\
&\geq 1-\frac{\delta(p+1,j-1)\{(1+\beta)(1-B)+(p+1-j)(A-B)\}a}{\left[\delta(p+1,j-1)\{(1+\beta)(1-B)+(p+1-j)(A-B)\}a+\delta(p,j)(A-B)c\right]} r^p \\
&- \frac{\delta(p,j)(A-B)c}{\left[\delta(p+1,j-1)\{(1+\beta)(1-B)+(p+1-j)(A-B)\}a+\delta(p,j)(A-B)c\right]} r^p \times \\
&\times \sum_{k=1}^{\infty} \frac{\delta(p+k,j-1)\{k(1+\beta)(1-B)+(p+1-j)(A-B)\}}{\delta(p,j)(A-B)} \frac{(a)_k}{(c)_k} a_{p+k} \\
&= 1-r^p > 0 \quad (|z|=r<1),
\end{aligned}$$

where we have made use of the assertion (2.4) of Theorem 2. This proves the subordination result (5.4). Letting $g(z)=z^p/(1-z)=z^p+\sum_{k=1}^{\infty} z^{p+k}$ $(z \in \mathcal{U})$ in (5.4), we easily get the inequality (5.5).

To prove the sharpness of the constant ρ, we consider the function $f_0 \in \mathcal{T}_{p,j}(a,c,\beta,A,B)$ given by

$$f_0(z) = z^p - \frac{\delta(p,j)(A-B)c}{\delta(p+1,j-1)\{(1+\beta)(1-B)+(p+1-j)(A-B)\}a} z^{p+1} \quad (z \in \mathcal{U}).$$

Thus, from (5.4), we have

$$\frac{\delta(p+1,j-1)\{(1+\beta)(1-B)+(p+1-j)(A-B)\}a}{2\left[\delta(p+1,j-1)\{(1+\beta)(1-B)+(p+1-j)(A-B)\}a+\delta(p,j)(A-B)c\right]} f_0(z) \prec \frac{z^p}{1-z} \ (z \in \mathcal{U}).$$

Moreover, it can be easily verified that for the function f_0

$$\min_{z\in\mathcal{U}}\left\{\mathrm{Re}\left(\frac{\delta(p+1,j-1)\{(1+\beta)(1-B)+(p+1-j)(A-B)\}a}{2\left[\delta(p+1,j-1)\{(1+\beta)(1-B)+(p+1-j)(A-B)\}a+\delta(p,j)(A-B)c\right]}\right)\right\} = -\frac{1}{2},$$

so that the constant factor ρ in (5.4) cannot be replaced by a larger number.

6. Modified Hadamard product

For functions f, defined by (1.10) and g given by (4.6), we define the modified Hadamard product or quasi Hadamard product of f and g by

$$(f \star g)(z) = z^p - \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^{p+k} = (g \star f)(z) \quad (z \in \mathcal{U}).$$

We now prove

Theorem 10. Let $a \geq c > 0$. If the functions $f, g \in \mathcal{T}_{p,j}(a,c,\beta,A,B)$, then $f \star g \in \mathcal{T}_{p,j}(a,c,\beta,\widetilde{A},B)$, where

$$\widetilde{A} = B + \frac{(p+1-j)(p+2-j)(1+\beta)(1-B)(A-B)^2\, c}{(p+1)\{(1+\beta)(1-B)+(p+1-j)(A-B)\}^2 a - (p+2-j)\{(p+1-j)(A-B)\}^2\, c}.$$

The result is sharp.

Proof. Let the functions f be defined by (1.10) and g be given by (4.6). We need to find the largest value of $\widetilde{A}$ such that

$$\sum_{k=1}^{\infty} \frac{\delta(p+k,j-1)\{k(1+\beta)(1-B)+(p+1-j)(\widetilde{A}-B)\}}{\delta(p,j)(\widetilde{A}-B)} \frac{(a)_k}{(c)_k} a_{p+k} b_{p+k} \leq 1. \tag{6.1}$$

Since $f, g \in \mathcal{T}_{p,j}(a, c, \beta, A, B)$, by Theorem 2 we get

$$\sum_{k=1}^{\infty} \frac{\delta(p+k, j-1)\{k(1+\beta)(1-B)+(p+1-j)(A-B)\}}{\delta(p,j)(A-B)} \frac{(a)_k}{(c)_k} a_{p+k} \leq 1$$

and

$$\sum_{k=1}^{\infty} \frac{\delta(p+k, j-1)\{k(1+\beta)(1-B)+(p+1-j)(A-B)\}}{\delta(p,j)(A-B)} \frac{(a)_k}{(c)_k} b_{p+k} \leq 1.$$

Hence, by using Cauchy-Schwarz inequality, we deduce that

$$\sum_{k=1}^{\infty} \frac{\delta(p+k, j-1)\{k(1+\beta)(1-B)+(p+1-j)(A-B)\}}{\delta(p,j)(A-B)} \frac{(a)_k}{(c)_k} \sqrt{a_{p+k}b_{p+k}} \leq 1 \tag{6.2}$$

which implies that

$$\sqrt{a_{p+k}b_{p+k}} \leq \frac{\delta(p,j)(A-B)}{\delta(p+k, j-1)\{k(1+\beta)(1-B)+(p+1-j)(A-B)\}} \frac{(c)_k}{(a)_k} \quad (k \in \mathbb{N}). \tag{6.3}$$

In view of (6.1) and (6.2), it is sufficient to show that

$$\begin{aligned} &\frac{\delta(p+k, j-1)\{k(1+\beta)(1-B)+(p+1-j)(\widetilde{A}-B)\}}{\delta(p,j)(\widetilde{A}-B)} a_{p+k}b_{p+k} \\ &\quad \leq \frac{\delta(p+k, j-1)\{k(1+\beta)(1-B)+(p+1-j)(A-B)\}}{\delta(p,j)(A-B)} \sqrt{a_{p+k}b_{p+k}} \quad (k \in \mathbb{N}), \end{aligned}$$

or, equivalently

$$\sqrt{a_{p+k}b_{p+k}} \leq \frac{(\widetilde{A}-B)\{k(1+\beta)(1-B)+(p+1-j)(A-B)\}}{(A-B)\{k(1+\beta)(1-B)+(p+1-j)(\widetilde{A}-B)\}} \quad (k \in \mathbb{N}).$$

In the light of the inequality (6.3), we need to show that

$$\begin{aligned} &\frac{\delta(p,j)(A-B)}{\delta(p+k, j-1)\{k(1+\beta)(1-B)+(p+1-j)(A-B)\}} \frac{(c)_k}{(a)_k} \\ &\quad \leq \frac{(\widetilde{A}-B)\{k(1+\beta)(1-B)+(p+1-j)(A-B)\}}{(A-B)\{k(1+\beta)(1-B)+(p+1-j)(\widetilde{A}-B)\}} \quad (k \in \mathbb{N}). \end{aligned} \tag{6.4}$$

It follows from (6.4) that for $k \in \mathbb{N}$

$$\begin{aligned} &\widetilde{A} \geq B+ \\ &\frac{k\delta(p,j)(1+\beta)(1-B)(A-B)^2(c)_k}{\delta(p+k, j-1)\{k(1+\beta)(1-B)+(p+1-j)(A-B)\}^2(a)_k - \delta(p,j)(p+1-j)(A-B)^2(c)_k}. \end{aligned}$$

Denoting the right side of the above inequality by $\psi(k)$, we note that ψ is a decreasing function of $k\,(k \in \mathbb{N})$. Therefore, we conclude that

$$\begin{aligned} &\widetilde{A} \geq \psi(1) = B+ \\ &+ \frac{(p+1-j)(p+2-j)(1+\beta)(1-B)(A-B)^2\,c}{(p+1)\{(1+\beta)(1-B)+(p+1-j)(A-B)\}^2 a - (p+2-j)\{(p+1-j)(A-B)\}^2\,c} \end{aligned}$$

which evidently completes the proof of Theorem 10.

The result is sharp for the functions f_0 and g_0 defined by

$$f_0(z) = g_0(z) = z^p - \frac{(p+1-j)(p+2-j)(A-B)c}{(p+1)\{(1+\beta)(1-B)+(p+1-j)(A-B)\}a} z^{p+1} \quad (-1 \leq B < 0; z \in \mathcal{U}). \tag{6.5}$$

Setting $\beta = 0, A = 1 - \{2\alpha/(p+1-j)\}\,(0 \leq \alpha < p+1-j-\alpha)$ and $B = -1$ in Theorem 10, we get the following result which, in turn, yield the corresponding works for the classes $\mathcal{T}^*(p,\alpha,j)$ and $\mathcal{C}(p,\alpha,j)$ obtained by Aouf [4, Theorem 10 and Corollary 9] when $a = c$ and $a = p+2-j, c = p+1-j$ respectively.

Corollary 5. If $a \geq c > 0$ and the functions $f, g \in \mathcal{T}_{p,j}(a,c,\beta,\alpha)$, then the function $(f \star g) \in \mathcal{T}_{p,j}(a,c,\beta,\varkappa)$, where

$$\varkappa = (p+1-j) - \frac{(1+\beta)(p+2-j)(p+1-j-\alpha)^2\, c}{(p+1)(p+2+\beta-j-\alpha)^2\, a - (p+2-j)(p+1-j-\alpha)^2\, c}.$$

The result is sharp.

Theorem 11. Let $a \geq c > 0$ and $-1 \leq B < 0$. If the functions f, defined by (1.10) and g given by (4.6) are in the class $\mathcal{T}_{p,j}(a,c,\beta,A,B)$, then the function

$$h(z) = z^p - \sum_{k=1}^{\infty} \left(a_{p+k}^2 + b_{p+k}^2\right) z^{p+k} \quad (z \in \mathcal{U})$$

belongs to the class $\mathcal{T}_{p,j}(a,c,\beta,\widetilde{A},B)$, where

$$C = \frac{2(p+1-j)(p+2-j)(1+\beta)(1-B)(A-B)^2\, c}{(p+1)\{(1+\beta)(1-B)+(p+1-j)(A-B)\}^2\, a - 2(p+2-j)\{(p+1-j)(A-B)\}^2\, c}.$$

The result is sharp.

Proof. By virtue of Theorem 2, we obtain

$$\begin{aligned}\sum_{k=1}^{\infty} &\left[\frac{\delta(p+k,j-1)\{k(1+\beta)(1-B)+(p+1-j)(A-B)\}(a)_k}{\delta(p,j)(A-B)(c)_k}\right]^2 a_{p+k}^2 \\ &\leq \left[\sum_{k=1}^{\infty} \frac{\delta(p+k,j-1)\{k(1+\beta)(1-B)+(p+1-j)(A-B)\}(a)_k}{\delta(p,j)(A-B)(c)_k} a_{p+k}\right]^2 \leq 1. \end{aligned} \tag{6.6}$$

Similarly,

$$\begin{aligned}\sum_{k=1}^{\infty} &\left[\frac{\delta(p+k,j-1)\{k(1+\beta)(1-B)+(p+1-j)(A-B)\}(a)_k}{\delta(p,j)(A-B)(c)_k}\right]^2 b_{p+k}^2 \\ &\leq \left[\sum_{k=1}^{\infty} \frac{\delta(p+k,j-1)\{k(1+\beta)(1-B)+(p+1-j)(A-B)\}(a)_k}{\delta(p,j)(A-B)(c)_k} b_{p+k}\right]^2 \leq 1. \end{aligned} \tag{6.7}$$

It follows from (6.6) and (6.7) that

$$\frac{1}{2}\sum_{k=1}^{\infty} \left[\frac{\delta(p+k,j-1)\{k(1+\beta)(1-B)+(p+1-j)(A-B)\}(a)_k}{\delta(p,j)(A-B)(c)_k}\right]^2 (a_{p+k}^2 + b_{p+k}^2) \leq 1.$$

Thus, we need to find the largest value of $\widetilde{A}$ such that

$$\begin{aligned}&\frac{\delta(p+k,j-1)\{k(1+\beta)(1-B)+(p+1-j)(\widetilde{A}-B)\}(a)_k}{\delta(p,j)(\widetilde{A}-B)(c)_k} \\ &\qquad \leq \frac{1}{2}\left[\frac{\delta(p+k,j-1)\{k(1+\beta)(1-B)+(p+1-j)(A-B)\}(a)_k}{\delta(p,j)(A-B)(c)_k}\right]^2 \quad (k \in \mathbb{N}).\end{aligned}$$

On simplifying the above inequality, we deduce that

$$\begin{aligned}&\widetilde{A} \geq B + \\ &\frac{2\delta(p,j)k(1+\beta)(1-B)(A-B)^2(c)_k}{\delta(p+k,j-1)\{k(1+\beta)(1-B)+(p+1-j)(A-B)\}^2(a)_k - 2\delta(p,j)(p+1-j)(A-B)^2(c)_k}\end{aligned} \tag{6.8}$$

for $k \in \mathbb{N}$. Since the right side of (6.8) is an decreasing function of $k\,(k \in \mathbb{N})$, putting $k = 1$ in (6.8), we get

$$\widetilde{A} \geq B + \frac{2(p+1-j)(p+2-j)(1+\beta)(1-B)(A-B)^2\,c}{(p+1)\{(1+\beta)(1-B)+(p+1-j)(A-B)\}^2\,a - 2(p+2-j)\{(p+1-j)(A-B)\}^2\,c}.$$

and Theorem 11 follows at once.

The result is sharp for the functions f_0 and g_0 defined by (6.5).

Taking $\beta = 0, A = 1 - \{2\alpha/(p+1-j)\}\ (0 \leq \alpha < p+1-j-\alpha)$ and $B = -1$ in Theorem 11, we obtain the following result which yield the corresponding works for the classes $\mathcal{T}^*(p, \alpha, j)$ and $\mathcal{C}(p, \alpha, j)$ due to Aouf [4, Theorem 12 and Corollary 11] for $a = c$ and $a = p+2-j, c = p+1-j$ respectively.

Corollary 6. Let $a \geq c > 0$ and $\beta \geq 0$. If the functions f, defined by (1.10) and g given by (3.6) are in the class $\mathcal{T}_{p,j}(a, c, \beta, \alpha)$, then the function

$$h(z) = z^p - \sum_{k=1}^{\infty} \left(a_{p+k}^2 + b_{p+k}^2\right) z^{p+k} \quad (z \in \mathcal{U})$$

belongs to the class $\mathcal{T}_{p,j}(a, c, \beta, \eta)$, where

$$\eta = (p+1-j) - \frac{2(1+\beta)(p+2-j)(p+1-j-\alpha)^2\,c}{(p+1)(p+2+\beta-j-\alpha)^2\,a - 2(p+2-j)(p+1-j-\alpha)^2\,c}.$$

The result is sharp.

Remark. The results obtained in various section can suitably be extended to hold true for the new subclasses of $\mathcal{A}_p$ and $\mathcal{T}_p$ defined in the introduction.

Acknowledgements. This work was supported by a Research Grant of Pukyong National University(2014 year) and the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (No. 2011-0007037).

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 19, NO.5, 2015, COPYRIGHT 2015 EUDOXUS PRESS, LLC

The characterizations of McShane integral and Henstock integrals for fuzzy-number-valued functions with a small Riemann sum on a small set†

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Abstract In this paper, we shall characterize McShane integral and Henstock integral of fuzzy-number-valued functions by Riemann-type integral of fuzzy-number-valued functions with a small Riemann sum on a small set, and the results show that McShane integral (Henstock integrals) of fuzzy-number-valued functions could be represented by Riemann integral (McShane integral) with a small Riemann sum on a small set, respectively.
Keywords: Fuzzy numbers; fuzzy integrals; small Riemann sum.

1. Introduction

Since the concept of fuzzy sets was firstly introduced by Zadeh in 1965[19], it has been studied extensively from many different aspects of the theory and applications, such as fuzzy topology, fuzzy analysis [16], fuzzy decision making and fuzzy logic, information science and so on. Recently, fuzzy integrals of fuzzy-number-valued functions have been studied by many authors from different points of views, including Goetschel [8], Nanda [15], Kaleva [12], Wu [17,18] and other authors [1-7]. For the constructive definition of fuzzy integrals, Goetschel defined and discussed the fuzzy integral of fuzzy-number-valued functions by first taking the sum and then the limit which is known as a Riemann type definition in 1986 [8]. The Mcshane integrals of fuzzy-valued functions were defined and characterized, and it shown that the Mcshane integrals of fuzzy-valued functions are the Riemann-type definitions of (K) integral [6]. In 2001, Wu and Gong defined and discussed the fuzzy Henstock integral for fuzzy-number-valued functions and several necessary and sufficient conditions of integrability for fuzzy-number-valued functions were given by means of abstract function theory and using a concrete structure into which they embed the fuzzy number space [17]. After that, Hsien-Chung Wu [18] discussed the fuzzy Riemann integral (improper the fuzzy Riemann integral) and its numerical integration by using the probabilistic Monte Carlo method, the Riemann integral (improper the fuzzy Riemann integral) of real-valued functions and its numerical integration respectively, and Barnabbas Bede [1] introduced the quadrature rules of fuzzy Henstock integral on finite interval. In this paper, we shall characterize McShane integral and Henstock integrals of fuzzy-number-valued functions by Riemann integral with a small Riemann sum on a small set, and the results show that McShane integral (Henstock integrals) of fuzzy-number-valued functions could be represented by Riemann integral (McShane integral) with a small Riemann sum on a small set, respectively.

The rest of this paper is organized as follows. To make our analysis possible, in Section 2 we shall review the relevant concepts and properties of fuzzy sets and the definition of McShane integral and Henstock integrals for fuzzy-number-valued functions. Section 3 is devoted to discussing McShane integral of fuzzy-number-valued functions by Riemann integral of the fuzzy-number-valued functions with a small Riemann sum on a small set. Section 4 we shall investigate Henstock integrals of fuzzy-number-valued

†Supported by the Natural Scientific Fund of China (11461062, 61262022), the Natural Scientific Fund of Gansu Province of China (1208RJZA251),and the Scientific Research Project of Northwest Normal University (No. NWNU-KJCXGC-03-61) of China.

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functions by McShane integral of fuzzy-number-valued functions with a small Riemann sum on a small set.

2. Preliminaries

Fuzzy set $\tilde{u} \in E^1$ is called a fuzzy number if $\tilde{u}$ is a normal, convex fuzzy set, upper semi-continuous and supp $u = \overline{\{x \in \mathbb{R} \mid u(x) > 0\}}$ is compact. Here $\bar{A}$ denotes the closure of A. We use E^1 to denote the fuzzy number space [8].

Let $\tilde{u}, \tilde{v} \in E^1, k \in \mathbb{R}$, the addition and scalar multiplication are defined by

$$[\tilde{u} + \tilde{v}]_\lambda = [\tilde{u}]_\lambda + [\tilde{v}]_\lambda, \qquad [k\tilde{u}]_\lambda = k[\tilde{u}]_\lambda,$$

respectively, where $[\tilde{u}]_\lambda = \{x : u(x) \geqslant \lambda\} = [u_\lambda^-, u_\lambda^+]$ for any $\lambda \in [0, 1]$.

We use the Hausdorff distance between fuzzy numbers given by $D : E^1 \times E^1 \to [0, +\infty)$ as in [8]

$$D(\tilde{u}, \tilde{v}) = \sup_{\lambda \in [0,1]} d([\tilde{u}]_\lambda, [\tilde{v}]_\lambda) = \sup_{\lambda \in [0,1]} \max\{|u_\lambda^- - v_\lambda^-|, |u_\lambda^+ - v_\lambda^+|\},$$

where d is the Hausdorff metric. $D(\tilde{u}, \tilde{v})$ is called the distance between $\tilde{u}$ and $\tilde{v}$. Furthermore, we write

$$\|\tilde{u}\|_{E^1} = D(\tilde{u}, \tilde{0}) = \sup_{\lambda \in [0,1]} \max\{|u_\lambda^-|, |u_\lambda^+|\}.$$

Notice that $\|\cdot\|_{E^1} = D(\cdot, \tilde{0})$ doesn't stands for the norm of E^1.

Definition 2.1 [8]. A fuzzy-number-valued function $\tilde{f}(x)$ is said to be fuzzy Riemann integrable on $[a, b]$ if there exists a fuzzy number $\tilde{H} \in E^1$ such that for every $\varepsilon > 0$, there is a $\delta > 0$ such that for any division $T : a = x_0 \leq x_1 \leq ... \leq x_n = b$ satisfying $\|T\| = \max_{i=1}^n \{x_i - x_{i-1}\} < \delta$, and for any $\xi_i \in [x_{i-1}, x_i]$, we have

$$D(\sum_{i=1}^n \tilde{f}(\xi_i)(x_i - x_{i-1}), \tilde{H}) < \varepsilon.$$

We write $(RH) \int_a^b \tilde{f}(x)\mathrm{d}x = \tilde{H}$ and $\tilde{f} \in RH[a, b]$.

Definition 2.2[10, 13]. Let $\delta : [a, b] \to \mathbb{R}^+$ be a positive real-valued function. $T = \{[x_{i-1}, x_i]; \xi_i\}_{i=1}^n$ is said to be a δ-fine division, if the following conditions are satisfied:

(1) $a = x_0 < x_1 < x_2 < ... < x_n = b$;

(2) $\xi_i \in [x_{i-1}, x_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))(i = 1, 2, \cdots, n)$.

If we replace above (2) by

(2)' $[x_{i-1}, x_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))(i = 1, 2, \cdots, n)$,

then $T = \{[x_{i-1}, x_i]; \xi_i\}_{i=1}^n$ is said to be a δ-fine (M) division. For brevity, we write $T = \{[u, v]; \xi\}$, where $[u, v]$ denotes a typical interval in T and ξ is the associated point of $[u, v]$.

Definition 2.3 [6, 17]. A fuzzy-number-valued function $\tilde{f}(x)$ is said to be fuzzy Henstock integrable (fuzzy McShane integrable) on $[a, b]$ if there exists a fuzzy number $\tilde{H} \in E^1$ such that for every $\varepsilon > 0$, there is a function $\delta(x) > 0$ such that for any δ-fine division (δ-fine (M) division) $T = \{[x_{i-1}, x_i]; \xi_i\}_{i=1}^n$, we have

$$D(\sum_{i=1}^n \tilde{f}(\xi_i)(x_i - x_{i-1}), \tilde{H}) < \varepsilon.$$

We write $(FH) \int_a^b \tilde{f}(x)\mathrm{d}x = \tilde{H}$ ($(FM) \int_a^b \tilde{f}(x)\mathrm{d}x = \tilde{H}$) and $\tilde{f} \in FH[a, b]$ ($\tilde{f} \in FM[a, b]$).

Obviously, $\tilde{f} \in RH[a, b]$ implies that $\tilde{f} \in FH[a, b]$ and $\tilde{f} \in FM[a, b]$. Furthermore, from the results of [13] and [6], we have the following Lemma.

Lemma 2.1[6, 17]. A fuzzy-number-valued function $\tilde{f}(x)$ is fuzzy Henstock integrable (fuzzy McShane integrable) on $[a, b]$ if and only if $f_\lambda^-(x), f_\lambda^+(x)$ are (RH) integrable for any $\lambda \in [0, 1]$ uniformly, i.e., for

every $\varepsilon > 0$, there is a function $\delta(x) > 0$ (independent of $\lambda \in [0,1]$) such that for any δ-fine division (δ-fine (M) division) $T = \{[x_{i-1}, x_i]; \xi_i\}_{i=1}^n$, and for any $\lambda \in [0,1]$ we have

$$|\sum_{i=1}^{n} f_\lambda^-(\xi_i)(x_i - x_{i-1}) - H_\lambda^-| < \varepsilon.$$

$f_\lambda^+(x)$ is similar. Here H_λ^- and H_λ^+ denote respectively the (H) integrals ((M) integrals) of f_λ^- and f_λ^+ on $[a,b]$ for any $\lambda \in [0,1]$.

Lemma 2.2 [7]. Let $\tilde{f} : [a,b] \to E^1$ be a fuzzy-number-valued function, and $\tilde{f} \in FH[a,b]$. Then $\tilde{f} \in FM[a,b]$ if and only if $\|\tilde{f}\|_{E^1}$ is Lebesgue integrable, and

$$\|(FH)\int_a^b \tilde{f}\|_{E^1} \le (L)\int_a^b \|\tilde{f}\|_{E^1}.$$

Lemma 2.3. Let $\tilde{f} : [a,b] \to E^1$ be a fuzzy-number-valued function and $\tilde{f} \in FM[a,b]$. Then for any $\varepsilon > 0$ there exists a positive integer N such that for every $n \ge N$ there is a δ_n-fine division δ-fine (M) division $T = \{[x_{i-1}, x_i]; \xi_i\}_{i=1}^n$, we have

$$\| \sum_{\|\tilde{f}\|_{E^1} > n} \tilde{f}(\xi_i)(x_i - x_{i-1})\|_{E^1} \le \varepsilon,$$

where the sum is taken over T for which $\|\tilde{f}\|_{E^1} > n$.

Proof. Since $\tilde{f} \in FM[a,b]$, by Lemma 2.2 we know that $\|\tilde{f}\|_{E^1}$ is Lebesgue integrable. Define a real function

$$g_n(x) = \begin{cases} \|\tilde{f}\|_{E^1}, & \text{if } \|\tilde{f}\|_{E^1} \le n, \\ 0, & \text{if } \|\tilde{f}\|_{E^1} > n. \end{cases}$$

Then $g_n(x)$ is Lebesgue integrable on $[a,b]$ and $g_n(x) \to \|\tilde{f}\|_{E^1}$. Let $G_n(a,b)$ denote the integration of $g_n(x)$, and $F(a,b)$ the integration of $\|\tilde{f}\|_{E^1}$, then $G_n(a,b) \to F(a,b)$. That is to say, for given $\varepsilon > 0$, there is a function $\delta_n(x) > 0$ such that for any δ_n-fine division $T = \{[x_{i-1}, x_i]; \xi_i\}_{i=1}^m$, we have

$$|\sum_{i=1}^{m} g_n(\xi_i)(x_i - x_{i-1}) - G_n(a,b))| < \varepsilon,$$

$$|\sum_{i=1}^{m} \|\tilde{f}\|_{E^1}(\xi_i)(x_i - x_{i-1}) - F(a,b)| < \varepsilon.$$

Choose N so that whenever $n \ge N$

$$|G_n(a,b) - F(a,b)| < \varepsilon.$$

Therefore for $n \ge N$ and δ_n-fine division $T = \{[x_{i-1}, x_i]; \xi_i\}_{i=1}^m$, we have

$$\begin{aligned} &\| \sum_{\|\tilde{f}\|_{E^1} > n} \tilde{f}(\xi_i)(x_i - x_{i-1})\|_{E^1} \\ &\le \sum_{\|\tilde{f}\|_{E^1} > n} \|\tilde{f}(\xi_i)\|_{E^1}(x_i - x_{i-1}) \\ &\le |\sum g_n(\xi_i)(x_i - x_{i-1}) - \sum \|\tilde{f}\|_{E^1}(\xi_i)(x_i - x_{i-1})| \\ &\le |\sum g_n(\xi_i)(x_i - x_{i-1}) - G_n(a,b)| + |G_n(a,b) - F(a,b)| \\ &< 3\varepsilon. \end{aligned}$$

This complete the proof.

3. The characterization of McShane integral of fuzzy-number-valued functions by Riemann integral and a small Riemann sum on a small set

Theorem 3.1. Let $\tilde{f} : [a, b] \to E^1$. Then the following statements are equivalent:

(1) $\widetilde{f}$ is fuzzy McShane integrable on $[a, b]$;

(2) for any $\lambda \in [0, 1]$, f_λ^- and f_λ^+ are McShane integrable on $[a, b]$ for any $\lambda \in [0, 1]$ uniformly ($\delta(x)$ is independent of $\lambda \in [0, 1]$), and

$$[(FM)\int_a^b \tilde{f}(x)\mathrm{d}x]_\lambda = [(M)\int_a^b f_\lambda^-(x)\mathrm{d}x, (M)\int_a^b f_\lambda^+(x)\mathrm{d}x];$$

(3) there exists a fuzzy number $\widetilde{A}$ such that for any $\epsilon > 0, \eta > 0$, we have

(i) there exists a constant $\delta > 0$ and a open set G satisfying $|G| < \eta$, such that for any δ fine division $T : a = x_0 \leq x_1 \leq ... \leq x_n = b$ satisfying $\|T\| = \max_{i=1}^n\{x_i - x_{i-1}\} < \delta$, i.e., $0 < x_i - x_{i-1} < \delta$ and $\xi_i \in [a, b]\backslash G$, we have

$$D(\sum_{\xi_i \notin G} \widetilde{f}(\xi_i)(x_i - x_{i-1}), \widetilde{A}) < \epsilon,$$

and

(ii) there exists a $\delta(\xi_i) > 0$, such that for any interval $[x_i - x_{i-1}]$ satisfying $[x_i - x_{i-1}] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i)) \subset G$, we have

$$\|\sum_{\xi_i \in G} \widetilde{f}(\xi_i)(x_i - x_{i-1})\|_{E^1} < \epsilon.$$

Proof. From the results of [17], (1) and (2) are equivalent, we only prove that (1) and (3) are equivalent. Let $\widetilde{f}$ is fuzzy McShane integrable on $[a, b]$. By using of the Definition 2.3, there exists a fuzzy number $\tilde{A} \in E^1$ such that for every $\varepsilon > 0$, there is a function $\delta(x) > 0$ such that for any δ-fine (M) division $T = \{[x_{i-1}, x_i]; \xi_i\}_{i=1}^n$, we have

$$D(\sum_{i=1}^n \tilde{f}(\xi_i)(x_i - x_{i-1}), \tilde{A}) < \varepsilon.$$

For any $\eta > 0$, due to the results of [14], we could select a $\delta(\xi)$ which is measurable and an open set G_1 satisfying $|G_1| < \frac{\eta}{2}$ such that $\{x|0 < \delta(x) < \delta_0\} \subset G_1$, and there exists an open set G_2 and a natural number N satisfying $|G_2| < \frac{\eta}{2}$ and $\{x|\|\widetilde{f}(x)\|_{E^1} > N\} \subset G_2$.

Let $G = G_1 \cup G_2$. Then $|G| < \eta$. Define

$$\delta^*(x) = \begin{cases} \delta_0, & \text{if } \xi \notin G, \\ \delta(x), \text{ such that } (x - \delta(x), x + \delta(x)) \subset G, & \text{if } \xi \in G, \end{cases}$$

then according to Lemma 2.3, for all $\delta^*(\xi)$-fine Mcshane division, we have

$$\begin{aligned}
&\|\sum_{\xi_i \in G} \widetilde{f}(\xi_i)(x_i - x_{i-1})\|_{E^1} \leqslant \sum_{\xi_i \in G} \|\widetilde{f}(\xi_i)\|_{E^1}(x_i - x_{i-1}) \\
&\leqslant \sum_{\|\tilde{f}\|_{E^1} > N} \|\widetilde{f}(\xi_i)\|_{E^1}(x_i - x_{i-1}) + \sum_{\xi_i \in G \cap \{x|\|\tilde{f}\|_{E^1} \leq N\}} \|\widetilde{f}(\xi_i)\|_{E^1}(x_i - x_{i-1}); \\
&< \epsilon + N\eta,
\end{aligned}$$

and

$$\begin{aligned}
&D(\sum_{\xi_i \notin G} \widetilde{f}(\xi_i)(x_i - x_{i-1}), \widetilde{A}) \\
&\leq D(\sum \widetilde{f}(\xi_i)(x_i - x_{i-1}), \widetilde{A}) + D(\sum_{\xi_i \in G} \widetilde{f}(\xi_i)(x_i - x_{i-1}), \tilde{0}) \\
&< \varepsilon + \|\sum_{\xi_i \in G} \widetilde{f}(\xi_i)(x_i - x_{i-1})\|_{E^1} \leqslant \varepsilon + \sum_{\xi_i \in G} \|\widetilde{f}(\xi_i)\|_{E^1}(x_i - x_{i-1}) \\
&< 2\epsilon + N\eta.
\end{aligned}$$

Conversely, Define

$$\delta^*(x) = \begin{cases} \delta_0, & \text{if } x \notin G, \\ \delta(x), & \text{if } x \in G, \end{cases}$$

for any δ-fine (M) division $T = \{[x_{i-1}, x_i]; \xi_i\}_{i=1}^n$, we have

$$\begin{aligned} & D(\sum_{i=1}^{n} \tilde{f}(\xi_i)(x_i - x_{i-1}), \tilde{A}) \\ & \le D(\sum_{\xi_i \notin G} \widetilde{f}(\xi_i)(x_i - x_{i-1}), \widetilde{A}) + D(\sum_{\xi_i \in G} \widetilde{f}(\xi_i)(x_i - x_{i-1}), \widetilde{0}) \\ & < \varepsilon + \| \sum_{\xi_i \in G} \widetilde{f}(\xi_i)(x_i - x_{i-1}) \|_{E^1} \leqslant \varepsilon + \sum_{\xi_i \in G} \| \widetilde{f}(\xi_i) \|_{E^1} (x_i - x_{i-1}) \\ & < 2\epsilon. \end{aligned}$$

This complete the proof.

4. The characterization of Henstock integral of fuzzy-number-valued functions by McShane integral and a small Riemann sum on a small set

Lemma 4.1[9]. Let $f : [a, b] \to R$ be a real function. f is Henstock integrable on $[a, b]$, then for each $\eta > 0$ there exists a measurable set $E \subset [a, b]$ such that $\mu(E) < \eta$, f is Lebesgue integrable on E, and $(L) \int_{[a,b] \setminus E} f(x)dx = (H) \int_a^b f(x)dx$.

Lemma 4.2[11]. Let $f : [a, b] \to R$ be a real function. f is Henstock integrable on $[a, b]$, then there exists an increasing sequence E_n of measurable sets whose union is $[a, b]$ and such that f is Lebesgue integrable on E_n, and $\lim_{n \to +\infty} (L) \int_{E_n} f(x)dx = (H) \int_a^b f(x)dx$.

Definition 4.1 Let $f, g : [a, b] \to R$ be real functions, and Henstock integrable on $[a, b]$. f and g are said to be of equi-Riemann tails on E, if for each $\eta > 0$ there exists a measurable set $E \subset [a, b]$ such that $\mu(E) < \eta$, f and g are Lebesgue integrable on E, and

$$(L) \int_{[a,b] \setminus E} f(x)dx = (H) \int_a^b f(x)dx,$$

$$(L) \int_{[a,b] \setminus E} g(x)dx = (H) \int_a^b g(x)dx.$$

Theorem 4.1. Let $\tilde{f} : [a, b] \to E^1$, and f_0^- and f_0^+ be of equi-Riemann tails on E. Then the following statements are equivalent:

(1) $\widetilde{f}$ is fuzzy Henstock integrable on $[a, b]$;

(2) for any $\lambda \in [0, 1]$, f_λ^- and f_λ^+ are Henstock integrable on $[a, b]$ for any $\lambda \in [0, 1]$ uniformly ($\delta(x)$ is independent of $\lambda \in [0, 1]$), and

$$[(FH) \int_a^b \tilde{f}(x)\mathrm{d}x]_\lambda = [(H) \int_a^b f_\lambda^-(x)\mathrm{d}x, (H) \int_a^b f_\lambda^+(x)\mathrm{d}x];$$

(3) there exists a fuzzy number $\widetilde{A}$ such that the following conditions are satisfied for any given $\epsilon > 0, \eta > 0$:

(i) there exists a measurable set E satisfying $\mu(E) < \eta$, such that $\tilde{f}(x)$ is fuzzy McShane integrable on $[a, b] \setminus E$;

(ii) there is a function $\delta(x) > 0$ such that for any δ fine division $T = \{[x_{i-1}, x_i]; \xi_i\}_{i=1}^n$ and $\xi_i \in [a, b] \setminus E$, we have

$$D(\sum_{\xi_i \in [a,b] \setminus E} \widetilde{f}(\xi_i)(x_i - x_{i-1}), \widetilde{A}) < \epsilon,$$

and

$$\|\sum_{\xi_i\in E}\widetilde{f}(\xi_i)(x_i-x_{i-1})\|_{E^1}<\epsilon.$$

Proof. From the results of [17], (1) and (2) are equivalent, we only prove that (1) and (3) are equivalent.

(1) implies (3): If $\widetilde{f}$ is fuzzy McShane integrable on $[a,b]$, then (3) holds, we only define $E=[a,b]$. Let $\widetilde{f}$ is fuzzy Henstock integrable on $[a,b]$ but it is not fuzzy McShane integrable on $[a,b]$. By Lemma 2.2, $\|\tilde{f}\|_{E^1}$ is not Lebesgue integrable on $[a,b]$. On the other hand, we note that

$$\|\tilde{f}\|_{E^1}=\sup_{\lambda\in[0,1]}\max\{|f_\lambda^-|,|f_\lambda^+|\}=\sup_{\lambda_n\in[0,1]}\max\{|f_{\lambda_n}^-|,|f_{\lambda_n}^+|\}\le\max\{|f_0^-|,|f_0^+|\},$$

where $\{\lambda_n\}$ is the set of all rational numbers on $[0,1]$. This means that at least one of f_0^- and f_0^+ is Henstock integrable on $[a,b]$ but Lebesgue. Since f_0^- and f_0^+ are of equi-Riemann tails on E, by Lemma 4.1, for any given $\epsilon>0,\eta>0$ there exists a measurable set $E\subset[a,b]$ such that $\mu(E)<\eta$, f_0^- and f_0^+ are Lebesgue integrable on $[a,b]\setminus E$. Furthermore,

$$|(L)\int_{[a,b]\setminus E}f_0^-(x)dx-(H)\int_a^b f_0^-(x)dx|<\varepsilon,$$

$$|(L)\int_{[a,b]\setminus E}f_0^+(x)dx-(H)\int_a^b f_0^+(x)dx|<\varepsilon.$$

Applies the following formula

$$\|\tilde{f}\|_{E^1}=\sup_{\lambda\in[0,1]}\max\{|f_\lambda^-|,|f_\lambda^+|\}=\sup_{\lambda_n\in[0,1]}\max\{|f_{\lambda_n}^-|,|f_{\lambda_n}^+|\}\le\max\{|f_0^-|,|f_0^+|\}$$

and the mesurability of $f_{\lambda_n}^-$ and $f_{\lambda_n}^+$, we have $\|\tilde{f}\|_{E^1}$ is Lebesgue integrable on $[a,b]\setminus E$, where $\{\lambda_n\}$ is the set of all rational numbers on $[a,b]\setminus E$. By using Lemma 2.2 again, we obtain that $\widetilde{f}$ is fuzzy McShane integrable on $[a,b]\setminus E$. By the Henstock integrability of f_0^- and f_0^+ on $[a,b]$, there is a function $\delta(x)>0$ such that for any δ fine division $T=\{[x_{i-1},x_i];\xi_i\}_{i=1}^n$

$$\begin{aligned}
&\|\sum_{\xi_i\in E}\widetilde{f}(\xi_i)(x_i-x_{i-1})\|_{E^1}\\
=&\sup_{\lambda\in[0,1]}\max\{|\sum_{\xi_i\in E}f_\lambda^-(\xi_i)(x_i-x_{i-1})|,|\sum_{\xi_i\in E}f_\lambda^+(\xi_i)(x_i-x_{i-1})|\}\\
\le&\max\{|\sum_{\xi_i\in E}f_0^-(\xi_i)(x_i-x_{i-1})|,|\sum_{\xi_i\in E}f_0^+(\xi_i)(x_i-x_{i-1})|\}\\
<&\ \varepsilon.
\end{aligned}$$

Since $\widetilde{f}$ is fuzzy McShane integrable on $[a,b]\setminus E$, there is a function $\delta(x)>0$ such that for any δ fine division $T=\{[x_{i-1},x_i];\xi_i\}_{i=1}^n$, we have

$$\begin{aligned}
&D(\sum_{\xi_i\in[a,b]\setminus E}\widetilde{f}(\xi_i)(x_i-x_{i-1}),\widetilde{A})\\
\le&\ D(\sum_{\xi_i\in[a,b]}\widetilde{f}(\xi_i)(x_i-x_{i-1}),\widetilde{A})+\|\sum_{\xi_i\in E}\widetilde{f}(\xi_i)(x_i-x_{i-1})\|_{E^1}\\
<&\ 2\varepsilon.
\end{aligned}$$

(3) implies (1): $\epsilon>0,\eta>0$, there exists a measurable set E satisfying $\mu(E)<\eta$, such that $\tilde{f}(x)$ is fuzzy McShane integrable on $[a,b]\setminus E$, and there is a function $\delta(x)>0$ such that for any δ fine division $T=\{[x_{i-1},x_i];\xi_i\}_{i=1}^n$, we have

$$D(\sum_{\xi_i\in[a,b]\setminus E}\widetilde{f}(\xi_i)(x_i-x_{i-1}),\widetilde{A})<\epsilon,$$

and

$$\|\sum_{\xi_i\in E}\widetilde{f}(\xi_i)(x_i-x_{i-1})\|_{E^1}<\epsilon.$$

It follows that

$$\begin{aligned}
& D(\sum_{\xi_i\in[a,b]\setminus E}\widetilde{f}(\xi_i)(x_i-x_{i-1}),\widetilde{A}) \\
\leq\ & D(\sum_{\xi_i\in[a,b]\setminus E}\widetilde{f}(\xi_i)(x_i-x_{i-1}),\widetilde{A})+\|\sum_{\xi_i\in E}\widetilde{f}(\xi_i)(x_i-x_{i-1})\|_{E^1} \\
<\ & 2\varepsilon.
\end{aligned}$$

This prove that $\widetilde{f}$ is fuzzy Henstock integrable on $[a,b]$.

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BARNES-TYPE NARUMI OF THE SECOND KIND AND POISSON-CHARLIER MIXED-TYPE POLYNOMIALS

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Abstract. In this paper, we consider the Barnes-type Narumi of the second kind and Poisson-Charlier mixed-type polynomials. We present several explicit formulas and recurrence relations for these polynomials. Also, we establish a connection between our polynomials and several known families of polynomials.

1. Introduction

The aim of this paper is to use umbral calculus to obtain several new and interesting identities of Barnes-type Narumi of the second kind and Poisson-Charlier mixed-type polynomials. Umbral calculus has been used in numerous problems of mathematics and applied mathematics (for instance, see [1, 6, 7, 12, 13, 14, 15, 16]). Di Bucchianico and Loeb [8] give more than five hundred old and new findings related to the study of Sheffer polynomial sequences. For instance, applications of umbral calculus to the physics of gases can be found in [22] and umbral techniques have been used in group theory and quantum mechanics by Biedenharn et al. [2, 3] (for other examples, see [4, 5, 6, 9, 17, 18, 23]).

In mathematics, the *Narumi polynomials* $N_n(x)$ are polynomials introduced by Narumi [19] (also, see [21, Section 4.4]) given by the generating function $\sum_{n\geq 0} N_n(x)\frac{x^n}{n!} = \frac{t}{\log(1+t)}(1+t)^x$. We recall here that the *Barnes-type Narumi polynomials of the first kind* $N_n(x|a_1,\ldots,a_r)$ are given by the generating function as

$$\prod_{j=1}^{r}\frac{(1+t)^{a_j}-1}{\log(1+t)}(1+t)^x = \sum_{n\geq 0} N_n(x|a_1,\ldots,a_r)\frac{t^n}{n!}.$$

Here we consider the polynomials $\hat{N}C_n(x|a_1,\ldots,a_r;b)$ where the corresponding generating function is given by

$$e^{-t}\prod_{j=1}^{r}\left(\frac{(1+t/b)^{a_j}-1}{(1+t/b)^{a_j}\log(1+t/b)}\right)(1+t/b)^x = \sum_{n\geq 0}\hat{N}C_n(x|a_1,\ldots,a_r;b)\frac{t^n}{n!}, \tag{1.1}$$

where $r\in\mathbb{Z}_{>0}$, and $a_1,\ldots,a_r,b\neq 0$. For simple notation, we define

$$\hat{N}C_n(x) = \hat{N}C_n(x|a_1,\ldots,a_r;b) \text{ and } \hat{N}C_{n;j}(x) = \hat{N}C_n(x|a_1,\ldots,a_{j-1},a_{j+1},\ldots,a_r;b).$$

We recall first that the Barnes-type Narumi polynomials of the second kind $\hat{N}_n(x) = \hat{N}_n(x|a_1,\ldots,a_r)$ are given by the generating function as

$$\prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\log(1+t)}\right)(1+t)^x = \sum_{n\geq 0}\hat{N}_n(x)\frac{t^n}{n!}.$$

2000 *Mathematics Subject Classification.* 05A40.

Key words and phrases. Barnes-type Narumi polynomials, Poisson-Charlier polynomials, Umbral calculus.

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When $x = 0$, $\hat{N}_n = \hat{N}_n(0)$ are called the Barnes-type Narumi numbers of the second kind. Next, the *Poisson-Charlier polynomial* $c_n(x;b)$ $(b \neq 0)$ are defined by the generating function as

$$e^{-t}(1+t/b)^x = \sum_{n\geq 0} c_n(x;b)\frac{t^n}{n!}.$$

In order to study the Barnes-type Narumi polynomials o the second kind, we need the use of the Umbral algebra and Umbral calculus. Let Π be the algebra of polynomials in a single variable x over $\mathbb{C}$ and let Π^* be the vector space of all linear functionals on Π. We denote the action of a linear functional L on a polynomial $p(x)$ by $\langle L|p(x)\rangle$, and we define the vector space structure on Π^* by

$$\langle cL + c'L'|p(x)\rangle = c\langle L|p(x)\rangle + c'\langle L'|p(x)\rangle,$$

where $c, c' \in \mathbb{C}$ (see [10, 11, 20, 21]). The algebra of formal power series in a single variable t is given by

$$\mathcal{H} = \left\{ f(t) = \sum_{k\geq 0} a_k \frac{t^k}{k!} \mid a_k \in \mathbb{C} \right\}. \tag{1.2}$$

Each formal power series in the variable t defines a linear functional on Π as

$$\langle f(t)|x^n\rangle = a_n, \text{ for all } n \geq 0, \text{ (see [10, 11, 20, 21])}. \tag{1.3}$$

By (1.2) and (1.3), we have

$$\langle t^k|x^n\rangle = n!\delta_{n,k}, \text{ for all } n, k \geq 0, \text{ (see [10, 11, 20, 21])}, \tag{1.4}$$

where $\delta_{n,k}$ is the Kronecker's symbol.

Let $f_L(t) = \sum_{n\geq 0}\langle L|x^n\rangle \frac{t^n}{n!}$, thus by (1.4), we have $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$. Therefore, the map $L \mapsto f_L(t)$ defines a vector space isomorphism from Π^* onto $\mathcal{H}$, which implies that $\mathcal{H}$ is thought of as set of both formal power series and linear functionals. We call $\mathcal{H}$ the *umbral algebra*. The *umbral calculus* is the study of umbral algebra.

The smallest integer k for which the coefficient of t^k does not vanish in the non-zero power series $f(t)$ is called the *order* $O(f(t))$ of $f(t)$ (see [10, 11, 20, 21]). If $O(f(t)) = 1$ $(O(f(t)) = 0)$, then $f(t)$ is called a *delta* (*invertable*) series. Suppose that $O(f(t)) = 1$ and $O(g(t)) = 0$, then there exists a unique sequence $s_n(x)$ of polynomials such that $\langle g(t)(f(t))^k|s_n(x)\rangle = n!\delta_{n,k}$, where $n, k \geq 0$. The sequence $s_n(x)$ is called the *Sheffer* sequence for $(g(t), f(t))$ which is denoted by $s_n(x) \sim (g(t), f(t))$ (see [10, 11, 20, 21]). For $f(t) \in \mathcal{H}$ and $p(x) \in \Pi$, we have $\langle e^{yt}|p(x)\rangle = p(y)$, $\langle f(t)g(t)|p(x)\rangle = \langle g(t)|f(t)p(x)\rangle$, and

$$f(t) = \sum_{n\geq 0}\langle f(t)|x^n\rangle\frac{t^n}{n!}, \quad p(x) = \sum_{n\geq 0}\langle t^n|p(x)\rangle\frac{x^n}{n!}, \tag{1.5}$$

(see [10, 11, 20, 21]). From (1.5), we obtain

$$\langle t^k|p(x)\rangle = p^{(k)}(0), \quad \langle 1|p^{(k)}(x)\rangle = p^{(k)}(0), \tag{1.6}$$

where $p^{(k)}(0)$ denotes the k-th derivative of $p(x)$ with respect to x at $x = 0$. So, by (1.6), we get that $t^k p(x) = p^{(k)}(x) = \frac{d^k}{dx^k}p(x)$, for all $k \geq 0$, (see [10, 11, 20, 21]).

Let $s_n(x) \sim (g(t), f(t))$. Then we have

$$\frac{1}{g(\bar{f}(t))}e^{y\bar{f}(t)} = \sum_{n\geq 0} s_n(y)\frac{t^n}{n!}, \tag{1.7}$$

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for all $y \in \mathbb{C}$, where $\bar{f}(t)$ is the compositional inverse of $f(t)$ (see [10, 11, 20, 21]). For $s_n(x) \sim (g(t), f(t))$ and $r_n(x) \sim (h(t), \ell(t))$, let $s_n(x) = \sum_{k=0}^{n} c_{n,k} r_k(x)$, then

$$c_{n,k} = \frac{1}{k!}\left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))}(\ell(\bar{f}(t)))^k | x^n \right\rangle, \tag{1.8}$$

(see [10, 11, 20, 21]).

It is immediate from (1.1) to see that $\hat{N}C_n(x)$ is the Sheffer sequence for the pair

$$\hat{N}C_n(x) \sim \left(e^{b(e^t-1)} \prod_{j=1}^{r} \frac{te^{a_j t}}{e^{a_j t}-1}, b(e^t-1) \right). \tag{1.9}$$

The polynomials $\hat{N}C_n(x)$ will be called the Barnes-type Narumi of the second kind and Poisson-Carlier mixed-type polynomials. When $x = 0$, $\hat{N}C_n = \hat{N}C_n(a_1, \dots, a_r; b) = \hat{N}C_n(0)$ are called the Barnes-type Narumi of the second kind and Poisson-Charlier mixed-type numbers,

The aim of the present paper is to present several new identities for the polynomials $\hat{N}C_n(x)$ by the use of umbral calculus. At first, in the next section, we present explicit formulas for such polynomials. Then, we present several recurrence relations for these polynomials. At the end, we establish a connection between our polynomials and several known families of polynomials.

2. Explicit formulas

By the definitions and (1.9), we have

$$\begin{aligned}
\hat{N}C_n(y) &= \left\langle \sum_{i\ge 0} \hat{N}C_i(y)\frac{t^i}{i!} \mid x^n \right\rangle = \left\langle e^{-t}\prod_{j=1}^{r}\left(\frac{(1+t/b)^{a_j}-1}{(1+t/b)^{a_j}\log(1+t/b)}\right)(1+t/b)^y \mid x^n \right\rangle \\
&= \left\langle e^{-t} \mid \prod_{j=1}^{r}\left(\frac{(1+t/b)^{a_j}-1}{(1+t/b)^{a_j}\log(1+t/b)}\right)(1+t/b)^y x^n \right\rangle \\
&= \left\langle e^{-t} \mid \sum_{\ell\ge 0}\hat{N}_\ell(y)\frac{t^\ell}{b^\ell \ell!}x^n \right\rangle = \sum_{\ell=0}^{n}\binom{n}{\ell}\frac{1}{b^\ell}\hat{N}_\ell(y)\langle e^{-t}|x^{n-\ell}\rangle = \sum_{\ell=0}^{n}\binom{n}{\ell}\frac{(-1)^{n-\ell}}{b^\ell}\hat{N}_\ell(y).
\end{aligned}$$

Also,

$$\begin{aligned}
\hat{N}C_n(y) &= \left\langle \sum_{i\ge 0} \hat{N}C_i(y)\frac{t^i}{i!} \mid x^n \right\rangle = \left\langle e^{-t}\prod_{j=1}^{r}\left(\frac{(1+t/b)^{a_j}-1}{(1+t/b)^{a_j}\log(1+t/b)}\right)(1+t/b)^y \mid x^n \right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(\frac{(1+t/b)^{a_j}-1}{(1+t/b)^{a_j}\log(1+t/b)}\right) \mid e^{-t}(1+t/b)^y x^n \right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(\frac{(1+t/b)^{a_j}-1}{(1+t/b)^{a_j}\log(1+t/b)}\right) \mid \sum_{\ell\ge 0} c_\ell(y;b)\frac{t^\ell}{\ell!}x^n \right\rangle \\
&= \sum_{\ell=0}^{n}\binom{n}{\ell}c_\ell(y;b)\left\langle \sum_{i\ge 0}\hat{N}_i\frac{t^i}{b^i i!} \mid x^{n-\ell} \right\rangle = \sum_{\ell=0}^{n}\frac{1}{b^{n-\ell}}\binom{n}{\ell}c_\ell(y;b)\hat{N}_{n-\ell}.
\end{aligned}$$

Thus, we can state the following identities.

Theorem 1. *For all $n \geq 0$,*

$$\hat{N}C_n(x) = \sum_{\ell=0}^{n} \binom{n}{\ell} \frac{(-1)^{n-\ell}}{b^\ell} \hat{N}_\ell(x) = \sum_{\ell=0}^{n} \binom{n}{\ell} \frac{\hat{N}_{n-\ell}}{b^{n-\ell}} c_\ell(x; b).$$

By (1.9), we have

$$e^{b(e^t-1)} \prod_{j=1}^{r} \left(\frac{te^{a_j t}}{e^{a_j t} - 1} \right) \hat{N}C_n(x) \sim (1, b(e^t - 1)). \tag{2.1}$$

It is well known that $(x)_n = x(x-1)\cdots(x-n+1) = \sum_{k=0}^{n} S_1(n,k)x^n$, where $S_1(n,k)$ is the Stirling number of the first kind, which implies that $b^{-n}(x)_n \sim (1, b(e^t - 1))$. Thus

$$\hat{N}C_n(x) = b^{-n} \prod_{j=1}^{r} \left(\frac{e^{a_j t} - 1}{te^{a_j t}} \right) (-b(e^t-1))^\ell (x)_n = b^{-n} \prod_{j=1}^{r} \left(\frac{e^{a_j t} - 1}{te^{a_j t}} \right) \sum_{\ell=0}^{n} \frac{(-b)^\ell}{\ell!} (e^t - 1)^\ell (x)_n.$$

By the facts that $(x)_n \sim (1, e^t - 1)$ and $(e^t - 1)^\ell (x)_n = (n)_\ell (x)_{n-\ell}$, we obtain

$$\begin{aligned}
\hat{N}C_n(x) &= b^{-n} \prod_{j=1}^{r} \left(\frac{e^{a_j t} - 1}{te^{a_j t}} \right) \sum_{\ell=0}^{n} (-b)^\ell \binom{n}{\ell} (x)_{n-\ell} \\
&= b^{-n} \prod_{j=1}^{r} \left(\frac{e^{a_j t} - 1}{te^{a_j t}} \right) \sum_{\ell=0}^{n} (-b)^\ell \binom{n}{\ell} \sum_{m=0}^{n-\ell} S_1(n-\ell, m) x^m \\
&= b^{-n} \sum_{\ell=0}^{n} \sum_{m=0}^{n-\ell} (-b)^\ell \binom{n}{\ell} S_1(n-\ell, m) \prod_{j=1}^{r} \left(\frac{e^{a_j t} - 1}{te^{a_j t}} \right) x^m \\
&= b^{-n} \sum_{\ell=0}^{n} \sum_{m=0}^{n-\ell} (-b)^\ell \binom{n}{\ell} S_1(n-\ell, m) \sum_{k=0}^{m} \hat{F}_k \frac{t^k}{k!} x^m \\
&= b^{-n} \sum_{\ell=0}^{n} \sum_{m=0}^{n-\ell} (-b)^\ell \binom{n}{\ell} S_1(n-\ell, m) \sum_{k=0}^{m} \binom{m}{k} \hat{F}_k x^{m-k} \\
&= b^{-n} \sum_{j=0}^{n} \left(\sum_{\ell=0}^{n-j} \sum_{m=j}^{n-\ell} (-b)^\ell \binom{n}{\ell} \binom{m}{j} S_1(n-\ell, m) \hat{F}_{m-j} \right) x^j,
\end{aligned} \tag{2.2}$$

see (2.3), for the definition $\hat{F}_k$. Clearly, from (2.2), we have

$$\hat{N}C_n(x) = b^{-n} \sum_{\ell=0}^{n} \sum_{m=0}^{n-\ell} (-b)^\ell \binom{n}{\ell} S_1(n-\ell, m) \hat{F}_m(x).$$

Here, we introduced the Appell numbers $\hat{F}_n = \hat{F}_n(a_1, \ldots, a_r)$ by

$$\hat{F}_n(a_1, \ldots, a_r) \sim \left(\prod_{j=1}^{r} \left(\frac{te^{a_j t}}{e^{a_j t} - 1} \right), t \right), \tag{2.3}$$

so that

$$\prod_{j=1}^{r} \left(\frac{e^{a_j t} - 1}{te^{a_j t}} \right) e^{xt} = \sum_{n \geq 0} \hat{F}_n(x|a_1, \ldots, a_r) \frac{t^n}{n!} = \sum_{n \geq 0} \hat{F}_n(x) \frac{t^n}{n!}.$$

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When $x=0$, $\hat{F}_n(0)$ will be simply denoted by $\hat{F}_n$, so that $\prod_{j=1}^r\left(\frac{e^{a_jt}-1}{te^{a_jt}}\right)=\sum_{n\ge0}\hat{F}_n\frac{t^n}{n!}$. It is easy to see that $\hat{F}_n(x)=\sum_{m=0}^n\binom{n}{m}\hat{F}_mx^{n-m}$. In [13], the Appell numbers $F_n\sim\left(\prod_{j=1}^r\left(\frac{t}{e^{a_jt}-1}\right),t\right)$, so that $\prod_{j=1}^r\left(\frac{e^{a_jt}-1}{t}\right)=\sum_{n\ge0}F_n\frac{t^n}{n!}$. It was shown that

$$F_n(x)=\sum_{i=0}^n\sum_{\ell_1+\dots+\ell_r=i}\frac{i!}{(i+r)!}\binom{n}{i}\binom{i+r}{\ell_1,\dots,\ell_r}\prod_{j=1}^r a_j^{\ell_j+1}x^{n-i}.$$

Also, from

$$\begin{aligned}\sum_{n\ge0}\hat{F}_n(x)\frac{t^n}{n!}&=\prod_{j=1}^r\left(\frac{e^{a_jt}-1}{te^{a_jt}}\right)e^{xt}=e^{-\sum_{j=1}^r a_jt}\prod_{j=1}^r\left(\frac{e^{a_jt}-1}{t}\right)e^{xt}\\&=\sum_{\ell\ge0}\frac{(-\sum_{j=1}^r a_j)^\ell t^\ell}{\ell!}\sum_{n\ge0}F_n(x)\frac{t^n}{n!}=\sum_{n\ge0}\left(\sum_{\ell=0}^n\binom{n}{\ell}(-\sum_{j=1}^r a_j)^\ell F_{n-\ell}(x)\right)\frac{t^n}{n!},\end{aligned}$$

we see that

$$\hat{F}_n(x)=\sum_{\ell=0}^n\binom{n}{\ell}(-\sum_{j=1}^r a_j)^\ell F_{n-\ell}(x).$$

Now we invoke the conjugation formula (1.7) with $g(\bar{f}(t))^{-1}=e^{-t}\prod_{j=1}^r\left(\frac{(1+t/b)^{a_j}-1}{(1+t/b)^{a_j}\log(1+t/b)}\right)$ and $\bar{f}(t)=\log(1+t/b)$:

$$\begin{aligned}\langle g(\bar{f}(t))^{-1}\bar{f}(t)^j|x^n\rangle&=\left\langle e^{-t}\prod_{j=1}^r\left(\frac{(1+t/b)^{a_j}-1}{(1+t/b)^{a_j}\log(1+t/b)}\right)(\log(1+t/b))^j\mid x^n\right\rangle\\&=\left\langle e^{-t}\prod_{j=1}^r\left(\frac{(1+t/b)^{a_j}-1}{(1+t/b)^{a_j}\log(1+t/b)}\right)\mid(\log(1+t/b))^jx^n\right\rangle\\&=\left\langle e^{-t}\prod_{j=1}^r\left(\frac{(1+t/b)^{a_j}-1}{(1+t/b)^{a_j}\log(1+t/b)}\right)\mid\sum_{\ell\ge j}j!S_1(\ell,j)\frac{t^\ell}{b^\ell\ell!}x^n\right\rangle\\&=j!\sum_{\ell=j}^n\left[b^{-\ell}S_1(\ell,j)\binom{n}{\ell}\left\langle e^{-t}\prod_{j=1}^r\left(\frac{(1+t/b)^{a_j}-1}{(1+t/b)^{a_j}\log(1+t/b)}\right)\mid x^{n-\ell}\right\rangle\right]\\&=j!\sum_{\ell=j}^n b^{-\ell}S_1(\ell,j)\binom{n}{\ell}\hat{N}C_{n-\ell}.\end{aligned}$$

So we can state the following result.

Theorem 2. *For all $n\ge0$,*

$$\hat{N}C_n(x)=\sum_{j=0}^n\left[\sum_{\ell=j}^{n-j}b^{\ell-n}S_1(\ell,j)\binom{n}{\ell}\hat{N}C_\ell\right]x^j.$$

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By the fact that $x^n \sim (1,t)$ and by (2.1), we obtain

$$e^{b(e^t-1)}\prod_{j=1}^{r}\left(\frac{te^{a_jt}}{e^{a_jt}-1}\right)\hat{N}C_n(x) = x\left(\frac{t}{b(e^t-1)}\right)^n x^{n-1} = b^{-n}x\sum_{k\geq 0}B_k^{(n)}\frac{t^k}{k!}x^{n-1}$$
$$= b^{-n}x\sum_{k=0}^{n-1}\binom{n-1}{k}B_k^{(n)}x^{n-1-k} = b^{-n}\sum_{k=0}^{n-1}\binom{n-1}{k}B_k^{(n)}x^{n-k}.$$

where $B_k^{(n)}$ is the k-th Bernoulli polynomial of order n. Thus,

$$\begin{aligned}
\hat{N}C_n(x) &= b^{-n}\sum_{k=0}^{n}\left[\binom{n-1}{k}B_k^{(n)}\prod_{j=1}^{r}\left(\frac{e^{a_jt}-1}{te^{a_jt}}\right)e^{-b(e^t-1)}x^{n-k}\right]\\
&= b^{-n}\sum_{k=0}^{n}\left[\binom{n-1}{k}B_k^{(n)}\prod_{j=1}^{r}\left(\frac{e^{a_jt}-1}{te^{a_jt}}\right)\sum_{\ell=0}^{n-k}\frac{(-b)^\ell}{\ell!}(e^-1)^\ell x^{n-k}\right]\\
&= b^{-n}\sum_{k=0}^{n}\left[\binom{n-1}{k}B_k^{(n)}\prod_{j=1}^{r}\left(\frac{e^{a_jt}-1}{te^{a_jt}}\right)\sum_{\ell=0}^{n-k}\sum_{j=\ell}^{n-k}\frac{\ell!(-b)^\ell}{j!}S_2(j,\ell)t^jx^{n-k}\right]\\
&= b^{-n}\sum_{k=0}^{n}\sum_{\ell=0}^{n-k}\sum_{j=\ell}^{n-k}\left[\binom{n-1}{k}\binom{n-k}{j}(-b)^\ell S_2(j,\ell)B_k^{(n)}\prod_{j=1}^{r}\left(\frac{e^{a_jt}-1}{te^{a_jt}}\right)x^{n-k-j}\right]\\
&= b^{-n}\sum_{k=0}^{n}\sum_{\ell=0}^{n-k}\sum_{j=0}^{n-k-\ell}\left[\binom{n-1}{k}\binom{n-k}{j}(-b)^\ell S_2(n-k-j,\ell)B_k^{(n)}\hat{F}_j(x|a_1,\ldots,a_r)\right]\\
&= b^{-n}\sum_{k=0}^{n}\sum_{\ell=0}^{n-k}\sum_{j=0}^{n-k-\ell}\sum_{i=0}^{j}\left[\binom{n-1}{k}\binom{n-k}{j}\binom{j}{i}(-b)^\ell S_2(n-k-j,\ell)B_k^{(n)}\hat{F}_ix^{j-i}\right],
\end{aligned}$$

which leads the following result.

Theorem 3. *For all $n \geq 0$, the polynomial $\hat{N}C_n(x)$ is given by*

$$b^{-n}\sum_{m=0}^{n}\left[\sum_{k=0}^{n-m}\sum_{\ell=0}^{n-m-k}\sum_{j=m}^{n-k-\ell}\binom{n-1}{k}\binom{n-k}{j}\binom{j}{m}(-b)^\ell S_2(n-k-j,\ell)B_k^{(n)}\hat{F}_{j-m}\right]x^m.$$

3. Recurrence relations

By (1.9), we have $\prod_{j=1}^{r}\left(\frac{te^{a_jt}}{e^{a_jt}-1}\right)e^{b(e^t-1)}\hat{N}C_n(x) = b^{-n}(x)_n \sim (1, b(e^t-1))$. Thus,

$$\hat{N}C_n(x+y) = \sum_{j=0}^{n}\binom{n}{j}\hat{N}C_j(x)b^{j-n}(y)_{n-j} = b^{-n}\sum_{j=0}^{n}\binom{n}{j}\hat{N}C_j(x)b^j(y)_{n-j}.$$

For $s_n(x) \sim (g(t), f(t))$, $f(t)s_n(x) = ns_{n-1}(x)$. Here $b(e^t-1)\hat{N}C_n(x) = n\hat{N}C_{n-1}(x)$, that is, we have the following recurrence relation.

Theorem 4. *We have*

$$\hat{N}C_n(x+1|a_1,\ldots,a_r;b) - \hat{N}C_n(x|a_1,\ldots,a_r;b) = \frac{n}{b}\hat{N}C_{n-1}(x|a_1,\ldots,a_r;b).$$

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For $s_n(x) \sim (g(t), f(t))$, $s_{n+1}(x) = (x - g'(t)/g(t))\frac{1}{f'(t)}s_n(x)$. Thus,

$$\hat{N}C_{n+1}(x) = b^{-1}\left(x\hat{N}C_n(x-1) - e^{-t}\frac{g'(t)}{g(t)}\hat{N}C_n(x)\right).$$

Observe that

$$\begin{aligned} g'(t)/g(t) = (\log g(t))' &= \left(b(e^t-1) + r\log t + t\sum_{j=1}^r a_j - \sum_{j=1}^r \log(e^{a_jt}-1)\right)' \\ &= be^t + \frac{r}{t} + \sum_{j=1}^r a_j - \sum_{j=1}^r \frac{a_je^{a_jt}}{e^{a_jt}-1} = be^t + \sum_{j=1}^r a_j \frac{r - \sum_{j=1}^r \frac{a_jte^{a_jt}}{e^{a_jt}-1}}{t}, \end{aligned}$$

where

$$\begin{aligned} r - \sum_{j=1}^r \frac{a_jte^{a_jt}}{e^{a_jt}-1} &= \frac{\sum_{j=1}^r \prod_{i\neq j}(e^{a_it}-1)(e^{a_jt}-1-a_jte^{a_jt})}{\prod_{i=1}^r (e^{a_it}-1)} \\ &= -\frac{1}{2}\frac{a_1\cdots a_r\sum_{j=1}^r a_jt^{t+1} + \cdots}{a_1a_2\cdots a_rt^r + \cdots} = -\frac{\sum_{j=1}^r a_j}{2}t + \cdots \end{aligned}$$

has order at least 1. Now, by Theorem 2, we obtain

$$\begin{aligned} \frac{g'(t)}{g(t)}\hat{N}C_n(x) &= \left(be^t + \sum_{j=1}^r a_j \frac{r - \sum_{j=1}^r \frac{a_jte^{a_jt}}{e^{a_jt}-1}}{t}\right)\hat{N}C_n(x) \\ &= b\hat{N}C_n(x+1) + \sum_{j=1}^r a_j\hat{N}C_n(x) + \sum_{k=0}^n\sum_{\ell=0}^{n-k}\binom{n}{\ell}b^{\ell-n}S_1(n-\ell,k)\hat{N}C_\ell\frac{r - \sum_{j=1}^r \frac{a_jte^{a_jt}}{e^{a_jt}-1}}{t}x^k. \end{aligned} \tag{3.1}$$

But

$$\begin{aligned} \frac{r - \sum_{j=1}^r \frac{a_jte^{a_jt}}{e^{a_jt}-1}}{t}x^k &= \left(r - \sum_{j=1}^r \frac{a_jte^{a_jt}}{e^{a_jt}-1}\right)\frac{x^{k+1}}{k+1} = \frac{1}{k+1}\sum_{j=1}^r\left(1 - \frac{-e^{a_jt}}{e^{-a_jt}-1}\right)x^{k+1} \\ &= \frac{1}{k+1}\sum_{j=1}^r\left(1 - \sum_{m\geq 0}\frac{(-a_j)^mB_mt^m}{m!}\right)x^{k+1} \\ &= \frac{1}{k+1}\sum_{j=1}^r\left(x^{k+1} - \sum_{m=0}^{k+1}\binom{k+1}{m}(-a_j)^mB_mx^{k+1-m}\right) \\ &= \frac{-1}{k+1}\sum_{j=1}^r\sum_{m=1}^{k+1}\binom{k+1}{m}(-a_j)^mB_mx^{k+1-m}. \end{aligned}$$

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Thus, by (3.1), we derive

$$\begin{aligned}
&\frac{g'(t)}{g(t)}\hat{N}C_n(x)\\
&= b\hat{N}C_n(x+1) + \sum_{j=1}^{r} a_j \hat{N}C_n(x)\\
&- \sum_{k=0}^{n}\sum_{\ell=0}^{n-k}\sum_{j=1}^{r}\sum_{m=1}^{k+1}\frac{1}{k+1}\binom{n}{\ell}\binom{k+1}{m}(-a_j)^m b^{\ell-n} S_1(n-\ell,k)B_m\hat{N}C_\ell x^{k+1-m}\\
&= b\hat{N}C_n(x+1) + \sum_{j=1}^{r} a_j \hat{N}C_n(x)\\
&- \sum_{k=0}^{n}\sum_{\ell=0}^{n-k}\sum_{j=1}^{r}\sum_{m=0}^{k}\frac{1}{k+1}\binom{n}{\ell}\binom{k+1}{m}(-a_j)^{k+1-m} b^{\ell-n} S_1(n-\ell,k)B_{k+1-m}\hat{N}C_\ell x^{m}\\
&= b\hat{N}C_n(x+1) + \sum_{j=1}^{r} a_j \hat{N}C_n(x)\\
&- \sum_{m=0}^{n}\sum_{k=m}^{n}\sum_{\ell=0}^{n-k}\sum_{j=1}^{r}\frac{1}{k+1}\binom{n}{\ell}\binom{k+1}{m}(-a_j)^{k+1-m} b^{\ell-n} S_1(n-\ell,k)B_{k+1-m}\hat{N}C_\ell x^{m}.
\end{aligned}$$

Altogether, we obtain the following result.

Theorem 5. *For $n \geq 0$,*

$$\begin{aligned}
\hat{N}C_{n+1}(x) + \hat{N}C_n(x) &= b^{-1}x\hat{N}C_n(x-1) - b^{-1}\sum_{j=1}^{r} a_j\hat{N}C_n(x-1)\\
&- \sum_{m=0}^{n}\sum_{k=m}^{n}\sum_{\ell=0}^{n-k}\sum_{j=1}^{r}\frac{(-a_j)^{k+1-m}b^{\ell-n-1}}{k+1}\binom{n}{\ell}\binom{k+1}{m}S_1(n-\ell,k)B_{k+1-m}\hat{N}C_\ell(x-1)^m.
\end{aligned}$$

Another recurrence can be obtained as follows. For $s_n(x) \sim (g(t), f(t))$, we have that $\frac{d}{dx}s_n(x) = \sum_{\ell=0}^{n-1}\binom{n}{\ell}\langle \bar{f}(t) \mid x^{n-\ell}\rangle s_\ell(x)$. By the definitions, we have

$$\langle \bar{f}(t) \mid x^{n-\ell}\rangle = \langle \log(1+t/b) \mid x^{n-\ell}\rangle = \sum_{m\geq 1}\frac{(-1)^{m-1}}{mb^m}\langle t^m | x^{n-\ell}\rangle = \frac{(-1)^{n-1-\ell}(n-1-\ell)!}{b^{n-\ell}},$$

which implies $\frac{d}{dx}\hat{N}C_n(x) = \sum_{\ell=0}^{n-1}\binom{n}{\ell}\frac{(-1)^{n-1-\ell}(n-1-\ell)!}{b^{n-\ell}}\hat{N}C_\ell(x)$, and hence the following result.

Theorem 6. *For all $n \geq 0$,*

$$\frac{d}{dx}\hat{N}C_n(x) = n!\sum_{\ell=0}^{n-1}\frac{(-1)^{n-1-\ell}}{\ell!(n-\ell)b^{n-\ell}}\hat{N}C_\ell(x).$$

Another recurrence relation can be derived as follows. By definitions, we have

$$\hat{N}C_n(y) = \left\langle \sum_{i\ge 0} \hat{N}C_i(y)\frac{t^i}{i!} \mid x^n \right\rangle = \left\langle e^{-t}\prod_{j=1}^{r}\left(\frac{(1+t/b)^{a_j}-1}{(1+t/b)^{a_j}\log(1+t/b)}\right)(1+t/b)^y \mid x^n \right\rangle \tag{3.2}$$

$$= \left\langle \frac{d}{dt}e^{-t}\prod_{j=1}^{r}\left(\frac{(1+t/b)^{a_j}-1}{(1+t/b)^{a_j}\log(1+t/b)}\right)(1+t/b)^y \mid x^{n-1} \right\rangle \tag{3.3}$$

$$+ \left\langle e^{-t}\frac{d}{dt}\prod_{j=1}^{r}\left(\frac{(1+t/b)^{a_j}-1}{(1+t/b)^{a_j}\log(1+t/b)}\right)(1+t/b)^y \mid x^{n-1} \right\rangle \tag{3.4}$$

$$+ \left\langle e^{-t}\prod_{j=1}^{r}\left(\frac{(1+t/b)^{a_j}-1}{(1+t/b)^{a_j}\log(1+t/b)}\right)\frac{d}{dt}(1+t/b)^y \mid x^{n-1} \right\rangle. \tag{3.5}$$

The first term in (3.2), namely (3.3), is

$$\left\langle \frac{d}{dt}e^{-t}\prod_{j=1}^{r}\left(\frac{(1+t/b)^{a_j}-1}{(1+t/b)^{a_j}\log(1+t/b)}\right)(1+t/b)^y \mid x^{n-1} \right\rangle = -\hat{N}C_{n-1}(y). \tag{3.6}$$

The third term in (3.2), namely (3.5), is

$$\frac{y}{b}\left\langle e^{-t}\prod_{j=1}^{r}\left(\frac{(1+t/b)^{a_j}-1}{(1+t/b)^{a_j}\log(1+t/b)}\right)\frac{d}{dt}(1+t/b)^{y-1} \mid x^{n-1} \right\rangle = \frac{y}{b}\hat{N}C_{n-1}(y-1). \tag{3.7}$$

Define $L = \sum_{j=1}^{r}\left(\frac{a_j t/b(1+t/b)^{a_j}}{(1+t/b)^{a_j}-1} - \frac{t/b}{\log(1+t/b)}\right)$. It is easy to see that

$$\frac{d}{dt}\prod_{j=1}^{r}\left(\frac{(1+t/b)^{a_j}-1}{(1+t/b)^{a_j}\log(1+t/b)}\right) = \prod_{j=1}^{r}\left(\frac{(1+t/b)^{a_j}-1}{(1+t/b)^{a_j}\log(1+t/b)}\right)\left(\frac{L}{t(1+t/b)} - \frac{\sum_{j=1}^{r}a_j}{b(1+t/b)}\right),$$

with L has order at least one. Thus the second term in (3.2), namely (3.4), is

$$\begin{aligned}
&\left\langle e^{-t}\frac{d}{dt}\prod_{j=1}^{r}\left(\frac{(1+t/b)^{a_j}-1}{(1+t/b)^{a_j}\log(1+t/b)}\right)(1+t/b)^y \mid x^{n-1} \right\rangle \\
&= -\frac{\sum_{j=1}^{r}a_j}{b}\hat{N}C_{n-1}(y-1) + \frac{1}{n}\left\langle e^{-t}\prod_{j=1}^{r}\left(\frac{(1+t/b)^{a_j}-1}{(1+t/b)^{a_j}\log(1+t/b)}\right)(1+t/b)^{y-1} \mid Lx^n \right\rangle \\
&= -\frac{\sum_{j=1}^{r}a_j}{b}\hat{N}C_{n-1}(y-1) \\
&+ \frac{1}{n}\sum_{j=1}^{r}a_j\left\langle e^{-t}\frac{(1+t/b)^{a_j}\log(1+t/b)}{(1+t/b)^{a_j}-1}\prod_{i=1}^{r}\left(\frac{(1+t/b)^{a_i}-1}{(1+t/b)^{a_i}\log(1+t/b)}\right)(1+t/b)^{y-1} \mid \frac{t/b}{\log(1+t/b)}x^n \right\rangle \\
&- \frac{r}{n}\sum_{j=1}^{r}a_j\left\langle e^{-t}\prod_{i=1}^{r}\left(\frac{(1+t/b)^{a_i}-1}{(1+t/b)^{a_i}\log(1+t/b)}\right)(1+t/b)^{y-1} \mid \frac{t/b}{\log(1+t/b)}x^n \right\rangle.
\end{aligned}$$

By the fact that $\frac{t/b}{\log(1+t/b)}=\sum_{\ell\geq 0}c_\ell\frac{t^\ell}{b^\ell \ell!}$, we get that

$$\begin{aligned}
&\left\langle e^{-t}\frac{d}{dt}\prod_{j=1}^r\left(\frac{(1+t/b)^{a_j}-1}{(1+t/b)^{a_j}\log(1+t/b)}\right)(1+t/b)^y \mid x^{n-1}\right\rangle\\
&=-\frac{1}{b}\sum_{j=1}^r a_j\hat{N}C_{n-1}(y-1)+\frac{1}{n}\sum_{j=1}^r a_j\sum_{\ell=0}^n\binom{n}{\ell}\frac{c_\ell}{b^\ell}\hat{N}C_{n-\ell;j}(y-1)\\
&-\frac{r}{n}\sum_{\ell=0}^n\binom{n}{\ell}\frac{c_\ell}{b^\ell}\hat{N}C_{n-\ell}(y-1).
\end{aligned}\tag{3.8}$$

By (3.2), (3.6)-(3.8), we obtain the following result.

Theorem 7. *For $n\geq 1$,*

$$\begin{aligned}
\hat{N}C_n(x)&=-\hat{N}C_{n-1}(x)+\frac{x-\sum_{j=1}^r a_j}{b}\hat{N}C_{n-1}(x-1)\\
&+\frac{1}{n}\sum_{\ell=0}^n\binom{n}{\ell}\frac{c_\ell}{b^\ell}\left(\sum_{j=1}^r a_j\hat{N}C_{n-\ell;j}(x-1)-\frac{r}{n}\hat{N}C_{n-\ell}(x-1)\right).
\end{aligned}$$

Now we compute the following expression in two different ways:

$$A=\left\langle e^{-t}\prod_{j=1}^r\left(\frac{(1+t/b)^{a_j}-1}{(1+t/b)^{a_j}\log(1+t/b)}\right)(\log(1+t/b))^m \mid x^n\right\rangle.\tag{3.9}$$

On the one hand, it is equal to

$$\begin{aligned}
A&=\left\langle e^{-t}\prod_{j=1}^r\left(\frac{(1+t/b)^{a_j}-1}{(1+t/b)^{a_j}\log(1+t/b)}\right)\mid(\log(1+t/b))^m x^n\right\rangle\\
&=m!\left\langle e^{-t}\prod_{j=1}^r\left(\frac{(1+t/b)^{a_j}-1}{(1+t/b)^{a_j}\log(1+t/b)}\right)\mid\sum_{\ell\geq m}S_1(\ell,m)\frac{t^\ell}{b^\ell\ell!}x^n\right\rangle\\
&=m!\sum_{\ell=m}^n\frac{\binom{n}{\ell}}{b^\ell}S_1(\ell,m)\left\langle e^{-t}\prod_{j=1}^r\left(\frac{(1+t/b)^{a_j}-1}{(1+t/b)^{a_j}\log(1+t/b)}\right)\mid x^{n-\ell}\right\rangle\\
&=m!\sum_{\ell=m}^n\frac{\binom{n}{\ell}}{b^\ell}S_1(\ell,m)\hat{N}C_{n-\ell}.
\end{aligned}\tag{3.10}$$

On the other hand, it is equal to

$$A=\left\langle\frac{d}{dt}e^{-t}\prod_{j=1}^r\left(\frac{(1+t/b)^{a_j}-1}{(1+t/b)^{a_j}\log(1+t/b)}\right)(\log(1+t/b))^m \mid x^n\right\rangle\tag{3.11}$$

$$+\left\langle e^{-t}\frac{d}{dt}\prod_{j=1}^r\left(\frac{(1+t/b)^{a_j}-1}{(1+t/b)^{a_j}\log(1+t/b)}\right)(\log(1+t/b))^m \mid x^n\right\rangle\tag{3.12}$$

$$+\left\langle e^{-t}\prod_{j=1}^r\left(\frac{(1+t/b)^{a_j}-1}{(1+t/b)^{a_j}\log(1+t/b)}\right)\frac{d}{dt}(\log(1+t/b))^m \mid x^n\right\rangle.\tag{3.13}$$

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By (3.9), the right side of (3.11) is equal to

$$-m!\sum_{\ell=m}^{n-1}\frac{\binom{n-1}{\ell}}{b^\ell}S_1(\ell,m)\hat{N}C_{n-1-\ell}. \tag{3.14}$$

Also, by (3.9), the expression in (3.13) is equal to

$$\begin{aligned}
&\frac{m}{b}\left\langle e^{-t}\prod_{j=1}^{r}\left(\frac{(1+t/b)^{a_j}-1}{(1+t/b)^{a_j}\log(1+t/b)}\right)(\log(1+t/b))^{-1}\mid(\log(1+t/b))^{m-1}x^{n-1}\right\rangle\\
&=\frac{m!}{b}\sum_{\ell=m-1}^{n-1}\frac{\binom{n-1}{\ell}}{b^\ell}S_1(\ell,m-1)\left\langle e^{-t}\prod_{j=1}^{r}\left(\frac{(1+t/b)^{a_j}-1}{(1+t/b)^{a_j}\log(1+t/b)}\right)(1+t/b)^{-1}\mid x^{n-1-\ell}\right\rangle\\
&=\frac{m!}{b}\sum_{\ell=m-1}^{n-1}\frac{\binom{n-1}{\ell}}{b^\ell}S_1(\ell,m-1)\hat{N}C_{n-1-\ell}(-1). \qquad (3.15)
\end{aligned}$$

The term in (3.12) is equal to

$$\begin{aligned}
&\frac{1}{n}\left\langle e^{-t}\prod_{j=1}^{r}\left(\frac{(1+t/b)^{a_j}-1}{(1+t/b)^{a_j}\log(1+t/b)}\right)(1+t/b)^{-1}L\mid(\log(1+t/b))^{m}x^{n}\right\rangle\\
&-\frac{\sum_{j=1}^r a_j}{b}\left\langle e^{-t}\prod_{j=1}^{r}\left(\frac{(1+t/b)^{a_j}-1}{(1+t/b)^{a_j}\log(1+t/b)}\right)(1+t/b)^{-1}\mid(\log(1+t/b))^{m}x^{n-1}\right\rangle,
\end{aligned}$$

and by using the fact $(\log(1+t/b))^m=m!\sum_{i\geq m}\frac{S_1(i.m)t^i}{b^i i!}$ and $\frac{t/b}{\log(1+t/b)}=\sum_{i\geq 0}\frac{c_\ell t^\ell}{b^\ell \ell!}$ and defining $L'=-r+\sum_{j=1}^r a_j\frac{(1+t/b)^{a_j}\log(1+t/b)}{(1+t/b)^{a_j}-1}$, we obtain

$$\begin{aligned}
&\frac{m!}{n}\sum_{i=m}^{n}\frac{\binom{n}{i}}{b^i}S_1(i,m)\left\langle e^{-t}\prod_{j=1}^{r}\left(\frac{(1+t/b)^{a_j}-1}{(1+t/b)^{a_j}\log(1+t/b)}\right)(1+t/b)^{-1}\mid Lx^{n-i}\right\rangle\\
&-\frac{m!\sum_{j=1}^r a_j}{b}\sum_{i=m}^{n-1}\frac{\binom{n-1}{i}}{b^i}S_1(i,m)\left\langle e^{-t}\prod_{j=1}^{r}\left(\frac{(1+t/b)^{a_j}-1}{(1+t/b)^{a_j}\log(1+t/b)}\right)(1+t/b)^{-1}\mid x^{n-1-i}\right\rangle\\
&=\frac{m!}{n}\sum_{i=m}^{n}\frac{\binom{n}{i}}{b^i}S_1(i,m)\left\langle e^{-t}\prod_{j=1}^{r}\left(\frac{(1+t/b)^{a_j}-1}{(1+t/b)^{a_j}\log(1+t/b)}\right)(1+t/b)^{-1}L'\mid \frac{t/b}{\log(1+t/b)}x^{n-i}\right\rangle\\
&-\frac{m!\sum_{j=1}^r a_j}{b}\sum_{i=m}^{n-1}\frac{\binom{n-1}{i}}{b^i}S_1(i,m)\hat{N}C_{n-1-i}(-1)\\
&=\frac{m!}{n}\sum_{i=m}^{n}\sum_{\ell=0}^{n-i}\frac{\binom{n}{i}\binom{n-i}{\ell}}{b^{\ell+i}}c_\ell S_1(i,m)\left(\sum_{j=1}^r a_j\hat{N}C_{n-i-\ell;j}(-1)-r\hat{N}C_{n-i-\ell}(-1)\right)\\
&-\frac{m!\sum_{j=1}^r a_j}{b}\sum_{i=m}^{n-1}\frac{\binom{n-1}{i}}{b^i}S_1(i,m)\hat{N}C_{n-1-i}(-1). \qquad (3.16)
\end{aligned}$$

Hence, by (3.10), (3.11)-(3.13) and (3.14)-(3.16), we obtain the following result.

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Theorem 8. *For all* $1 \leq m \leq n-1$,

$$\begin{aligned}\sum_{\ell=m}^{n} \frac{\binom{n}{\ell}}{b^\ell} S_1(\ell,m)\hat{N}C_{n-\ell} &= -\sum_{\ell=m}^{n-1} \frac{\binom{n-1}{\ell}}{b^\ell} S_1(\ell,m)\hat{N}C_{n-1-\ell} + \sum_{\ell=m-1}^{n-1} \frac{\binom{n-1}{\ell}}{b^{1+\ell}} S_1(\ell,m-1)\hat{N}C_{n-1-\ell}(-1)\\ &+ \frac{1}{n}\sum_{i=m}^{n}\sum_{\ell=0}^{n-i} \frac{\binom{n}{i}\binom{n-i}{\ell}c_\ell}{b^{\ell+i}} S_1(i,m)\left(\sum_{j=1}^{r} a_j \hat{N}C_{n-i-\ell;j}(-1) - r\hat{N}C_{n-i-\ell}(-1)\right)\\ &- \sum_{j=1}^{r} a_j \sum_{i=m}^{n-1} \frac{\binom{n-1}{i}}{b^{1+i}} S_1(i,m)\hat{N}C_{n-1-i}(-1).\end{aligned}$$

4. Connections with families of polynomials

Now let $\hat{N}C_n(x) = \sum_{m=0}^{n} c_{n,m} b^{-m}(x)_m$. Then by (1.8), (1.9) and $b^{-n}(x)_n \sim (1, b(e^t-1))$, we obtain

$$\begin{aligned}c_{n,m} &= \frac{1}{m!}\left\langle \frac{1}{e^{b(e^{\log(1+t/b)}-1)}\prod_{j=1}^{r}\frac{\log(1+t/b)e^{a_j\log(1+t/b)}}{e^{a_j\log(1+t/b)}-1}} (b(e^{\log(1+t/b)}-1))^m \mid x^n \right\rangle\\ &= \frac{1}{m!}\left\langle e^{-t}\prod_{j=1}^{r}\frac{(1+t/b)^{a_j}-1}{(1+t/b)^{a_j}\log(1+t/b)} \mid t^m x^n\right\rangle\\ &= \binom{n}{m}\left\langle e^{-t}\prod_{j=1}^{r}\frac{(1+t/b)^{a_j}-1}{(1+t/b)^{a_j}\log(1+t/b)} \mid x^{n-m}\right\rangle = \binom{n}{m}\hat{N}C_{n-m},\end{aligned}$$

which implies the following result.

Theorem 9. *For all* $n \geq 0$,

$$\hat{N}C_n(x) = \sum_{m=0}^{n}\binom{n}{m} b^{-m}\hat{N}C_{n-m}(x)_m.$$

Now, let $\hat{N}C_n(x) = \sum_{m=0}^{n} c_{n,m} H_m^{(s)}(x|\lambda)$. Then by (1.8), (1.9) and $H_n^{(s)}(x|\lambda) \sim \left(\left(\frac{e^t-\lambda}{1-\lambda}\right)^s, t\right)$, we obtain

$$\begin{aligned}c_{n,m} &= \frac{1}{m!}\left\langle \frac{1}{e^{b(e^{\log(1+t/b)}-1)}\prod_{j=1}^{r}\frac{\log(1+t/b)e^{a_j\log(1+t/b)}}{e^{a_j\log(1+t/b)}-1}} \left(\frac{e^{\log(1+t/b)}-\lambda}{1-\lambda}\right)^s (\log(1+t/b))^m \mid x^n\right\rangle\\ &= \frac{1}{m!(1-\lambda)^s}\left\langle e^{-t}\prod_{j=1}^{r}\frac{(1+t/b)^{a_j}-1}{(1+t/b)^{a_j}\log(1+t/b)}(\log(1+t/b))^m \mid (1-\lambda++t/b)^s x^n\right\rangle\\ &= \frac{1}{m!(1-\lambda)^s}\sum_{j=0}^{n-m}\frac{\binom{s}{j}(1-\lambda)^{s-j}}{b^j}(n)_j\left\langle e^{-t}\prod_{j=1}^{r}\frac{(1+t/b)^{a_j}-1}{(1+t/b)^{a_j}\log(1+t/b)}(\log(1+t/b))^m \mid x^{n-j}\right\rangle.\end{aligned}$$

Thus, by (3.10), we can state the following result.

Theorem 10. *For all* $n \geq 0$,

$$\hat{N}C_n(x) = \sum_{m=0}^{n}\left(\sum_{j=0}^{n-m}\sum_{\ell=0}^{n-m-j}\binom{s}{j}\binom{n-j}{\ell}(1-\lambda)^{-j}b^{\ell-n}(n)_j S_1(n-j-\ell,m)\hat{N}C_\ell\right)H_m^{(s)}(x|\lambda).$$

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By similar arguments, using (1.8), (1.9) and $B_n^{(s)} \sim \left(\left(\frac{e^t-1}{t}\right)^s, t\right)$, we obtain the following identity.

Theorem 11. *For all $n \geq 0$,*

$$\hat{N}C_n(x) = \sum_{m=0}^{n} \left(\sum_{j=0}^{n-m} \sum_{\ell=0}^{n-m-j} \binom{n}{j} \binom{n-j}{\ell} b^{\ell-n} (n)_j S_1(n-j-\ell, m) c_i^{(s)} \hat{N}C_\ell \right) B_m^{(s)}(x).$$

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Local control and approximation properties of a C^2 rational interpolating curve

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Abstract

A new piecewise smooth rational quintic interpolation with a shape parameter is presented based on the function values only. The interpolation has a simple and explicit mathematical representation, and is of C^2 continuity without solving a system of consistency equations for the derivative values at the knots. In order to meet the needs of practical design, a local control method is employed to control the shape of curves. The advantage of the method is that it can be applied to modify the local shape of an interpolating curve by selecting suitable parameter. Also when $f(x) \in C^2[a,b]$, the error estimation formula of the interpolator is obtained.

Keywords: C^2-spline, rational interpolation, local control, error estimates

1 Introduction

In many problems of industrial design and manufacturing, it is usually needed to generate a smooth function. Spline interpolation is a useful and powerful tool to settle the problem, such as the polynomial spline, the rational spline and others [3, 5, 7, 11, 17, 20, 21]. In recent years, the rational spline and its application to shape preserving and shape control have received attention. Since the parameters in the interpolation function are selective according to the control need, the constrained control of the shape becomes possible. In References [4, 6, 13, 14, 15, 16, 18], positivity preserving, monotonic preserving and convexity preserving of the interpolating curves were discussed. In References [8, 9], the region control and convexity control of the interpolating curves were studied.

In fact, some practical designs in CAGD need only local control, but there are reports on only a few methods for local control. In Reference [12], a local control method of interval tension using weighted splines was given. In References [1, 2, 10], the local control methods of the rational cubic spline were discussed. For many applications, C^1 smoothness is generally sufficient. However, curvature continuity sometimes is needed and this leads to the need for C^2 or higher order of continuity. These rational interpolations above are also C^2 continuous by solving a system of consistency equations for the derivative values at the knots.

In this paper, we will describe a piecewise rational spline function which is of C^2 continuity without solving a systems of consistency equations. The interpolant contains a free parameter α_i, and can be used to control the local shape of interpolating curve.

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This paper is arranged as follows. In Section 2, a piecewise expression of the C^2 rational quintic interpolation with the parameter is described. In Section 3, a local control method of the interpolating curve is developed, including the value control and the derivative control of interpolation function at a point, and the numerical examples show the performance of the method. Section 4 is about the bases and error estimates, the interpolant can be represented by using basis functions clearly, and the error estimate formula is derived when the interpolated function $f(x) \in C^2[a, b]$.

2 Piecewise rational interpolation

Let $\{(x_i, f_i); i = 1, \cdots, n\}$ be a given set of data points, $a = x_1 < x_2 < \cdots < x_n = b$ be the knot spacing, and d_i be chosen derivative values at the knots x_i. We denote $h_i = x_{i+1} - x_i, t = (x - x_i)/h_i$ for $i = 1, 2, \cdots, n-1$, and $\Delta_i = \frac{f_{i+1} - f_i}{h_i}$. Let α_i be a positive parameter.

For $x \in [x_i, x_{i+1}], i = 1, 2, \cdots, n-1$, we construct the interpolant

$$P(x) = \frac{(1-t)^3 f_i + t(1-t)^2(W_i + \varphi(t)) + t^2(1-t)(V_i + \psi(t)) + t^3 f_{i+1}}{(1-t)^2 + t(1-t)\alpha_i + t^2}, \tag{1}$$

where both $\varphi(t)$ and $\psi(t)$ are the polynomial functions, and

$$W_i = (\alpha_i + 1) f_i + h_i d_i,$$
$$V_i = (\alpha_i + 1) f_{i+1} - h_i d_{i+1}.$$

In the special case, when $\varphi(t) \equiv 0, \psi(t) \equiv 0$, the interpolant $P(x)$ defined by (1) is a piecewise rational cubic interpolation, and which satisfies

$$P(x_i) = f_i, P(x_i) = f_{i+1}, P'(x_i) = d_i, P'(x_{i+1}) = d_{i+1}.$$

In the follows, we consider the case that both $\varphi(t)$ and $\psi(t)$ are the quadratic polynomials. Let

$$\varphi(t) = (t - t(1-t)\alpha_i)(f_{i+1} - f_i - h_i d_i),$$
$$\psi(t) = ((1-t) - t(1-t)\alpha_i)(f_i - f_{i+1} + h_i d_{i+1}),$$

the interpolation function $P(x)$ defined by (1) is called a piecewise rational quintic interpolation. $P(x)$ is C^1 continuous in interpolating interval $[a, b]$ for any given $d_i, i = 1, 2, \cdots, n$, and which satisfies

$$P(x_i) = f_i, P(x_i) = f_{i+1}, P'(x_i) = d_i, P'(x_{i+1}) = d_{i+1}.$$

It is easy to test that when $d_i = f'(x_i)$ and $\alpha_i \to +\infty$, the interpolation is the well-known standard cubic Hermite interpolation. That is to say, in this special case, the interpolant $P(x)$ defined by (1) will give approximately a standard cubic Hermite interpolation.

Further, if the partial derivative values d_i at the data sites are estimated using the arithmetic mean method:

$$d_i = \frac{1}{h_{i-1} + h_i}(h_{i-1}\Delta_i + h_i \Delta_{i-1}), \quad i = 2, 3, \cdots, n-1, \tag{2}$$

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then $P(x)$ is C^2 continuous in $[a,b]$, and which satisfies

$$P''(x_i) = \frac{2}{h_{i-1}+h_i}(\Delta_i - \Delta_{i-1}), \quad i = 2, 3, \cdots, n-1.$$

It is means that the interpolation function $P(x)$ defined by (1) can be C^2-continuous without solving a system of consistency equations for the derivative values an the knots.

Remark. At the end knots x_1, x_n, the derivative values are given as

$$\begin{aligned} d_1 &= \Delta_1 - \frac{h_1}{h_1+h_2}(\Delta_2 - \Delta_1), \\ d_n &= \Delta_{n-1} + \frac{h_{n-1}}{h_{n-1}+h_{n-2}}(\Delta_{n-1} - \Delta_{n-2}). \end{aligned} \tag{3}$$

Example 1. The interpolated function $f(x) = \cos^4(2x/5)$, $x \in [0,1]$ with interpolating knots at $x_i = (i-1)h, i = 1, 2, \cdots, 6$ and $h = 0.2$, the values of d_i at the knots x_i are conducted by using (2) and (3). Let $P(x)$ be the interpolation function defined by (1) over interval $[0,1]$, $Q(x)$ be the interpolation function provided in [8] over interval $[0, 0.8]$. Since both of the interpolants $P(x)$ and $Q(x)$ only based on function values are local, we consider the interpolation over subinterval $[0.2, 0.4]$ in order to compare their approximation. Let $\alpha_i = 0.6$, $\beta_i = 0.8$. Figure 1 is the graphs of curves $f(x)$ and $P(x)$, it is obvious that the two curves almost coincide. Figure 2 shows the curve of the error $f(t) - P(t)$. Figure 3 is the graphs of curves $f(x)$ and $Q(x)$, Figure 4 shows the curve of the error $f(t) - Q(t)$.

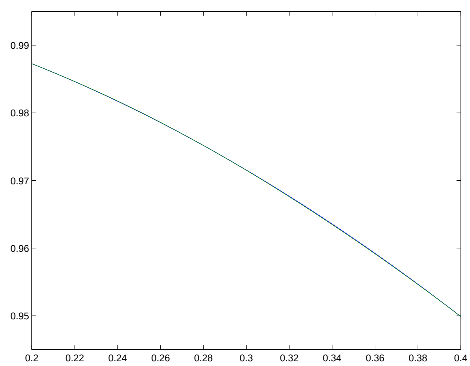

Figure 1: Graphs of curves $f(x)$ and $P(x)$.

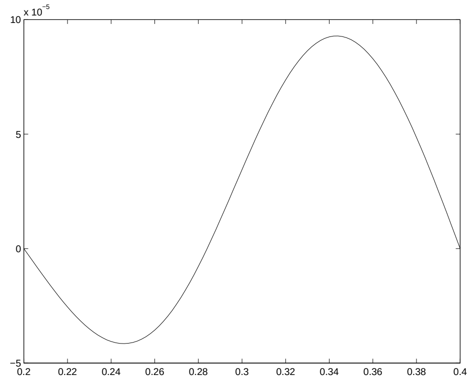

Figure 2: Curve of error $f(x) - P(x)$.

From Figure 1 to Figure 4, it is easy to see that the interpolator $P(x)$ defined by (1) approximates $f(x)$ better than that of the interpolator $Q(x)$ defined in [8].

3 Constrained control of the interpolating curve

The shape of interpolating curves on an interpolating interval depends on the interpolating data. Generally speaking, when the interpolating data are given, the shape of the interpolating curve is fixed. For the interpolation function defined by (1), since there is a parameter α_i, when the

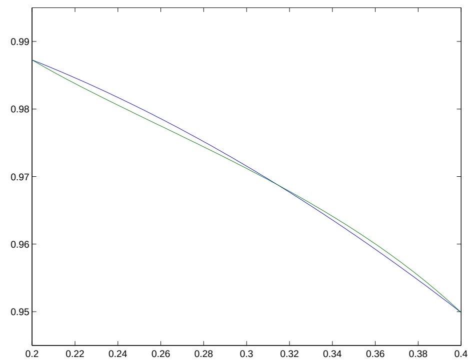

Figure 3: Graphs of curves $f(x)$ and $Q(x)$.

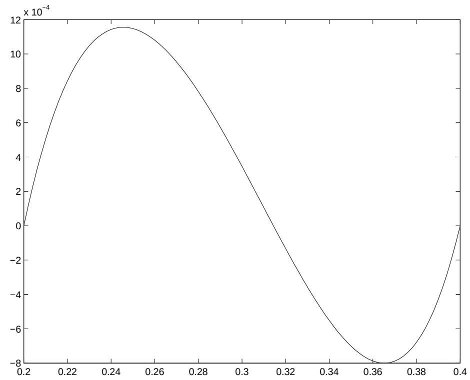

Figure 4: Curve of error $f(x) - Q(x)$.

parameter vary, the interpolation function can be changed for the unchanged interpolating data. On the basis of this, the shape of interpolating curve can be modified by selecting suitable parameter. We consider a method to control the shape of the interpolating curve at a point. Two issues are discussed in the following.

The first issue is the value control of interpolation function at a point. Let t be the local co-ordinate for a point $x \in [x_i, x_{i+1}]$. If the practical design requires the function value of the interpolation at the point x to be equal to a real number M, and $M \in (\min\{f_i, f_{i+1}\}, \max\{f_i, f_{i+1}\})$, this kind of control is called the value control of the interpolation at a point. Denoting

$$M = \lambda f_i + (1 - \lambda) f_{i+1},$$

with $\lambda \in [0, 1]$, then

$$\lambda = \frac{M - f_{i+1}}{f_i - f_{i+1}}. \tag{4}$$

Thus, the equation $P(x) = M$ can be written as

$$m_1 \alpha_i + m_2 = 0, \tag{5}$$

where

$$m_1 = (t(1-t)^3(1+2t) - t(1-t)\lambda)(f_i - f_{i+1}) + t^2(1-t)^2((1-t)d_i - td_{i+1})h_i,$$
$$m_2 = ((1-t)^2 - (1-2t+2t^2)\lambda)(f_i - f_{i+1}) + t(1-t)((1-t)^2 d_i - t^2 d_{i+1})h_i.$$

Obviously, if $m_1 m_2 < 0$, then there must exist positive parameter α_i satisfying Eq. (5). Thus, we have the following function value control theorem.

Theorem 1. *Let $P(x)$ be the interpolation function over $[x_i, x_{i+1}]$ defined by (1), and let x be a point in $[x_i, x_{i+1}]$, M is a real number, then the sufficient condition for existence of the positive parameter α_i to satisfy $P(x) = M$ is that $m_1 m_2 < 0$, and the positive parameter α_i can be selected by Eq. (5).*

The second issue is the derivative control of interpolation function at a point. If the practical design requires the first-order derivative of the interpolation function at the point x to be equal to

a real number N, how this can be achieved. Let

$$P'(x) = N, \tag{6}$$

if there exist a parameter α_i to satisfy this equation, α_i is called the solution of the control equation (6). This control is called the derivative control of interpolation function. Eq. (6) can be written as

$$A\alpha_i^2 + B\alpha_i + C = 0, \tag{7}$$

where

$$\begin{aligned}
A &= 6t^3(1-t)^2(f_i - f_{i+1}) + t^2(1-t)^2(N + (t(2-3t)d_{i+1} - (1-4t+3t^2)d_i)h_i),\\
B &= 2t^2(1-t)^2(5-6t+6t^2)(f_i - f_{i+1}) + 2t(1-t)(1-2t+2t^2)N\\
&\quad + t(1-t)[t(3-7t+8t^2-6t^3)d_{i+1} - (1-t)(2-9t+10t^2-6t^3)d_i]h_i,\\
C &= 2t(1-t)(f_i - f_{i+1}) + (1-2t+2t^2)^2N + t^2(3-8t+8t^2-4t^3)d_{i+1}h_i\\
&\quad -(1-t)^2(1-4t+4t^2-4t^3)d_ih_i.
\end{aligned}$$

Eq. (7) is a quadratic algebraic equation of α_i, if it exists a positive root, then Eq. (6) holds. This can be stated in the following derivative control theorem.

Theorem 2. *Let $P(x)$ be the interpolation function over $[x_i, x_{i+1}]$ defined by (1), and let x be a point in $[x_i, x_{i+1}]$, N is a real number, then the sufficient condition for existence of the positive parameter α_i to satisfy $P'(x) = N$ is that Eq. (7) has a positive root.*

Example 2. Given the interpolating data shown in Table 1, let $P(t)$ be the interpolation function defined by (1) in the interpolating interval $[0, 5]$. This example will show how the constrained control of the interpolating curves can be achieved by selecting a suitable parameter for unchanged given data.

Table 1: Set of the interpolating data.

x_i	0.0	1.0	2.0	3.0	4.0	5.0
$f(x_i)$	2.0	5.0	3.0	3.0	6.0	4.0

Without loss of generality, we consider the interpolation in $[1, 2]$ only, and all the figures below are in $[1, 2]$. The values of d_i at the knots x_i are conducted by using (2). Let $\alpha_i = 0.4$, we denote the interpolation function by $P_1(x)$, and it can be shown that $P_1(1.2) = \frac{19221}{3875} = 4.96026$. For the given interpolating data, if the design requires $P(1.2) = 4.9$, then $\lambda = 0.95$ by (4). Thus $\alpha_i = \frac{95}{4} = 23.75$ by (5). Denoting the interpolation function by $P_2(t)$. Figure 5 shows the graphs of the interpolating curves $P_1(t)$ and $P_2(t)$, and it can be tested that $P_2(1.2) = 4.9$.

Also, it can be computed that $P_1'(1.2) = -\frac{78251}{72075} = -1.08569$. If the practical design requires the derivative of the interpolation function at the point $x = 1.2$ to be equal to -1.4, then $\alpha_i = \frac{1}{8}(-27 + \sqrt{6349}) = 6.58508$ by Eq. (7). We denote the interpolation function by $P_3(x)$. Figure 6 shows the graphs of the interpolation curves $P_1(x)$ and $P_3(t)$, and it can be tested that $P_3'(1.2) = -1.4$.

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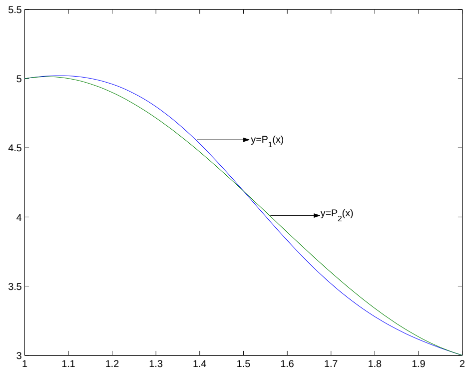

Figure 5: Graphs of the curves $P_1(x)$ and $P_2(x)$.

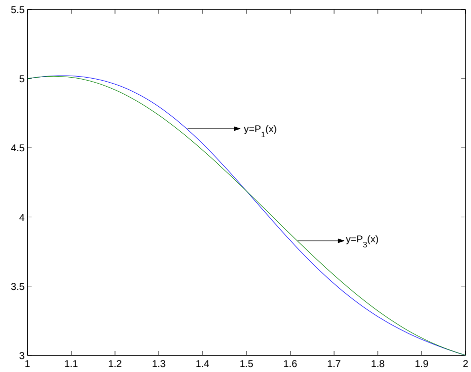

Figure 6: Graphs of the curves $P_1(x)$ and $P_3(x)$.

4 The bases and error estimates of the interpolation

In what follows in this paper, we consider the case of equally spaced knots, namely, $h_i = h_j$ for all $i, j \in \{1, 2, \cdots, n-1\}$. When d_i is given by (2) and (3), the interpolation function $P_i(x)$ defined by (1) can be written as follows:

$$P(x) = \omega_1(t) f_{i-1} + \omega_2(t) f_i + \omega_3(t) f_{i+1} + \omega_4(t) f_{i+2}, \tag{8}$$

where for $x \in [x_1, x_2]$,

$$\omega_1(t) = 0, \quad \omega_2(t) = \frac{1}{2}(2 - 3t + t^2), \quad \omega_3(t) = t(2-t), \quad \omega_4(t) = -\frac{1}{2}t(1-t); \tag{9}$$

for $x \in [x_i, x_{i+1}]$, $i = 2, 3, \cdots, n-2$,

$$\begin{aligned}
\omega_1(t) &= -\frac{t(1-t)^3(1+t\alpha_i)}{2((1-t)^2 + t^2 + t(1-t)\alpha_i)}, \\
\omega_2(t) &= \frac{(1-t)(2 - 2t + t^3 + t(2 - 5t + 3t^2)\alpha_i)}{2((1-t)^2 + t^2 + t(1-t)\alpha_i)}, \\
\omega_3(t) &= \frac{t(1 - t + 3t^2 - t^3 + t(1 + 3t - 7t^2 + 3t^3)\alpha_i)}{2((1-t)^2 + t^2 + t(1-t)\alpha_i)}, \\
\omega_4(t) &= -\frac{t^3(1-t)(1 + (1-t)\alpha_i)}{2((1-t)^2 + t^2 + t(1-t)\alpha_i)};
\end{aligned} \tag{10}$$

and for $x \in [x_{n-1}, x_n]$,

$$\omega_1(t) = -\frac{1}{2}t(1-t), \quad \omega_2(t) = 1 - t^2, \quad \omega_3(t) = \frac{1}{2}t(1+t), \quad \omega_4(t) = 0. \tag{11}$$

The terms $\omega_j(t)$ $(j = 1, 2, 3, 4)$ are called the basis of the interpolant defined by (8), and which satisfy $\sum_{k=1}^{4} \omega_k(t) = 1$, and

$$\sum_{k=1}^{4} |\omega_k(t)| = 1 + t - t^2. \tag{12}$$

We denote

$$M = \max\{|f_k|, k = i-1, i, i+1, i+2\}.$$

From (8) and (12), it is easy to see that the following theorem holds.

Theorem 3. *Let $P(x)$ is the interpolation function defined by (8) in $[x_i, x_{i+1}]$, then whatever the positive value of the parameter α_i might be, the values of $P(x)$ in $[x_i, x_{i+1}]$ satisfy*

$$|P(x)| \leq \frac{5}{4}M.$$

In order to proceed error estimates of the interpolation function $P(x)$ defined by (8), without loss of generality, it is only necessary to consider the subinterval $[x_i, x_{i+1}]$. As $f(x) \in C^2[a,b]$, using the Peano-Kernel Theorem [19] gives the following:

$$f(x) - P(x) = \int_{x_{i-1}}^{x_{i+2}} f^{(2)}(\tau) R_x[(x-\tau)_+]\, d\tau. \tag{13}$$

(I) For $x \in [x_i, x_{i+1}]$, $i = 2, 3, \cdots, n-2$, from (8) and (10), we have

$$R_x[(x-\tau)_+] = \begin{cases} p_i(\tau), & x_{i-1} < \tau < x_i < x, \\ q_i(\tau), & x_i < \tau < x, \\ r_i(\tau), & x < \tau < x_{i+1}, \\ s_i(\tau), & x_{i+1} < \tau < x_{i+2}, \end{cases}$$

where

$$\begin{aligned} p_i(\tau) &= x - \tau - \omega_2(t)(x_i - \tau) - \omega_3(t)(x_{i+1} - \tau) - \omega_4(t)(x_{i+2} - \tau), \\ q_i(\tau) &= x - \tau - \omega_3(t)(x_{i+1} - \tau) - \omega_4(t)(x_{i+2} - \tau), \\ r_i(\tau) &= -\omega_3(t)(x_{i+1} - \tau) - \omega_4(t)(x_{i+2} - \tau), \\ s_i(\tau) &= -\omega_4(t)(x_{i+2} - \tau). \end{aligned}$$

Now, we consider the properties of the kernel function $R_x[(x-\tau)_+]$ of the variable τ in the interval $[x_{i-1}, x_{i+2}]$. For all $\tau \in [x_{i+1}, x_{i+2}]$, it is easy to see that

$$s_i(\tau) = \frac{t^3(1-t)(1+(1-t)\alpha_i)(x_{i+2}-\tau)}{2((1-t)^2 + t^2 + t(1-t)\alpha_i)} \geq 0.$$

For $p_i(\tau)$ in $[x_{i-1}, x_i]$, it is easy to test that

$$\begin{aligned} p_i(x_{i-1}) &= 0, \\ p_i(x_i) &= \frac{t(1-t)^3(1+t\alpha_i)h_i}{2 - 4t + 4t^2 + 2t(1-t)\alpha_i} \geq 0. \end{aligned}$$

Thus, $p_i(\tau) \geq 0$ for all $\tau \in [x_{i-1}, x_i]$.

For $q_i(\tau)$ in $[x_i, x]$, we have

$$\begin{aligned} q_i(x_i) &= p_i(x_i) \geq 0, \\ q_i(x) &= -\frac{t(1-t)(1-t+t^2+t(1+t-4t^2+2t^3)\alpha_i)h_i}{2-4t+4t^2+2t(1-t)\alpha_i} \leq 0. \end{aligned}$$

The root of $q_i(\tau)$ in $[x_i, x]$ can be found,

$$\tau^* = x_i + \frac{t(1-t)^2(1+t\alpha_i)h_i}{2-3t+2t^2+t(2-t-3t^2+2t^3)\alpha_i}.$$

Similarly, for $r_i(\tau)$, we can derive

$$\begin{aligned} r_i(x) &= q(x) \le 0, \\ r_i(x_{i+1}) &= \frac{t^3(1-t)(1+(1-t)\alpha_i)h_i}{2-4t+4t^2+2t(1-t)\alpha_i} \ge 0. \end{aligned}$$

The root of $r_i(\tau)$ in $[x, x_{i+1}]$ is

$$\tau_* = x_i + \frac{(1-t+t^2+t^3+t(1+t-3t^2+t^3)\alpha_i)h_i}{1-t+2t^2+t(1+2t-5t^2+2t^3)\alpha_i}.$$

From (13), we obtain that

$$\begin{aligned} |f(x)-P(x)| &\le \|f^{(2)}(x)\|[\int_{x_{i-1}}^{x_i} p_i(\tau)d\tau + \int_{x_i}^{x} |q_i(\tau)|d\tau + \int_{x}^{x_{i+1}} |r_i(\tau)|d\tau + \int_{x_{i+1}}^{x_{i+2}} s_i(\tau)d\tau] \\ &= h_i^2\|f^{(2)}(x)\|W_i(t), \end{aligned}$$

where

$$W_i(t) = \frac{t(1-t)(1-t+t^2+t(1+t-4t^2+2t^3)\alpha_i)^2}{(1-t+2t^2+t(1+2t-5t^2+2t^3)\alpha_i)(2-3t+2t^2+t(2-t-3t^2+2t^3)\alpha_i)}. \tag{14}$$

(II) For $x \in [x_1, x_2]$, similar to the case (I), we have from (8), (9) and (13) that

$$\begin{aligned} |f(x)-P(x)| &\le \|f^{(2)}(x)\|[\int_{x_1}^{x} |p_1(\tau)|d\tau + \int_{x}^{x_2} |q_1(\tau)|d\tau + \int_{x_2}^{x_3} |r_1(\tau)|d\tau] \\ &= h_i^2\|f^{(2)}(x)\|W_1(t), \end{aligned}$$

where

$$W_1(t) = \frac{t(2-3t+t^2)}{3-t}. \tag{15}$$

(III) For $x \in [x_{n-1}, x_n]$, from (8), (11) and (13), it is easy to derive that

$$\begin{aligned} |f(x)-P(x)| &\le \|f^{(2)}(x)\|[\int_{x_{n-2}}^{x_{n-1}} |p_{n-1}(\tau)|d\tau + \int_{x_{n-1}}^{x} |q_{n-1}(\tau)|d\tau + \int_{x_{n-1}}^{x_n} |r_{n-1}(\tau)|d\tau] \\ &= h_i^2\|f^{(2)}(x)\|W_{n-1}(t), \end{aligned}$$

where

$$W_{n-1}(t) = \frac{t(1-t^2)}{2+t}. \tag{16}$$

For the fixed α_i, let

$$\mu_i = \max_{0\le t\le 1} W_i(t), \tag{17}$$

where $W_i(t)$, $i = 1, 2, \cdots, n-1$ are defined by (15), (14) and (16), respectively. Based on the analysis above, we have the following theorem.

Theorem 4. *Let $f(x) \in C^2[a, b]$ and $a = x_1 < x_2 < \cdots < x_n = b$ be an equal-knot spacing. For the given positive parameter α_i, $P(x)$ is the corresponding rational interpolation function defined by (8). Then for $x \in [x_i, x_{i+1}]$,*

$$|f(x) - P(x)| \leq h_i^2 \|f^{(2)}(x)\| \mu_i$$

where μ_i defined by (17).

The error coefficient μ_i is called the optimal error constant. It is evident that the optimal error constant does not depend on the subinterval $[x_i, x_{i+1}]$. Since $W_i(t)$ is a continuous function of the variate t in the interval $[0, 1]$, so the coefficient μ_i is bounded. The following theorem about the optimal error constant μ_i holds.

Theorem 5. *For any given positive parameter α_i, the optimal error constant μ_i satisfies*

$$\begin{aligned} &\mu_1 = \mu_{n-1} = 0.150644, \\ &\mu_i = \frac{9}{64}, i = 2, 3, \cdots, n-2. \end{aligned} \tag{18}$$

Proof. From (15), (16) and (17), it is easy to derive that $\mu_1 = \mu_{n-1} = 0.150644$. We consider μ_i, $i = 2, 3, \cdots, n-2$. (14) can be rewritten as

$$W_i(t) = \frac{k_1 + k_2 + k_3}{k_4 + k_5 + k_6}, \tag{19}$$

where

$$\begin{aligned} &k_1 = t(1-t)(1-t+t^2)^2, \\ &k_2 = 2t^2(1-t)^2(1-t+t^2)(1+2t-2t^2)\alpha_i, \\ &k_3 = t^3(1-t)^3(1+2t-2t^2)^2\alpha_i^2, \\ &k_4 = (1-t+2t^2)(2-3t+2t^2), \\ &k_5 = 2t(1-t)(2+t-5t^2+8t^3-4t^4)\alpha_i, \\ &k_6 = t^2(1-t)^2(1+3t-2t^2)(2+t-2t^2)\alpha_i^2. \end{aligned}$$

For $t \in (0, 1)$ and any $\alpha_i > 0$, then $k_i > 0, i = 1, 2, \cdots, 6$. Also, we can obtain that

$$k_1(k_5 + k_6) - k_4(k_2 + k_3) = t^4(1-t)^4(1-2t)^2\alpha_i^2(2(1-t+t^2) + 3t^2(1-t)^2\alpha_i) \geq 0.$$

Thus, from (19), we have

$$W_i(t) \leq \frac{k_1}{k_4} = \frac{t(1-t)(1-t+t^2)^2}{(2-3t+2t^2)(1-t+2t^2)} \leq \frac{9}{64}. \tag{20}$$

Further, it is easy to test that

$$k_5(k_1 + k_3) \geq k_2(k_4 + k_6),$$

it means that

$$W_i(t) \geq \frac{k_2}{k_5} = \frac{t(1-t)(1-t+t^2)(1+2t-2t^2)}{2+t-5t^2+8t^3-4t^4}.$$

Since

$$\max_{t\in[0,1]}\{\frac{t(1-t)(1-t+t^2)(1+2t-2t^2)}{2+t-5t^2+8t^3-4t^4}\}=\frac{9}{64},$$

from (20), we can see that $\mu_i=\frac{9}{64}, i=2,3,\cdots,n-2.$

The proof is completed. $\square$

Theorem 5 shows that the maximum error of the interpolation function defined by (8) does not depend on the positive parameter α_i, it means the interpolation is stable for the parameter.

5 Concluding remarks

In this paper, an explicit representation of a rational quintic interpolator is presented. The interpolation function, which can be represented by the basis functions, contains a free positive parameter α_i, and is of C^2 continuity in the interpolating interval without solving a system of consistency equations for the derivative values at the knots. Also, a local control method of the interpolating curve is developed, including the value control and the derivative control. When the given data are not changed, on the basis of this method, the shape of the interpolating curve can be modified merely by selecting suitable parameter. Convergence analysis shows that the interpolator gives a good approximation to the interpolated function, and is stable for the free positive parameter α_i.

Acknowledgment

The author acknowledge supports provided by the National Nature Science Foundation of China (No.61070096) and the Natural Science Foundation of Shandong Province (No.ZR2012FL05).

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CERTAIN NEW GRÜSS TYPE INEQUALITIES INVOLVING SAIGO FRACTIONAL q-INTEGRAL OPERATOR

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Abstract

In the present paper, we aim to investigate a new q-integral inequality of Grüss type for the Saigo fractional q-integral operator. Some special cases of our main results are also provided. The results presented in this paper improve and extend some recent results.

2010 Mathematics Subject Classification: 26D10; 26D15; 26A33; 05A30.
Key words and phrases: Integral inequalities; Grüss inequality, q-Saigo fractional integral operator.

1 Introduction and preliminaries

Fractional q-calculus, which combines fractional calculus with q-calculus, has the advantage of both. Its origin dates back to the works of Al-Salam [1] and Agarwal [2]. Currently, Fractional q-calculus had been proven to have important applications on many fields, such as physics, quantum theory, theory of relativity, Combinatorics, basic hyper-geometric functions, orthogonal polynomials, mechanics, chemicals and engineering. For some new development on the topic, see book [3] and the papers [4, 5, 6, 7, 8, 9, 10, 11, 12] .

In the preset investigation, we assume that f and g be two functions which are defined and integrable on [a, b]. Then the following inequality holds (see also [21], [23, p. 296]):

$$l \leq f(t) \leq L, \quad m \leq g(t) \leq M, \tag{1.1}$$

Let, for each $t \in [a,b]$, l, L, m and, M be real constants satisfying the inequalities (1.1). Then the following Grüss type inequalities involving Riemann-Liouville fractional integrals holds (see, for example, [27]):

$$\left|\frac{t^\alpha}{\Gamma_q(\alpha+1)} I_q^\alpha(f\,g)(t) - I_q^\alpha(f)(t) I_q^\alpha(g)(t)\right| \leq \left(\frac{t^\alpha}{\Gamma_q(\alpha+1)}\right)^2 (L-l)(M-m), \tag{1.2}$$

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The inequality (1.2) has various generalizations that have appeared in the literature, for example, e.g. [13, 15, 16, 18, 19, 22, 23, 24, 25, 26] and the references cited therein.

Very recently, Zhu *et al.*[27] gave certain Grüss type inequalities involving the fractional q-integral operator. Using the same technique, in this paper, we establish certain new Grüss type fractional q-integral inequalities involving the Saigo fractional q-integral operator due to Garg and Chanchlani[20]. Moreover, we also consider their relevant connections with other known results.

Throughout the present paper, we shall investigate a fractional integral over the space C_λ introduced in [17] and defined as follows.

Definition 1.1 The space of functions $C_\lambda, \lambda \in \mathbb{R}$, the set of real numbers, consists of all functions $f(t), t > 0$, that can be represented in the form $f(t) = t^p f_1(t)$ with $p > \lambda$ and $f_1 \in C[0,\infty)$, where $C[0,\infty)$ is the set of continuous functions in the interval $[0,\infty)$.

Here, we define a new (presumable) fractional integral operator $K_q^{\alpha,\beta,\eta}$ associated with the Saigo fractional q-integral operator as follows.

Definition 1.2 Let $0 < q < 1$, $f \in C_\lambda$. Then for $\alpha > 0, \alpha+\eta > -1$ and $\beta > 1$ we define a fractional integral $K_q^{\alpha,\beta,\eta}$ as follows:

$$\left(K_q^{\alpha,\beta,\eta} f\right)(t) = \frac{\Gamma_q(1-\beta)\Gamma_q(\alpha+\eta+1)}{\Gamma_q(\eta-\beta+1)} t^\beta \left(I_q^{\alpha,\beta,\eta} f\right)(t), \tag{1.3}$$

where $I_q^{\alpha,\beta,\eta} f$ is the Saigo fractional q-integral operator of order α and is defined in the following.

Definition 1.3 Let $\Re(\alpha) > 0$, β and η be real or complex numbers. Then a q-analogue of Saigo's fractional integral $I_q^{\alpha,\beta,\eta}$ is given for $|\frac{\tau}{t}| < 1$ by (see [20, p. 172, Eq.(2.1)]):

$$\begin{aligned} I_q^{\alpha,\beta,\eta}\{f(t)\} :=& \frac{t^{-\beta-1}}{\Gamma_q(\alpha)} \int_0^t (q\,\tau/t;q)_{\alpha-1} \sum_{m=0}^{\infty} \frac{\left(q^{\alpha+\beta};q\right)_m \left(q^{-\eta};q\right)_m}{\left(q^\alpha;q\right)_m (q;q)_m} \\ & \cdot q^{(\eta-\beta)m} (-1)^m q^{-\binom{m}{2}} \; (\tau/t-1)_q^m \; f(\tau)\, d_q\tau. \end{aligned} \tag{1.4}$$

The integral operator $I_q^{\alpha,\beta,\eta}$ includes both the q-analogues of the Riemann-Liouville and Erdélyi-Kober fractional integral operators given by the following relationships:

$$\begin{aligned} I_q^{\alpha}\{f(t)\} &: \left(= I_q^{\alpha,-\alpha,0}\{f(t)\}\right) \\ &= \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^t (q\,\tau/t;q)_{\alpha-1} f(\tau) d_q\tau \quad (\alpha > 0;\, 0 < q < 1). \end{aligned} \tag{1.5}$$

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and

$$I_q^{\eta,\alpha}\{f(t)\} : \left(= I_q^{\alpha,0,\eta}\{f(t)\}\right) \\ = \frac{t^{-\eta-1}}{\Gamma_q(\alpha)} \int_0^t (q\,\tau/t;q)_{\alpha-1}\,\tau^\eta f(\tau) d_q\tau \quad (\alpha > 0;\, 0 < q < 1), \tag{1.6}$$

where $(a;q)_\alpha$ is the *q-shifted factorial.*
The *q-shifted factorial* $(a;q)_n$ is defined by

$$(a;q)_n := \begin{cases} 1 & (n = 0) \\ \prod_{k=0}^{n-1} \left(1 - a\,q^k\right) & (n \in \mathbb{N}), \end{cases} \tag{1.7}$$

where $a,\, q \in \mathbb{C}$ and it is assumed that $a \neq q^{-m}$ $(m \in \mathbb{N}_0)$.

The *q-shifted factorial* for negative subscript is defined by

$$(a;q)_{-n} := \frac{1}{(1 - a\,q^{-1})\,(1 - a\,q^{-2}) \cdots (1 - a\,q^{-n})} \quad (n \in \mathbb{N}_0)\,. \tag{1.8}$$

We also write

$$(a;q)_\infty := \prod_{k=0}^{\infty} \left(1 - a\,q^k\right) \quad (a,\, q \in \mathbb{C};\ |q| < 1). \tag{1.9}$$

It follows from (1.7), (1.8) and (1.9) that

$$(a;q)_n = \frac{(a;q)_\infty}{(a\,q^n;q)_\infty} \quad (n \in \mathbb{Z}), \tag{1.10}$$

which can be extended to $n = \alpha \in \mathbb{C}$ as follows:

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(a\,q^\alpha;q)_\infty} \quad (\alpha \in \mathbb{C};\ |q| < 1), \tag{1.11}$$

where the principal value of q^α is taken.

For $f(t) = t^\mu$ in (1.4), we get the known formula [12]:

$$I_q^{\alpha,\beta,\eta}\{t^\mu\} := \frac{\Gamma_q(\mu+1)\Gamma_q(\mu-\beta+\eta+1)}{\Gamma_q(\mu-\beta+1)\Gamma_q(\mu+\alpha+\eta+1)} x^\mu - \beta \tag{1.12}$$

Lemma 1.4(Choi and Agarwal [14]): *Let* $0 < q < 1$ *and* $f : [0,\, \infty) \to \mathbb{R}$ *be a continuous function with* $f(t) \geq 0$ *for all* $t \in [0,\, \infty)$. *Then we have the following inequalities*:

(i) *The Saigo fractional q-integral operator of the function $f(t)$ in* (1.4)

$$I_q^{\alpha,\beta,\eta}\{f(t)\} \geq 0, \tag{1.13}$$

for all $\alpha > 0$ and β, $\eta \in \mathbb{R}$ with $\alpha + \beta > 0$ and $\eta < 0$;

(ii) *The q-analogue of Riemann-Liouville fractional integral operator of the function $f(t)$ of a order α in* (1.5)

$$I_q^{\alpha}\{f(t)\} \geq 0, \tag{1.14}$$

for all $\alpha > 0$;

(iii) *The q-analogue of Erdélyi-Kober fractional integral operator of the function $f(t)$ in* (1.6)

$$I_q^{\eta,\alpha}\{f(t)\} \geq 0, \tag{1.15}$$

for all $\alpha > 0$ and $\eta \in \mathbb{R}$.

Remark 1.5: We can present a large number of special cases of our main fractional integral operator $K_q^{\alpha,\beta,\eta}$. Out of which, here, we give only few ones as follows:

(i) For $\Re(\alpha) > 0$, β and η be real or complex numbers and $\Re(\mu) > -1$, $\mu-\eta+\beta > -1$, we have

$$K_q^{\alpha,\beta,\eta}(t^{\mu}) = \frac{\Gamma_q(1-\beta)\Gamma_q(\alpha+\eta+1)\Gamma_q(\mu+1)\Gamma_q(\mu-\beta+\eta+1)}{\Gamma_q(\eta-\beta+1)\Gamma_q(\mu-\beta+1)\Gamma_q(\mu+\alpha+\eta+1)\,\Gamma_q(\eta-\beta+1)}t^{\mu} \tag{1.16}$$

and

$$K_q^{\alpha,\beta,\eta}(C) = C \tag{1.17}$$

where C is constant.

(ii) For $\Re(\alpha) > 0$, and η be real or complex number and $\Re(\mu) > -1$, $\mu + \eta > -1$, we have

$$K_q^{\eta,\alpha}(t^{\mu}) = \frac{\Gamma_q(\alpha+\eta+1)\,\Gamma_q(\mu+\eta+1)}{\Gamma_q(\eta+1)\Gamma_q(\mu+\alpha+\eta+1)}t^{\mu} \tag{1.18}$$

(iii) For $\Re(\alpha) > 0$and $\Re(\mu) > -1$, we have

$$K_q^{\alpha}(t^{\mu}) = \frac{\Gamma_q(\alpha+1)}{\Gamma_q(\mu+\alpha+1)}t^{\mu} \tag{1.19}$$

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2 Grüss type inequalities for the Saigo fractional q-integral operator

To state the main results in this section, first we establish the following Grüss type lemmas involving the fractional integral operator $K_q^{\alpha,\beta,\eta}$ (1.3), some of which are presumably (new) ones.

Lemma 2.1 *Let $0 < q < 1$, $h \in C_\lambda$ and $m, M \in \mathbb{R}$ with $m \le h(t) \le M$. Then, for all $t > 0$ we have*

$$\begin{aligned} &K_q^{\alpha,\beta,\eta} h^2(t) - \left(K_q^{\alpha,\beta,\eta}\, h(t)\right)^2 \\ &\quad = \left(M - K_q^{\alpha,\beta,\eta} h(t)\right)\left(K_q^{\alpha,\beta,\eta}\, h(t) - m\right) - K_q^{\alpha,\beta,\eta}\left(M - h(t)\right)\left(h(t) - m\right), \end{aligned} \tag{2.1}$$

for all $\alpha > 0$, and β, $\eta \in \mathbb{R}$ with $\alpha + \beta > 0$, and $\eta < 0$.

Proof. Let $h \in C_\lambda$ and $m, M \in R; m \le h(t) \le M$, for all $t \in [0, \infty)$, then, for any $u, v, \in [0, \infty)$, we have

$$\begin{aligned} &(M - h(u))(h(v) - m) + (M - h(v))(h(u) - m) - (M - h(u))(h(u) - m) \\ &\qquad - (M - h(v))(h(v) - m) = h^2(u) + h^2(v) - 2h(u)h(v). \end{aligned} \tag{2.2}$$

Multiplying both sides of (2.2) by

$$\frac{t^{-\beta-1}}{\Gamma_q(\alpha)}(q\,u/t;q)_{\alpha-1}\sum_{m=0}^{\infty}\frac{\left(q^{\alpha+\beta};q\right)_m\,(q^{-\eta};q)_m}{\left(q^{\alpha};q\right)_m\,(q;q)_m}\cdot q^{(\eta-\beta)m}\,(-1)^m\,q^{-\binom{m}{2}}\;(u/t-1)_q^m$$

and taking q-integration of the resulting inequality with respect to u from 0 to t with the aid of Definition 1.2, we get

$$\begin{aligned} &\left(M - K_q^{\alpha,\beta,\eta} h(t)\right)(h(v) - m) + (M - h(v))\left(K_q^{\alpha,\beta,\eta} h(t) - m\right) \\ &\qquad - K_q^{\alpha,\beta,\eta}\left(M - h(t)\right)\left(h(t) - m\right) - (M - h(v))(h(v) - m) \\ &\qquad\qquad = K_q^{\alpha,\beta,\eta} h^2(t) + h^2(v) - 2K_q^{\alpha,\beta,\eta} h(t)h(v). \end{aligned} \tag{2.3}$$

Again multiplying both sides of (2.3) by

$$\frac{t^{-\beta-1}}{\Gamma_q(\alpha)}(q\,v/t;q)_{\alpha-1}\sum_{m=0}^{\infty}\frac{\left(q^{\alpha+\beta};q\right)_m\,(q^{-\eta};q)_m}{\left(q^{\alpha};q\right)_m\,(q;q)_m}\cdot q^{(\eta-\beta)m}\,(-1)^m\,q^{-\binom{m}{2}}\;(v/t-1)_q^m$$

and taking q-integration of the resulting inequality with respect to v from 0 to t with the aid of Definition 1.2, we get the required result (2.1). This is complete the proof of the Lemma 2.1. □

Lemma 2.2 *Let $0 < q < 1$, $f, q \in C_\lambda$. Then, for all $t > 0$,we have*

$$\Big(K_f^{\alpha,\beta,\eta} q(t) + K_f^{\gamma,\delta,\zeta} q(t) - K_q^{\alpha,\beta,\eta} f(t)\, K_q^{\gamma,\delta,\zeta} g(t) - K_q^{\alpha,\beta,\eta} g(t)\, K_q^{\gamma,\delta,\zeta} f(t)\Big)^2$$
$$\leq \Big(K_q^{\alpha,\beta,\eta} f^2(t) + K_q^{\gamma,\delta,\zeta} f^2(t) - K_q^{\alpha,\beta,\eta} f(t)\, K_q^{\gamma,\delta,\zeta} f(t)\Big)$$
$$- \Big(K_q^{\alpha,\beta,\eta} g^2(t) + K_q^{\gamma,\delta,\zeta} g^2(t) - K_q^{\alpha,\beta,\eta} g(t)\, K_q^{\gamma,\delta,\zeta} g(t)\Big), \tag{2.4}$$

for all $\alpha, \gamma > 0$, and $\beta, \eta, \delta, \zeta \in \mathbb{R}$ with $\alpha + \beta > 0, \gamma + \delta > 0,$, and $\eta, \zeta < 0$.

Proof. Let f and g be two continuous and synchronous functions on $[0, \infty)$. Then, for all $u,\ v \in (0, t)$ with $0 < q < 1$, we have

$$A(u, v) = (f(u) - f(v))\,(g(u) - g(v)) \;\; (u, v \in [0, t)). \tag{2.5}$$

Multiplying (2.5) by

$$\frac{t^{-\beta-1}\, t^{-\delta-1}}{(\Gamma_q(\alpha))\ (\Gamma_q(\gamma))}\,(q\,u/t;q)_{\alpha-1}\,(q\,v/t;q)_{\gamma-1}$$
$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\left(q^{\alpha+\beta};q\right)_m\ (q^{-\eta};q)_m}{(q^{\alpha};q)_m\ (q;q)_m}\ \frac{\left(q^{\gamma+\delta};q\right)_n\ \left(q^{-\zeta};q\right)_n}{(q^{\gamma};q)_n\ (q;q)_n}$$
$$\cdot\, q^{(\eta-\beta)m}\,(-1)^m\, q^{-\binom{m}{2}}\, q^{(\zeta-\delta)n}\,(-1)^n\, q^{-\binom{n}{2}}\ \ (u/t-1)_q^m\ \ (v/t-1)_q^n\,. \tag{2.6}$$

Integrating (2.6) twice with respect to u and v from 0 to t, we obtain the following result with the help of (1.3) and (1.4):

$$\frac{t^{-\beta-1}\, t^{-\delta-1}}{(\Gamma_q(\alpha))\ (\Gamma_q(\gamma))}\int_0^t\int_0^t (q\,u/t;q)_{\alpha-1}\,(q\,v/t;q)_{\gamma-1}$$
$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\left(q^{\alpha+\beta};q\right)_m\ (q^{-\eta};q)_m}{(q^{\alpha};q)_m\ (q;q)_m}\ \frac{\left(q^{\gamma+\delta};q\right)_n\ \left(q^{-\zeta};q\right)_n}{(q^{\gamma};q)_n\ (q;q)_n}$$
$$\cdot\, q^{(\eta-\beta)m}\,(-1)^m\, q^{-\binom{m}{2}}\, q^{(\zeta-\delta)n}\,(-1)^n\, q^{-\binom{n}{2}}\ \ (u/t-1)_q^m\ \ (v/t-1)_q^n\, A(u,v)dudv$$
$$= K_q^{\alpha,\beta,\eta} fg(t) + K_q^{\gamma,\delta,\zeta} fg(t) - K_q^{\alpha,\beta,\eta} f(t) K_q^{\gamma,\delta,\zeta} g(t) - K_q^{\alpha,\beta,\eta} g(t) K_q^{\gamma,\delta,\zeta} f(t). \tag{2.7}$$

By making use of the well known Cauchy-Schwarz inequality for double q-integrals [27, pp. 4-5], we obtain the required result (2.4)involving the $K_q^{(.,.,.)}$ (1.3), type fractional integral operator. This is complete the proof of Lemma 2.2. □

Lemma 2.3 *Let $0 < q < 1$, $h \in C_\lambda$ and $m, M \in \mathbb{R}$ with $m \leq h(t) \leq M$. Then, for all $t > 0$ we have*

$$K_q^{\alpha,\beta,\eta} h^2(t) + K_q^{\gamma,\delta,\zeta} h^2(t) - 2K_q^{\alpha,\beta,\eta}\, h(t) K_q^{\gamma,\delta,\zeta} h(t)$$
$$= \Big(M - K_q^{\alpha,\beta,\eta} h(t)\Big)\Big(K_q^{\alpha,\beta,\eta}\, h(t) - m\Big) + \Big(M - K_q^{\gamma,\delta,\zeta} h(t)\Big)\Big(K_q^{\gamma,\delta,\zeta}\, h(t) - m\Big)$$
$$- K_q^{\alpha,\beta,\eta}\,(M - h(t))\,(h(t) - m) - K_q^{\gamma,\delta,\zeta}\,(M - h(t))\,(h(t) - m)\,, \tag{2.8}$$

for all $\alpha, \gamma > 0$ *and* $\beta,\ \eta, \delta, \zeta \in \mathbb{R}$ *with* $\alpha+\beta > 0, \gamma+\delta > 0$ *and* $\eta, \zeta < 0$.

Proof. Multiplying both sides of (2.3) by

$$\frac{t^{-\delta-1}}{\Gamma_q(\gamma)}\,(q\,v/t;q)_{\gamma-1}\sum_{n=0}^{\infty}\frac{\left(q^{\gamma+\delta};q\right)_n\,\left(q^{-\zeta};q\right)_n}{(q^{\gamma};q)_n\,(q;q)_n}\cdot q^{(\zeta-\delta)n}\,(-1)^n\,q^{-\binom{n}{2}}\,\left(v/t-1\right)_q^n$$

and taking q-integration of the resulting inequality with respect to v from 0 to t with the aid of Definition 1.2, we get the required result (2.8). This is complete the proof of the Lemma 2.3. □

Definition 2.4. Two functions f and g are said to be synchronous function on $[0,\infty)$ if

$$A(u,v) = (f(u)-f(v))(g(u)-g(v)) \geq 0;\quad u,v \in [0,\infty). \tag{2.9}$$

Theorem 2.5. *Let* $0 < q < 1$, $f, g \in C_\lambda$ *and satisfying the condition* (1.1) *on* $[0,\infty)$. *Then the following inequality holds true*:

$$\left|K_q^{\alpha,\beta,\eta}fg(x) - K_q^{\alpha,\beta,\eta}f(t)K_q^{\alpha,\beta,\eta}g(t)\right| \leq \frac{1}{4}(L-l)(M-m), \tag{2.10}$$

for all $\alpha > 0$, *and* $\beta,\ \eta \in \mathbb{R}$ *with* $\alpha+\beta > 0$ *and* $\eta < 0$.

Proof. Let f and g be two continuous and synchronous functions on $[0,\ \infty)$. Then, for all $u,\ v \in (0,t)$ with $0 < q < 1$, we have

$$A(u,v) = (f(u)-f(v))\,(g(u)-g(v))\ \ (u,v \in [0,t)). \tag{2.11}$$

Multiplying (2.11) by

$$\frac{t^{-2\beta-2}}{\left(\Gamma_q(\alpha)\right)^2}\,(q\,u/t;q)_{\alpha-1}\,(q\,v/t;q)_{\alpha-1}$$
$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\left(q^{\alpha+\beta};q\right)_m\,(q^{-\eta};q)_m}{(q^{\alpha};q)_m\,(q;q)_m}\,\frac{\left(q^{\alpha+\beta};q\right)_n\,(q^{-\eta};q)_n}{(q^{\alpha};q)_n\,(q;q)_n}$$
$$\cdot\, q^{(\eta-\beta)m}\,(-1)^m\,q^{-\binom{m}{2}}\,q^{(\eta-\beta)n}\,(-1)^n\,q^{-\binom{n}{2}}\,\left(u/t-1\right)_q^m\,\left(v/t-1\right)_q^n. \tag{2.12}$$

Integrating (2.12) twice with respect to u and v from 0 to t, we obtain the following result with the help of (1.3) and (1.4)

$$\frac{t^{-2\beta-2}}{\left(\Gamma_q(\alpha)\right)^2}\int_0^t\int_0^t(q\,u/t;q)_{\alpha-1}\,(q\,v/t;q)_{\alpha-1}$$
$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\left(q^{\alpha+\beta};q\right)_m\,(q^{-\eta};q)_m}{(q^{\alpha};q)_m\,(q;q)_m}\,\frac{\left(q^{\alpha+\beta};q\right)_n\,(q^{-\eta};q)_n}{(q^{\alpha};q)_n\,(q;q)_n}$$
$$\cdot\, q^{(\eta-\beta)m}\,(-1)^m\,q^{-\binom{m}{2}}\,q^{(\eta-\beta)n}\,(-1)^n\,q^{-\binom{n}{2}}\,\left(u/t-1\right)_q^m\,\left(v/t-1\right)_q^n.A(u,v)dudv$$
$$= 2K_q^{\alpha,\beta,\eta}fg(t) - 2K_q^{\alpha,\beta,\eta}f(t)K_q^{\alpha,\beta,\eta}g(t). \tag{2.13}$$

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Making use of the well known Cauchy-Schwarz inequality for linear operator [23, Eq. (1.3), p. 296], we find that

$$\begin{aligned}&\left(K_q^{\alpha,\beta,\eta}fg(t)-K_q^{\alpha,\beta,\eta}f(t)K_q^{\alpha,\beta,\eta}g(t)\right)^2\leq\\&\qquad\left(K_q^{\alpha,\beta,\eta}f^2(t)-\left(K_q^{\alpha,\beta,\eta}f(t)\right)^2\right)\left(K_q^{\alpha,\beta,\eta}g^2(t)-\left(K_q^{\alpha,\beta,\eta}g(t)\right)^2\right).\end{aligned}\tag{2.14}$$

Since

$$(L-f(t))\,(f(t)-l)\geq 0 \quad\text{and}\quad (M-g(t))\,(g(t)-m)\geq 0,$$

therefore, we have

$$K_q^{\alpha,\beta,\eta}\,(L-f(t))\,(f(t)-l)\geq 0 \;\;\text{and}\;\; K_q^{\alpha,\beta,\eta}\,(M-g(t))\,(g(t)-m)\geq 0.\tag{2.15}$$

Thus by using Lemma 2.1, we get

$$K_q^{\alpha,\beta,\eta}f^2(t)-\left(K_q^{\alpha,\beta,\eta}f(t)\right)^2\leq\left(L-K_q^{\alpha,\beta,\eta}f(t)\right)\left(K_q^{\alpha,\beta,\eta}f(t)-l\right)\tag{2.16}$$

and

$$K_q^{\alpha,\beta,\eta}g^2(t)-\left(K_q^{\alpha,\beta,\eta}g(t)\right)^2\leq\left(M-K_q^{\alpha,\beta,\eta}g(t)\right)\left(K_q^{\alpha,\beta,\eta}g(t)-m\right).\tag{2.17}$$

Applying (2.16) and (2.17) in to (2.14), Equation (2.14) reduces to the following form:

$$\begin{aligned}&\left(K_q^{\alpha,\beta,\eta}fg(t)-K_q^{\alpha,\beta,\eta}f(t)K_q^{\alpha,\beta,\eta}g(t)\right)^2\leq\\&\quad\left(L-K_q^{\alpha,\beta,\eta}f(t)\right)\left(K_q^{\alpha,\beta,\eta}f(t)-l\right)\left(M-K_q^{\alpha,\beta,\eta}g(t)\right)\left(K_q^{\alpha,\beta,\eta}g(t)-m\right).\end{aligned}\tag{2.18}$$

Applying the well known inequality $4ab\leq(a+b)^2;\quad a,b\in\mathbb{R}$ in the right hand side of the inequality (2.18)and simplifying it, we obtain the required result (2.10). This is complete the proof of Theorem 2.5. □

Theorem 2.6. *Let $0<q<1$, f and g be two synchronous functions on $[0,\infty)$. Then the following inequality holds:*

$$K_q^{\alpha,\beta,\eta}fg(t)\geq K_q^{\alpha,\beta,\eta}f(t)K_q^{\alpha,\beta,\eta}g(t),\tag{2.19}$$

for all $\alpha>0$, and β, $\eta\in\mathbb{R}$ with $\alpha+\beta>0$ and $\eta<0$.

Proof. For the synchronous function f and g, the inequality (2.9) holds for all $u,v\in[0,\infty)$.

This implies that

$$f(u)g(u)-f(v)g(v)\geq f(u)g(v)+f(v)g(u).\tag{2.20}$$

Following the procedure of the Lemma 2.1 for applying the fractional integral $K_q^{\alpha,\beta,\eta}$, after a little simplification, we arrive at the required result (2.19). This completes the proof of Theorem 2.6. □

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Theorem 2.7. *Let $0 < q < 1$ and $f, g \in C_\lambda$. Then, for all $t > 0$ we have:*

$$\begin{aligned}&\left(K_q^{\alpha,\beta,\eta} fg(t) + K_q^{\gamma,\delta,\zeta} fg(t) - K_q^{\alpha,\beta,\eta} f(t) K_q^{\gamma,\delta,\zeta} g(t) - K_q^{\alpha,\beta,\eta} g(t) K_q^{\gamma,\delta,\zeta} f(t)\right)^2\\ &\leq \left\{\left(M - K_q^{\alpha,\beta,\eta} f(t)\right)\left(K_q^{\gamma,\delta,\zeta} f(t) - m\right)\left(M - K_q^{\gamma,\delta,\zeta} f(t)\right)\left(K_q^{\alpha,\beta,\eta} f(t) - m\right)\right\}\\ &\times \left\{\left(L - K_q^{\alpha,\beta,\eta} g(t)\right)\left(K_q^{\gamma,\delta,\zeta} g(t) - l\right)\left(L - K_q^{\gamma,\delta,\zeta} g(t)\right)\left(K_q^{\alpha,\beta,\eta} g(x) - l\right)\right\},\end{aligned} \tag{2.21}$$

for all $\alpha, \gamma > 0$ and $\beta, \eta, \delta, \zeta \in \mathbb{R}$ with $\alpha + \beta > 0, \gamma + \delta > 0$ and $\eta, \zeta < 0$.

Proof. Since

$$(L - f(t))(f(t) - l) \geq 0 \quad \text{and} \quad (M - g(t))(g(t) - m) \geq 0,$$

therefore, we have

$$\begin{aligned}&- K_q^{\alpha,\beta,\eta}(L - f(t))(f(t) - l) - K_q^{\gamma,\delta,\zeta}(L - f(t))(f(t) - l) \geq 0,\\ &\text{and}\\ &- K_q^{\alpha,\beta,\eta}(M - g(t))(g(t) - m) - K_q^{\gamma,\delta,\zeta}(M - g(t))(g(t) - m) \geq 0.\end{aligned} \tag{2.22}$$

Thus by using Lemma 2.3 to f and g, then by using Lemma 2.2 and the formula (2.22), we get the desired result (2.21). $\square$

3 Special Cases and Concluding Remarks

We conclude our present investigation by remarking further that we can present a large number of special cases of our main inequalities in Theorems 2.5, 2.6 and 2.7. Here we give only three examples: Setting $\beta = 0$ in (2.10), (2.19) and $\beta = \delta = 0$ in (2.21), we obtain an interesting inequalities involving Erdélyi-Kober fractional integral operator fractional integral operator as follows.

Corollary 3.1. *Let $0 < q < 1$, $f, g \in C_\lambda$ and satisfying the condition (1.1) on $[0, \infty)$. Then the following inequality holds true:*

$$\left|K_q^{\eta,\alpha} fg(x) - K_q^{\eta,\alpha} f(t) K_q^{\eta,\alpha} g(t)\right| \leq \frac{1}{4}(L - l)(M - m), \tag{3.1}$$

for all $\alpha > 0$, and $\eta \in \mathbb{R}$ with $\eta < 0$.

Corollary 3.2. *Let $0 < q < 1$, f and g be two synchronous functions on $[0, \infty)$. Then the following inequality holds:*

$$K_q^{\eta,\alpha} fg(t) \geq K_q^{\eta,\alpha} f(t) K_q^{\eta,\alpha} g(t), \tag{3.2}$$

for all $\alpha > 0$ and $\eta \in \mathbb{R}$ with $\eta < 0$.

Corollary 3.3. *Let $0 < q < 1$ and $f, g \in C_\lambda$. Then, for all $t > 0$ we have:*

$$\begin{aligned}&\left(K_q^{\eta,\alpha} fg(t) + K_q^{\zeta,\gamma} fg(t) - K_q^{\eta,\alpha} f(t) K_q^{\zeta,\gamma} g(t) - K_q^{\zeta,\gamma} g(t) K_q^{\eta,\alpha} f(t)\right)^2 \\ &\leq \left\{\left(M - K_q^{\eta,\alpha} f(t)\right)\left(K_q^{\zeta,\gamma} f(t) - m\right)\left(M - K_q^{\zeta,\gamma} f(t)\right)\left(K_q^{\eta,\alpha} f(t) - m\right)\right\} \\ &\times \left\{\left(L - K_q^{\eta,\alpha} g(t)\right)\left(K_q^{\zeta,\gamma} g(t) - l\right)\left(L - K_q^{\zeta,\gamma} g(t)\right)\left(K_q^{\eta,\alpha} g(x) - l\right)\right\},\end{aligned} \tag{3.3}$$

for all $\alpha, \gamma > 0$ and $\eta, \zeta \in \mathbb{R}$ with $\eta, \zeta < 0$.

We conclude our present investigation by remarking further that the results obtained here are useful in deriving various fractional integral inequalities involving such relatively more familiar fractional integral operators. For example, if we consider $\beta = -\alpha$ in (2.10) and $\beta = -\alpha$ and $\delta = -\gamma$ in (2.21) and making use of the relation (1.5), Theorems 2.5 and 2.7 provide, respectively, the known fractional q-integral inequalities due to Zhu *et.al.*[27, pp. 6-7, Theorems 1 and 2].

Competing interests

The authors declare that they have no competing interests.

Authors contributions

All authors have equal contributions.

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ULTRA BESSEL SEQUENCES OF SUBSPACES IN HILBERT SPACES

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Abstract. In this paper, we introduce ultra Bessel sequences of subspaces in Hilbert spaces and we establish some new results about ultra Bessel sequences of subspaces and their perturbation.

1. Introduction

Frames for Hilbert spaces were defined by Duffin and Schaeffer [4] in 1952, and were reintroduced and developed in 1986 by Daubechies, Grossmann and Meyer [3].

A sequence $\{f_i\}_{i=1}^{\infty}$ in Hilbert space $\mathcal{H}$ is called a frame for $\mathcal{H}$ if there exist two constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \leq B\|f\|^2 \tag{1.1}$$

for all $f \in \mathcal{H}$. The constants A and B are called frame bounds. If $A = B$ then we call the frame $\{f_i\}_{i=1}^{\infty}$ a tight frame and if $A = B = 1$ then it is called a Parseval frame. If the right-hand inequality of (1.1) holds for all $f \in \mathcal{H}$, then we call $\{f_i\}_{i=1}^{\infty}$ a Bessel sequence for $\mathcal{H}$. See also [2, 6, 7, 9, 10] for more information on frames and Bessel sequences.

In 2008, the concept of ultra Bessel sequences in Hilbert spaces (i.e., Bessel sequences with uniform convergence property) was introduced by Faroughi and Najati [5].

Definition 1.1. Let $\mathcal{H}_0$ be an inner product space. The sequence $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}_0$ is called an ultra Bessel sequence in $\mathcal{H}_0$ if

$$\sup_{\|f\|=1} \sum_{i=n}^{\infty} |\langle f, f_i\rangle|^2 \to 0 \tag{1.2}$$

as $n \to \infty$, i.e., the series $\sum_{i=n}^{\infty} |\langle f, f_i\rangle|^2$ converges uniformly in unit sphere of $\mathcal{H}_0$.

2010 *Mathematics Subject Classification.* Primary 46C99; 42C15.
Keywords: Frame of subspaces; Perturbation; Ultra Bessel sequence of subspaces.
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In 2004, frame of subspaces as a generalization of ordinary frame was introduced and investigated by Casazza and Kutyniok [1].

Definition 1.2. Let $\{v_i\}_{i=1}^{\infty}$ be a sequence of weights, i.e., for all $i \geq 1$, $v_i > 0$. A family of closed subspaces $\{W_i\}_{i=1}^{\infty}$ of a Hilbert space $\mathcal{H}$ is a frame of subspaces with respect to $\{v_i\}_{i=1}^{\infty}$ for $\mathcal{H}$, if there exist constants $0 < C \leq D < \infty$ such that

$$C\|f\|^2 \leq \sum_{i=1}^{\infty} v_i^2 \|\pi_{W_i}(f)\|^2 \leq D\|f\|^2 \qquad f \in \mathcal{H}, \tag{1.3}$$

where π_{W_i} is the orthogonal projection of $\mathcal{H}$ onto W_i. If $v = v_i > 0$ for all $i \geq 1$ in (1.3), then we say that $\{W_i\}_{i=1}^{\infty}$ is a frame of subspaces with respect to v for $\mathcal{H}$.

If in (1.3) the right hand inequality holds for all $f \in \mathcal{H}$, then $\{W_i\}_{i=1}^{\infty}$ is called a Bessel sequence of subspaces with respect to $\{v_i\}_{i=1}^{\infty}$ with Bessel bound D.

Definition 1.3. Let $\mathcal{H}$ be a Hilbert space. For each sequence of subspaces $\{W_i\}_{i=1}^{\infty}$ of $\mathcal{H}$, we define the set

$$\Big(\sum_{i=1}^{\infty} \oplus W_i\Big)_{\ell^2} = \big\{\{f_i\}_{i=1}^{\infty} | f_i \in W_i, \quad \sum_{i=1}^{\infty} \|f_i\|^2 < \infty\big\}.$$

It is clear that $\Big(\sum_{i=1}^{\infty} \oplus W_i\Big)_{\ell^2}$ is a Hilbert space with the point wise operations and with the inner product given by

$$\langle \{f_i\}_{i=1}^{\infty}, \{g_i\}_{i=1}^{\infty}\rangle = \sum_{i=1}^{\infty} \langle f_i, g_i\rangle.$$

It is proved in [1] that if $\{W_i\}_{i=1}^{\infty}$ is a frame of subspaces with respect to $\{v_i\}_{i=1}^{\infty}$ for $\mathcal{H}$, then the operator

$$T : \Big(\sum_{i=1}^{\infty} \oplus W_i\Big)_{\ell^2} \to \mathcal{H}, \quad T(\{f_i\}_{i=1}^{\infty}) = \sum_{i=1}^{\infty} v_i f_i$$

is bounded and onto and its adjoint is

$$T^* : \mathcal{H} \to \Big(\sum_{i=1}^{\infty} \oplus W_i\Big)_{\ell^2}, \quad T^*(f) = \{v_i \pi_{W_i}(f)\}_{i=1}^{\infty}.$$

The operators T and T^* are called the synthesis and analysis operators for $\{W_i\}_{i=1}^{\infty}$ and $\{v_i\}_{i=1}^{\infty}$, respectively.

Also, it is proved in [1] that if $\{W_i\}_{i=1}^{\infty}$ is a frame of subspaces with respect to $\{v_i\}_{i=1}^{\infty}$, the operator

$$S : \mathcal{H} \to \mathcal{H}, \quad S(f) = TT^*(f)$$

is a positive, self-adjoint and invertible operator on $\mathcal{H}$ and we have the reconstruction formula

$$f = \sum_{i=1}^{\infty} v_i^2 S^{-1} \pi_{W_i}(f), \qquad f \in \mathcal{H}.$$

The operator S is called the frame operator for $\{W_i\}_{i=1}^{\infty}$ and $\{v_i\}_{i=1}^{\infty}$.

2. Ultra Bessel sequence of subspaces

In this section we intend to introduce ultra Bessel sequences of subspaces and investigate some of their properties.

Definition 2.1. Let $\mathcal{H}_0$ be an inner product space. Let $\{W_i\}_{i=1}^{\infty}$ be a sequence of closed subspaces of $\mathcal{H}_0$. Then $\{W_i\}_{i=1}^{\infty}$ is called an ultra Bessel sequence of subspaces in $\mathcal{H}_0$, if

$$\sup_{\|f\|=1} \sum_{i=n}^{\infty} v_i^2 \|\pi_{W_i}(f)\|^2 \to 0, \tag{2.1}$$

as $n \to \infty$, i.e., the series $\sum_{i=1}^{\infty} v_i^2 \|\pi_{W_i}(f)\|^2$ converges uniformly in the unit sphere of $\mathcal{H}_0$.

It is clear that each ultra Bessel sequence of subspaces is a Bessel sequence of subspaces.

Example 2.2. (1) Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for some Hilbert space $\mathcal{H}$. We define the subspaces W_i by

$$W_i = span\{e_i\}, \quad i = 1, 2, 3, ...$$

Then $\{W_i\}_{i=1}^{\infty}$ is a Bessel sequence of subspaces with respect to each $v > 0$. since

$$\sum_{i=1}^{\infty} v^2 \|\pi_{W_i}(f)\|^2 = \sum_{i=1}^{\infty} v^2 |\langle f, e_i \rangle|^2 = v^2 \|f\|^2, \quad f \in \mathcal{H}.$$

But $\{W_i\}_{i=1}^{\infty}$ is not an ultra Bessel sequence of subspaces with respect to each $v > 0$. In fact,

$$\sup_{\|f\|=1} \sum_{i=n}^{\infty} v^2 \|\pi_{W_i}(f)\|^2 \geq v^2 \|\pi_{W_n}(e_n)\|^2 = v^2 \|e_n\|^2 = v^2 > 0.$$

(2) If

$$W_i = span\{i^{-1} e_i\},$$

then $\{W_i\}_{i=1}^{\infty}$ is an ultra Bessel sequence of subspaces, since

$$\sup_{\|f\|=1} \sum_{i=n}^{\infty} v^2 \|\pi_{W_i}(f)\|^2 = v^2 \sup_{\|f\|=1} \sum_{i=n}^{\infty} |\langle f, i^{-1} e_i \rangle|^2 \leq v^2 \sum_{i=n}^{\infty} \frac{1}{i^2} \to 0,$$

as $n \to \infty$.

Proposition 2.3. *Let $\{W_i\}_{i=1}^{\infty}$ be a sequence of closed subspaces in Hilbert space $\mathcal{H}$ and $\{v_i\}_{i=1}^{\infty}$ be a sequence of weights such that $\sum_{i=1}^{\infty} v_i^2 < \infty$. Then $\{W_i\}_{i=1}^{\infty}$ is an ultra Bessel sequence of subspaces in $\mathcal{H}$.*

Proof. For each $n \in \mathbb{N}$ and each $f \in \mathcal{H}$ we have

$$\sum_{i=n}^{\infty} v_i^2 \|\pi_{W_i}(f)\|^2 \leq \|f\|^2 \sum_{i=n}^{\infty} v_i^2.$$

So

$$\sup_{\|f\|=1} \sum_{i=n}^{\infty} v_i^2 \|\pi_{W_i}(f)\|^2 \leq \sum_{i=n}^{\infty} v_i^2 \to 0,$$

as $n \to \infty$. □

The converse of Proposition 2.3 is not true in general. We give an example.

Example 2.4. Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$ and put

$$W_i = span\{e_i\}.$$

Then $\{W_i\}_{i=1}^{\infty}$ is an ultra Bessel sequence of subspaces for $\mathcal{H}$ with respect to weights $v_i := \frac{1}{\sqrt{i}}$ for all $i \in \mathbb{N}$. In fact,

$$\begin{aligned}\sup_{\|f\|=1} \sum_{i=n}^{\infty} \frac{1}{i} \|\pi_{W_i}(f)\|^2 &= \sup_{\|f\|=1} \sum_{i=n}^{\infty} \frac{1}{i} |\langle f, e_i \rangle|^2 \\ &\leq \sup_{\|f\|=1} \Big(\sum_{i=n}^{\infty} \frac{|\langle f, e_i \rangle|^2}{i^2} \Big)^{\frac{1}{2}} \Big(\sum_{i=n}^{\infty} |\langle f, e_i \rangle|^2 \Big)^{\frac{1}{2}} \\ &\leq \Big(\sum_{i=n}^{\infty} \frac{1}{i^2} \Big)^{\frac{1}{2}} \to 0,\end{aligned}$$

as $n \to \infty$. But this series $\sum_{i=1}^{\infty} v_i^2 = \sum_{i=1}^{\infty} \frac{1}{i}$ does not converge.

Now, we prove the next simple lemma which has an important role in proof of Theorem 2.6.

Lemma 2.5. *Let $\{W_i\}_{i=1}^{\infty}$ be a Bessel sequence of subspaces with respect to $\{v_i\}_{i=1}^{\infty}$ in Hilbert space $\mathcal{H}$. Then*

$$\sup_{\|f\|\leq 1} \sum_{i=n}^{\infty} v_i^2 \|\pi_{W_i}(f)\|^2 \leq \sup_{\|f\|=1} \sum_{i=n}^{\infty} v_i^2 \|\pi_{W_i}(f)\|^2, \quad n \in \mathbb{N}.$$

Proof. Let $\alpha = \sup_{\|f\|=1} \sum_{i=n}^{\infty} v_i^2 \|\pi_{W_i}(f)\|^2$. It is clear that if $\|f\| = 1$ or $\|f\| = 0$, then

$$\sum_{i=n}^{\infty} v_i^2 \|\pi_{W_i}(f)\|^2 \leq \alpha.$$

So we consider $0 < \|f\| = k < 1$. In this case,

$$\frac{1}{k} \sum_{i=n}^{\infty} v_i^2 \|\pi_{W_i}(f)\|^2 = \sum_{i=n}^{\infty} v_i^2 \|\pi_{W_i}(\frac{f}{k})\|^2 \leq \alpha.$$

Therefore, $\sum_{i=n}^{\infty} v_i^2 \|\pi_{W_i}(f)\|^2 \leq k\alpha < \alpha$, and we conclude that

$$\sup_{\|f\|\leq 1} \sum_{i=n}^{\infty} v_i^2 \|\pi_{W_i}(f)\|^2 \leq \alpha.$$

□

Theorem 2.6. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and $T : \mathcal{H} \to \mathcal{K}$ be a bounded invertible operator. If $\{W_i\}_{i=1}^{\infty}$ is an ultra Bessel sequence of subspaces in $\mathcal{H}$, then $\{T(W_i)\}_{i=1}^{\infty}$ also is an ultra Bessel sequence of subspaces in $\mathcal{K}$.*

Proof. Similar to the proof of Lemma 2.3 of [8] we show that

$$\pi_{TW_i} = \pi_{TW_i} (T^*)^{-1} \pi_{W_i} T^*. \tag{2.2}$$

In fact, if $f \in \mathcal{H}$ then $f = \pi_{W_i} f + g$, for some $g \in W_i^{\perp}$. Therefore

$$(T^*)^{-1} f = (T^*)^{-1} \pi_{W_i} f + (T^*)^{-1} g.$$

On the other hand

$$0 = \langle w, g\rangle = \langle T^{-1}Tw, g\rangle = \langle Tw, (T^{-1})^* g\rangle, \quad w \in W_i.$$

Hence $(T^{-1})^* g = (T^*)^{-1} g \in (TW_i)^{\perp}$ and so for all $f \in \mathcal{H}$,

$$\pi_{TW_i}(T^*)^{-1} f = \pi_{TW_i}(T^*)^{-1}\pi_{W_i} f + \pi_{TW_i}(T^*)^{-1} g = \pi_{TW_i}(T^*)^{-1}\pi_{W_i} f.$$

Now, (2.2) implies that for all $f \in \mathcal{H}$, $\|\pi_{TW_i}(f)\| \leq \|(T^*)^{-1}\| \|\pi_{W_i} T^* f\|$. If $\|f\| = 1$, then by Lemma 2.5

$$\begin{aligned}
\sum_{i=n}^{\infty} v_i^2 \|\pi_{T(W_i)}(f)\|^2 &\leq \|(T^*)^{-1}\|^2 \sum_{i=n}^{\infty} v_i^2 \|\pi_{W_i} T^* f\|^2 \\
&\leq \|(T^*)^{-1}\|^2 \|T^*\|^2 \sum_{i=n}^{\infty} v_i^2 \|\pi_{W_i}(\frac{T^* f}{\|T^*\|})\|^2 \\
&\leq \|(T^*)^{-1}\|^2 \|T^*\|^2 \sup_{\|g\|\leq 1} \sum_{i=n}^{\infty} v_i^2 \|\pi_{W_i}(g)\|^2 \\
&\leq \|(T^*)^{-1}\|^2 \|T^*\|^2 \sup_{\|g\|= 1} \sum_{i=n}^{\infty} v_i^2 \|\pi_{W_i}(g)\|^2,
\end{aligned}$$

as desired. □

Theorem 2.7. *Suppose that $\{W_i\}_{i=1}^{\infty}$ is a sequence of closed subspaces of Hilbert space $\mathcal{H}$. Let V be a dense subset of the unit sphere of $\mathcal{H}$ and*

$$\sup\Big\{ \sum_{i=n}^{\infty} v_i^2 \|\pi_{W_i}(f)\|^2 : f \in V \Big\} \to 0, \tag{2.3}$$

as $n \to \infty$. Then $\{W_i\}_{i=1}^{\infty}$ is an ultra Bessel sequence of subspaces in $\mathcal{H}$.

Proof. Suppose

$$A(n) := \sup\Big\{ \sum_{i=n}^{\infty} v_i^2 \|\pi_{W_i}(f)\|^2 : f \in \mathcal{H} \ \textit{and} \ \|f\| = 1 \Big\}$$

does not tend to zero as $n \to \infty$. Then there exists $\beta > 0$ and a subsequence $\{A(n_k)\}_{k=1}^{\infty}$ of $\{A(n)\}_{n=1}^{\infty}$ such that $A(n_k) > \beta$ for each $k \geq 1$. Hence, for some $f^k \in \mathcal{H}$ with $\|f^k\| = 1$,

$$\sum_{i=n_k}^{\infty} v_i^2 \|\pi_{W_i}(f^k)\|^2 > \beta, \qquad k \geq 1. \tag{2.4}$$

Choose a subspaces $\{f_j^k\}_{j=1}^{\infty}$ of member of V such that $f_j^k \to f^k$ as $j \to \infty$. By (2.3) there exists $k_0 > 0$ such that

$$\sum_{i=n_{k_0}}^{\infty} v_i^2 \|\pi_{W_i}(f_j^{k_0})\|^2 < \beta, \qquad j \geq 1.$$

Also, (2.4) implies that there exists $l \geq n_{k_0}$ such that

$$\sum_{i=n_{k_0}}^{l} v_i^2 \|\pi_{W_i}(f^{k_0})\|^2 > \beta.$$

Since

$$\sum_{i=n_{k_0}}^{l} v_i^2 \|\pi_{W_i}(f_j^{k_0})\|^2 \to \sum_{i=n_{k_0}}^{l} v_i^2 \|\pi_{W_i}(f^{k_0})\|^2$$

as $j \to \infty$, for sufficiently large j, we have

$$\beta > \sum_{i=n_{k_0}}^{\infty} v_i^2 \|\pi_{W_i}(f_j^{k_0})\|^2 \geq \sum_{i=n_{k_0}}^{l} v_i^2 \|\pi_{W_i}(f_j^{k_0})\|^2 > \beta,$$

which is a contradiction. □

3. Perturbation of Ultra Bessel sequence of subspaces

In this section we present some perturbation results for ultra Bessel sequences of subspaces.

Proposition 3.1. *Let $\{W_i\}_{i=1}^{\infty}$ be an ultra Bessel sequence of subspaces in $\mathcal{H}$ with respect to $\{v_i\}_{i=1}^{\infty}$ and $\{\widetilde{W}_i\}_{i=1}^{\infty}$ be a sequence of closed subspaces of $\mathcal{H}$. If there exist $\lambda \geq 0, \mu < 1$ such that*

$$\Big\| \sum_{i\in J} v_i^2 \Big(\pi_{W_i}(f) - \pi_{\widetilde{W}_i}(f)\Big) \Big\| \leq \lambda \Big\| \sum_{i\in J} v_i^2 \pi_{W_i}(f) \Big\| + \mu \Big\| \sum_{i\in J} v_i^2 \pi_{\widetilde{W}_i}(f) \Big\|, \tag{3.1}$$

for all $f \in \mathcal{H}$ and all finite $J \subseteq \mathbb{N}$, then $\{\widetilde{W}_i\}_{i=1}^{\infty}$ is an ultra Bessel sequence of subspaces in $\mathcal{H}$ with respect to $\{v_i\}_{i=1}^{\infty}$.

Proof. We first prove that $\{\widetilde{W}_i\}_{i=1}^{\infty}$ is a Bessel sequence of subspaces. Let J be a finite subset of $\mathbb{N}$. By (3.1) we have

$$\Big\| \sum_{i\in J} v_i^2 \pi_{\widetilde{W}_i}(f) \Big\| \leq \frac{1+\lambda}{1-\mu} \Big\| \sum_{i\in J} v_i^2 \pi_{W_i}(f) \Big\|, \quad f \in \mathcal{H}.$$

Let B be the Bessel bound of $\{W_i\}_{i=1}^{\infty}$. Then

$$\begin{aligned}
\Big\| \sum_{i\in J} v_i^2 \pi_{W_i}(f) \Big\| &= \sup_{\|g\|=1} \Big| \Big\langle \sum_{i\in J} v_i^2 \pi_{W_i}(f), g \Big\rangle \Big| \\
&= \sup_{\|g\|=1} \Big| \sum_{i\in J} v_i^2 \Big\langle \pi_{W_i}(f), \pi_{W_i}(g) \Big\rangle \Big| \\
&\leq \sup_{\|g\|=1} \Big(\sum_{i\in J} v_i^2 \|\pi_{W_i}(g)\|^2 \Big)^{\frac{1}{2}} \Big(\sum_{i\in J} v_i^2 \|\pi_{W_i}(f)\|^2 \Big)^{\frac{1}{2}} \\
&\leq \sqrt{B} \Big(\sum_{i\in J} v_i^2 \|\pi_{W_i}(f)\|^2 \Big)^{\frac{1}{2}}.
\end{aligned}$$

So

$$\Big\| \sum_{i\in J} v_i^2 \pi_{\widetilde{W}_i}(f) \Big\| \leq \frac{1+\lambda}{1-\mu} \sqrt{B} \Big(\sum_{i\in J} v_i^2 \|\pi_{W_i}(f)\|^2 \Big)^{\frac{1}{2}}, \quad f \in \mathcal{H}.$$

Hence $\sum_{i=1}^{\infty} v_i^2 \pi_{\widetilde{W}_i}(f)$ converges unconditionally for all $f \in \mathcal{H}$, and (3.1) holds for all infinite subsets of $\mathbb{N}$. Let us consider

$$\Gamma : \mathcal{H} \to \mathcal{H}, \quad \Gamma(f) = \sum_{i=1}^{\infty} v_i^2 \pi_{\widetilde{W}_i}(f).$$

Then Γ is a well defined bounded operator with $\|\Gamma\| \leq \dfrac{1+\lambda}{1-\mu} B$ and

$$\sum_{i=1}^{\infty} v_i^2 \|\pi_{\widetilde{W}_i}(f)\|^2 \leq \|\Gamma\| \|f\|^2, \quad f \in \mathcal{H}.$$

Therefore $\{\widetilde{W}_i\}_{i=1}^{\infty}$ is a Bessel sequence of subspaces for $\mathcal{H}$ with respect to $\{v_i\}_{i=1}^{\infty}$. Let $f \in \mathcal{H}$ and $\|f\| = 1$. Similar to above we have

$$\begin{aligned}\sum_{i=n}^{\infty} v_i^2 \|\pi_{\widetilde{W}_i} f\|^2 =& \Big\langle \sum_{i=n}^{\infty} v_i^2 \pi_{\widetilde{W}_i}(f), f \Big\rangle \leq \Big\| \sum_{i=n}^{\infty} v_i^2 \pi_{\widetilde{W}_i}(f) \Big\| \\ \leq& \frac{1+\lambda}{1-\mu} \Big\| \sum_{i=n}^{\infty} v_i^2 \pi_{W_i}(f) \Big\| \\ \leq& \frac{1+\lambda}{1-\mu} \sqrt{B} \Big(\sum_{i=n}^{\infty} v_i^2 \|\pi_{W_i}(f)\|^2 \Big)^{\frac{1}{2}} \to 0,\end{aligned}$$

as $n \to \infty$, and this completes the proof. □

Corollary 3.2. *Let $\{W_i\}_{i=1}^{\infty}$ be an ultra Bessel sequence of subspaces in $\mathcal{H}$ with respect to $\{v_i\}_{i=1}^{\infty}$, and $\{\widetilde{W}_i\}_{i=1}^{\infty}$ be a sequence of closed subspaces of $\mathcal{H}$. If there exists $\lambda \geq 0$ such that*

$$\Big\| \sum_{i \in J} v_i^2 \Big(\pi_{W_i}(f) - \pi_{\widetilde{W}_i}(f) \Big) \Big\| \leq \lambda \Big\| \sum_{i \in J} v_i^2 \pi_{W_i}(f) \Big\|,$$

for all $f \in \mathcal{H}$ and all finite $J \subseteq \mathbb{N}$, then $\{\widetilde{W}_i\}_{i=1}^{\infty}$ is an ultra Bessel sequence of subspaces in $\mathcal{H}$ with respect to $\{v_i\}_{i=1}^{\infty}$

Theorem 3.3. *Let $\{W_i\}_{i=1}^{\infty}$ be an ultra Bessel sequence of subspaces in $\mathcal{H}$ with respect to $\{v_i\}_{i=1}^{\infty}$ and $\{\widetilde{W}_i\}_{i=1}^{\infty}$ be a sequence of closed subspaces of $\mathcal{H}$. If there exists $\lambda \geq 0$ such that*

$$\Big\| \sum_{i \in J} v_i (f_i - g_i) \Big\| \leq \lambda \Big\| \sum_{i \in J} v_i f_i \Big\|, \ f_i \in W_i, \ g_i \in \widetilde{W}_i \tag{3.2}$$

for all $f \in \mathcal{H}$ and all finite $J \subseteq \mathbb{N}$, then $\{\widetilde{W}_i\}_{i=1}^{\infty}$ is an ultra Bessel sequence of subspaces in $\mathcal{H}$ with respect to $\{v_i\}_{i=1}^{\infty}$.

Proof. Let J be a finite subset of $\mathbb{N}$. By (3.2) we have

$$\Big\| \sum_{i \in J} v_i g_i \Big\| \leq (1+\lambda) \Big\| \sum_{i \in J} v_i f_i \Big\|, \quad f_i \in W_i, g_i \in \widetilde{W}_i.$$

Let B be the Bessel bound for $\{W_i\}_{i=1}^{\infty}$. We have

$$\begin{aligned}\Big\|\sum_{i\in J} v_i f_i\Big\| &= \sup_{\|g\|=1}\Big|\Big\langle \sum_{i\in J} v_i f_i, g\Big\rangle\Big| \\ &= \sup_{\|g\|=1}\Big|\sum_{i\in J}\Big\langle f_i, v_i\pi_{W_i}(g)\Big\rangle\Big| \\ &\le \sup_{\|g\|=1}\Big(\sum_{i\in J} v_i^2\|\pi_{W_i}(g)\|^2\Big)^{\frac{1}{2}}\Big(\sum_{i\in J}\|f_i\|^2\Big)^{\frac{1}{2}} \\ &\le \sqrt{B}\Big(\sum_{i\in J}\|f_i\|^2\Big)^{\frac{1}{2}}.\end{aligned}$$

So

$$\Big\|\sum_{i\in J} v_i g_i\Big\| \le (1+\lambda)\sqrt{B}\Big(\sum_{i\in J}\|f_i\|^2\Big)^{\frac{1}{2}}, \quad f_i\in W_i,\ g_i\in \widetilde{W}_i.$$

Hence $\sum_{i=1}^{\infty} v_i g_i$ converges unconditionally for all $\{g_i\}_{i=1}^{\infty}\in\Big(\sum_{i=1}^{\infty}\oplus\widetilde{W}_i\Big)_{\ell^2}$, and (3.2) holds for all $\{f_i\}_{i=1}^{\infty}\in\Big(\sum_{i=1}^{\infty}\oplus W_i\Big)_{\ell^2}$ and $\{g_i\}_{i=1}^{\infty}\in\Big(\sum_{i=1}^{\infty}\oplus\widetilde{W}_i\Big)_{\ell^2}$. In this case the operator

$$K:\Big(\sum_{i=1}^{\infty}\oplus\widetilde{W}_i\Big)_{\ell^2}\to\mathcal{H},\quad K(\{g_i\}_{i=1}^{\infty})=\sum_{i=1}^{\infty} v_i g_i$$

is well defined and bounded with $\|K\|\le(1+\lambda)\sqrt{B}$ and its adjoint

$$K^*:\Big(\sum_{i=1}^{\infty}\oplus\widetilde{W}_i\Big)_{\ell^2}\to\mathcal{H},\quad K^*(f)=\{v_i\pi_{\widetilde{W}_i}(f)\}_{i=1}^{\infty}$$

will be bounded and

$$\sum_{i=1}^{\infty} v_i^2\|\pi_{\widetilde{W}_i}(f)\|^2\le\|K^*\|\|f\|^2,\quad f\in\mathcal{H}.$$

Therefore $\{\widetilde{W}_i\}_{i=1}^{\infty}$ is a Bessel sequence of subspaces for $\mathcal{H}$ with respect to $\{v_i\}_{i=1}^{\infty}$. Let $f\in\mathcal{H}$ and $\|f\|=1$. Then

$$\{v_i\pi_{W_i}(f)\}_{i=1}^{\infty}\in\Big(\sum_{i=1}^{\infty}\oplus W_i\Big)_{\ell^2},\ \{v_i\pi_{\widetilde{W}_i}(f)\}_{i=1}^{\infty}\in\Big(\sum_{i=1}^{\infty}\oplus\widetilde{W}_i\Big)_{\ell^2},$$

and

$$\begin{aligned}\sum_{i=n}^{\infty} v_i^2\|\pi_{\widetilde{W}_i}(f)\|^2 &\le \Big\|\sum_{i=n}^{\infty} v_i^2\pi_{\widetilde{W}_i}(f)\Big\| \\ &\le \Big\|\sum_{i=n}^{\infty} v_i^2\Big(\pi_{\widetilde{W}_i}(f)-\pi_{W_i}(f)\Big)\Big\|+\Big\|\sum_{i=n}^{\infty} v_i^2\pi_{W_i}f\Big\| \\ &\le (1+\lambda)\Big\|\sum_{i=n}^{\infty} v_i^2\pi_{W_i}(f)\Big\| \\ &\le (1+\lambda)\sqrt{B}\Big(\sum_{i=n}^{\infty} v_i^2\|\pi_{W_i}(f)\|^2\Big)^{\frac{1}{2}}\end{aligned}$$

and this completes the proof. □

Proposition 3.4. *Let $\{W_i\}_{i=1}^{\infty}$ be an ultra Bessel sequence of subspaces in $\mathcal{H}$ with respect to $\{v_i\}_{i=1}^{\infty}$ and $\{\widetilde{W_i}\}_{i=1}^{\infty}$ be a sequence of closed subspaces of $\mathcal{H}$. If there exist $M > 0$ such that*

$$\sum_{i\in J} v_i^2 \|\pi_{W_i}(f) - \pi_{\widetilde{W_i}}(f)\|^2 \leq \sum_{i\in J} v_i^2 \|\pi_{W_i}(f)\|^2, \quad f \in \mathcal{H},$$

all finite $J \subseteq \mathbb{N}$, then $\{\widetilde{W_i}\}_{i=1}^{\infty}$ is an ultra Bessel sequence of subspaces in $\mathcal{H}$ with respect to $\{v_i\}_{i=1}^{\infty}$.

Proof. For $n \in \mathbb{N}$ and for $f \in \mathcal{H}$ we have

$$\begin{aligned}\sum_{i=n}^{\infty} v_i^2 \|\pi_{\widetilde{W_i}}(f)\|^2 &\leq 2\sum_{i=n}^{\infty} v_i^2 \|\pi_{W_i}(f) - \pi_{\widetilde{W_i}}(f)\|^2 + 2\sum_{i=n}^{\infty} v_i^2 \|\pi_{W_i}(f)\|^2 \\ &\leq 2(M+1)\sum_{i=n}^{\infty} v_i^2 \|\pi_{W_i}(f)\|^2,\end{aligned}$$

as desired. □

Acknowledgments

C. Park was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2012R1A1A2004299), and D. Y. Shin was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2010-0021792)

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 19, NO.5, 2015, COPYRIGHT 2015 EUDOXUS PRESS, LLC

On Fixed Point Generalizations to Partial b-metric Spaces

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Abstract

We show that many of the recent proved fixed point results in partial $b-$metric spaces can be obtained by the corresponding ones in $b-$metric spaces by introducing a simple method. On the other hand, we prove point out that, in general, our method is not applicable for certain partial b-metric fixed point theorems.

1 Introduction and Preliminaries

Starting from the Banach contraction principle huge number of fixed point theorem generalisations have appeared in literature (see [20, 21, 22, 23] and the references therein). Such generalisations required to work on more general metric type spaces such as partial metric spaces (see [14, 15, 16, 17, 18, 24, 25, 26, 27, 28] and the references therein), $G-$metric spaces [13], cone metric spaces (see [14, 9, 10] and the references therein). On the other hand and after then, several articles have been published where simple methods were presented to reobtain the fixed point theorems in these general metric type spaces by using corresponding ones in metric spaces. However, those simple methods have failed in some cases and hence it remained of interest to work is such spaces. Sometimes, these spaces have been provided

with a partial ordering (see [29] and the references therein). In this article, We show that many of the recent proved fixed point results in partial $b-$metric spaces can be obtained by the corresponding ones in $b-$metric spaces by introducing a simple method. On the other hand, we conclude that our method is not applicable for certain partial b-metric fixed point theorems. The details will be presented inside the main result and final conclusions sections.

We recall some definitions of partial metric spaces, $b-$metric spaces and partial $b-$metric spaces and state some of their properties.

A partial metric space (PMS) is a pair (X, p), where $p : X \times X \to \mathbb{R}^+$ and $\mathbb{R}^+$ denotes the set of all non negative real numbers, such that

(P1) $p(x, y) = p(y, x)$ (symmetry);

(P2) If $p(x, x) = p(x, y) = p(y, y)$ then $x = y$ (equality);

(P3) $p(x, x) \leq p(x, y)$ (small self–distances);

(P4) $p(x, z) + p(y, y) \leq p(x, y) + p(y, z)$ (triangularity);

for all $x, y, z \in X$.

For a partial metric p on X, the function $p^s : X \times X \to \mathbb{R}^+$ given by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y) \tag{1}$$

is a (usual) metric on X. Each partial metric p on X generates a T_0 topology τ_p on X with a base of the family of open p-balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Definition 1. [24]

(i) *A sequence $\{x_n\}$ in a PMS (X, p) converges to $x \in X$ if and only if $p(x, x) = \lim_{n\to\infty} p(x, x_n)$.*

(ii) *A sequence $\{x_n\}$ in a PMS (X, p) is called Cauchy if and only if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists (and finite).*

(iii) *A PMS (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.*

(iv) *A mapping $T : X \to X$ is said to be continuous at $x_0 \in X$, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $T(B_p(x_0, \delta)) \subset B_p(T(x_0), \varepsilon)$.*

Lemma 1.1. [24]

(a1) *A sequence $\{x_n\}$ is Cauchy in a PMS (X, p) if and only if $\{x_n\}$ is Cauchy in a metric space (X, p^s).*

(a2) *A PMS (X, p) is complete if and only if the metric space (X, p^s) is complete. Moreover,*

$$\lim_{n\to\infty} p^s(x, x_n) = 0 \Leftrightarrow p(x, x) = \lim_{n\to\infty} p(x, x_n) = \lim_{n,m\to\infty} p(x_n, x_m). \tag{2}$$

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Definition 2. *[5] Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $d_b : X \times X \to \mathbb{R}^+$ is a $b-$metric if, for all $x, y, z \in X$, the following conditions hold:*

- *(b_1) $x = y$ if and only if $d_b(x, y) = 0$,*
- *$b_2)$ $d_b(x, y) = d_b(y, x)$,*
- *(b_3) $d_b(x, y) \leq s[d_b(x, z) + d_b(z, y)]$.*

The pair (X, d_b) is called a $b-$metric space.

Example 1. *Let (X, d) be a cone metric space over a closed normal cone P with constant $s \geq 1$, in a real Banach space $(E, \|.\|)$. Then $d_b(x, y) = \|d(x, y)\|$, for all $x, y \in X$, defines a $b-$metric on X.*

Definition 3. *[1] Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $p_b : X \times X \to \mathbb{R}^+$ is a partial $b-$metric if, for all $x, y, z \in X$, the following conditions are satisfied:*

- *(p_{b_1}) $x = y$ if and only if $p_b(x, y) = p_b(x, x) = p_b(y, y)$,*
- *(p_{b_2}) $p_b(x, x) \leq p_b(x, y)$,*
- *(p_{b_3}) $p_b(x, y) = p_b(y, x)$,*
- *(p_{b_4}) $p_b(x, y) \leq s[p_b(x, z) + p_b(z, y) - p_b(z, z)] - \frac{1-s}{2}[p_b(x, x) + p_b(y, y)]$.*

The pair (X, p_b) is called a partial $b-$metric space.

Since $s \geq 1$, from (p_{b_4}) we have

$$p_b(x, y) \leq s[p_b(x, z) + p_b(z, y) - p_b(z, z)] \leq s[p_b(x, z) + p_b(z, y)] - p_b(z, z).$$

Proposition 1.2. *[1] Every partial $b-$metric p_b defines a $b-$metric d_{p_b}, where*

$$d_{p_b}(x, y) = 2p_b(x, y) - p_b(x, x) - p_b(y, y),$$

for all $x, y \in X$.

For more information about partial $b-$metric spaces and some fixed point theorems, see [1, 3].

Let $(X, \preceq, d_b)$ be an ordered $b-$metric space and let $f : X \to X$ be a mapping. Set

$$M_s(x, y) = \max\{d_b(x, y), d_b(x, fx), d_b(y, fy), \frac{1}{2s}[d_b(x, fy) + d_b(y, fx)]\} \tag{3}$$

and

$$N_{d_b}(x, y) = \min\{d_b(x, fx), d_b(x, fy)\}. \tag{4}$$

Definition 4. *[6]A function $\psi : [0, \infty) \to [0, \infty)$ is called an alternating distance function if the following properties are satisfied:*

- *ψ is continuous and nondecreasing, and*
- *$\psi(t) = 0$ if and only if $t = 0$.*

Definition 5. *[2] Let (X, d_b) be a $b-$metric space. We say that a mapping $f : X \to X$ is an almost generalized $(\psi, \varphi)_s-$contractive mapping if there exist $L > 0$ and two altering distance functions ψ and ϕ such that*

$$\psi(sd_b(fx, fy)) \leq \psi(M_s(x, y)) - \varphi(M_s(x, y)) + L\psi(N_{d_b}(x, y)), \tag{5}$$

for all $x, y \in X$.

The following is Theorem 3 in [2].

Theorem 1.3. *[2] Let $(X, d_b, \preceq)$ be a partially ordered complete b-metric space. Let $f : X \to X$ be a non-decreasing continuous mapping with respect to $\preceq$. Suppose that f satisfies condition 5, for all comparable elements $x, y \in X$. If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.*

2 Main result

In this section we prove the following two useful lemmas that allow us to prove partial b-metric fixed point results by $b-$metric ones.

Lemma 2.1. *Let (X, p_b) be a partial $b-$metric space. Then the function $d_b : X \times X \to [0, \infty)$ defined by*

$$d_b(x, y) = \{ \begin{array}{cc} 0 & \textit{if } x = y \\ p_b(x, y) & \textit{if } x \neq y, \end{array} \tag{6}$$

is a b-metric on X such that $\tau_{d_{p_b}} \subseteq \tau_{d_b}$. Moreover, (X, d_b) is complete if and only if (X, p_b) is $0-$complete.

Proof. It is clear that $d_b(x, y) = 0$ if and only if $x = y$ and $d_b(x, y) = d_b(y, x)$ for all $x, y \in X$. To prove the triangle inequality, let $x, y, z \in X$. Then $d_b(x, y) \leq p_b(x, y) \leq s[p_b(x, z) + p_b(y, z) - p_b(z, z)]$ and the following cases are observed

- if $x \neq y$ and $x = z$ then $d_b(x, y) = p_b(x, y) \leq s[p_b(z, z) + p_b(z, y) - p_b(z, z)] = d_b(z, y) = d_b(x, y)$.
- if $x \neq y$ and $y = z$ then $d_b(x, y) = p_b(x, y) \leq s[p_b(z, z) + p_b(x, z) - p_b(z, z)] = d_b(x, z) = d_b(x, y)$.
- if $x = y$ then $d_b(x, y) = 0 \leq s[d_b(x, z) + d_b(z, y)]$.

Thus (x, d_b) is a $b-$metric space. The other parts are similar to those in Proposition 2.1 in [4]. □

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Let (X, p_b) be a partial metric space and (x, d_p) be its corresponding $b-$metric space, as defined in Lemma 2.1, and $x, y \in X$. Define

$$M_{d_b}(x,y) = \max\{d_p(x,y), d_b(x,Tx), d_b(y,Ty), \frac{1}{2s}[d_b(x,Ty) + d_b(y,Tx)]\} \tag{7}$$

and

$$M_{p_b}(x,y) = \max\{p_b(x,y), p_b(x,Tx), p_b(y,Ty), \frac{1}{2s}[p_b(x,Ty) + p_b(y,Tx)]\}. \tag{8}$$

Lemma 2.2. *$M_{p_b}(x,y) = M_{d_b}(x,y)$ for all $x, y \in X$ with $x \neq y$.*

Proof. First, we show that $M_{p_b}(x,y) \leq M_{d_b}(x,y)$ for all $x, y \in X$ with $x \neq y$. The cases that $M_{p_b}(x,y) = p_b(x,y)$ or $M_{p_b}(x,y) = p_b(x,Tx)$ or $M_{p_b}(x,y) = p_b(y,Ty)$ can be easily proved by using that $p_b(x,x) \leq p(x,y)$ as done in Lemma 2.2 in [4]. If $M_{p_b}(x,y) = \frac{1}{2s}[p_b(x,Ty) + p_b(y,Tx)]$ then the we have the following cases

- if $x = Ty$ and $y = Tx$, then $M_{p_b}(x,y) = \frac{1}{2s}[p_b(x,x) + p_b(y,y)] \leq \frac{1}{s}p_b(x,y) \leq d_b(x,y) \leq M_{d_b}(x,y)$.
- if $x = Ty$ and $y \neq Tx$, then by the help of triangle inequality in (X, p_b) we have $M_{p_b}(x,y) = \frac{1}{2s}[p_b(x,x) + p_b(y,Tx)] \leq \frac{1}{2s}[sp_b(x,x) + p_b(y,Tx)] \leq \frac{1}{2}[p_b(x,y) + p_b(x,Tx)]$ which is $= \frac{1}{2}[d_b(x,y) + d_b(x,Tx)]$ if $Tx \neq x$ or $\leq p_b(x,y) = d_b(x,y)$ if $Tx = x$. In both cases we conclude that $M_{p_b}(x,y) \leq M_{d_b}(x,y)$.
- if $x \neq Ty$ and $y = Tx$, then by the help of triangle inequality in (X, p_b) we have $M_{p_b}(x,y) = \frac{1}{2s}[p_b(x,Ty) + p_b(y,y)] \leq \frac{1}{2s}[p_b(x,Ty) + sp_b(y,y)] \leq \frac{1}{2}[p_b(x,y) + p_b(y,Ty)]$ which is $= \frac{1}{2}[d_b(x,y) + d_b(y,Ty)]$ if $Ty \neq y$ or $\leq p_b(x,y) = d_b(x,y)$ if $Ty = y$. In both cases we conclude that $M_{p_b}(x,y) \leq M_{d_b}(x,y)$.

The other way inequality "$M_{d_b}(x,y) \leq M_{p_b}(x,y)$" is direct and similar to that in the proof of Lemma 2.2 in [4] or follows by noting that $d_b(x,y) \leq p_b(x,y)$ for all $x, y \in X$. In closing, assume that $x \neq Ty$ and $y \neq Tx$. It is not difficult to see that $p_b(x,y) = d_b(x,y)$ for all $x \neq y$. Hence, $M_{p_b}(x,y) = M_{d_b}(x,y)$ as desired. $\square$

Definition 6. *Let (X, p_b) be an ordered partial b-metric space. We say that the mapping $T : X \to X$ is a generalized $(\psi, \varphi)_s$-weakly contractive mapping if there exist two altering distance functions ψ and φ such that*

$$\psi(sp_b(Tx,Ty)) \leq \psi(M_{p_b}(x,y)) - \varphi(M_{p_b}(x,y))$$

for all comparable $x, y \in X$.

Next theorem was proved in [1], but we give a much shorter proof using our previous two lemmas.

Theorem 2.3. *Let $(X, \preceq, p_b)$ be a p_b-complete ordered partial b-metric space. Let $T : X \to X$ be a nondecreasing, with respect to $\preceq$, continuous mapping. Suppose that T is a generalized $(\psi, \varphi)_s$-weakly contractive mapping. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point.*

Proof. Using the facts that $d_b(x,y) \le p_b(x,y)$ and the results of Lemma 2.2 we obtain:

$$\begin{aligned}\psi(sd_b(x,y)) &\le \psi(sp_b(x,y))\\ &\le \psi(M_{p_b}(x,y)) - \varphi(M_{p_b}(x,y))\\ &\le \psi(M_{p_b}(x,y)) - \varphi(M_{p_b}(x,y)) + L\psi(N_{d_b}(x,y))\\ &= \psi(N_{d_b}(x,y)) - \varphi(M_{d_b}(x,y)) + L\psi(N_{d_b}(x,y)).\end{aligned}$$

Therefore, all the hypothesis of Theorem1.3, are satisfied. Thus, T has a fixed point as required. □

In our next theorem we prove the results of Theorem 2.3, by replacing the continuity condition.

Theorem 2.4. *Let $(X, \preceq, p_b)$ be a p_b-complete ordered partial b-metric space. Let $T : X \to X$ be a nondecreasing, with respect to $\preceq$, and whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x \in X$, we have $x_n \preceq x$ for all natural numbers n. Suppose that T is a generalized $(\psi,\varphi)_s$-weakly contractive mapping. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point.*

Proof. Similarly to the argument in the proof of Theorem 2.3, we have

$$\psi(d_b(x,y)) \le \psi(M_{d_b}(x,y)) - \varphi(M_{d_b}(x,y)) + L\psi(N_{d_b}(x,y)).$$

Hence, all the hypothesis of Theorem 4 in [2], are satisfied. Thus, T has a fixed point as desired. □

Next, we give the following corollary.

Corollary 2.1. *Let $(X, \preceq, p_b)$ be a p_b-complete ordered partial b-metric space. Let $T : X \to X$ be a nondecreasing, with respect to $\preceq$, continuous mapping. Suppose there exists $k \in [0,1)$ such that*

$$p_b(Tx,Ty) \le \frac{k}{2}\max\{p_b(x,y), p_b(x,Tx), p_b(y,Ty), \frac{p_b(x,Tx)+p_b(y,Ty)}{2s}\}$$

for all comparable elements $x, y \in X$. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point.

Proof. If we take $\psi(t) = t$ and $\varphi(t) = (1-k)t$ for all $t \in [0,\infty)$, then the result follows from Theorem 2.3. □

Now, we state this trivial corollary.

Corollary 2.2. *Assume all the hypothesis of Corollary 2.1 without the continuity condition, and suppose that whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x \in X$, we have $x_n \preceq x$ for all natural numbers n. Then T has a fixed point in X.*

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Remark 1. *Notice that the equality*

$$N_{d_b}(x,y) = \min\{d_b(x,fx), d_b(x,fy)\} \le N_{p_b}(x,y) = \min\{p_b(x,fx), p_b(x,fy)\} \tag{9}$$

does not hold in general. For example, if $fy = x$ then $N_{d_b}(x,y) = 0 \neq N_{p_b}(x,y)$ in general. Take $\mathbb{R}^+$ with $p_b(x,y) = \max\{x,y\}$. Thus, our method can not be applied to generalise Theorem 1.3 from partially ordered b-metric spaces to partially ordered partial b-metric spaces. It would be interesting to go though this generalisation.

3 Final Conclusion

In the literature of fixed point theory many fixed point theorems have been generalized from the metric space to more general structures of metric spaces such as cone metric spaces, partial metric spaces, cone b-metric spaces, $G-$metric spaces and partial $b-$metric spaces. After then, some authors have started to develop easier techniques to reprove these generalisations by using the corresponding fixed point theorems in metric or $b-$metric spaces (see [4, 7, 8, 9, 10, 11, 12, 13]). However, those techniques fail to work in some cases and there still some fixed point theorem of special interest in these general metric space structures (see for example [14, 15, 16, 17, 18, 19]). Along with the above, In this article, we have introduced a method to reprove some fixed point theorems in partially ordered $b-$ metric spaces [1] using the corresponding ones in partially ordered $b-$metric spaces [2] and in Remark 1 we mentioned that the partially ordered $b-$metric fixed point theorems can not be generalised to partially ordered $b-$metric spaces by simply applying our method. In the same direction, it would be interesting to generalise the contraction principles in [15] or more generally in [16] from partial metric spaces to partial $b-$metric spaces.

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Existence of solutions for fractional four point boundary value problems with p-Laplacian operator

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June 10, 2014

Abstract

In this paper, we are concerned with proving the existence of solutions for four point fractional differential equations with p-Laplacian operator. We obtain the existence of at least one solution for the problem applying Schauder fixed-point theorem.

Keywords: Fractional differential equations, p-Laplacian operator, Boundary value problem, Existence of solutions.

1 Introduction

Fractional differential equations have been of greet interest due to its numerous applications in physics, chemistry, engineering, economics and other fields. We can find many applications in viscoelasticity, electrochemistry, systems control theory, electromagnetism, bioscience and diffusion processes etc. There has been a significant development in the study of fractional differential equations in recent years [5, 6, 7, 8, 9, 10, 16]. On the other hand, p-Laplacian problems with two-point, three-point and multi-point boundary value conditions for ordinary differential equations and finite difference equations have been studied extensively, see [17, 18, 19] and the references therein. Besides them, integer order p-Laplacian boundary value problems have been studied to its importance in theory and application of mathematics, physics and so on, see for example [11, 12, 13, 14, 15] and references therein. However, there are few articles dealing with the existence of solutions to boundary value problems for fractional differential equation with p-Laplacian operator.

In [3], applying Krasnosel'skii and Leggett-Williams fixed point theorems, the authors obtained some sufficient conditions for the existence of positive solutions for three point boundary value problems

$$D_{0^+}^{\beta}(\varphi_p(D_{0^+}^{\alpha}u))(t) + f(t,u(t)) = 0, \quad t \in (0,1),$$

$$u(0) = 0, \quad u(1) = au(\xi), \quad D_{0^+}^{\alpha}u(0) = 0.$$

In [1], applying the Schauder fixed point theorem, the authors investigated the existence of solutions for a system of nonlinear fractional differential equations with three point boundary conditions

$$D^{\alpha}u(t) = f(t, v(t), D^{p}v(t)), \quad t \in (0,1),$$
$$D^{\beta}v(t) = f(t, u(t), D^{q}u(t)), \quad t \in (0,1),$$
$$u(0) = 0, \quad u(1) = \gamma u(\eta), \quad v(0) = 0, \quad v(1) = \gamma v(\eta),$$

where $1 < \alpha, \beta < 2, p, q, \gamma > 0, 0 < \eta < 1, \alpha - p \geq 1, \beta - q \geq 1, \gamma\eta^{\alpha-1} < 1, \gamma\eta^{\beta-1} < 1, f, g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.

In [4], by using upper and lower solutions method, the authors investigated the existence of positive solutions for nonlocal four point problem.

$$D_{0^+}^{\beta}(\varphi_p(D_{0^+}^{\alpha}u))(t) + f(t, u(t)) = 0, \quad t \in (0,1),$$
$$u(0) = 0, \quad u(1) = au(\xi), \quad D_{0^+}^{\alpha}u(0) = 0, \quad D_{0^+}^{\alpha}u(1) = bD_{0^+}^{\alpha}u(\eta)$$

In this paper, we consider the existence of at least one solution for the following four point fractional differential equations with p-Laplacian operator

$$\begin{cases} D_{0^+}^{\beta}(\varphi_p(D_{0^+}^{\alpha}u))(t) + f(t, u(t), D_{0^+}^{\gamma}u(t)) = 0, \quad t \in (0,1), \\ u(0) = 0, \quad D_{0^+}^{r_1}u(0) - au(\xi) = 0, \quad D_{0^+}^{r_2}u(1) + bu(\eta) = 0, \quad D_{0^+}^{\alpha}u(0) = 0, \end{cases} \tag{1.1}$$

where $D_{0^+}^{\beta}$, $D_{0^+}^{\alpha}$, $D_{0^+}^{r_1}$, $D_{0^+}^{r_2}$ and $D_{0^+}^{\gamma}$ are the standard Riemann-Liouville derivative with $\alpha \in (2,3]$, $\beta \in (0,1]$,$r_1, r_2 \in (0,1]$, $\alpha - r_1 - 2 \geq 0$, $\alpha - r_2 - 2 \geq 0$, $\alpha - \gamma - 2 \geq 0$, the costants ξ, η, a, b are positive numbers with $0 < \xi \leq \eta < 1$ and $\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-r_2-1)} + b\eta^{\alpha-2} > 0$, φ_p is p-Laplacian operator, i.e $\varphi_p(s) = |s|^{p-2}s$, $p > 1$ such that $(\varphi_p)^{-1} = \varphi_q$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is given continuous function.

In section 2, we present some necessary definitions and preliminary results that will be used to prove our main result. In Section 3, we put forward and prove our main result.

2 Preliminaries

In this section we collect some preliminary definitions and results that will be used in subsequent section. Firstly, for convenience of the reader, we give some definitions and fundamental results of fractional calculus theory.

Definition 2.1. *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : (0,\infty) \to \mathbb{R}$ is given by*

$$I_{a^+}^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}y(s)ds,$$

where Γ denotes the gamma function, provided that the right side integral exists.

Definition 2.2. *The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $y : (0, \infty) \to \mathbb{R}$ is given by*

$$D_{a+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{y(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n = [\alpha] + 1$ with $[\alpha]$ denotes the integer part of α and Γ denotes the gamma function, provided that the right side is pointwise defined on $(0, \infty)$.

Lemma 2.1. *Let $\alpha, \beta > 0$, $y : (0, \infty) \to \mathbb{R}$ is a continuous function and assume that the Riemann-Liouville fractional integral and derivative of y exist , then*

$$I_{0+}^{\alpha} I_{0+}^{\beta} y(t) = I_{0+}^{\alpha+\beta} y(t).$$

Lemma 2.2. *Let $\alpha > 0$. If $y \in C(0,1) \cap L(0,1)$ possesses a fractional derivative of order α that belongs to $C(0,1) \cap L(0,1)$, then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} y(t) = y(t) + c_1 t^{\alpha-1} + c_{21} t^{\alpha-2} + ... c_n t^{\alpha-n},$$

for some $c_i \in \mathbb{R}$, $i = 1, 2, ..., n$, where $n = [\alpha] + 1$.

For finding a solution of the problem (1.1), we first consider the following fractional differential equation

$$D_{0+}^{\alpha} u(t) + h(t) = 0, \quad t \in (0,1),$$
$$u(0) = 0, \quad D_{0+}^{r_1} u(0) - au(\xi) = 0, \quad D_{0+}^{r_2} u(1) + bu(\eta) = 0,$$

where $h \in C([0,1], \mathbb{R})$.

Lemma 2.3. *Given $h(t) \in \mathcal{C}(0,1)$ and $2 < \alpha < 3$. Then the unique solution of*

$$\begin{cases} D_{0+}^{\alpha} u(t) + h(t) = 0, \quad t \in (0,1), \\ u(0) = 0, \quad D_{0+}^{r_1} u(0) - au(\xi) = 0, \quad D_{0+}^{r_2} u(1) + bu(\eta) = 0, \end{cases} \tag{2.2}$$

is given by

$$u(t) = \int_0^1 G(t,s) h(s) ds,$$

where

$$G(t,s) = \begin{cases} G_1(t,s), & t \le s; \\ G_2(t,s), & s \le t, \end{cases} \tag{2.3}$$

$$G_1(t,s) = \begin{cases} \frac{a}{d\Gamma(\alpha)} \{ \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-r_2-1)} (t - \frac{\alpha-1}{\alpha-r_2-1}) + b\eta^{\alpha-2}(t-\eta) \} t^{\alpha-2} (\xi - s)^{\alpha-1} + \\ \frac{a\xi^{\alpha-2}}{d\Gamma(\alpha-r_2)} (\xi - t) t^{\alpha-2} (1-s)^{\alpha-r_2-1} + \frac{ab\xi^{\alpha-2}}{d\Gamma(\alpha)} (\xi - t) t^{\alpha-2} (\eta - s)^{\alpha-1}, & s \le \xi; \\ \frac{a\xi^{\alpha-2}}{d\Gamma(\alpha-r_2)} (\xi - t) t^{\alpha-2} (1-s)^{\alpha-r_2-1} + \frac{ab\xi^{\alpha-2}}{d\Gamma(\alpha)} (\xi - t) t^{\alpha-2} (\eta - s)^{\alpha-1}, & \xi \le s \le \eta; \\ \frac{a\xi^{\alpha-2}}{d\Gamma(\alpha-r_2)} (\xi - t) t^{\alpha-2} (1-s)^{\alpha-r_2-1}, & \eta \le s, \end{cases} \tag{2.4}$$

and

$$G_2(t,s)=\begin{cases} \frac{a}{d\Gamma(\alpha)}\{\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-r_2-1)}(t-\frac{\alpha-1}{\alpha-r_2-1})+b\eta^{\alpha-2}(t-\eta)\}t^{\alpha-2}(\xi-s)^{\alpha-1} & \\ +\frac{a\xi^{\alpha-2}}{d\Gamma(\alpha-r_2)}(\xi-t)t^{\alpha-2}(1-s)^{\alpha-r_2-1}+\frac{ab\xi^{\alpha-2}}{d\Gamma(\alpha)}(\xi-t)t^{\alpha-2}(\eta-s)^{\alpha-1} & \\ -\frac{a}{d\Gamma(\alpha)}(t-s)^{\alpha-1}, & s\le\xi; \\ \frac{a\xi^{\alpha-2}}{d\Gamma(\alpha-r_2)}(\xi-t)t^{\alpha-2}(1-s)^{\alpha-r_2-1}+\frac{ab\xi^{\alpha-2}}{d\Gamma(\alpha)}(\xi-t)t^{\alpha-2}(\eta-s)^{\alpha-1} & \\ -\frac{a}{d\Gamma(\alpha)}(t-s)^{\alpha-1}, & \xi\le s\le\eta; \\ \frac{a\xi^{\alpha-2}}{d\Gamma(\alpha-r_2)}(\xi-t)t^{\alpha-2}(1-s)^{\alpha-r_2-1}-\frac{a}{d\Gamma(\alpha)}(t-s)^{\alpha-1}, & \eta\le s. \end{cases} \tag{2.5}$$

Proof. Applying Lemma 2.2 to reduce the first equation of (2.2) to an equivalent integral equation

$$u(t)=C_1t^{\alpha-1}+C_2t^{\alpha-2}+C_3t^{\alpha-3}-I_{+}^{\alpha}h(t)$$

for some $C_1,C_2,C_3\in\mathbb{R}$. Hence, the general solution of Eq. (2.2) is

$$u(t)=C_1t^{\alpha-1}+C_2t^{\alpha-2}+C_3t^{\alpha-3}-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}h(s)ds.$$

It follows from $u(0)=0$ that $C_3=0$. By using the second boundary condition $D_{0+}^{r_1}u(0)-au(\xi)=0$ we have

$$a\xi^{\alpha-1}C_1+a\xi^{\alpha-2}C_2=a\frac{1}{\Gamma(\alpha)}\int_0^{\xi}(\xi-s)^{\alpha-1}h(s)ds \tag{2.6}$$

and by using the third boundary condition $D_{0+}^{r_2}u(1)+bu(\eta)=0$ we have

$$\begin{cases} \left\{\frac{\Gamma(\alpha)}{\Gamma(\alpha-r_2)}+b\eta^{\alpha-1}\right\}C_1+\left\{\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-r_2-1)}+b\eta^{\alpha-2}\right\}C_2=\frac{1}{\Gamma(\alpha-r_2)}\int_0^1(1-s)^{\alpha-r_2-1}h(s)ds \\ \qquad +b\frac{1}{\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-1}h(s)ds. \end{cases} \tag{2.7}$$

Solving equation (2.6) and (2.7) we find that

$$\begin{aligned} C_1 &= \frac{1}{d}\left\{\left(\frac{\Gamma(\alpha)}{\Gamma(\alpha-r_2)}+b\eta^{\alpha-2}\right)\frac{a}{\Gamma(\alpha)}\int_0^{\xi}(\xi-s)^{\alpha-1}h(s)ds\right. \\ &\left. -a\xi^{\alpha-2}\left(\frac{1}{\Gamma(\alpha-r_2)}\int_0^1(1-s)^{\alpha-r_2-1}h(s)ds+b\frac{1}{\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-1}h(s)ds\right)\right\} \end{aligned}$$

and

$$\begin{aligned} C_2 &= \frac{1}{d}\left\{a\xi^{\alpha-1}\left(\frac{1}{\Gamma(\alpha-r_2)}\int_0^1(1-s)^{\alpha-r_2-1}h(s)ds+\frac{b}{\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-1}h(s)ds\right)\right. \\ &\left. -\left(\frac{\Gamma(\alpha)}{\Gamma(\alpha-r_2)}+b\eta^{\alpha-1}\right)\frac{a}{\Gamma(\alpha)}\int_0^{\xi}(\xi-s)^{\alpha-1}h(s)ds\right\} \end{aligned}$$

where $d=a\xi^{\alpha-2}\left\{\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-r_2-1)}(\xi-\frac{\alpha-1}{\alpha-r_2-1})+b\eta^{\alpha-2}(\xi-\eta)\right\}$ and it is easy to see that $d<0$.

Thus the unique solution of (2.2) is given by

$$\begin{aligned} u(t) &= \frac{a}{d\Gamma(\alpha)}\left\{\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-r_2-1)}(t-\frac{\alpha-1}{\alpha-r_2-1})+b\eta^{\alpha-2}(t-\eta)\right\}\int_0^\xi t^{\alpha-2}(\xi-s)^{\alpha-1}h(s)ds \\ &\quad +\frac{a\xi^{\alpha-2}}{\Gamma(\alpha-r_2)}(\xi-t)\int_0^1 t^{\alpha-2}(1-s)^{\alpha-r_2-1}h(s)ds+\frac{ab\xi^{\alpha-2}}{d\Gamma(\alpha)}(\xi-t)\int_0^\eta t^{\alpha-2}(\eta-s)^{\alpha-1}h(s)ds \\ &\quad -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds \\ &= \int_0^1 G(t,s)h(s)ds \end{aligned}$$

where $G(t,s)$ is defined by (2.3).

Let $D^{\alpha}_{0+}u=\phi$, $\varphi_p(\phi)=\omega$. By Lemma 2.2, the solution of initial value problem

$$D^{\beta}_{0+}\omega(t)+v(t)=0,\quad t\in(0,1),$$
$$\omega(0)=0,$$

is given by $\omega(t)=c_1t^{\beta-1}-I^{\beta}_{0+}v(t),\quad t\in(0,1)$.

From the boundary condition and $\beta\in(0,1]$, it follows that $c_1=0$, and so

$\omega(t)=-I^{\beta}_{0+}v(t),\quad t\in(0,1)$.

Noting that $D^{\alpha}_{0+}u=\phi$ and $\phi=\varphi_p^{-1}(\omega)=\varphi_q(\omega)$, we know that the solution of (1.1) satisfies

$$\begin{cases} D^{\alpha}_{0+}u(t)=-\varphi_q(I^{\beta}_{0+}v)(t),\quad t\in(0,1), \\ u(0)=0,\quad D^{r_1}_{0+}u(0)-au(\xi)=0,\quad D^{r_2}_{0+}u(1)+bu(\eta)=0. \end{cases} \tag{2.8}$$

Thus the solution of the problem (2.8) can be written as

$$u(t)=\int_0^1 G(t,s)\varphi_q(I^{\beta}_{0+}v)(s)ds,\quad t\in(0,1).$$

Since $v(t)>0$ for $t\in(0,1)$, we have $\varphi_q(I^{\beta}_{0+}v)(s)=(I^{\beta}_{0+}v)^{q-1}(s)$ and so

$$u(t)=\int_0^1 G(t,s)(I^{\beta}_{0+}v)^{q-1}(s)ds=\frac{1}{(\Gamma(\beta))^{q-1}}\int_0^1 G(t,s)(\int_0^s(s-\tau)^{\beta-1}v(\tau)d\tau)^{q-1}ds.$$

So, we have obtained the following lemma.

Lemma 2.4. *Let $f\in\mathcal{C}([0,1]\times[0,+\infty)\times\mathbb{R},[0,+\infty))$. Then BVP (1.1) has a unique solution of the problem (1.1) if and only if u is a soluton of*

$$u(t)=\frac{1}{(\Gamma(\beta))^{q-1}}\int_0^1 G(t,s)\left(\int_0^s(s-\tau)^{\beta-1}f(\tau,u(\tau),D^{\gamma}_{0+}u(\tau))d\tau\right)^{q-1}ds.$$

Proof. The proof can be seen from Lemma 2.3 immediately, so we omit it.

Let $C(J)$ denote the space of all continuous functions defined on $J=[0,1]$. Let $X=\{u(t):u\in C(J)\}$ and $D^{\gamma}_{0+}u\in C(J)$ be a Banach space endowed with the norm $\|u\|=\max_{t\in J}|u(t)|+\max_{t\in J}|D^{\gamma}_{0+}u(t)|$.

Let us define an operator $T : C(J) \to C(J)$ as

$$Tu(t) = \frac{1}{(\Gamma(\beta))^{q-1}} \int_0^1 G(t,s) \left(\int_0^s (s-\tau)^{\beta-1} f(\tau, u(\tau), D_{0+}^{\gamma} u(\tau)) d\tau \right)^{q-1} ds.$$

Because of the continuity of $G(t,s)$ and f, the operator T is continuous. By Lemma 2.4, we know that the fixed point of operator T coincides with the solution of (1.1).

The main result of this paper is as follows:

Theorem 2.1. *Let $f \in \mathcal{C}([0,1] \times [0,+\infty) \times \mathbb{R}, [0,+\infty))$ and assume that there exists a nonnegative function $g(t) \in L(0,1)$ such that*

$$|f(t,x,y)| \leq g(t) + (\epsilon_1 |x|^{\rho_1} + \epsilon_2 |y|^{\rho_2})^{\frac{1}{q-1}}$$

where $\epsilon_1, \epsilon_2 > 0$, $0 < \rho_1, \rho_2 < 1$ and q is a number with $\frac{1}{p} + \frac{1}{q} = 1$ such that $p > 1$ then the problem (1.1) has a solution.

Proof. Define a ball as $B = \{u(t) : \|u(t)\| \leq R, t \in [0,1]\}$, where $R \geq \max\{(3\Lambda\epsilon_1)^{\frac{1}{1-\rho_1}}, (3\Lambda\epsilon_2)^{\frac{1}{1-\rho_2}}, 3\Upsilon\}$,

$$\begin{aligned} \Upsilon : &= \frac{\max\{2^{q-2},1\}}{(\Gamma(\beta))^{q-1}} \left\{ (\Gamma(\beta))^{q-1} \max_{t\in[0,1]} \left\{ \int_0^1 |G(t,s)| (I_{0_+}^{\beta}(g(s)))^{q-1} ds + I_{0_+}^{\alpha} (I_{0_+}^{\beta}(g(t)))^{q-1} \right\} \right. \\ &+ \left. \frac{a\Gamma(\alpha-1)}{|d|\Gamma(\alpha-\gamma-1)\beta^{q-1}} \left\{ e(g(\xi))^{q-1} + \xi^{\alpha-2} \left(\xi + \frac{\alpha-1}{\alpha-\gamma-1} \right) (g(1))^{q-1} + b(g(\eta))^{q-1} \right\} \right\}, \end{aligned}$$

$$\Lambda := \frac{\max\{2^{q-2},1\}}{(\Gamma(\beta+1))^{q-1}} \left\{ \varepsilon + \frac{a\Gamma(\alpha-1)}{|d|\Gamma(\alpha-\gamma-1)\Gamma(\alpha+1)} \left\{ e\xi^{\beta(q-1)} + \xi^{\alpha-2} \left(\xi + \frac{\alpha-1}{\alpha-\gamma-1} \right) (1 + b\eta^{\beta(q-1)}) + 1 \right\} \right\},$$

$$e := (\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-r_2-1)} + b\eta^{\alpha-2}) \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\gamma-1)} + (\frac{\alpha-1}{\alpha-r_2-1} + \eta)$$

and

$$\varepsilon := \left(1 + \frac{\alpha-1}{\alpha-r_2-1}\right) \frac{\eta^{\beta(q-1)+2}}{(\eta-\xi)} + \frac{1}{(\eta-\xi)\left\{ \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-r_2-1)} + b\eta^{\alpha-2} \right\}} \left(1 + b\eta^{\alpha+\beta(q-1)}\right) + 1.$$

Firstly, we will prove that $T : B \to B$.

By using the inequality $(a+b)^{\gamma} \leq \max\{2^{\gamma-1},1\}(a^{\gamma}+b^{\gamma})$, for $a,b \geq 0$, $\gamma > 0$, we get

$$\begin{aligned} |Tu(t)| &\leq \frac{1}{(\Gamma(\beta))^{q-1}} \int_0^1 |G(t,s)| \left(\int_0^s (s-\tau)^{\beta-1} |f(\tau, u(\tau), D_{0+}^{\gamma} u(\tau))| d\tau \right)^{q-1} ds \\ &\leq \frac{1}{(\Gamma(\beta))^{q-1}} \int_0^1 |G(t,s)| \left(\int_0^s (s-\tau)^{\beta-1} (g(\tau) + (\epsilon_1 |R|^{\rho_1} + \epsilon_2 |R|^{\rho_2})^{\frac{1}{q-1}}) d\tau \right)^{q-1} ds \\ &= \frac{1}{(\Gamma(\beta))^{q-1}} \int_0^1 |G(t,s)| \left(\int_0^s (s-\tau)^{\beta-1} g(\tau) d\tau + (\epsilon_1 |R|^{\rho_1} + \epsilon_2 |R|^{\rho_2})^{\frac{1}{q-1}}) \int_0^s (s-\tau)^{\beta-1} d\tau \right)^{q-1} ds \\ &= \frac{1}{(\Gamma(\beta))^{q-1}} \int_0^1 |G(t,s)| \left(\int_0^s (s-\tau)^{\beta-1} g(\tau) d\tau + (\epsilon_1 |R|^{\rho_1} + \epsilon_2 |R|^{\rho_2})^{\frac{1}{q-1}}) \frac{s^{\beta}}{\beta} \right)^{q-1} ds \\ &\leq \frac{\max\{2^{q-2},1\}}{(\Gamma(\beta))^{q-1}} \left\{ \int_0^1 |G(t,s)| \left(\int_0^s (s-\tau)^{\beta-1} g(\tau) d\tau \right)^{q-1} ds \right. \\ &\quad + \left. \frac{(\epsilon_1 |R|^{\rho_1} + \epsilon_2 |R|^{\rho_2})}{\beta^{q-1}} \int_0^1 |G(t,s)| s^{\beta(q-1)} ds \right\}. \end{aligned}$$

We consider the integral as by using $\int_0^t (t-s)^\alpha s^\beta ds = t^{\alpha+\beta+1}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}$,

$$\begin{aligned}\int_0^1 |G(t,s)|s^{\beta(q-1)}ds \quad &\leq \quad \frac{a}{|d|\Gamma(\alpha)}|\{\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-r_2-1)}(t-\frac{\alpha-1}{\alpha-r_2-1})\\ &+b\eta^{\alpha-2}(t-\eta)\}|t^{\alpha-2}\xi^{\alpha+\beta(q-1)}\frac{\Gamma(\alpha)\Gamma(\beta(q-1)+1)}{\Gamma(\alpha+\beta(q-1)+1)}\\ &+\frac{a\xi^{\alpha-2}}{|d|\Gamma(\alpha-r_2)}|\xi-t|t^{\alpha-2}\frac{\Gamma(\alpha-r_2)\Gamma(\beta(q-1)+1)}{\Gamma(\alpha-r_2+\beta(q-1)+1)}\\ &+\frac{ab\xi^{\alpha-2}}{|d|\Gamma(\alpha)}|\xi-t|t^{\alpha-2}\eta^{\alpha+\beta(q-1)}\frac{\Gamma(\alpha)\Gamma(\beta(q-1)+1)}{\Gamma(\alpha+\beta(q-1)+1)}\\ &+\frac{1}{\Gamma(\alpha)}t^{\alpha+\beta(q-1)}\frac{\Gamma(\alpha)\Gamma(\beta(q-1)+1)}{\Gamma(\alpha+\beta(q-1)+1)}.\end{aligned}$$

Since $|d| = a\xi^{\alpha-2}\left\{\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-r_2-1)}(\frac{\alpha-1}{\alpha-r_2-1}-\xi)+b\eta^{\alpha-2}(\eta-\xi)\right\}$ and $\frac{\alpha-1}{\alpha-r_2-1}-\xi > \eta-\xi$, we get

$$|d| > \left|a\xi^{\alpha-2}\left\{\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-r_2-1)}(\eta-\xi)+b\eta^{\alpha-2}(\eta-\xi)\right\}\right| = a\xi^{\alpha-2}(\eta-\xi)\left\{\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-r_2-1)}+b\eta^{\alpha-2}\right\}.$$

Also, since $1<\beta(q-1)+1\leq q$, $3<\beta(q-1)+\alpha+1\leq 3+q$ and $2<\beta(q-1)+\alpha-r_2+1<3+q$ then $\Gamma(\beta(q-1)+1)<\Gamma(\beta(q-1)+\alpha+1)$, $\Gamma(\beta(q-1)+1)<\Gamma(\beta(q-1)+\alpha-r_2+1)$. Together with the above inequalities, we have

$$\int_0^1 |G(t,s)|s^{\beta(q-1)}ds < \left(1+\frac{\alpha-1}{\alpha-r_2-1}\right)\frac{\eta^{\beta(q-1)+2}}{(\eta-\xi)} + \frac{1}{(\eta-\xi)\left\{\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-r_2-1)}+b\eta^{\alpha-2}\right\}}\left(1+b\eta^{\alpha+\beta(q-1)}\right)+1.$$

If we define $\varepsilon := \left(1+\frac{\alpha-1}{\alpha-r_2-1}\right)\frac{\eta^{\beta(q-1)+2}}{(\eta-\xi)} + \frac{1}{(\eta-\xi)\left\{\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-r_2-1)}+b\eta^{\alpha-2}\right\}}\left(1+b\eta^{\alpha+\beta(q-1)}\right)+1$,
we get

$$|Tu(t)| \leq \frac{\max\{2^{q-2},1\}}{(\Gamma(\beta))^{q-1}}\left\{\int_0^1 |G(t,s)|(\Gamma(\beta)I^\beta(g(s)))^{q-1}ds + \frac{(\epsilon_1|R|^{\rho_1}+\epsilon_2|R|^{\rho_2})}{\beta^{q-1}}\times\varepsilon\right\}ds.$$

And also, we get

$$\begin{aligned}|D_{0+}^\gamma Tu(t)| \quad &= \quad |D_{0+}^\gamma\{\frac{1}{(\Gamma(\beta))^{q-1}}\int_0^1 G(t,s)\left(\int_0^s (s-\tau)^{\beta-1}f(\tau,u(\tau),D_{0+}^\gamma u(\tau))d\tau\right)^{q-1}ds\}|\\ &= \quad \frac{1}{(\Gamma(\beta))^{q-1}}\left|D_{0+}^\gamma\left\{\frac{a}{d\Gamma(\alpha)}\left\{\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-r_2-1)}(t-\frac{\alpha-1}{\alpha-r_2-1})+b\eta^{\alpha-2}(t-\eta)\right\}\right.\right.\\ &\int_0^\xi t^{\alpha-2}(\xi-s)^{\alpha-1}\left(\int_0^s (s-\tau)^{\beta-1}f(\tau,u(\tau),D_{0+}^\gamma u(\tau))d\tau\right)^{q-1}ds\\ &+\frac{a\xi^{\alpha-2}}{\Gamma(\alpha-r_2)}(\xi-t)\int_0^1 t^{\alpha-2}(1-s)^{\alpha-r_2-1}\left(\int_0^s (s-\tau)^{\beta-1}f(\tau,u(\tau),D_{0+}^\gamma u(\tau))d\tau\right)^{q-1}ds\\ &+\frac{ab\xi^{\alpha-2}}{d\Gamma(\alpha)}(\xi-t)\int_0^\eta t^{\alpha-2}(\eta-s)^{\alpha-1}\left(\int_0^s (s-\tau)^{\beta-1}f(\tau,u(\tau),D_{0+}^\gamma u(\tau))d\tau\right)^{q-1}ds\\ &\left.\left.-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(\int_0^s (s-\tau)^{\beta-1}f(\tau,u(\tau),D_{0+}^\gamma u(\tau))d\tau\right)^{q-1}ds\right\}\right|\\ &= \quad \frac{1}{(\Gamma(\beta))^{q-1}}\left|\frac{a}{d}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\gamma-1)}t^{\alpha-\gamma-2}\left[\left(\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-r_2-1)}+b\eta^{\alpha-2}\right)\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\gamma-1)}t\right.\right.\\ &-\left.\left(\frac{\alpha-1}{\alpha-r_2-1}+\eta\right)\right]I_{0_+}^\alpha\left(\int_0^\xi(\xi-\tau)^{\beta-1}f(\tau,u(\tau),D_{0+}^\gamma u(\tau))d\tau\right)^{q-1}\end{aligned}$$

$$
\begin{aligned}
&+\frac{a\xi^{\alpha-2}}{d}\left(\xi-\frac{\alpha-1}{\alpha-\gamma-1}t\right)t^{\alpha-\gamma-2}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\gamma-1)}I_{0_+}^{\alpha-r_2}\left(\int_0^1(1-\tau)^{\beta-1}f(\tau,u(\tau),D_{0^+}^{\gamma}u(\tau))d\tau\right)^{q-1}\\
&+\frac{ab\xi^{\alpha-2}}{d}\left(\xi-\frac{\alpha-1}{\alpha-\gamma-1}t\right)t^{\alpha-\gamma-2}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\gamma-1)}I_{0_+}^{\alpha}\left(\int_0^\eta(\eta-\tau)^{\beta-1}f(\tau,u(\tau),D_{0^+}^{\gamma}u(\tau))d\tau\right)^{q-1}\\
&-I_{0_+}^{\alpha}\left(\int_0^t(t-\tau)^{\beta-1}f(\tau,u(\tau),D_{0^+}^{\gamma}u(\tau))d\tau\right)^{q-1}\Bigg|\\
\leq\;&\frac{1}{(\Gamma(\beta))^{q-1}}\left\{\frac{a}{|d|}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\gamma-1)}\left[\left(\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-r_2-1)}+b\eta^{\alpha-2}\right)\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\gamma-1)}\right.\right.\\
&\left.-\left(\frac{\alpha-1}{\alpha-r_2-1}+\eta\right)\right]I_{0_+}^{\alpha}\left(\int_0^\xi(\xi-\tau)^{\beta-1}f(\tau,u(\tau),D_{0^+}^{\gamma}u(\tau))d\tau\right)^{q-1}\\
&+\frac{a\xi^{\alpha-2}}{|d|}\left(\xi-\frac{\alpha-1}{\alpha-\gamma-1}\right)\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\gamma-1)}I_{0_+}^{\alpha-r_2}\left(\int_0^1(1-\tau)^{\beta-1}f(\tau,u(\tau),D_{0^+}^{\gamma}u(\tau))d\tau\right)^{q-1}\\
&+\frac{ab\xi^{\alpha-2}}{|d|}\left(\xi-\frac{\alpha-1}{\alpha-\gamma-1}\right)\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\gamma-1)}I_{0_+}^{\alpha}\left(\int_0^\eta(\eta-\tau)^{\beta-1}f(\tau,u(\tau),D_{0^+}^{\gamma}u(\tau))d\tau\right)^{q-1}\\
&\left.-I_{0_+}^{\alpha}\left(\int_0^t(t-\tau)^{\beta-1}f(\tau,u(\tau),D_{0^+}^{\gamma}u(\tau))d\tau\right)^{q-1}\right\}.
\end{aligned}
$$

By using the properties $I_{0_+}^{\alpha}(f+g)(t)=I_{0_+}^{\alpha}f(t)+I_{0_+}^{\alpha}g(t)$ and if $f\leq g$ then $I_{0_+}^{\alpha}f(t)\leq I_{0_+}^{\alpha}g(t)$, we have

$$
\begin{aligned}
&I_{0_+}^{\alpha}\left(\int_0^\xi(\xi-\tau)^{\beta-1}f(\tau,u(\tau),D_{0^+}^{\gamma}u(\tau))d\tau\right)^{q-1}\\
\leq\;&I_{0_+}^{\alpha}\left(\int_0^\xi(\xi-\tau)^{\beta-1}[g(\tau)+(\epsilon_1|R|^{\rho_1}+\epsilon_2|R|^{\rho_2})^{\frac{1}{q-1}}]d\tau\right)^{q-1}\\
\leq\;&I_{0_+}^{\alpha}\left(\int_0^\xi(\xi-\tau)^{\beta-1}g(\tau)d\tau+(\epsilon_1|R|^{\rho_1}+\epsilon_2|R|^{\rho_2})^{\frac{1}{q-1}}\int_0^\xi(\xi-\tau)^{\beta-1}d\tau\right)^{q-1}\\
\leq\;&\max\{2^{q-2},1\}\left\{I_{0_+}^{\alpha}\left(\int_0^\xi(\xi-\tau)^{\beta-1}g(\tau)d\tau\right)^{q-1}+I_{0_+}^{\alpha}\left((\epsilon_1|R|^{\rho_1}+\epsilon_2|R|^{\rho_2})^{\frac{1}{q-1}}\,d\tau\right)^{q-1}\right\}\\
\leq\;&\max\{2^{q-2},1\}\left\{I_{0_+}^{\alpha}(\Gamma(\beta)I_{0_+}^{\beta}(g(\xi)))^{q-1}+\left((\epsilon_1|R|^{\rho_1}+\epsilon_2|R|^{\rho_2})^{\frac{1}{q-1}}\frac{\xi^{\beta}}{\beta}\right)^{q-1}I_{0_+}^{\alpha}(1)\right\}\\
\leq\;&\max\{2^{q-2},1\}\left\{\Gamma(\beta)^{q-1}I_{0_+}^{\alpha}(I_{0_+}^{\beta}(g(\xi)))^{q-1}+\left((\epsilon_1|R|^{\rho_1}+\epsilon_2|R|^{\rho_2})\frac{\xi^{\beta(q-1)}}{\beta^{q-1}}\right)\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\right\}.
\end{aligned}
$$

If we define $e:=(\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-r_2-1)}+b\eta^{\alpha-2})\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\gamma-1)}+(\frac{\alpha-1}{\alpha-r_2-1}+\eta)$, we get

$$
\begin{aligned}
|D_{0^+}^{\gamma}Tu(t)|\leq\;&\frac{\max\{2^{q-2},1\}}{(\Gamma(\beta))^{q-1}}\{\frac{a}{|d|}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\gamma-1)}e\{\Gamma(\beta)^{q-1}I_{0_+}^{\alpha}(I_{0_+}^{\beta}(g(\xi)))^{q-1}\\
&+((\epsilon_1|R|^{\rho_1}+\epsilon_2|R|^{\rho_2})\frac{\xi^{\beta(q-1)}}{\beta^{q-1}})\frac{t^{\alpha}}{\Gamma(\alpha+1)}\}\}\\
&+\frac{a\xi^{\alpha-2}}{|d|}(\xi+\frac{\alpha-1}{\alpha-\gamma-1})\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\gamma-1)}\{\Gamma(\beta)^{q-1}I_{0_+}^{\alpha}(I_{0_+}^{\beta}(g(1)))^{q-1}\\
&+((\epsilon_1|R|^{\rho_1}+\epsilon_2|R|^{\rho_2})\frac{1}{\beta^{q-1}})\frac{t^{\alpha}}{\Gamma(\alpha+1)}\}
\end{aligned}
$$

$$+\frac{ab\xi^{\alpha-2}}{|d|}(\xi+\frac{\alpha-1}{\alpha-\gamma-1})\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\gamma-1)}\{\Gamma(\beta)^{q-1}I^{\alpha}_{0_+}(I^{\beta}_{0_+}(g(\eta)))^{q-1}$$
$$+((\epsilon_1|R|^{\rho_1}+\epsilon_2|R|^{\rho_2})\frac{\eta^{\beta(q-1)}}{\beta^{q-1}})\frac{t^{\alpha}}{\Gamma(\alpha+1)}\}$$
$$+\{\Gamma(\beta)^{q-1}I^{\alpha}_{0_+}(I^{\beta}_{0_+}(g(t)))^{q-1}+((\epsilon_1|R|^{\rho_1}+\epsilon_2|R|^{\rho_2})I^{\alpha}_{0_+}(\frac{t^{\beta(q-1)}}{\beta^{q-1}}))\}$$

$$\leq \frac{\max\{2^{q-2},1\}}{(\Gamma(\beta))^{q-1}}\{\frac{a}{|d|}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\gamma-1)}e\{(\frac{g(\xi)}{\beta})^{q-1}\frac{\Gamma(\beta(q-1)+1)}{\Gamma(\alpha+\beta(q-1)+1)}t^{\alpha+\beta(q-1)}$$
$$+\frac{1}{\Gamma(\alpha+1)}(\epsilon_1|R|^{\rho_1}+\epsilon_2|R|^{\rho_2})\frac{\xi^{\beta(q-1)}}{\beta^{q-1}}\}\}$$

$$+\frac{a\xi^{\alpha-2}}{|d|}(\xi+\frac{\alpha-1}{\alpha-\gamma-1})\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\gamma-1)}\{(\frac{g(1)}{\beta})^{q-1}\frac{\Gamma(\beta(q-1)+1)}{\Gamma(\alpha-r_2+\beta(q-1)+1)}t^{\alpha-r_2+\beta(q-1)}$$
$$+\frac{1}{\Gamma(\alpha+1)}((\epsilon_1|R|^{\rho_1}+\epsilon_2|R|^{\rho_2})\frac{1}{\beta^{q-1}})\}$$

$$+\frac{ab\xi^{\alpha-2}}{|d|}(\xi+\frac{\alpha-1}{\alpha-\gamma-1})\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\gamma-1)}\{(\frac{g(\eta)}{\beta})^{q-1}\frac{\Gamma(\beta(q-1)+1)}{\Gamma(\alpha+\beta(q-1)+1)}t^{\alpha+\beta(q-1)}$$
$$+\frac{1}{\Gamma(\alpha+1)}((\epsilon_1|R|^{\rho_1}+\epsilon_2|R|^{\rho_2})\frac{\eta^{\beta(q-1)}}{\beta^{q-1}})\}$$
$$+\{\Gamma(\beta)^{q-1}I^{\alpha}_{0_+}(I^{\beta}_{0_+}(g(t)))^{q-1}+((\epsilon_1|R|^{\rho_1}+\epsilon_2|R|^{\rho_2})\frac{1}{\beta^{q-1}}\frac{\Gamma(\beta(q-1)+1)}{\Gamma(\alpha+\beta(q-1)+1)}t^{\alpha+\beta(q-1)})\}.$$

Since $\frac{\Gamma(\beta(q-1)+1)}{\Gamma(\alpha+\beta(q-1)+1)}<1$ and $\frac{\Gamma(\beta(q-1)+1)}{\Gamma(\alpha-r_2+\beta(q-1)+1)}<1$, then we have

$$\begin{aligned}
\leq\ & \frac{\max\{2^{q-2},1\}}{(\Gamma(\beta))^{q-1}}\left\{\frac{a}{|d|}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\gamma-1)}e\left\{(\frac{g(\xi)}{\beta})^{q-1}+\frac{1}{\Gamma(\alpha+1)}\left(\epsilon_1|R|^{\rho_1}+\epsilon_2|R|^{\rho_2}\right)\frac{\xi^{\beta(q-1)}}{\beta^{q-1}}\right\}\right.\\
&+\frac{a\xi^{\alpha-2}}{|d|}\left(\xi+\frac{\alpha-1}{\alpha-\gamma-1}\right)\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\gamma-1)}\left\{(\frac{g(1)}{\beta})^{q-1}+\frac{1}{\Gamma(\alpha+1)}\left(\epsilon_1|R|^{\rho_1}+\epsilon_2|R|^{\rho_2}\right)\frac{1}{\beta^{q-1}}\right\}\\
&+\frac{ab\xi^{\alpha-2}}{|d|}\left(\xi+\frac{\alpha-1}{\alpha-\gamma-1}\right)\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\gamma-1)}\left\{(\frac{g(\eta)}{\beta})^{q-1}+\frac{1}{\Gamma(\alpha+1)}\left(\epsilon_1|R|^{\rho_1}+\epsilon_2|R|^{\rho_2}\right)\frac{\eta^{\beta(q-1)}}{\beta^{q-1}}\right\}\\
&\left.+\Gamma(\beta)^{q-1}I^{\alpha}_{0_+}(I^{\beta}_{0_+}(g(t)))^{q-1}+\left(\epsilon_1|R|^{\rho_1}+\epsilon_2|R|^{\rho_2}\right)\frac{1}{\beta^{q-1}}\right\}
\end{aligned}$$

$$\begin{aligned}
\leq\ & \frac{\max\{2^{q-2},1\}}{(\beta\Gamma(\beta))^{q-1}}\left\{\frac{a\Gamma(\alpha-1)}{|d|\Gamma(\alpha-\gamma-1)}\left\{e(g(\xi))^{q-1}+\xi^{\alpha-2}\left(\xi+\frac{\alpha-1}{\alpha-\gamma-1}\right)(g(1))^{q-1}+b(g(\eta))^{q-1}\right\}\right.\\
&\left.+\Gamma(\beta)^{q-1}I^{\alpha}_{0_+}(I^{\beta}_{0_+}(g(t)))^{q-1}\right\}+\left(\epsilon_1|R|^{\rho_1}+\epsilon_2|R|^{\rho_2}\right)\frac{\max\{2^{q-2},1\}}{(\beta\Gamma(\beta))^{q-1}}\\
&\left\{\frac{a\Gamma(\alpha-1)}{|d|\Gamma(\alpha-\gamma-1)\Gamma(\alpha+1)}\left(e\xi^{\beta(\alpha-1)}+\xi^{\alpha-2}\left(\xi+\frac{\alpha-1}{\alpha-\gamma-1}\right)\left(1+b\eta^{\beta(\alpha-1)}\right)+1\right)\right\}.
\end{aligned}$$

Thus we get

$$\begin{aligned}
\|Tu(t)\| &= \max_{t\in[0,1]}|Tu(t)|+\max_{t\in[0,1]}|D^{\gamma}_{0^+}Tu(t)|\\
&\leq \Upsilon+\left(\epsilon_1|R|^{\rho_1}+\epsilon_2|R|^{\rho_2}\right)\Lambda.
\end{aligned}$$

Since $R \geq \max\{(3\Lambda\epsilon_1)^{\frac{1}{1-\rho_1}}, (3\Lambda\epsilon_2)^{\frac{1}{1-\rho_2}}, 3\Upsilon\}$ then we have

$$\|Tu(t)\| \quad \leq \frac{R}{3} + \frac{R}{3} + \frac{R}{3} = R.$$

Hence, we conclude that $\|Tu(t)\| \leq R$. Since $Tu(t)$ and $D^{\gamma}_{0+}Tu(t)$ are continuous operator on $[0,1]$, therefore $T : B \to B$.

Secondly, we will show that T is completely continuous operator.

For that we fix $M := \max_{t\in[0,1]} |f(t, u(t), D^{\gamma}_{0+})|$. Thus we get

$$\left|\int_0^s (s-\tau)^{\beta-1} |f(\tau, u(\tau), D^{\gamma}_{0+}u(\tau))| d\tau\right|^{q-1} = |\Gamma(\beta) I^{\beta}(f)|^{q-1} \leq \left(\frac{M}{\beta}\right)^{q-1} s^{q-1}.$$

For $t_1, t_2 \in [0,1]$ such that $t_1 < t_2$ and $u \in B$, we have

$$\begin{aligned}
|Tu(t_2) - Tu(t_1)| \quad &\leq \quad \frac{1}{(\Gamma(\beta))^{q-1}} \int_0^1 |G(t_2,s)| \left(\int_0^s (s-\tau)^{\beta-1} |f(\tau, u(\tau), D^{\gamma}_{0+}u(\tau))| d\tau\right)^{q-1} ds \\
&\quad - \frac{1}{(\Gamma(\beta))^{q-1}} \int_0^1 |G(t_1,s)| \left(\int_0^s (s-\tau)^{\beta-1} |f(\tau, u(\tau), D^{\gamma}_{0+}u(\tau))| d\tau\right)^{q-1} ds \\
&\leq \left(\frac{M}{\beta\Gamma(\beta)}\right)^{q-1} \frac{a}{|d|} \Gamma(q) \xi^{\alpha-1} \left\{ \left(\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-r_2-1)} + b\eta^{\alpha-2}\right) \frac{\xi^q}{\Gamma(\alpha+q)} |t_2^{\alpha-1} - t_1^{\alpha-1}| \right. \\
&\quad + \left(\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-r_2)} + b\eta^{\alpha-2}\right) \frac{\xi^q}{\Gamma(\alpha+q)} |t_2^{\alpha-2} - t_1^{\alpha-2}| \\
&\quad + \frac{1}{\Gamma(\alpha-r_2+q)} |t_2^{\alpha-2} - t_1^{\alpha-2}| + \frac{1}{\xi\Gamma(\alpha-r_2+q)} |t_2^{\alpha-1} - t_1^{\alpha-1}| \\
&\quad \left. + \frac{b\eta^{\alpha+q-1}}{\Gamma(\alpha+q)} |t_2^{\alpha-2} - t_1^{\alpha-2}| + \frac{b\eta^{\alpha+q-1}}{\xi\Gamma(\alpha-r_2+q)} |t_2^{\alpha-1} - t_1^{\alpha-1}| \right\} \\
&\quad + \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_2} (t_2-s)^{\alpha-1} \left(\int_0^s (s-\tau)^{\beta-1} f(\tau, u(\tau), D^{\gamma}_{0+}u(\tau)) d\tau\right)^{q-1} ds \right. \\
&\quad \left. - \int_0^{t_1} (t_1-s)^{\alpha-1} \left(\int_0^s (s-\tau)^{\beta-1} f(\tau, u(\tau), D^{\gamma}_{0+}u(\tau)) d\tau\right)^{q-1} ds \right| \\
&\leq \left(\frac{M}{\beta\Gamma(\beta)}\right)^{q-1} \frac{a}{|d|} \Gamma(q) \xi^{\alpha-1} \left\{ \left(\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-r_2-1)} + b\eta^{\alpha-2}\right) \frac{\xi^q}{\Gamma(\alpha+q)} |t_2^{\alpha-1} - t_1^{\alpha-1}| \right. \\
&\quad + \left(\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-r_2)} + b\eta^{\alpha-2}\right) \frac{\xi^q}{\Gamma(\alpha+q)} |t_2^{\alpha-2} - t_1^{\alpha-2}| \\
&\quad + \frac{1}{\Gamma(\alpha-r_2+q)} |t_2^{\alpha-2} - t_1^{\alpha-2}| + \frac{1}{\xi\Gamma(\alpha-r_2+q)} |t_2^{\alpha-1} - t_1^{\alpha-1}| \\
&\quad \left. + \frac{b\eta^{\alpha+q-1}}{\Gamma(\alpha+q)} |t_2^{\alpha-2} - t_1^{\alpha-2}| + \frac{b\eta^{\alpha+q-1}}{\xi\Gamma(\alpha-r_2+q)} |t_2^{\alpha-1} - t_1^{\alpha-1}| \right\} \\
&\quad + \left(\frac{M}{\beta\Gamma(\alpha)}\right)^{q-1} \left\{ |(t_2-c)^{\alpha-1} - (t_1-c)^{\alpha-1}| + |t_2-t_1| \right\},
\end{aligned}$$

and

$$\begin{aligned}
&|D^{\gamma}_{0+}Tu(t_2) - D^{\gamma}_{0+}Tu(t_1)| \\
&\leq \frac{a}{|d|} \left(\frac{M}{\Gamma(\beta)}\right)^{q-1} \left\{ \frac{1}{\Gamma(\alpha+1)} \left(\frac{\xi^{\beta}}{\beta}\right)^{q-1} \left[\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma)} \left(\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-r_2-1)} + b\eta^{\alpha-2}\right) |t_2^{\alpha-\gamma-1} - t_1^{\alpha-\gamma-1}| \right.\right. \\
&\quad \left. + \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\gamma-1)} \left(\frac{\alpha-1}{\alpha-r_2-1} + \eta\right) |t_2^{\alpha-\gamma-2} - t_1^{\alpha-\gamma-2}| \right]
\end{aligned}$$

$$+\frac{\xi^{\alpha-1}}{\Gamma(\alpha-r_2+1)}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\gamma-1)}|t_2^{\alpha-\gamma-2}-t_1^{\alpha-\gamma-2}|+\frac{\xi^{\alpha-2}}{\xi\Gamma(\alpha-r_2+1)}\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma)}|t_2^{\alpha-\gamma-1}-t_1^{\alpha-\gamma-1}|$$
$$+\frac{b\xi^{\alpha-1}}{\Gamma(\alpha+1)}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\gamma-1)}\left(\frac{\eta^\beta}{\beta}\right)^{q-1}|t_2^{\alpha-\gamma-2}-t_1^{\alpha-\gamma-2}|+\frac{b\xi^{\alpha-2}}{\Gamma(\alpha-1)}\left(\frac{\eta^\beta}{\beta}\right)^{q-1}|t_2^{\alpha-\gamma-1}-t_1^{\alpha-\gamma-1}|\Bigg\}$$
$$+\left(\frac{M}{\beta\Gamma(\alpha)}\right)^{q-1}\left\{|(t_2-c)^{\alpha-1}-(t_1-c)^{\alpha-1}|+|t_2-t_1|\right\},$$

where $c\in(0,t_1)$. Since the function $t^{\alpha-1}$, $t^{\alpha-2}$, $t^{\alpha-\gamma-1}$, $t^{\alpha-\gamma-2}$ and $(t-c)^{\alpha-1}$ are uniformly continuous on $[0,1]$, therefore it follows from the above estimates that TB is an equicontinuous set. Also, it is uniformly bounded as $TB\subset B$. Thus, we conclude that T is a completely continuous operator. Hence, by Schauder fixed point theorem, there exists a solutions of (1.1). This completes the proof. □

Example 2.1. *Consider the four point boundary value problem*

$$\begin{cases} D_{0^+}^{\frac{1}{3}}(\varphi_3(D_{0^+}^{\frac{5}{2}}u))(t)+f(t,u(t),D_{0^+}^{\frac{1}{4}}u(t))=0, & t\in(0,1), \\ u(0)=0, \quad D_{0^+}^{\frac{1}{3}}u(0)-\frac{4}{5}u(\frac{3}{8})=0, \quad D_{0^+}^{\frac{1}{5}}u(1)+2u(\frac{3}{5})=0, \quad D_{0^+}^{\frac{5}{2}}u(0)=0, \end{cases} \tag{2.9}$$

and $f(t,u,D^{\frac{1}{4}}(u))=a+(t+\frac{1}{2})^4[u^{2\rho_1}(t)+(D^{\frac{1}{4}}u(t))^{2\rho_2}]$ *where* a *is constant different from 0 and* $0<\rho_1,\rho_2<1$. *Obviously, it follows by Theorem 2.1 that there exists a solution of 2.9.*

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 19, NO.5, 2015, COPYRIGHT 2015 EUDOXUS PRESS, LLC

Differential subordinations obtained with some new integral operators

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Abstract

In this paper we define some new integral operators on classes of analytic functions and we study certain differential subordinations obtained by using those operators.

Keywords: Analytic function, univalent function, differential subordination, dominant, best dominant.
2000 Mathematical Subject Classification: 30C80, 30C20, 30C45, 30A40.

1 Introduction and preliminaries

The concept of differential subordination was introduced in [1], [2] and developed in [3], by S.S. Miller and P.T. Mocanu.

Denote by U the unit disc of the complex plane $U = \{z \in \mathbb{C} : \ |z| < 1\}$. Let $\mathcal{H}(U)$ be the space of holomorphic functions in U and let $A_n = \{f \in \mathcal{H}(U) : \ f(z) = z + a_{n+1}z^{n+1} + \dots, \ z \in U\}$, with $A_1 = A$, and $S = \{f \in A : \ f \text{ is univalent in } U\}$ be the class of holomorphic and univalent functions in the open unit disc U, with conditions $f(0) = 0$, $f'(0) = 1$, that is the holomorphic and univalent functions with the following power series development $f(z) = z + a_2z^2 + \dots, \quad z \in U$. For $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$, we denote by $\mathcal{H}[a, n] = \{f \in \mathcal{H}(U) : \ f(z) = a + a_nz^n + a_{n+1}z_{n+1} + \dots, \ z \in U\}$. Let $A(p)$ denote the subclass of the functions $f \in \mathcal{H}(U)$ of the form $f(z) = z^p + \sum_{k=p+1}^{\infty} a_kz^k, \quad p \in \mathbb{N}$ and set $A = A(1)$. Denote by $K = \left\{f \in A : \ \operatorname{Re} \dfrac{zf''(z)}{f'(z)} + 1 > 0, \ z \in U\right\}$ the class of normalized convex functions in U.

If f and g are analytic functions in U, then we say that f is subordinate to g, written $f \prec g$, if there is a function w analytic in U, with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$ such that $f(z) = g(w(z))$ for $z \in U$. If g is univalent then $f \prec g$ if and only if $f(0) = 0$ and $f(U) \subseteq g(U)$.

Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the differential subordination

(i) $\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z)$, $z \in U$,

then p is called a solution of the differential subordination.

The univalent function q is called a dominant of the solutions of the differential subordination, or more simply, a dominant, if $p \prec q$ for all p satisfying (i).

A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec q$ for all dominants q of (i) is said to be the best dominant of (i). (Note that the best dominant is unique up to a rotation of U.)

To prove our main results, we need the following lemmas.

Lemma A. (Hallenbeck and Ruscheweyh [3, Th. 3.1.6, p. 71]) *Let h be a convex function with $h(0) = a$, and let $\gamma \in \mathbb{C}^*$ be a complex number with $\operatorname{Re}\gamma \geq 0$. If $p \in \mathcal{H}[a, n]$ and $p(z) + \dfrac{1}{\gamma}zp'(z) \prec h(z), \quad z \in U$ then $p(z) \prec q(z) \prec h(z), \quad z \in U$ where $q(z) = \dfrac{\frac{\gamma}{n}}{z^{\frac{\gamma}{n}}}\displaystyle\int_0^z h(t)t^{\frac{\gamma}{n}-1}dt, \quad z \in U$.*

Lemma B. [3, Th. 3.44, p. 132] *Let the function q be univalent in the unit disc U and let θ and φ be analytic in a domain D containing $q(U)$, with $\varphi(w) \neq 0$, when $w \in q(U)$. Set $Q(z) = zq'(z) \cdot \varphi[q(z)]$ and $h(z) = \theta[q(z)] + Q(z)$.*

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Suppose that
(i) $Q(z)$ *is starlike univalent in* U,
(ii) $\mathrm{Re}\,\dfrac{zh'(z)}{Q(z)} > 0$, *for* $z \in U$.
If p *is analytic with* $p(0) = q(0)$, $p(U) \subseteq D$ *and* $\theta[p(z)] + zp'(z) \cdot \varphi[p(z)] \prec \theta[q(z)] + zq'(z) \cdot \varphi[z(q)]$, *then* $p(z) \prec q(z), \quad z \in U$ *and* $q(z)$ *is the best dominant.*

2 Main results

We introduce the following new integral operators:
Definition 1. For $f \in A_n$, $n \in \mathbb{N}^*$, $m \in \mathbb{N}$, $\gamma \in \mathbb{C}$, let L_γ be the integral operator given by $L_\gamma : A_n \to A_n$,

$$\begin{aligned} &L_\gamma^0 f(z) = f(z),\\ &L_\gamma^1 f(z) = \frac{\gamma+1}{z^\gamma}\int_0^z L_\gamma^0(t)t^{\gamma-1}dt, \dots\\ &L_\gamma^m f(z) = \frac{\gamma+1}{z^\gamma}\int_0^z L_\gamma^{m-1} f(t)\cdot t^{\gamma-1}dt. \end{aligned}$$

By using Definition 1, we can prove the following properties for this integral operator:
Property 1. For $f \in A_n$, $n \in \mathbb{N}^*$, $m \in \mathbb{N}$, $\gamma \in \mathbb{C}$, we have

$$L_\gamma^m f(z) = z + \sum_{k=n+1}^{\infty} \frac{(\gamma+1)^m}{(\gamma+k)^m}\cdot a_k z^k, \quad z \in U. \tag{2.1}$$

Property 2. For $f \in A_n$, $n \in \mathbb{N}^*$, $m \in \mathbb{N}$, $\gamma \in \mathbb{C}$, we obtain

$$z \cdot [L_\gamma^m f(z)]' = (\gamma+1)L_\gamma^{m-1} f(z) - \gamma L_\gamma^m f(z), \quad z \in U. \tag{2.2}$$

Remark 1. For $\gamma = 1$, $n \in \mathbb{N}^*$, $m \in \mathbb{N}$, $\gamma \in \mathbb{C}$, we obtain the integral operator $L_1^m := I^m : A_n \to A_n$, given by

$$\begin{aligned} &I^0 f(z) = f(z),\\ &I^1 f(z) = \frac{2}{z}\int_0^z I^0 f(t)dt, \dots\\ &I^m f(z) = \frac{2}{z}\int_0^z I^{m-1} f(t)dt. \end{aligned}$$

In this case, (2.1) and (2.2) become:

$$I^m f(z) = z + \sum_{k=n+1}^{\infty} \frac{2^m}{(k+1)^m} a_k z^k \tag*{(2.1)$'$}$$

and

$$z \cdot [I^m f(z)]' = 2 \cdot I^{m-1} f(z) - I^m f(z), \quad z \in U. \tag*{(2.2)$'$}$$

Remark 2. For $\gamma = 0$, $n = 1$, $m \in \mathbb{N}$, $f \in A$, we get the integral operator $L_0^m := I_s : A \to A$, introduced by G.St. Sălăgean in [4], given by

$$\begin{aligned} &I_s^0 f(z) = f(z)\\ &I_s^1 f(z) = \int_0^z f(t)\cdot t^{-1}dt, \dots\\ &I_s^m f(z) = \int_0^z I_s^{m-1} f(t)\cdot t^{-1}dt. \end{aligned}$$

In this case, (2.1) and (2.2) become:

$$I_s^m f(z) = z + \sum_{k=2}^{\infty} \frac{1}{k^m} a_k z^k, \quad z \in U \tag*{(2.1)$''$}$$

and

$$z \cdot [I_s^m f(z)]' = I_s^{m-1} f(z), \quad z \in U. \tag{2.2)''$$

Remark 3. For $n \in \mathbb{N}^*$, $m \in \mathbb{N}$, $\gamma \in \mathbb{C}$, $f \in \mathcal{H}[a,n]$, we obtain the integral operator $L_\gamma := L_{\gamma,a} : \mathcal{H}[a,n] \to \mathcal{H}[a,n]$ given by

$$\begin{aligned} L_{\gamma,a}^0 f(z) &= f(z), \\ L_{\gamma,a}^1 f(z) &= \frac{\gamma}{z^\gamma} \int_0^z L_\gamma^0 f(t) \cdot t^{\gamma-1} dt, \dots \\ L_{\gamma,a}^m f(z) &= \frac{\gamma}{z^\gamma} \int_0^z L_\gamma^{m-1} f(t) \cdot t^{\gamma-1} dt. \end{aligned}$$

In this case, (2.1) and (2.2) become:

$$L_{\gamma,a}^m f(z) = a + \sum_{k=n}^{\infty} \frac{\gamma^m}{(\gamma+k)^m} a_k z^k, \quad z \in U \tag{2.3}$$

and

$$z \cdot [L_{\gamma,a}^m f(z)]' = \gamma L_{\gamma,a}^{m-1} f(z) - \gamma L_{\gamma,a}^m f(z), \quad z \in U. \tag{2.4}$$

Remark 4. For $a = 0$, $n \in \mathbb{N}^*$, $m \in \mathbb{N}$, $\gamma \in \mathbb{C}$, $f \in \mathcal{H}[0,n]$, we have the integral operator $L_\gamma := G_\gamma : \mathcal{H}[0,n] \to \mathcal{H}[0,n]$, given by

$$\begin{aligned} G_\gamma^0 f(z) &= f(z), \\ G_\gamma^1 f(z) &= \frac{\gamma+n}{z^\gamma} \int_0^z G_\gamma^0 f(t) \cdot t^{\gamma-1} dt, \dots \\ G_\gamma^m f(z) &= \frac{\gamma+n}{z^\gamma} \int_0^z G_\gamma^{m-1} f(t) t^{\gamma-1} dt. \end{aligned}$$

In this case, (2.1) and (2.2) become:

$$G_\gamma^m f(z) = a_n z^n + \sum_{k=n+1}^{\infty} \frac{(\gamma+n)^n}{(\gamma+k)^m} a_k z^k, \quad z \in U \tag{2.5}$$

and

$$z \cdot [G_\gamma^m f(z)]' = (\gamma+n) G_\gamma^{m-1} f(z) - \gamma G_\gamma^m f(z), \quad z \in U. \tag{2.6}$$

Remark 5. For $a \in \mathbb{C}$, $n \in \mathbb{N}^*$, $m \in \mathbb{N}$, $\gamma = 1$, we get the integral operator $L_{\gamma,a} := T : \mathcal{H}[a,n] \to \mathcal{H}[a,n]$, given by

$$\begin{aligned} T^0 f(z) &= f(z) \\ T^1 f(z) &= \frac{1}{z} \int_0^z T^0 f(t) dt, \dots \\ T^m f(z) &= \frac{1}{z} \int_0^z T^{m-1} f(t) dt. \end{aligned}$$

In this case, (2.1) and (2.2) become:

$$T^m f(z) = a + \sum_{k=n}^{\infty} \frac{1}{(k+1)^m} \cdot a_k z^k, \quad z \in U \tag{2.7}$$

and

$$[z \cdot t^m f(z)]' = T^{m-1} f(z), \quad z \in U. \tag{2.8}$$

3

Definition 2. For $p \in \mathbb{N}$, $m \in \mathbb{N}$, $f \in A(p)$, let H be the integral operator given by $H : A(p) \to A(p)$,

$$\begin{aligned} H^0 f(z) &= f(z) \\ H^1 f(z) &= \frac{p+1}{z}\int_0^z H^0 f(t)dt, \dots \\ H^m f(z) &= \frac{p+1}{z}\int_0^z H^{m-1} f(t)dt. \end{aligned}$$

Property 3. For $f \in A(p)$, $m \in \mathbb{N}$, $p \in \mathbb{N}$, we have

$$H_m f(z) = z^p + \sum_{k=p+1}^{\infty} \frac{(p+1)^m}{(k+1)^m} a_k z^k, \quad z \in U. \tag{2.9}$$

Property 4. For $f \in A(p)$, $m \in \mathbb{N}$, $p \in \mathbb{N}$, we have

$$[z \cdot H^m f(z)]' = (p+1)H^{m-1} f(z), \quad z \in U. \tag{2.10}$$

Remark 6. If $p = 0$, $f \in \mathcal{H}[1, n]$, then $H^m := T^m$, the integral operator seen in Remark 5.
Remark 7. If $p = 1$, $f \in A$, then $H^m := I^m$, the integral operator shown in Remark 1.
Remark 8. These operators are also studied in [5], [6] and [1].

We next give certain differential subordinations obtained by using the integral operators we have defined here.

Theorem 1. *Let* $h(z) = \dfrac{1 + (2\alpha - 1)z}{1+z}$, *be convex in* U, *with* $h(0) = 1$, $0 \le \alpha < 1$.
Assume $\gamma \in \mathbb{C}$, $\operatorname{Re}\gamma > 0$ *and that* $f \in \mathcal{H}[a, n]$, *satisfies the differential subordination*

$$\frac{1}{\gamma} z \cdot [T^{m-1} f(z)]' + T^{m-1} f(z) \prec h(z). \tag{2.11}$$

Then $[z \cdot T^{m-1} f(z)]' \prec q(z)$, *where* q *is given by*

$$q(z) = (2\alpha - 1) + (2 - 2\alpha)\frac{\gamma}{n} \cdot \frac{\sigma(z)}{z^{\gamma/n}}, \tag{2.12}$$

and

$$\sigma(z) = \int_0^z \frac{t^{\frac{\gamma}{n}-1}}{1+t} dt. \tag{2.13}$$

The function q *is convex and is the best dominant.*

Proof. By using the property (2.8) of the integral operator T seen in Definition 1, Remark 5, the differential subordination (2.11) becomes:

$$\frac{1}{\gamma} z \cdot [zT^m f(z)]'' + [z \cdot T^m f(z)]' \prec h(z), \quad z \in U. \tag{2.14}$$

We let

$$p(z) = [z \cdot T^m f(z)]' = a + \sum_{k=n}^{\infty} \frac{1}{(k+1)^{m-1}} a_k z^k, \quad p \in \mathcal{H}[a, n]. \tag{2.15}$$

Using the notation (2.15) in (2.14), the differential subordination (2.11) becomes

$$p(z) + \frac{1}{\gamma} z p'(z) \prec g(z), \quad z \in U. \tag{2.16}$$

By using Lemma A, we have $p(z) \prec q(z) = \dfrac{\gamma}{n z^{\gamma/n}} \displaystyle\int_0^z \frac{1 + (2\alpha - 1)t}{1+t} t^{\gamma/n - 1} dt = \frac{\gamma}{n z^{\gamma/n}} \int_0^z [(2\alpha - 1)t^{\gamma/n-1} + \frac{2-2\alpha}{1+t} t^{\frac{\gamma}{n}-1}]dt = 2\alpha - 1 + (2 - 2\alpha) \cdot \frac{\gamma}{n} \cdot \frac{\sigma(z)}{z^{\gamma/n}}$, where σ is given by (2.13). □

Theorem 2. *Let* q *be univalent in* U, *with* $q(0) = 0$ *and* $q(z) \neq 0$, *for all* $z \in U$, *and suppose that*

(i) $\operatorname{Re} q(z) > 0$,
(ii) $\operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right) > 0,\ z \in U.$
Let $n \in \mathbb{N}^*$, $\gamma \in \mathbb{C}$, $\frac{z[G_\gamma^m f(z)]' + z^2[G_\gamma^m f(z)]''}{(\gamma+n)G_\gamma^{m-1}f(z) - \gamma G_\gamma^m f(z)} \neq 0,\ z \in U,$ *and*

$$(2.17)\qquad (\gamma+n)G_\gamma^{m-1}f(z) - \gamma G_\gamma^m f(z) + \frac{z[G_\gamma^m f(z)]' + z^2[G_\gamma^m f(z)]''}{(\gamma+n)G_\gamma^{m-1}f(z) - \gamma G_\gamma^m f(z)} \prec q(z) + \frac{zq'(z)}{q(z)}.$$

Then $z \cdot [G_\gamma^{m-1} f(z)]' \prec q(z)$ *and* $q(z)$ *is the best dominant.*
Proof. By using (2.6) we have

$$(2.18)\qquad (\gamma+n)G_\gamma^{m+1}f(z) - \gamma G_\gamma^m f(z) = z \cdot [G_\gamma^m f(z)]'.$$

We let

$$(2.19)\qquad p(z) = z \cdot [G_\gamma^m f(z)]'.$$

Using (2.5) in (2.19) we obtain

$$(2.20)\qquad p(z) = z \cdot \left[1 + \sum_{k=n+1}^{\infty} \frac{(\gamma+1)^m}{(\gamma+k)^m} k z^{k-1} a_k\right] = z + \sum_{k=n+1}^{\infty} \frac{(\gamma+1)^m}{(\gamma+k)^m} k z^k z_k.$$

Since $p(0) = 0$, we obtain that $p \in \mathcal{H}[0, n]$.
Differentiating (2.19) and after a short calculus, we obtain

$$(2.21)\qquad zp'(z) = z[G_\gamma^m f(z)]' + z^2[G_\gamma^m f(z)]''.$$

Using (2.18), (2.19) and (2.21), differential subordination (2.17) becomes

$$(2.22)\qquad p(z) + \frac{zp'(z)}{p(z)} \prec q(z) + \frac{zq'(z)}{q(z)}, \quad z \in U.$$

In order to prove the theorem, we shall use Lemma B.
For that, we show that the necessary conditions are satisfied.
Let the function $\theta : \mathbb{C} \to \mathbb{C}$ and $\varphi : \mathbb{C} \to \mathbb{C}$ with

$$(2.23)\qquad \theta(w) = w$$

and

$$(2.24)\qquad \varphi(w) = \frac{1}{w}, \quad \varphi(w) \neq 0.$$

We check the conditions from the hypothesis of Lemma B.
Using (2.24), we have

$$(2.25)\qquad Q(z) = zq'(z) \cdot \varphi(q(z)) = \frac{zq'(z)}{q(z)}.$$

Differentiating (2.25) and after a short calculus, we obtain

$$(2.26)\qquad \frac{zQ'(z)}{Q(z)} = 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}.$$

From (ii) we have

$$(2.27)\qquad \operatorname{Re}\frac{zQ'(z)}{Q(z)} = \operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right) > 0, \quad z \in U,$$

hence function Q is starlike.

Using (2.23) we have

$$h(z) = \theta[q(z)] + Q(z) = q(z) + Q(z). \tag{2.28}$$

Differentiating (2.28) and after a short calculus, we obtain $\dfrac{zh'(z)}{Q(z)} = q(z) + \dfrac{zQ'(z)}{Q(z)}$. From (i) and (2.27) we have

$$\operatorname{Re} \frac{zh'(z)}{Q(z)} = \operatorname{Re} \left[q(z) + \frac{zQ'(z)}{Q(z)} \right] > 0. \tag{2.29}$$

Using (2.23) and (2.24), we get $\theta[p(z)] = p(z)$, $\varphi(p(z)) = \dfrac{1}{p(z)}$, $\theta[q(z)] = q(z)$, $\varphi(q(z)) = \dfrac{1}{q(z)}$, and differential subordination (2.17) becomes $\theta[p(z)] + zp'(z)\varphi(p(z)) \prec \theta[q(z)] + zq'(z) \times (q(z))$.

Using Lemma B, we obtain $p(z) \prec q(z)$ i.e. $z[G_\gamma^m f(z)]' \prec q(z)$, and q is the best dominant. □

Theorem 3. *Let q be univalent in U, with $q(0) = 1$ and $q(z) \neq 0$ for all $z \in U$ and suppose that*
(j) $\operatorname{Re} \left(1 + \dfrac{zq''(z)}{q'(z)} + \dfrac{zq'(z)}{q(z)} \right) > 0,\ z \in U.$
Let $f \in A(p)$, $p \in \mathbb{N}$. If the differential subordination

$$1 + \frac{zH^m f(z)[H^m f(z)]' - p[H^m f(z)]^2}{z^{2p}} \prec 1 + zq(z)q'(z) \tag{2.30}$$

is satisfied, then $\dfrac{H^m f(z)}{z^p} \prec q(z)$ *and $q(z)$ is the best dominant.*

Proof. We let

$$p(z) = \frac{H^m f(z)}{z^p}, \quad z \in U. \tag{2.31}$$

Using (2.9) in (2.31) we have $p(z) = \dfrac{z^p + \sum_{k=p+1}^{\infty} \frac{(p+1)^m}{(k+1)^m} a_k z^k}{z^p} = 1 + \sum_{k=p+1}^{\infty} \dfrac{(p+1)^m}{(k+1)^m} a_k z^{k-p}$. Since $p(0) = 1$, we get that $p \in \mathcal{H}[1, k-p]$.

Differentiating (2.31), we have

$$[H^m f(z)]' = pz^{p-1} p(z) + z^p p'(z). \tag{2.32}$$

Using (2.31) and (2.32), we have

$$1 + \frac{zH^m f(z)[H^m f(z)]' - p[H^m f(z)]^2}{z^{2p}} = 1 + zp(z)p'(z). \tag{2.33}$$

Using (2.33), differential subordination (2.30) becomes

$$1 + zp(z)p'(z) \prec 1 + zq(z)q'(z). \tag{2.34}$$

In order to prove the theorem, we shall use Lemma B.
For that, we show that the necessary conditions are satisfied.
Let the functions $\theta : \mathbb{C} \to \mathbb{C}$ and $\varphi : \mathbb{C} \to \mathbb{C}$ with

$$\theta[w] = 1 \tag{2.35}$$

and

$$\varphi[w] = w. \tag{2.36}$$

Then

$$Q(z) = zq'(z)\varphi[q(z)] = zq'(z)q(z) \tag{2.37}$$

and

$$h(z) = \theta[q(z)] + Q(z) = 1 + Q(z). \tag{2.38}$$

Differentiating (2.37) we have $\dfrac{zQ'(z)}{Q(z)} = 1 + \dfrac{zq'(z)}{q(z)} + \dfrac{zq''(z)}{q'(z)}, \quad z \in U$. Using (j), we have

$$\operatorname{Re} \frac{zQ'(z)}{Q(z)} = \operatorname{Re} \left(1 + \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right) > 0, \quad z \in U. \tag{2.39}$$

Differentiating (2.38), we have $h'(z) = Q'(z)$, and using (2.39) we obtain:

$$\operatorname{Re} \frac{zh'(z)}{Q(z)} = \operatorname{Re} \frac{zQ'(z)}{Q(z)} > 0. \tag{2.40}$$

From (2.35) and (2.36), differential subordination (2.34) can be written as follows:

$$\theta[p(z)] + zp'(z)\varphi(p(z)) \prec \theta[q(z)] + zq'(z)\varphi(q(z)). \tag{2.41}$$

Using Lemma B, we have $p(z) \prec q(z)$, i.e. $\dfrac{H^m f(z)}{z^p} \prec q(z), \quad z \in U$ and $q(z)$ is the best dominant.

Example 1. Let $m \in \mathbb{N}$, $p \in \mathbb{N}$, $q(z) = 1 + z$, $z \in U$, with

$$\operatorname{Re} \left(1 + \frac{zq''(z)}{q'(z)} + \frac{zq'(z)}{q(z)}\right) = \operatorname{Re} \left(1 + \frac{z}{1+z}\right) = \operatorname{Re} \frac{1+2z}{1+z} > 0, \quad z \in U.$$

If $f \in A(p)$, $p \in \mathbb{N}$, and $1 + \dfrac{zH^m f(z)[H^m f(z)]' - p[H^m f(z)]^2}{z^{2p}} \prec 1 + z + z^2$ then $\dfrac{H^m f(z)}{z^p} \prec 1 + z$, $z \in U$ and $q(z) = 1 + z$ is the best dominant.

Remark 9. An open problem is the study of starlikeness, convexity and subordination preserving through those operators.

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Preconditioned SAOR iterative methods for H-matrices linear systems

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Abstract: In this paper, we consider a preconditioned SAOR iterative method for solving linear system $Ax = b$. We prove its convergence and give more comparison results of the spectral radius for the case when A is an H-matrix or a strictly diagonally dominant matrix. Numerical example are also given to illustrate our method.

Key words: H-matrix; SAOR iterative methods; preconditioner; convergence; the spectral radius.,
2000 MR Subject Classification: 65F10, 65F50

1. Introduction

For the linear system

$$Ax = b, \tag{1.1}$$

where A is an $n \times n$ square matrix, and x and b are two n-dimensional vectors. The basic iterative method for solving equation (1.1) is

$$Mx^{k+1} = Nx^k + b, \ \ k = 0, 1, \cdots, \tag{1.2}$$

where $A = M - N$, and M is nonsingular. (1.2) can be written as

$$x^{k+1} = Tx^k + c, \ \ k = 0, 1, \cdots,$$

where $T = M^{-1}N$, $c = M^{-1}b$. Let $A = D - L - U$, where D, $-L$ and $-U$ are diagonal, strictly lower and strictly upper triangular parts of A, respectively. We split A into $A = M - N = P - Q$, here

$$M = \frac{1}{\omega}(D - rL), \ \ N = \frac{1}{\omega}[(1-\omega)D + (\omega - r)L + \omega U]$$

and

$$F = \frac{1}{\omega}(D - rU), \ \ Q = \frac{1}{\omega}[(1-\omega)D + (\omega - r)U + \omega L].$$

Then, the iteration matrix of the SAOR method is given by

$$\mathscr{L}_{r\omega} = F^{-1}QM^{-1}N,$$

where ω and r are real parameters with $\omega \neq 0$.

Transforming the original systems (1.1) into the preconditioned form

$$PAx = Pb, \tag{1.3}$$

then, we can define the basic iterative scheme

$$M_p x^{k+1} = N_p x^k + Pb, \ \ k = 0, 1, \cdots, \tag{1.4}$$

where $PA = M_p - N_p$, and M_p is nonsingular. Thus (1.4) can also be written as

$$x^{k+1} = Tx^k + c, \ \ k = 0, 1, \cdots,$$

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where $T = M_p^{-1}N_p$, $c = M_p^{-1}Pb$.

In the literature, various authors have suggested different models of $(I+S)$-type preconditioner [1-8] for linear systems (1.1). These preconditioners have reasonable effectiveness and low construction cost. For example. In this paper, we present the preconditioner of $(I+S)$-type with the following form

$$P = I + S = \begin{pmatrix} 1 & & & & \alpha_1 - a_{1m} & & \\ & 1 & & & & \alpha_2 - a_{2s} & \\ & & \ddots & & & & \\ & \alpha_r - a_{ru} & & & \ddots & & \\ & & & & & \ddots & \\ & & \alpha_n - a_{nt} & & & & 1 \end{pmatrix}, \tag{1.5}$$

where $m, s, u, t \neq 1$.

We consider the preconditioned SAOR iterative method for solving linear systems with preconditioner P. Let

$$PA = (I+S)(D-L-U) = \tilde{D} - \tilde{L} - \tilde{U},$$

where $\tilde{D}, -\tilde{L}$ and $-\tilde{U}$ are diagonally, strictly lower and strictly upper triangular parts of PA, respectively. We split PA into $PA = \tilde{M} - \tilde{N} = \tilde{P} - \tilde{Q}$, here

$$\tilde{M} = \frac{1}{\omega}(\tilde{D} - r\tilde{L}),\ \ \tilde{N} = \frac{1}{\omega}[(1-\omega)\tilde{D} + (\omega - r)\tilde{L} + \omega\tilde{U}]$$

and

$$\tilde{F} = \frac{1}{\omega}(\tilde{D} - r\tilde{U}),\ \ \tilde{Q} = \frac{1}{\omega}[(1-\omega)\tilde{D} + (\omega - r)\tilde{U} + \omega\tilde{L}].$$

If $\tilde{D}$ is nonsingular, then $(\tilde{D} - r\tilde{L})^{-1}$ and $(\tilde{D} - r\tilde{U})^{-1}$ exist, it is possible to define the SAOR iteration matrix for PA. Namely

$$\mathscr{L}_{r\omega} = \tilde{F}^{-1}\tilde{Q}\tilde{M}^{-1}\tilde{N},$$

Our work in the presentation are to prove Convergence of the SAOR method applied to H-matrix with preconditioner P. Also more comparison results of the spectral radius for the case when A is a nonnegative H-matrix are given. Numerical example shows that the results are valid.

2. Preliminaries

A matrix A is called nonnegative (positive) if each entry of A is nonnegative (positive), respectively. We denote them by $A \geq 0$ ($A > 0$). Similarly, for a n-dimensional vector x, we can also define $x \geq 0$ ($x > 0$). Additionally, we denote the spectral radius of A by $\rho(A)$. A^T denotes the transpose of A. A matrix $A = (a_{ij})$ is called a *Z-matrix* if for any $i \neq j$, $a_{ij} \leq 0$. A *Z-matrix* is a nonsingular *M-matrix* if A is nonsingular and if $A^{-1} \geq 0$, If $\langle A \rangle$ is a nonsingular *M-matrix,* then A is called an *H-matrix.* $A = M - N$ is said to be a *splitting* of A if M is nonsingular, $A = M - N$ is said to be *regular* if $M^{-1} \geq 0$ and $N \geq 0$, *M-splitting* if M is an M-matrix and $N \geq 0$, and *weak regular* if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$, respectively.

Some basic properties on special matrices introduced previously are given to be used in this paper.

Lemma 2.1 [5]. *Let A be a Z-matrix. Then the following statements are equivalent:*
(a) A is an M-matrix.
(b) There is a positive vector x such that $Ax > 0$.
(c) $A^{-1} \geq 0$.

Lemma 2.2 [5]. *Let A and B be two* $n \times n$ *matrices with* $0 \leq B \leq A$. *Then,* $\rho(B) \leq \rho(A)$.

Lemma 2.3 [8]. *If A is an H−matrix, then* $|A^{-1}| \leq \langle A \rangle^{-1}$.

Lemma 2.4 [9]. *Suppose that $A_1 = M_1 - N_1$ and $A_2 = M_2 - N_2$ are weak regular splitting of monotone matrices A_1 and A_2 respectively. Such that $M_2^{-1} \geq M_1^{-1}$. If there exists a positive vector x such that $0 \leq A_1 x \leq A_2 x$, then for the monotone norm associated with x,*

$$\| M_2^{-1} N_2 \|_x \leq \| M_1^{-1} N_1 \|_x . \tag{2.1}$$

In particular, if $M_1^{-1} N_1$ has a positive perron vector, then

$$\rho(M_2^{-1} N_2) \leq \rho(M_1^{-1} N_1). \tag{2.2}$$

Moreover if x is a Perron vector of $M_1^{-1} N_1$ and strictly inequality holds in (2.1)*, then strictly inequality holds in* (2.2).

Lemma 2.5 [17]. *Let $A^{-1} \geq 0$. If the splitting $A = M - N = F - Q$ are weak regular, then $\rho(T) < 1$ and the unique splitting $A = B - C$ induced by T is weak regular. Where $B = F(M + P - A)^{-1} M$ and $C = B - A$.*

Lemma 2.6 [5]. *Matrix A is a strictly diagonally dominant (SDD) matrix, if*

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|, \ i = 1, 2, \ldots, n.$$

3. Convergence results

For the convenience, let $t_i = \frac{(\langle A \rangle x)_i}{2x_m - (\langle A \rangle x)_m}, \quad i = 1, 2, \cdots, n$. Now we give main results as follows:

Theorem 3.1 *Let A be an $H-$matrix with unit diagonally elements. Assume that there exists a positive vector $x = (x_1, x_2, \cdots, x_n)^T$, such that $\langle A \rangle x > 0$. If*

$$a_{im} - t_i \leq \alpha_i \leq a_{im} + t_i, \quad i = 1, 2, \cdots, n, \tag{3.1}$$

then PA is an $H-$matrix.

Proof. Let $(PA)_{ij} = a_{ij} + (\alpha_i - a_{im}) a_{mj}, i, j = 1, 2, \cdots, n, m = r, s, \cdots, t$, and $x = \left(x_1, x_2, \cdots, x_n\right)^T$, then

$$\begin{aligned}
(\langle PA \rangle x)_i &= |1 + (\alpha_i - a_{im}) a_{mi}| x_i - |a_{im} + (\alpha_i - a_{im})| x_m \\
&\quad - \sum_{j \neq i, m} \left| a_{ij} + (\alpha_i - a_{im}) a_{mj} \right| x_j \\
&\geq x_i - |\alpha_i - a_{im}| |a_{mi}| x_i - |a_{im}| x_m - |\alpha_i - a_{im}| x_m - \sum_{j \neq i, m} \left| a_{ij} \right| x_j \\
&\quad - \sum_{j \neq i, m} |\alpha_i - a_{im}| \left| a_{mj} \right| x_j .
\end{aligned}$$

Case 1. $a_{im} \leq \alpha_i \leq a_{im} + t_i$

$$\begin{aligned}
(\langle PA \rangle x)_i &\geq x_i - (\alpha_i - a_{im}) |a_{mi}| x_i - |a_{im}| x_m - (\alpha_i - a_{im}) x_m - \sum_{j \neq i, m} \left| a_{ij} \right| x_j \\
&\quad - \sum_{j \neq i, m} (\alpha_i - a_{im}) \left| a_{mj} \right| x_j \\
&= x_i - |a_{im}| x_m - \sum_{j \neq i, m} \left| a_{ij} \right| x_j - (\alpha_i - a_{im}) |a_{mi}| x_i - (\alpha_i - a_{im}) x_m \\
&\quad - \sum_{j \neq i, m} (\alpha_i - a_{im}) \left| a_{mj} \right| x_j \\
&= (\langle A \rangle x)_i + (\alpha_i - a_{im})(-x_m - \sum_{j \neq m} \left| a_{mj} \right| x_j) \\
&= (\langle A \rangle x)_i + (\alpha_i - a_{im})(- \sum_{j \neq m} \left| a_{mj} \right| x_j + x_m - 2x_m) \\
&= (\langle A \rangle x)_i + (\alpha_i - a_{im}) [(\langle A \rangle x)_m - 2x_m] \\
&> 0.
\end{aligned}$$

Case 2. $a_{im} - t_i \le \alpha_i \le a_{im}$

$$\begin{aligned}
(\langle PA\rangle x)_i &\ge x_i + (\alpha_i - a_{im})|a_{mi}|x_i - |a_{im}|x_m + (\alpha_i - a_{im})x_m - \sum_{j\ne i,m} |a_{ij}|x_j \\
&\quad + \sum_{j\ne i,m} (\alpha_i - a_{im})|a_{mj}|x_j \\
&= (\langle A\rangle x)_i + (\alpha_i - a_{im})(\sum_{j\ne m} |a_{mj}|x_j - x_m + 2x_m) \\
&= (\langle A\rangle x)_i + (\alpha_i - a_{im})[2x_m - (\langle A\rangle x)_m] \\
&> 0.
\end{aligned}$$

Therefore, $\langle PA\rangle$ is an M-matrix, and PA is an H-matrix.

Theorem 3.2. *If A is an nonsingular H-matrix with unit diagonally elements, and $0 < r \le \omega \le 1$, and $\omega \ne 0$. If*

$$a_{im} - t_i \le \alpha_i \le a_{im} + t_i, \quad i = 1, 2, \cdots, n.$$

Then $\rho(\tilde{\mathscr{L}}_{r\omega}) < 1$.

Proof From Theorem 3.1, we know $\langle PA\rangle$ is an M-matrix, and $\langle PA\rangle = |\tilde{D}| - |\tilde{L}| - |\tilde{U}|$. Let $\langle PA\rangle = \langle\tilde{M}\rangle - |\tilde{N}| = \langle\tilde{P}\rangle - |\tilde{Q}|$, where

$$\langle\tilde{M}\rangle = \frac{1}{\omega}(|\tilde{D}| - r|\tilde{L}|),\ |\tilde{N}| = \frac{1}{\omega}[(1-\omega)|\tilde{D}| + (\omega - r)|\tilde{L}| + |\omega\tilde{U}|]$$

and

$$\langle\tilde{P}\rangle = \frac{1}{\omega}(|\tilde{D}| - r|\tilde{U}|),\ |\tilde{Q}| = \frac{1}{\omega}[(1-\omega)|\tilde{D}| + (\omega - r)|\tilde{U}| + \omega|\tilde{L}|].$$

Then the SAOR iteration matrix for $\langle PA\rangle$ is as follows.

$$\ddot{\mathscr{L}}_{r\omega} = \langle\tilde{P}\rangle^{-1}|\tilde{Q}|\langle\tilde{M}\rangle^{-1}|\tilde{N}|.$$

Obviously, $\langle\tilde{P}\rangle$ and $\langle\tilde{M}\rangle$ are two $M-$matrices, which follow from $\langle PA\rangle$ is an M-matrix, by Lemma 2.3, we know that $|\tilde{P}^{-1}| \le \langle\tilde{P}\rangle^{-1}$ and $|\tilde{M}^{-1}| \le \langle\tilde{M}\rangle^{-1}$. Thus

$$|\tilde{\mathscr{L}}_{r\omega}| = |\tilde{P}^{-1}\tilde{Q}\tilde{M}^{-1}\tilde{N}| \le |\tilde{P}^{-1}||\tilde{Q}||\tilde{M}^{-1}||\tilde{N}| \le \langle\tilde{P}\rangle^{-1}|\tilde{Q}|\langle\tilde{M}\rangle^{-1}|\tilde{N}| = \rho(\ddot{\mathscr{L}}_{r\omega}).$$

Since $\langle PA\rangle = \langle\tilde{M}\rangle - |\tilde{N}| = \langle\tilde{P}\rangle - |\tilde{Q}|$ are two weak regular splitting, by Lemma 2.5 $\rho(\ddot{\mathscr{L}}_{r\omega}) < 1$. Together with Lemma 2.2, we have

$$\rho(\tilde{\mathscr{L}}_{r\omega}) \le \rho(|\tilde{\mathscr{L}}_{r\omega}|) \le \rho(\ddot{\mathscr{L}}_{r\omega}) < 1.$$

Remark 3.1. Theorem 3.2 shows that presented preconditioned SAOR method is convergent by employing preconditioner P for H-matrices linear systems.

4. Comparison results of spectral radius

In what follows we will give some comparison results on the spectral radius of preconditioned SAOR iteration matrices with preconditioner P.

Let $\langle A\rangle = \hat{M} - \hat{N} = \hat{F} - \hat{Q}$, where

$$\hat{M} = \frac{1}{\omega}(|D| - r|L|),\ \hat{N} = \frac{1}{\omega}[(1-\omega)|D| + (\omega - r)|L| + \omega|U|]$$

and

$$\hat{F} = \frac{1}{\omega}(|D| - r|U|),\ \hat{Q} = \frac{1}{\omega}[(1-\omega)|D| + (\omega - r)|U| + \omega|L|].$$

Then the SAOR iteration matrix for $\langle A\rangle$ is as follows.

$$\mathscr{L}_{r\omega} = \hat{F}^{-1}\hat{Q}\hat{M}^{-1}\hat{N}.$$

It is easy to show $\langle A\rangle = \hat{M} - \hat{N} = \hat{F} - \hat{Q}$ are two weak regular splitting, from Lemma 2.5, the unique splitting $\langle A\rangle = \hat{B} - \hat{C}$ induced by $\hat{\mathscr{L}}_{r\omega}$ is weak regular splitting. Where $\hat{B} = \hat{F}(\hat{M} + \hat{F} - \langle A\rangle)^{-1}\hat{M}$ and $\hat{C} = \hat{B} - \langle A\rangle$.

From $\langle PA\rangle = \langle\tilde{M}\rangle - |\tilde{N}| = \langle\tilde{F}\rangle - |\tilde{Q}|$ are two weak regular splitting, similar to the above analysis, we have the unique splitting $\langle PA\rangle = \tilde{B} - \tilde{C}$ induced by $\ddot{\mathscr{L}}_{r\omega}$ is weak regular splitting. Where $\tilde{B} = \langle\tilde{F}\rangle(\langle\tilde{M}\rangle + \langle\tilde{F}\rangle - \langle PA\rangle)^{-1}\langle\tilde{M}\rangle$ and $\tilde{C} = \tilde{B} - \langle PA\rangle$.

Theorem 4.1. *Let A be a nonnegative SDD matrix with unit diagonally elements and $a_{mj} \neq 0$, $0 < r \leq \omega \leq 1$. If*

$$a_{im} - s_i \leq \alpha_i \leq a_{im} - w_i, \quad i = 1, 2, \cdots, n.$$

Then $\rho(\ddot{\mathscr{L}}_{r\omega}) \leq \rho(\hat{\mathscr{L}}_{r\omega})$, where $s = \min\{\frac{(\langle A\rangle x)_i}{2x_m - (\langle A\rangle x)_m}, \frac{2}{a_{mj}}\}$ and $w = \min\{\frac{(\langle A\rangle x)_i}{2x_m - (\langle A\rangle x)_m}, \frac{2a_{ij}}{a_{mj}}\}$.

Proof Since $\langle A\rangle$ and $\langle PA\rangle$ are M-matrices, and $\langle A\rangle = \hat{B} - \hat{C}$ and $\langle PA\rangle = \tilde{B} - \tilde{C}$ are two weak regular splittings. From

$$\tilde{B} = \langle\tilde{F}\rangle(\langle\tilde{M}\rangle + \langle\tilde{F}\rangle - \langle PA\rangle)^{-1}\langle\tilde{M}\rangle$$

we have

$$\tilde{B}^{-1} = \langle\tilde{M}\rangle^{-1}(\langle\tilde{M}\rangle + \langle\tilde{F}\rangle - \langle PA\rangle)\langle\tilde{F}\rangle^{-1}.$$

Since $\langle PA\rangle = (I + |S|)\langle A\rangle$. By simple calculation, we have $\tilde{B}^{-1} \geq \hat{B}^{-1} \geq 0$. Let $x = \langle A\rangle^{-1}e > 0$, then

$$(\langle PA\rangle - \langle A\rangle)x = (I + |S|)e > 0.$$

Since $\tilde{B}^{-1} \geq \hat{B}^{-1} \geq 0$, we get

$$\tilde{B}^{-1}\langle PA\rangle x = (I - \tilde{B}^{-1}\tilde{C})x \geq \hat{B}^{-1}\langle A\rangle x = (I - \hat{B}^{-1}\hat{C})x.$$

Thus, it follows that

$$||\tilde{B}^{-1}\tilde{C}||_x \leq ||\hat{B}^{-1}\hat{C}||_x,$$

as $\langle PA\rangle = \tilde{B} - \tilde{C}$ is a weak regular splitting, there exists a positive perron vectors y, thus, by Lemma 2.4, the following inequality holds:

$$\rho(\tilde{B}^{-1}\tilde{C}) \leq \rho(\hat{B}^{-1}\hat{C})$$

ie,

$$\rho(\ddot{\mathscr{L}}_{r\omega}) \leq \rho(\hat{\mathscr{L}}_{r\omega}).$$

Combining the above Theorems, we obtain the following conclusion:

Theorem 4.2. *Let A be a nonnegative SDD with unit diagonally elements and $a_{mj} \neq 0$, $0 < r \leq \omega \leq 1$. If*

$$a_{im} - s_i \leq \alpha_i \leq a_{im} - w_i, \quad i = 1, 2, \cdots, n.$$

Then, $\rho(\tilde{\mathscr{L}}_{r\omega}) \leq \rho(\ddot{\mathscr{L}}_{r\omega}) \leq \rho(\hat{\mathscr{L}}_{r\omega}) < 1$.

Theorem 4.3. *Let A be a nonnegative H-matrix with unit diagonally elements and $a_{mj} \neq 0$, $0 < r \leq \omega \leq 1$. If*

$$a_{im} - s_i \leq \alpha_i \leq a_{im} - w_i, \quad i = 1, 2, \cdots, n.$$

Then $\rho(\ddot{\mathscr{L}}_{r\omega}) \leq \rho(\hat{\mathscr{L}}_{r\omega})$.

Proof It is well known that each H-matrix A there exists a nonsingular diagonal matrix V such that AV is an SDD matrix. By simple calculation, we have iterative matrices of SAOR for A and AV are similar. Since similar matrices have the same eigenvalues, it follows that $\rho(\ddot{\mathscr{L}}_{r\omega}) \leq \rho(\hat{\mathscr{L}}_{r\omega})$.

Theorem 4.4. *Let A be a nonnegative H-matrix with unit diagonally elements and $a_{mj} \neq 0$, $0 < r \leq \omega \leq 1$. If*

$$a_{im} - s_i \leq \alpha_i \leq a_{im} - w_i, \quad i = 1, 2, \cdots, n.$$

Then, $\rho(\tilde{\mathscr{L}}_{r\omega}) \leq \rho(\ddot{\mathscr{L}}_{r\omega}) \leq \rho(\hat{\mathscr{L}}_{r\omega}) < 1$.

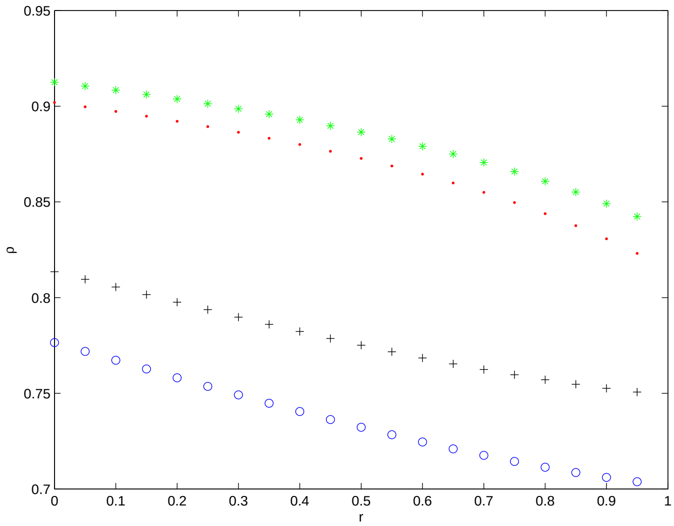
Proof The proof is similar to the proof of Theorem 4.3, so omitted.

5. Example.

For randomly generated nonsingular H-matrices for $n = 50,\ 100,\ 200, 1000$. Let $r = \langle A\rangle^{-1}e$, where $e = (1,1,\ldots,1)^T$. Taking $k_i = min\ j \in \{j|\max_j |a_{ij}|\}$ for $i \neq j$ in (1.5) and α_i, $(i = 1,2,\ldots,n)$ meet the inequality in (3.1). We have determined the spectral radius of the iteration matrices of SAOR method mentioned previously with preconditioner P.

For P, we make two group of experiments. In figure 1, we test the relation between r and ρ, when $n = 10$, $\omega = 1$, where "+","∗","·"and "∘"denote the spectral radius of $\langle PA\rangle$, $\langle A\rangle$, A and PA, respectively. In Table 1, the meaning of notations $\rho(\tilde{\mathscr{L}}_\omega)$, $\rho(\ddot{\mathscr{L}}_\omega)$, $\rho(\hat{\mathscr{L}}_\omega)$ and $\rho(\mathscr{L}_\omega)$ denote the spectral radius of PA, $\langle PA\rangle$, $\langle A\rangle$ and A, respectively.

Figure 1: The relation between r and ρ, when $n = 10$, $\omega = 1$



From Figure 1 and Table 1, we can conclude that the rate of convergence of the preconditioned SAOR method with preconditioner P is faster than others, which further illustrate the Theorem 4.2 is true.

Table 1: Comparison of spectral radius with preconditioner P

ω,r	n	$\rho(\tilde{\mathscr{L}}_\omega)$	$\rho(\ddot{\mathscr{L}}_\omega)$	$\rho(\hat{\mathscr{L}}_\omega)$	$\rho(\mathscr{L}_\omega)$
ω=0.8	5	0.6309	0.6899	0.7899	0.7293
r=0.6	10	0.8542	0.8755	0.9210	0.8638
	50	0.9488	0.9499	0.9665	0.9501
ω=0.6	5	0.7087	0.7301	0.7715	0.7520
r=0.5	50	0.9454	0.9659	0.9805	0.9721
	200	0.9804	0.9857	0.9948	0.9758
ω=1	10	0.7320	0.7615	0.7658	0.7644
r=0.8	50	0.9204	0.9508	0.9819	0.9606
	100	0.9411	0.9538	0.9799	0.9642
	500	0.9713	0.9806	0.9926	0.9893

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Volume 19, Number 6 December 2015
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

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(twelve times annually)

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Abstract: In this paper, we proved some common fixed point theorems of mappings satisfying almost generalized contractive condition in complete partially ordered $G-$metric spaces. These results extend, unify and generalize several well-known corresponding results in the iterature.

Keywords: Common fixed point, partially ordered set, $G-$metric space.

2010 AMS Subject Classification: 47H10.

1 Introduction and Preliminaries

The study of fixed points of mappings satisfying certain contractive conditions has been at the center of rigorous research activity. Mustafa and Sims [14] generalized the concept of a metric space. Based on the notion of generalized metric spaces, Mustafa et al.([13]-[17]) obtained some fixed point theorems for mappings satisfying different contractive conditions. Abbas and Rhoades [1] initiated the study of a common fixed point theory in generalized metric spaces (see [11]). Abbas et al. [2] and Chugh et al. [8] obtained some fixed point results for maps satisfying property P in $G-$metric spaces.

Recently, Shatanawi [21] proved some fixed point results for self-mapping in a complete $G-$ metric space under some contractive conditions related to a nondecreasing map $\phi : R^+ \to R^+$ with $\lim\limits_{n\to\infty} \phi^n(t) = 0$ for all $t \geq 0$. Saadati et al. [20] proved some fixed point results for contractive mappings in partially ordered $G-$metric spaces. Existence of fixed points in ordered metric spaces was first investigated in 2004 by Ran and Reurings [19] (see also, [18]). Several mathematicians then obtained fixed

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point and common fixed point theorems in partially ordered metric spaces ([4], [10], [12]).

The notion of almost contractive condition was introduced by Berinde ([5]-[7]). Recently, Ćirić et al. [9] obtained common fixed point theorems of almost generalized contractive mappings in ordered metric spaces.

The aim of this paper is to define almost generalized contractive condition for two mappings in the setting of generalized metric spaces and to obtain common fixed point theorems for such mappings in partially ordered complete $G-$metric spaces.

Consistent with Mustafa and Sims [14], the following definitions and results will be needed in the sequel.

Definition 1.1. Let X be a nonempty set. A mapping $G : X \times X \times X \to R^+$ is said to be a $G-$metric on X if

(a) $G(x, y, z) = 0$ if $x = y = z$;

(b) $0 < G(x, y, z)$ for all $x, y, z \in X$, with $x \neq y$;

(c) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$, with $y \neq z$;

(d) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$, (symmetry in all three variables); and

(e) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

Then (X, G) is called a $G-$metric space.

Definition 1.2. A sequence $\{x_n\}$ in a $G-$ metric space X is said to be

(a) $G-$*Cauchy* if for every $\varepsilon > 0$, there is an $n_0 \in N$ (the set of all natural numbers), such that for all $n, m, l \geq n_0$, $G(x_n, x_m, x_l) < \varepsilon$,

(b) $G-$*Convergent* if for any $\varepsilon > 0$, there is an $x \in X$ and an $n_0 \in N$, such that for all $n, m \geq n_0$, $G(x_n, x_m, x) < \varepsilon$.

A $G-$metric space on X is said to be $G-$complete if every $G-$Cauchy sequence in X is $G-$convergent in X. It is known that $\{x_n\}$ is $G-$converget to $x \in X$ if and only if $G(x_m, x_n, x) \to 0$ as $n, m \to \infty$.

Proposition 1.3. [14] *Let X be a $G-$metric space. Then the followings are equivalent:*

(1) The sequence $\{x_n\}$ is $G-$convergent to x.

(2) $G(x_n, x_n, x) \to 0$ as $n \to \infty$.

(3) $G(x_n, x, x) \to 0$ as $n \to \infty$.

(4) $G(x_n, x_m, x) \to 0$ as $n, m \to \infty$.

Proposition 1.4. [14] *Let X be a $G-$metric space. Then the followings are equivalent:*

(1) The sequence $\{x_n\}$ is $G-$Cauchy.

(2) For every $\varepsilon > 0$, there exists $n_0 \in N$, such that for all $n, m \geq n_0$, $G(x_n, x_m, x_m) < \varepsilon$; that is, $G(x_n, x_m, x_m) \to 0$ as $n, m \to \infty$.

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Definition 1.5. A $G-$metric on X is said to be symmetric if

$$G(x,y,y) = G(y,x,x)$$

for all $x, y \in X$.

Proposition 1.6. *Every $G-$metric on X defines a metric d_G on X by*

$$d_G(x,y) = G(x,y,y) + G(y,x,x), \tag{1.1}$$

for all $x, y \in X$.

For a symmetric $G-$ metric, we obtain

$$d_G(x,y) = 2G(x,y,y), \tag{1.2}$$

for all $x, y \in X$. However, if G is not symmetric, then we have

$$\frac{3}{2}G(x,y,y) \leq d_G(x,y) \leq 3G(x,y,y),$$

for all $x, y \in X$.

2 Common Fixed Point Theorems

Now, we obtain common fixed point theorems of mappings satisfying almost generalized contractive condition in partially ordered complete generalized metric spaces.

Definition 2.1. [3] Let $(X, \preceq)$ be a partially ordered set. Two maps $f, g : X \to X$ are said to be weakly increasing if $fx \preceq gfx$ and $gx \preceq fgx$ for all $x \in X$.

Theorem 2.2. *Let $(X, \preceq)$ be a partially ordered set equipped with a complete metric G and let $f, g : X \to X$ be two weakly increasing mappings for which there exist $\delta \in (0,1)$ and some $L \geq 0$ such that*

$$\begin{aligned} G(fx, gy, gy) \quad &\leq \quad \delta M(x,y,y) + L\min\big\{G(x,fx,fx), \\ &\quad\quad G(y,gy,gy), G(x,gy,gy), G(y,fx,fx)\big\}, \end{aligned} \tag{2.1}$$

for all comparable $x, y \in X$, where

$$\begin{aligned} M(x,y,y) \quad &= \quad a_1G(x,y,y) + a_2G(x,fx,fx) + a_3G(y,gy,gy) \\ &\quad\quad +a_4\big[G(x,gy,gy) + G(y,fx,fx)\big] \end{aligned}$$

and $a_i > 0$ for $i = \{1,2,3,4\}$ with $a_1 + a_2 + a_3 + 2a_4 < 1$. If either f or g is continuous or for a nondecreasing sequence $\{x_n\}$ in X which is $G-$convergent to z in X implies that $x_n \preceq z$ for all $n \in N$. Then f and g have a common fixed point.

Proof. First, we show that if f or g has a fixed point, then it is a common fixed point of f and g. Indeed, let u be a fixed point of f. Now assume that $G(u, gu, gu) > 0$, then from (2.1) with $x = y = u$, we obtain

$$\begin{aligned} G(u,gu,gu) \quad &= \quad G(fu,gu,gu) \\ &\leq \quad \delta M(u,u,u) + L\min\big\{G(u,fu,fu), \\ &\quad\quad G(u,gu,gu), G(u,gu,gu), G(u,fu,fu)\big\} \\ &= \quad \delta M(u,u,u), \end{aligned} \tag{2.2}$$

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where

$$\begin{aligned} M(u,u,u) &= a_1G(u,u,u)+a_2G(u,fu,fu)+a_3G(u,gu,gu) \\ &\quad +a_4\big[G(u,gu,gu)+G(u,fu,fu)\big] \\ &= (a_3+a_4)G(u,gu,gu), \end{aligned}$$

which implies that

$$\begin{aligned} G(u,gu,gu) &\leq \delta(a_3+a_4)G(u,gu,gu) \\ &< G(u,gu,gu), \end{aligned}$$

a contradiction. Therefore $G(u,gu,gu)=0$ and so u is a common fixed point of f and g. Similarly, if u is a fixed point of g, then it is also fixed point of f. Now let x_0 be an arbitrary point of X. If $fx_0=x_0$, then the proof is finished. We assume that $fx_0 \neq x_0$. Define a sequence $\{x_n\}$ in X as follows:

$$\begin{aligned} x_1 &= fx_0 \preceq gfx_0 = gx_1 = x_2, \\ x_2 &= gx_1 \preceq fgx_1 = fx_2 = x_3 \end{aligned}$$

and continuing this process, we have

$$x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots.$$

We may assume that $G(x_{2n},x_{2n+1},x_{2n+1})>0$, for every $n \in N$. If not, then $x_{2n}=x_{2n+1}$ for some n. Since x_{2n} and x_{2n+1} are comparable, using (2.1), we obtain

$$\begin{aligned} & G(x_{2n+1},x_{2n+2},x_{2n+2}) \\ = \;& G(fx_{2n},gx_{2n+1},gx_{2n+1}) \\ \leq \;& \delta M(x_{2n},x_{2n+1},x_{2n+1})+L\min\big\{G(x_{2n},fx_{2n},fx_{2n}),G(x_{2n+1},gx_{2n+1},gx_{2n+1}), \\ & G(x_{2n},gx_{2n+1},gx_{2n+1}),G(x_{2n+1},fx_{2n},fx_{2n})\big\} \\ = \;& \delta M(x_{2n},x_{2n+1},x_{2n+1})+L\min\big\{G(x_{2n},x_{2n+1},x_{2n+1}),G(x_{2n+1},x_{2n+2},x_{2n+2}), \\ & G(x_{2n},x_{2n+2},x_{2n+2}),G(x_{2n+1},x_{2n+1},x_{2n+1})\big\} \\ = \;& \delta M(x_{2n},x_{2n+1},x_{2n+1}), \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} & M(x_{2n},x_{2n+1},x_{2n+1}) \\ = \;& a_1G(x_{2n},x_{2n+1},x_{2n+1})+a_2G(x_{2n},fx_{2n},fx_{2n})+a_3G(x_{2n+1},gx_{2n+1},gx_{2n+1}) \\ & +a_4\big[G(x_{2n},gx_{2n+1},gx_{2n+1})+G(x_{2n+1},fx_{2n},fx_{2n})\big] \\ = \;& a_1G(x_{2n},x_{2n+1},x_{n+1})+a_2G(x_{2n},x_{2n+1},x_{2n+1})+a_3G(x_{2n+1},x_{2n+2},x_{2n+2}) \\ & +a_4\big[G(x_{2n},x_{2n+2},x_{2n+2})+G(x_{2n+1},x_{2n+1},x_{2n+1})\big] \\ = \;& a_3G(x_{2n+1},x_{2n+2},x_{2n+2})+a_4G(x_{2n},x_{2n+2},x_{2n+2}) \\ = \;& (a_3+a_4)G(x_{2n+1},x_{2n+2},x_{2n+2}). \end{aligned}$$

Hence

$$G(x_{2n+1},x_{2n+2},x_{2n+2}) \leq \delta(a_3+a_4)G(x_{2n+1},x_{2n+2},x_{2n+2}),$$

implies that $G(x_{2n+1},x_{2n+2},x_{2n+2})=0$ as $\delta(a_3+a_4)<1$ and hence $x_{2n+1}=x_{2n+2}$. Following the similar arguments, we obtain $x_{2n+2}=x_{2n+3}$ and hence x_{2n} is a common fixed point of f and g.

Now, take $G(x_{2n}, x_{2n+1}, x_{2n+1}) > 0$ for $n = 0, 1, 2, 3, \cdots$. Since x_{2n} and x_{2n+1} are comparable, therefore

$$\begin{aligned}
& G(x_{2n+1}, x_{2n+2}, x_{2n+2}) \\
= \; & G(fx_{2n}, gx_{2n+1}, gx_{2n+1}) \\
\leq \; & \delta M(x_{2n}, x_{2n+1}, x_{2n+1}) + L \min\big\{G(x_{2n}, fx_{2n}, fx_{2n}), G(x_{2n+1}, gx_{2n+1}, gx_{2n+1}), \\
& G(x_{2n}, gx_{2n+1}, gx_{2n+1}), G(x_{2n+1}, fx_{2n}, fx_{2n})\big\} \\
= \; & \delta M(x_{2n}, x_{2n+1}, x_{2n+1}) + L \min\big\{G(x_{2n}, x_{2n+1}, x_{2n+1}), G(x_{2n+1}, x_{2n+2}, x_{2n+2}), \\
& G(x_{2n}, x_{2n+2}, x_{2n+2}), G(x_{2n+1}, x_{2n+1}, x_{2n+1})\big\} \\
= \; & \delta M(x_{2n}, x_{2n+1}, x_{2n+1}), \qquad (2.4)
\end{aligned}$$

where

$$\begin{aligned}
& M(x_{2n}, x_{2n+1}, x_{2n+1}) \\
= \; & a_1 G(x_{2n}, x_{2n+1}, x_{2n+1}) + a_2 G(x_{2n}, fx_{2n}, fx_{2n}) + a_3 G(x_{2n+1}, gx_{2n+1}, gx_{2n+1}) \\
& + a_4 \big[G(x_{2n}, gx_{2n+1}, gx_{2n+1}) + G(x_{2n+1}, fx_{2n}, fx_{2n})\big] \\
= \; & a_1 G(x_{2n}, x_{2n+1}, x_{2n+1}) + a_2 G(x_{2n}, x_{2n+1}, x_{2n+1}) + a_3 G(x_{2n+1}, x_{2n+2}, x_{2n+2}) \\
& + a_4 \big[G(x_{2n}, x_{2n+2}, x_{2n+2}) + G(x_{2n+1}, x_{2n+1}, x_{2n+1})\big] \\
\leq \; & (a_1 + a_2) G(x_{2n}, x_{2n+1}, x_{2n+1}) + a_3 G(x_{2n+1}, x_{2n+2}, x_{2n+2}) \\
& + a_4 \big[G(x_{2n}, x_{2n+1}, x_{2n+1}) + G(x_{2n+1}, x_{2n+2}, x_{2n+2})\big] \\
= \; & (a_1 + a_2 + a_4) G(x_{2n}, x_{2n+1}, x_{2n+1}) + (a_3 + a_4) G(x_{2n+1}, x_{2n+2}, x_{2n+2}).
\end{aligned}$$

Now if $G(x_{2n+1}, x_{2n+2}, x_{2n+2}) \geq G(x_{2n}, x_{2n+1}, x_{2n+1})$ for some $n = 0, 1, 2, ...$, then $M(x_{2n}, x_{2n+1}, x_{2n+1}) \leq G(x_{2n+1}, x_{2n+2}, x_{2n+2})$ and from (2.4), we have

$$\begin{aligned}
G(x_{2n+1}, x_{2n+2}, x_{2n+2}) & \leq \delta M(x_{2n}, x_{2n+1}, x_{2n+1}) \\
& \leq \delta G(x_{2n+1}, x_{2n+2}, x_{2n+2}),
\end{aligned}$$

a contradiction. Therefore, for all $n \geq 0$,

$$G(x_{2n+1}, x_{2n+2}, x_{2n+2}) \leq G(x_{2n}, x_{2n+1}, x_{2n+1}).$$

Similarly, we have $G(x_{2n}, x_{2n+1}, x_{2n+1}) \leq G(x_{2n-1}, x_{2n}, x_{2n})$ for all $n \geq 0$. Hence for all $n \geq 0$

$$G(x_{n+1}, x_{n+2}, x_{n+2}) \leq \delta G(x_n, x_{n+1}, x_{n+1}).$$

Similarly, repetition of the above process n times gives

$$\begin{aligned}
G(x_{n+1}, x_{n+2}, x_{n+2}) & \leq \delta G(x_n, x_{n+1}, x_{n+1}) \\
& \leq \delta^2 G(x_{n-1}, x_n, x_n) \\
& \leq \cdots \\
& \leq \delta^{n+1} G(x_0, x_1, x_1).
\end{aligned}$$

For $m, n \in N$ with $m > n \geq 0$, using property (e) of $G-$metric, we have

$$\begin{aligned}
& G(x_n, x_m, x_m) \\
\leq \; & G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + G(x_{m-1}, x_m, x_m) \\
\leq \; & (\delta^n + \delta^{n+1} + \cdots + \delta^{m-1}) G(x_0, x_1, x_1) \\
\leq \; & \frac{\delta^n}{1 - \delta^n} G(x_0, x_1, x_1),
\end{aligned}$$

and so $G(x_n, x_m, x_m) \to 0$ as $n, m \to \infty$. It follows that $\{x_n\}$ is a $G-$Cauchy sequence and by the completeness of X, there exists $z \in X$ such that $\{x_n\}$ converges to z as $n \to \infty$. If f is continuous, then it is clear that $fz = z$. Using similar arguments to those given above, we conclude that z is a common fixed point of f and g. Consider the case, when f is not continuous. Since $\{x_{2n}\}$ is a nondecreasing sequence with $x_{2n} \to z$ in X implies $x_{2n} \preceq z$ for all $n \in N$. Now from (2.1)

$$\begin{aligned}
& G(x_{2n+1}, gz, gz) \\
= \; & G(fx_{2n}, gz, gz) \leq \delta M(x_{2n}, z, z) + L \min\{G(x_{2n}, fx_{2n}, fx_{2n}), \\
& G(z, gz, gz), G(x_{2n}, gz, gz), G(z, fx_{2n}, fx_{2n})\} \\
= \; & \delta M(x_{2n}, z, z) + L \min\{G(x_{2n}, x_{2n+1}, x_{2n+1}), \\
& G(z, gz, gz), G(x_{2n}, gz, gz), G(z, fx_{2n+1}, fx_{2n+1})\}, \qquad (2.5)
\end{aligned}$$

where

$$\begin{aligned}
& M(x_{2n}, z, z) \\
= \; & a_1 G(x_{2n}, z, z) + a_2 G(x_{2n}, fx_{2n}, fx_{2n}) + a_3 G(z, gz, gz) \\
& + a_4[G(x_{2n}, gz, gz) + G(z, fx_{2n}, fx_{2n})] \\
= \; & a_1 G(x_{2n}, z, z) + a_2 G(x_{2n}, x_{2n+1}, x_{2n+1}) + a_3 G(z, gz, gz) \\
& + a_4[G(x_{2n}, gz, gz) + G(z, x_{2n+1}, x_{2n+1})],
\end{aligned}$$

which on taking limit as $n \to \infty$, implies that $\lim\limits_{n\to\infty} M(x_{2n}, z, z) = (a_3+a_4)G(z, gz, gz)$. From (2.5)

$$G(z, gz, gz) = \lim_{n\to\infty} G(x_{2n+1}, gz, gz) \leq \delta(a_3 + a_4)G(z, gz, gz).$$

Hence, we have $G(z, gz, gz) = 0$, so $z = gz$ and z is common fixed point of f and g.

Corollary 2.3. *Let $(X, \preceq)$ be a partially ordered set equipped with a complete metric G and let $f, g : X \to X$ be two weakly increasing mappings for which there exist $\delta \in (0, 1)$ and some $L \geq 0$ such that*

$$\begin{aligned}
G(fx, gy, gy) \;\; \leq \;\; & \delta G(x, y, y) + L \min\big\{G(x, fx, fx), \\
& G(y, gy, gy), G(x, gy, gy), G(y, fx, fx)\big\}, \qquad (2.6)
\end{aligned}$$

for all comparable $x, y \in X$. If either f or g is continuous or for a nondecreasing sequence $\{x_n\}$ with $x_n \to z$ in X implies that $x_n \preceq z$ for all $n \in N$. Then f and g have a common fixed point.

Proof. As inequality (2.6) is a special case of (2.1), the result follows from Theorem 2.2.

Theorem 2.4. *Let $(X, \preceq)$ be a partially ordered set equipped with a complete metric G and let $f, g : X \to X$ be two weakly increasing mappings for which there exist $\delta \in (0, 1)$ and some $L \geq 0$ such that*

$$\begin{aligned}
G(fx, gy, gy) \;\; \leq \;\; & \delta M(x, y, y) + L \min\big\{G(x, fx, fx), \qquad (2.7) \\
& G(y, gy, gy), G(x, gy, gy), G(y, fx, fx)\big\},
\end{aligned}$$

for all comparable $x, y \in X$, where

$$\begin{aligned}
M(x, y, y) \;\; = \;\; & \max\big\{G(x, y, y), G(x, fx, fx), G(y, gy, gy), \\
& [G(x, gy, gy) + G(y, fx, fx)]/2\big\}.
\end{aligned}$$

If either f or g is continuous or for a nondecreasing sequence $\{x_n\}$ with $x_n \to z$ in X implies that $x_n \preceq z$ for all $n \in N$. Then f and g have a common fixed point.

Proof. First we show that if f or g has a fixed point, then it is a common fixed point of f and g. Indeed, let u be a fixed point of f and assume that $G(u, gu, gu) > 0$, then from (2.7) with $x = y = u$, we have

$$\begin{aligned} G(u, gu, gu) &= G(fu, gu, gu) \\ &\leq \delta M(u, u, u) + L \min\big\{G(u, fu, fu), \\ &\quad G(u, gu, gu), G(u, gu, gu), G(u, fu, fu)\big\} \\ &= \delta M(u, u, u), \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} M(u, u, u) &= \max\big\{G(u, u, u), G(u, fu, fu), G(u, gu, gu), \\ &\quad [G(u, gu, gu) + G(u, fu, fu)]/2\big\} \\ &= \max\big\{0, G(u, gu, gu), G(u, gu, gu)/2\big\} \\ &= G(u, gu, gu), \end{aligned}$$

that is,

$$\begin{aligned} G(u, gu, gu) &\leq \delta G(u, gu, gu) \\ &< G(u, gu, gu), \end{aligned}$$

a contradiction. Hence $G(u, gu, gu) = 0$ and so u is a common fixed point of f and g. Similarly, if u is a fixed point of g, then it is also fixed point of f. Now let x_0 be an arbitrary point of X. If $fx_0 = x_0$, then the proof is finished, so we assume that $fx_0 \neq x_0$. Define a sequence $\{x_n\}$ in X as follows:

$$\begin{aligned} x_1 &= fx_0 \preceq gfx_0 = gx_1 = x_2, \\ x_2 &= gx_1 \preceq fgx_1 = fx_2 = x_3 \end{aligned}$$

and continuing this process we have

$$x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots.$$

Assume that $G(x_{2n}, x_{2n+1}, x_{2n+1}) > 0$, for every $n \in N$. If not, then $x_{2n} = x_{2n+1}$ for some n. Since x_{2n} and x_{2n+1} are comparable using (2.7) we obtain

$$\begin{aligned} &G(x_{2n+1}, x_{2n+2}, x_{2n+2}) \\ =\ &G(fx_{2n}, gx_{2n+1}, gx_{2n+1}) \\ \leq\ &\delta M(x_{2n}, x_{2n+1}, x_{2n+1}) + L \min\big\{G(x_{2n}, fx_{2n}, fx_{2n}), G(x_{2n+1}, gx_{2n+1}, gx_{2n+1}), \\ &G(x_{2n}, gx_{2n+1}, gx_{2n+1}), G(x_{2n+1}, fx_{2n}, fx_{2n})\big\} \\ =\ &\delta M(x_{2n}, x_{2n+1}, x_{2n+1}) + L \min\big\{G(x_{2n}, x_{2n+1}, x_{2n+1}), G(x_{2n+1}, x_{2n+2}, x_{2n+2}), \\ &G(x_{2n}, x_{2n+2}, x_{2n+2}), G(x_{2n+1}, x_{2n+1}, x_{2n+1})\big\} \\ =\ &\delta M(x_{2n}, x_{2n+1}, x_{2n+1}), \end{aligned} \tag{2.9}$$

where

$$M(x_{2n}, x_{2n+1}, x_{2n+1})$$

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$$\begin{aligned}
&= \max\big\{G(x_{2n},x_{2n+1},x_{2n+1}),G(x_{2n},fx_{2n},fx_{2n}),G(x_{2n+1},gx_{2n+1},gx_{2n+1}),\\
&\quad [G(x_{2n},gx_{2n+1},gx_{2n+1})+G(x_{2n+1},fx_{2n},fx_{2n})]/2\big\}\\
&= \max\big\{G(x_{2n},x_{2n+1},x_{n+1}),G(x_{2n},x_{2n+1},x_{2n+1}),G(x_{2n+1},x_{2n+2},x_{2n+2}),\\
&\quad [G(x_{2n},x_{2n+2},x_{2n+2})+G(x_{2n+1},x_{2n+1},x_{2n+1})]/2\big\}\\
&= \max\big\{0,G(x_{2n+1},x_{2n+2},x_{2n+2}),G(x_{2n+1},x_{2n+2},x_{2n+2})/2\big\}\\
&= G(x_{2n+1},x_{2n+2},x_{2n+2}).
\end{aligned}$$

Therefore,

$$G(x_{2n+1},x_{2n+2},x_{2n+2}) \le \delta G(x_{2n+1},x_{2n+2},x_{2n+2}),$$

implies that $G(x_{2n+1},x_{2n+2},x_{2n+2}) = 0$ and $x_{2n+1} = x_{2n+2}$. Following the similar arguments, we obtain $x_{2n+2} = x_{2n+3}$ and hence x_{2n} becomes a common fixed point of f and g. Suppose that $G(x_{2n},x_{2n+1},x_{2n+1}) > 0$ for $n = 0,1,2,3,\cdots$. Since x_{2n} and x_{2n+1} are comparable, using (2.7) we obtain

$$\begin{aligned}
&G(x_{2n+1},x_{2n+2},x_{2n+2})\\
&= G(fx_{2n},gx_{2n+1},gx_{2n+1})\\
&\le \delta M(x_{2n},x_{2n+1},x_{2n+1})+L\min\big\{G(x_{2n},fx_{2n},fx_{2n}),G(x_{2n+1},gx_{2n+1},gx_{2n+1}),\\
&\quad G(x_{2n},gx_{2n+1},gx_{2n+1}),G(x_{2n+1},fx_{2n},fx_{2n})\big\}\\
&= \delta M(x_{2n},x_{2n+1},x_{2n+1})+L\min\big\{G(x_{2n},x_{2n+1},x_{2n+1}),G(x_{2n+1},x_{2n+2},x_{2n+2}),\\
&\quad G(x_{2n},x_{2n+2},x_{2n+2}),G(x_{2n+1},x_{2n+1},x_{2n+1})\big\}\\
&= \delta M(x_{2n},x_{2n+1},x_{2n+1}),
\end{aligned} \tag{2.10}$$

where

$$\begin{aligned}
&M(x_{2n},x_{2n+1},x_{2n+1})\\
&= \max\big\{G(x_{2n},x_{2n+1},x_{2n+1}),G(x_{2n},fx_{2n},fx_{2n}),G(x_{2n+1},gx_{2n+1},gx_{2n+1}),\\
&\quad [G(x_{2n},gx_{2n+1},gx_{2n+1})+G(x_{2n+1},fx_{2n},fx_{2n})]/2\big\}\\
&= \max\big\{G(x_{2n},x_{2n+1},x_{2n+1}),G(x_{2n},x_{2n+1},x_{2n+1}),G(x_{2n+1},x_{2n+2},x_{2n+2}),\\
&\quad [G(x_{2n},x_{2n+2},x_{2n+2})+G(x_{2n+1},x_{2n+1},x_{2n+1})]/2\big\}\\
&\le \max\big\{G(x_{2n},x_{2n+1},x_{2n+1}),G(x_{2n+1},x_{2n+2},x_{2n+2}),\\
&\quad [G(x_{2n},x_{2n+1},x_{2n+1})+G(x_{2n+1},x_{2n+2},x_{2n+2})]/2\big\}\\
&= \max\big\{G(x_{2n},x_{2n+1},x_{2n+1}),G(x_{2n+1},x_{2n+2},x_{2n+2})\big\}.
\end{aligned}$$

Now, if $G(x_{2n+1},x_{2n+2},x_{2n+2}) \ge G(x_{2n},x_{2n+1},x_{2n+1})$ for some $n = 0,1,2,\cdots$, then $M(x_{2n},x_{2n+1},x_{2n+1}) \le G(x_{2n+1},x_{2n+2},x_{2n+2})$ and from (2.10), we have

$$\begin{aligned}
G(x_{2n+1},x_{2n+2},x_{2n+2}) &\le \delta M(x_{2n},x_{2n+1},x_{2n+1})\\
&\le \delta G(x_{2n+1},x_{2n+2},x_{2n+2}),
\end{aligned}$$

a contradiction. Therefore, for all $n \ge 0$,

$$G(x_{2n+1},x_{2n+2},x_{2n+2}) \le G(x_{2n},x_{2n+1},x_{2n+1}).$$

Similarly, we have $G(x_{2n},x_{2n+1},x_{2n+1}) \le G(x_{2n-1},x_{2n},x_{2n})$ for all $n \ge 0$. Hence for all $n \ge 0$,

$$G(x_{n+1},x_{n+2},x_{n+2}) \le \delta G(x_n,x_{n+1},x_{n+1}).$$

Following similar argument to those in Theorem 2.2, it follows that $\{x_n\}$ is a $G-$Cauchy sequence and by the $G-$completeness of X, there exists $u \in X$ such that $\{x_n\}$ is $G-$convergent to u as $n \to \infty$. If f is continuous, then it is clear that $fu = u$ and hence u is common fixed point of f and g. Now consider the case when f is not continuous. Since $\{x_{2n}\}$ is a nondecreasing sequence with $x_{2n} \to u$ in X implies $x_{2n} \preceq u$ for all $n \in N$. Now from (2.7)

$$\begin{aligned}
& G(x_{2n+1}, gu, gu) \\
= \; & G(fx_{2n}, gu, gu) \le \delta M(x_{2n}, u, u) + L \min \{G(x_{2n}, fx_{2n}, fx_{2n}), \\
& G(u, gu, gu), G(x_{2n}, gu, gu), G(u, fx_{2n}, fx_{2n})\} \\
= \; & \delta M(x_{2n}, u, u) + L \min \{G(x_{2n}, x_{2n+1}, x_{2n+1}), \\
& G(u, gu, gu), G(x_{2n}, gu, gu), G(u, fx_{2n+1}, fx_{2n+1})\}, \qquad (2.11)
\end{aligned}$$

where

$$\begin{aligned}
& M(x_{2n}, u, u) \\
= \; & \max \{G(x_{2n}, u, u), G(x_{2n}, fx_{2n}, fx_{2n}), G(u, gu, gu), \\
& [G(x_{2n}, gu, gu) + G(u, fx_{2n}, fx_{2n})]/2\} \\
= \; & \max \{G(x_{2n}, u, u), G(x_{2n}, x_{2n+1}, x_{2n+1}), G(u, gu, gu), \\
& [G(x_{2n}, gu, gu) + G(u, x_{2n+1}, x_{2n+1})]/2\},
\end{aligned}$$

which implies that $\lim_{n\to\infty} M(x_{2n}, u, u) = G(u, gu, gu)$. From (2.11), we obtain that

$$G(u, gu, gu) = \lim_{n\to\infty} G(x_{2n+1}, gu, gu) \le \delta G(u, gu, gu),$$

and $G(u, gu, gu) = 0$ and so $u = gu$ and u is a common fixed point of f and g.

Corollary 2.5. *Let $(X, \preceq)$ be a partially ordered set equipped with a complete metric G and let $f, g : X \to X$ be two weakly increasing mappings for which there exist $\delta \in (0, 1)$ and some $L \ge 0$ such that*

$$\begin{aligned}
& G(fx, gy, gy) \\
\le \; & \delta \max \left\{ G(x, y, y), \frac{G(x, fx, fx) + G(y, gy, gy)}{2}, \frac{G(x, gy, gy) + G(y, fx, fx)}{2} \right\} \\
& + L \min \{G(x, fx, fx), G(y, gy, gy), G(x, gy, gy), G(y, fx, fx)\}, \qquad (2.12)
\end{aligned}$$

for all comparable $x, y \in X$. If either f or g is continuous or for a nondecreasing sequence $\{x_n\}$ with $x_n \to z$ in X implies that $x_n \preceq z$ for all $n \in N$. Then f and g have a common fixed point.

Proof. As inequality (2.12) is a special case of (2.7), the result follows from Theorem 2.4.

Corollary 2.6. *Let $(X, \preceq)$ be a partially ordered set equipped with a complete metric G and let $f, g : X \to X$ be two weakly increasing mappings for which there exist $\delta \in (0, 1)$ and some $L \ge 0$ such that*

$$\begin{aligned}
& G(fx, gy, gy) \\
\le \; & \delta \max \left\{ \frac{G(x, fx, fx) + G(y, gy, gy)}{2}, \frac{G(x, gy, gy) + G(y, fx, fx)}{2} \right\}, \qquad (2.13)
\end{aligned}$$

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for all comparable $x, y \in X$. If either f or g is continuous or for a nondecreasing sequence $\{x_n\}$ with $x_n \to z$ in X implies that $x_n \preceq z$ for all $n \in N$. Then f and g have a common fixed point.

Proof. As inequality (2.13) is a special case of (2.7), the result follows from Theorem 2.4.

Remark 2.7.

1. If $L = 0$ in Theorem 2.2, then [15, Theorem 2.1 and 2.9] and [17, Theorem 2.1-2.5, Corollary 2.6-2.8] are extended to two mappings in partially ordered $G-$metric spaces.

2. If $L = 0$ in Theorem 2.4, then [9, Theorem 2.1-2.2], [15, Theorem 2.3 and 2.9], [16, Theorem 2.1 and 2.6] and [17, Corollary 2.7-2.8] are extended to the common fixed point results in partially ordered $G-$metric spaces. Also, Theorem 2.4 extends and generalizes the results [9, Theorem 2.3, 2.6, Corollary 2.4-2.5] to the common fixed point result in partially ordered $G-$metric spaces.

3. Corollary 2.3 extends [9, Corollary 2.5] to the partially ordered $G-$ metric spaces. Also, if $L = 0$ in Corollary 2.3, then the results [15, Theorem 2.9] and [17, Corollary 2.7-2.8] are extended to the common fixed point results in partially ordered $G-$metric spaces.

4. If $L = 0$ in Corollary 2.5, then [17, Corollary 2.8] are extended to two mappings in partially ordered $G-$ metric spaces. Also Corollary 2.5 extends [9, Corollary 2.4-2.5] to partially ordered $G-$metric spaces.

5. Corollary 2.6 extends [17, Corollary 2.8] to the common fixed point result in partially ordered $G-$metric spaces.

Acknologements: This work was supported by the Basic Science Research Program through the National Research Foundation Grant funded by Ministry of Education of the republic of Korea(2013R1A1A2054617).

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ADDITIVE ρ-FUNCTIONAL INEQUALITIES IN MATRIX NORMED SPACES

AFSHAN BATOOL, CHOONKIL PARK, AND DONG YUN SHIN*

Abstract. In [22], Kim et al. introduced and investigated the following additive ρ-functional inequalities

$$\begin{aligned}\|f(x+y+z)-f(x)-f(y)-f(z)\| \qquad (0.1)\\ \leq \left\|\rho\left(2f\left(\frac{x+y}{2}+z\right)-f(x)-f(y)-2f(z)\right)\right\|,\end{aligned}$$

$$\begin{aligned}\left\|2f\left(\frac{x+y}{2}+z\right)-f(x)-f(y)-2f(z)\right\| \qquad (0.2)\\ \leq \left\|\rho\left(2f\left(\frac{x+y+z}{2}\right)-f(x)-f(y)-f(z)\right)\right\|\end{aligned}$$

in complex Banach spaces.

We prove the Hyers-Ulam stability of the additive ρ-functional inequalities (0.1) and (0.2) in complex matrix normed spaces and prove the Hyers-Ulam stability of additive ρ-functional equations associated with the additive ρ-functional inequalities (0.1) and (0.2) in complex matrix normed spaces.

1. Introduction and preliminaries

The abstract characterization given for linear spaces of bounded Hilbert space operators in terms of *matricially normed spaces* [29] implies that quotients, mapping spaces and various tensor products of operator spaces may again be regarded as operator spaces. Owing in part to this result, the theory of operator spaces is having an increasingly significant effect on operator algebra theory (see [5]).

The proof given in [29] appealed to the theory of ordered operator spaces [2]. Effros and Ruan [6] showed that one can give a purely metric proof of this important theorem by using a technique of Pisier [26] and Haagerup [11] (as modified in [4]).

We will use the following notations:

$M_n(X)$ is the set of all $n \times n$-matrices in X;

$e_j \in M_{1,n}(\mathbb{C})$ is that j-th component is 1 and the other components are zero;

$E_{ij} \in M_n(\mathbb{C})$ is that (i,j)-component is 1 and the other components are zero;

$E_{ij} \otimes x \in M_n(X)$ is that (i,j)-component is x and the other components are zero;

2010 *Mathematics Subject Classification.* Primary 47L25, 47H10, 39B82, 39B72, 46L07, 39B52, 39B62.

Key words and phrases. Hyers-Ulam stability; additive ρ-functional equation; additive ρ-functional inequality; matrix normed space.

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For $x \in M_n(X), y \in M_k(X)$,

$$x \oplus y = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}.$$

Note that $(X, \{\|\cdot\|_n\})$ is a matrix normed space if and only if $(M_n(X), \|\cdot\|_n)$ is a normed space for each positive integer n and $\|AxB\|_k \leq \|A\|\|B\|\|x\|_n$ holds for $A \in M_{k,n}(\mathbb{C})$, $x = (x_{ij}) \in M_n(X)$ and $B \in M_{n,k}(\mathbb{C})$, and that $(X, \{\|\cdot\|_n\})$ is a matrix Banach space if and only if X is a Banach space and $(X, \{\|\cdot\|_n\})$ is a matrix normed space.

A matrix normed space $(X, \{\|\cdot\|_n\})$ is called an L^∞*-matrix normed space* if $\|x \oplus y\|_{n+k} = \max\{\|x\|_n, \|y\|_k\}$ holds for all $x \in M_n(X)$ and all $y \in M_k(X)$.

Let E, F be vector spaces. For a given mapping $h : E \to F$ and a given positive integer n, define $h_n : M_n(E) \to M_n(F)$ by

$$h_n([x_{ij}]) = [h(x_{ij})]$$

for all $[x_{ij}] \in M_n(E)$.

Lemma 1.1. ([25]) *Let $(X, \{\|.\|_n\})$ be a matrix normed space.*

(1) $\| E_{kl} \otimes x \|_n = \| x \|$ *for* $x \in X$.

(2) $\| x_{kl} \| \leq \| [x_{ij}] \| n \leq \sum_{i,j=1}^{n} \| x_{ij} \|$ *for* $[x_{ij}] \in M_n(X)$.

(3) $lim_{n\to\infty} x_n = x$ *if and only if* $lim_{n\to\infty} x_{nij} = x_{ij}$ *for* $x_n = [x_{nij}], x = [x_{ij}] \in M_k(X)$.

The stability problem of functional equations originated from a question of Ulam [32] concerning the stability of group homomorphisms. Hyers [12] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [27] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [8] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. See [3, 13, 14, 15, 16, 17, 18, 19, 20, 21, 23, 30, 31] for more information on functional equations and their stability.

In [9], Gilányi showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(xy^{-1})\| \leq \|f(xy)\| \tag{1.1}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(xy) + f(xy^{-1}).$$

See also [28]. Gilányi [10] and Fechner [7] proved the Hyers-Ulam stability of the functional inequality (1.1). Park, Cho and Han [24] proved the Hyers-Ulam stability of additive functional inequalities. Kim et al. [22] solved the additive ρ-functional inequalities (0.1) and (0.2) in complex normed spaces and proved the Hyers-Ulam stability of the additive ρ-functional inequalities (0.1) and (0.2) in complex Banach spaces

In Section 2, we prove the Hyers-Ulam stability of the additive ρ-functional inequality (0.1) in complex matrix Banach spaces. We moreover prove the Hyers-Ulam stability of an additive ρ-functional equation associated with the additive ρ-functional inequality (0.1) in complex matrix Banach spaces.

In Section 3, we prove the Hyers-Ulam stability of the additive ρ-functional inequality (0.2) in complex matrix Banach spaces. We moreover prove the Hyers-Ulam stability

of an additive ρ-functional equation associated with the addtive ρ-functional inequality (0.2) in complex matrix Banach spaces.

Throughout this paper, let $(X, \{\|\cdot\|_n\})$ be a matrix normed space and $(Y, \{\|\cdot\|_n\})$ a matrix Banach space. Let ρ be a fixed complex number with $|\rho| < 1$.

2. Hyers-Ulam stability of the additive ρ-functional inequality (0.1) in matrix normed spaces

In this section, we prove the Hyers-Ulam stability of the additive ρ-functional inequality (0.1) in complex matrix normed spaces.

Theorem 2.1. *Let $r > 1$ and be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\begin{aligned} &\|f_n([x_{ij} + y_{ij} + z_{ij}]) - f_n([x_{ij}]) - f_n([y_{ij}]) - f_n([z_{ij}])\|_n \\ &\quad \leq \left\|\rho\left(2f_n\left(\frac{[x_{ij}] + [y_{ij}]}{2} + [z_{ij}]\right) - f_n([x_{ij}]) - f_n([y_{ij}]) - 2f_n([z_{ij}])\right)\right\|_n \\ &\qquad + \sum_{i,j=1}^{n} \theta\left(\| x_{ij} \|^r + \| y_{ij} \|^r + \| z_{ij} \|^r\right) \end{aligned} \tag{2.1}$$

for all $x = [x_{ij}], y = [y_{ij}], z = [z_{ij}] \in M_n(X)$. Then there exists a unique additive mapping $h : X \to Y$ such that

$$\|f_n([x_{ij}]) - h_n([x_{ij}])\|_n \leq \sum_{i,j=1}^{n} \frac{2\theta}{2^r - 2} \| x_{ij} \|^r \tag{2.2}$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. Let $n = 1$ in (2.1). Then we obtain

$$\begin{aligned} &\|f(a + b + c) - f(a) - f(b) - f(c)\| \\ &\quad \leq \left\|\rho\left(2f\left(\frac{a+b}{2} + c\right) - f(a) - f(b) - 2f(c)\right)\right\| \\ &\qquad + \theta\left(\|a\|^r + \|b\|^r + \|c\|^r\right) \end{aligned}$$

for all $a, b, c \in X$.

By [22, Theorem 2.3], there is a unique additive mapping $h : X \to Y$ such that

$$\|f(a) - h(a)\| \leq \frac{2\theta}{2^r - 2}\|a\|^r$$

for all $a \in X$.

By Lemma 1.1,

$$\|f_n([x_{ij}]) - h_n([x_{ij}])\|_n \leq \sum_{i,j=1}^{n} \frac{2\theta}{2^r - 2} \| x_{ij} \|^r$$

for all $x = [x_{ij}] \in M_n(X)$. Thus the mapping $h : X \to Y$ is a unique additive mapping satisfying (2.2). □

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Theorem 2.2. *Let $r < 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (2.1). *Then there exists a unique additive mapping $h : X \to Y$ such that*

$$\|f_n([x_{ij}]) - h_n([x_{ij}])\|_n \leq \sum_{i,j=1}^{n} \frac{2\theta}{2-2^r} \parallel x_{ij} \parallel^r \tag{2.3}$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. By [22, Theorem 2.4], there is a unique additive mapping $h : X \to Y$ such that

$$\|f(a) - h(a)\| \leq \frac{2\theta}{2-2^r}\|a\|^r$$

for all $a \in X$.

By Lemma 1.1,

$$\|f_n([x_{ij}]) - h_n([x_{ij}])\|_n \leq \sum_{i,j=1}^{n} \frac{2\theta}{2-2^r} \parallel x_{ij} \parallel^r$$

for all $x = [x_{ij}] \in M_n(X)$. Thus the mapping $h : X \to Y$ is a unique additive mapping satisfying (2.3). □

By the triangle inequality, we have

$$\begin{aligned}
&\|f_n([x_{ij}+y_{ij}+z_{ij}]) - f_n([x_{ij}]) - f_n([y_{ij}]) - f_n([z_{ij}])\|_n \\
&\quad - \left\|\rho\left(2f_n\left(\frac{[x_{ij}]+[y_{ij}]}{2} + [z_{ij}]\right) - f_n([x_{ij}]) - f_n([y_{ij}]) - 2f_n([z_{ij}])\right)\right\|_n \\
&\leq \|f_n([x_{ij}+y_{ij}+z_{ij}]) - f_n([x_{ij}]) - f_n([y_{ij}]) - f_n([z_{ij}]) \\
&\quad -\rho\left(2f_n\left(\frac{[x_{ij}]+[y_{ij}]}{2} + [z_{ij}]\right) - f_n([x_{ij}]) - f_n([y_{ij}]) - 2f_n([z_{ij}])\right)\Bigg\|_n .
\end{aligned}$$

As corollaries of Theorems 2.1 and 2.2, we obtain the Hyers-Ulam stability results for the additive ρ-functional equations associated with the additive-ρ-functional inequality (0.1) in complex matrix Banach spaces.

Corollary 2.3. *Let $r > 1$ and be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\begin{aligned}
&\|f_n([x_{ij}+y_{ij}+z_{ij}]) - f_n([x_{ij}]) - f_n([y_{ij}]) - f_n([z_{ij}]) \\
&\quad -\rho\left(2f_n\left(\frac{[x_{ij}]+[y_{ij}]}{2} + [z_{ij}]\right) - f_n([x_{ij}]) - f_n([y_{ij}]) - 2f_n([z_{ij}])\right)\Bigg\|_n \\
&\leq \sum_{i,j=1}^{n} \theta\left(\parallel x_{ij} \parallel^r + \parallel y_{ij} \parallel^r + \parallel z_{ij} \parallel^r\right)
\end{aligned} \tag{2.4}$$

for all $x = [x_{ij}], y = [y_{ij}], z = [z_{ij}] \in M_n(X)$. Then there exists a unique additive mapping $h : X \to Y$ satisying (2.2).

Corollary 2.4. *Let $r < 1$ and be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (2.4). *Then there exists a unique additive mapping $h : X \to Y$ satisfying* (2.3).

Remark 2.5. If ρ is a real number such that $-1 < \rho < 1$ and Y is a real Banach space, then all the assertions in this section remain valid.

Also, we prove the Hyers-Ulam stability of the following additive ρ-functional inequality

$$\left\|2f\left(\frac{x+y}{2}+z\right)-f(x)-f(y)-2f(z)\right\| \\ \leq \|\rho\left(f(x+y+z)-f(x)-f(y)-f(z)\right)\|$$

in complex matrix Banach spaces. The proof is similar to the proofs of Theorems 2.1 and 2.2 except for direction of the inequality, and so we will omit it.

3. Hyers-Ulam stability of the additive ρ-functional inequality (0.2) in matrix normed spaces

In this section, we prove the Hyers-Ulam stability of the additive ρ-functional inequality (0.2) in complex matrix normed spaces.

Theorem 3.1. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\left\|2f_n\left(\frac{[x_{ij}]+[y_{ij}]}{2}+[z_{ij}]\right)-f_n([x_{ij}])-f_n([y_{ij}])-2f_n([z_{ij}])\right\|_n \\ \leq \left\|\rho\left(2f_n\left(\frac{[x_{ij}]+[y_{ij}]+[z_{ij}]}{2}\right)-f_n([x_{ij}])-f_n([y_{ij}])-f_n([z_{ij}])\right)\right\|_n \\ +\sum_{i.j=1}^{n}\theta\left(\|x_{ij}\|^r+\|y_{ij}\|^r+\|z_{ij}\|^r\right) \tag{3.1}$$

for all $x=[x_{ij}], y=[y_{ij}], z=[z_{ij}]\in M_n(X)$. Then there exists a unique additive mapping $h : X \to Y$ satisfying

$$\|f_n([x_{ij}])-h_n([x_{ij}])\|_n \leq \sum_{i,j=1}^{n}\frac{2^{r-1}\theta}{(1-|\rho|)(2^r-2)}\parallel x_{ij}\parallel^r \tag{3.2}$$

Proof. Let $n=1$ in (3.1). Then we get

$$\left\|2f\left(\frac{a+b}{2}+c\right)-f(a)-f(b)-2f(c)\right\| \\ \leq \left\|\rho\left(2f\left(\frac{a+b+c}{2}\right)-f(a)-f(b)-f(c)\right)\right\|+\theta\left(\|a\|^r+\|b\|^r+\|c\|^r\right)$$

for all $a,b,c\in X$.

By [22, Theorem 3.3], there is a unique additive mapping $h : X \to Y$ such that

$$\|f(a)-h(a)\| \leq \frac{2^{r-1}\theta}{(1-|\rho|)(2^r-2)}\|a\|^r$$

for all $a\in X$.

By Lemma 1.1,

$$\|f_n([x_{ij}])-h_n([x_{ij}])\|_n \leq \sum_{i,j=1}^{n}\frac{2^{r-1}\theta}{(1-|\rho|)(2^r-2)}\parallel x_{ij}\parallel^r$$

for all $x = [x_{ij}] \in M_n(X)$. Thus the mapping $h : X \to Y$ is a unique additive mapping satisfying (3.2). □

Theorem 3.2. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying* (3.1). *Then there exists a unique mapping $h : X \to Y$ such that*

$$\|f_n([x_{ij}]) - h_n([x_{ij}])\|_n \leq \sum_{i,j=1}^{n} \frac{2^r\theta}{(1-|\rho|)(2-2^r)} \parallel x_{ij} \parallel^r \tag{3.3}$$

for all $x = [x_{ij}] \in M_n(X)$

Proof. By [22, Theorem 3.4], there is a unique additive mapping $h : X \to Y$ such that

$$\|f(a) - h(a)\| \leq \frac{2^r\theta}{(1-|\rho|)(2-2^r)}\|a\|^r$$

for all $a \in X$.

By Lemma 1.1,

$$\|f_n([x_{ij}]) - h_n([x_{ij}])\|_n \leq \sum_{i,j=1}^{n} \frac{2^r\theta}{(1-|\rho|)(2-2^r)} \parallel x_{ij} \parallel^r$$

for all $x = [x_{ij}] \in M_n(X)$. Thus the mapping $h : X \to Y$ is a unique additive mapping satisfying (3.3). □

By the triangle inequality, we have

$$\begin{aligned}
&\left\|2f_n\left(\frac{[x_{ij}]+[y_{ij}]}{2}+[z_{ij}]\right) - f_n([x_{ij}]) - f_n([y_{ij}]) - 2f_n([z_{ij}])\right\|_n \\
&\quad - \left\|\rho\left(2f_n\left(\frac{[x_{ij}]+[y_{ij}]+[z_{ij}]}{2}\right) - f_n([x_{ij}]) - f_n([y_{ij}]) - f_n([z_{ij}])\right)\right\|_n \\
&\leq \left\|2f_n\left(\frac{[x_{ij}]+[y_{ij}]}{2}+[z_{ij}]\right) - f_n([x_{ij}]) - f_n([y_{ij}]) - 2f_n([z_{ij}])\right. \\
&\quad \left. -\rho\left(2f_n\left(\frac{[x_{ij}]+[y_{ij}]+[z_{ij}]}{2}\right) - f_n([x_{ij}]) - f_n([y_{ij}]) - f_n([z_{ij}])\right)\right\|_n .
\end{aligned}$$

As corollaries of Theorems 3.1 and 3.2, we obtain the Hyers-Ulam stability results for the additive ρ-functional equations associated with the additive-ρ-functional inequality (0.2) in complex matrix Banach spaces.

Corollary 3.3. *Let $r > 1$ and be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\begin{aligned}
&\left\|2f_n\left(\frac{[x_{ij}]+[y_{ij}]}{2}+[z_{ij}]\right) - f_n([x_{ij}]) - f_n([y_{ij}]) - 2f_n([z_{ij}])\right. \\
&\left. -\rho\left(2f_n\left(\frac{[x_{ij}]+[y_{ij}]+[z_{ij}]}{2}\right) - f_n([x_{ij}]) - f_n([y_{ij}]) - f_n([z_{ij}])\right)\right\|_n \\
&\leq \sum_{i,j=1}^{n} \theta\left(\parallel x_{ij} \parallel^r + \parallel y_{ij} \parallel^r + \parallel z_{ij} \parallel^r\right)
\end{aligned} \tag{3.4}$$

for all $x = [x_{ij}], y = [y_{ij}], z = [z_{ij}] \in M_n(X)$. Then there exists a unique additive mapping $h : X \to Y$ satisying (3.2).

Corollary 3.4. *Let $r < 1$ and be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (3.4). *Then there exists a unique additive mapping $h : X \to Y$ satisfying* (3.3).

Remark 3.5. If ρ is a real number such that $-1 < \rho < 1$ and Y is a real Banach space, then all the assertions in this section remain valid.

Also, we prove the Hyers-Ulam stability of the following additive ρ-functional inequality

$$\begin{aligned}\left\|2f\left(\frac{x+y+z}{2}\right) - f(x) - f(y) - f(z)\right\| \\ \leq \left\|\rho\left(2f\left(\frac{x+y}{2} + z\right) - f(x) - f(y) - 2f(z)\right)\right\|\end{aligned}$$

in complex matrix Banach spaces. The proof is similar to the proofs of Theorems 3.1 and 3.2 except for direction of the inequality, and so we will omit it.

Acknowledgments

C. Park was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2012R1A1A2004299), and D. Y. Shin was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2010-0021792)

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 19, NO.6, 2015, COPYRIGHT 2015 EUDOXUS PRESS, LLC

New iterative algorithms for coupled matrix equations

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September 18, 2014

Abstract

Iterative algorithms are presented for solving coupled matrix equations including the well-known coupled Sylvester matrix equations and the discrete-time coupled Markovian jump Lyapunov matrix equations as special cases. The proposed methods remove the restriction that the considered matrix equation has a unique solution, which is required in the existing gradient based iterative algorithms and identification principle based methods.

Keywords: Coupled matrix equations; iterative algorithms; least squares solutions.

1 Introduction

For convenience, in this paper we use $I[m,n]$ to denote the set $\{m, m+1, \cdots, n\}$ for two integers m and n, $m \leq n$. In systems and control theory, some coupled Sylvester matrix equations play very vital roles. For example, pairs of generalized Sylvester equations $(A_1X - YB_1,\ A_2X - YB_2) = (C_1,\ C_2)$ are encountered in perturbation analysis of generalized eigenspaces of matrix pencils [1]. In the analysis of discrete-time jump linear systems with Markovian transitions, the following coupled discrete Markovian jump Lyapunov (CDMJL) equation is encountered ([2] [3] [4])

$$P_i = A_i^T \left(\sum_{l=1}^{N} p_{il} P_l \right) A_i + Q_i, Q_i > 0, i \in I[1,N] \tag{1}$$

where P_i, $i \in I[1,N]$ are the matrices to be determined. Due to their wide applications, they have attracted considerable attention of many researchers. It was pointed out in [4] that existence of positive definite solutions for the CDMJL equation (1) is related to the spectral radius of an augmented matrix being less than one.

A simple explicit solution for the CDMJL equation (1) was given in [4] in terms of matrix inversions by convert them into matrix-vector equations. However, computational difficulties arise when the dimensions of the coefficient matrices are high. To eliminate the high dimensionality problems, a parallel iterative scheme for solving the CDMJL equation was proposed in [3]. Under the condition that all the subsystems are Schur stable, this method was proven to converge to the exact solution if the initial condition is chosen as zero. In [5], the restriction of zero initial conditions of the method in [3] was removed, and a new iteration method was provided by using the solutions of discrete-time Lyapunov equations as its intermediate steps. Very recently, from an optimization point of view a gradient-based algorithm was developed in [6] to solve the CDMJL equation (1).

Alternative ways are to transform the matrix equations into forms for which solutions may be readily computed. In this area, Chu gave a numerical algorithm for solving the coupled Sylvester equations in [7].

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†This work is supported by the National Natural Science Foundation of China (No. U1204605, 11226239, 61402149), Excellent-Young-Backbone Teacher Project in high school of Henan Province (No. 2013GGJS-087), Scientific Research Key Project Fund of the Education Department of Henan Province(No. 12B120007).

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In [8], generalized Schur methods for solving the coupled Sylvester matrix equations was proposed. Recently, some iterative algorithms for solving the pairs of generalized matrix equations were proposed in [9] and [10] by applying the hierarchical identification principle, which has been used to construct iterative algorithms for noncoupled matrix equations in [11]. Different from the methods in [7] and [8], these algorithms do not require matrix transformation, and thus can be realized in terms of original coefficient matrices. Very recently, from an optimization point of view gradient based iteration was constructed in [12] to solve the general coupled matrix equation. A significant characteristic of the method in [12] is that a necessary and sufficient condition guaranteeing the convergence of the algorithm can be explicitly obtained. However, the methods in [10], [12] are only suitable for solving the coupled matrix equation having unique solutions.

In addition, it should be pointed out that complex matrix equations with the conjugate of the unknown matrices are attracting more and more researchers. In [13], the so-called Sylvester-polynomial-conjugate matrix equation was investigated, and an explicit solution was given. In [14], an iterative algorithm was given for coupled Sylvester-conjugate matrix equations by using hierarchical identification principle. In [15], an iterative algorithm was constructed for a kind of coupled Sylvester-conjugate matrix equations by using a real inner product as tools.

In this paper, we investigate iterative algorithms for solving general coupled matrix equations. The proposed methods do not require that the concerned matrix equations have unique solutions. Compared with the method of obtaining least squares solutions in [9], the method in the current paper removes the restriction that some coefficient matrices must have full column or row ranks, and does not involve matrix inversions.

Throughout this paper, the symbol $\mathrm{tr}(A)$ is used to denote the trace of A. For the space $\mathbb{R}^{m\times n}$ we define an inner product as $\langle A, B\rangle = \mathrm{tr}(B^T A)$ for all $A, B \in \mathbb{R}^{m\times n}$. The norm of a matrix A generated by this inner product is, obviously, a Frobenius norm, and denoted by $\|A\|$. In addition, the symbol $\|A\|_2$ denotes the spectral norm of matrix A. For two matrices M and N, $M \otimes N$ is their Kronecker product. For a matrix

$$X = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \in \mathbb{R}^{m\times n},$$

$\mathrm{vec}(X)$ is the column stretching operation of X, and defined as

$$\mathrm{vec}(X) = \begin{bmatrix} x_1^T & x_2^T & \cdots & x_n^T \end{bmatrix}^T.$$

A well-known property of Kronecker product is, for matrices M, N and X with appropriate dimension

$$\mathrm{vec}(MXN) = \left(N^T \otimes M\right)\mathrm{vec}(X).$$

2 The case of two unknown matrices

In this section, we consider the following coupled Sylvester matrix equation

$$\begin{cases} A_1XB_1 + C_1YD_1 = E_1 \\ A_2XB_2 + C_2YD_2 = E_2 \end{cases} \tag{2}$$

where $A_i \in \mathbb{R}^{m_i\times r}$, $B_i \in \mathbb{R}^{s\times n_i}$, $C_i \in \mathbb{R}^{m_i\times p}$, $D_i \in \mathbb{R}^{q\times n_i}$, and $E_i \in \mathbb{R}^{m_i\times n_i}$, $i = 1, 2$, are known matrices, and $X \in \mathbb{R}^{r\times s}$ and $Y \in \mathbb{R}^{p\times q}$ are the matrices to be determined. Obviously, this coupled matrix equation includes the pair of generalized Sylvester matrix equation in [1], [9] as a special case. It should be mentioned that the methods in [1], [9] require the unknown matrices X and Y to have the same dimension. In this paper, the algorithm to be developed removes such restriction.

In this paper, we solve it in the least squares sense. That is, we search matrix pair (X, Y) to minimize the following index function

$$J(X,Y) = \frac{1}{2}\|E_1 - A_1XB_1 - C_1YD_1\|^2 + \frac{1}{2}\|E_2 - A_2XB_2 - C_2YD_2\|^2. \tag{3}$$

It is easy to obtain that

$$\begin{aligned} \frac{\partial J}{\partial X} &= A_1^T\left(E_1 - A_1XB_1 - C_1YD_1\right)B_1^T + A_2^T\left(E_2 - A_2XB_2 - C_2YD_2\right)B_2^T, \\ \frac{\partial J}{\partial Y} &= C_1^T\left(E_1 - A_1XB_1 - C_1YD_1\right)D_1^T + C_2^T\left(E_2 - A_2XB_2 - C_2YD_2\right)D_2^T. \end{aligned}$$

Obviously, a least squares solution (X_*, Y_*) satisfies

$$\left.\frac{\partial J}{\partial X}\right|_{X=X_*} = 0, \ \left.\frac{\partial J}{\partial Y}\right|_{Y=Y_*} = 0.$$

In addition, denote

$$R(k) = \left.\frac{\partial J}{\partial X}\right|_{X=X(k)}, \ S(k) = \left.\frac{\partial J}{\partial Y}\right|_{Y=Y(k)}.$$

With the above prelimilaries, the iteration method to solve least squares solutions of the coupled matrix equation (2) can be constructed as the following algorithm.

Algorithm 1 *(Iterative algorithm for least squares solutions of (2))*

1. *Given initial values $X(0)$ and $Y(0)$, calculate*

$$\begin{array}{rcl}
\bar{R}(0) & = & E_1 - A_1X(0)B_1 - C_1Y(0)D_1; \\
\bar{S}(0) & = & E_2 - A_2X(0)B_2 - C_2Y(0)D_2; \\
R(0) & = & A_1^T\bar{R}(0)B_1^T + A_2^T\bar{S}(0)B_2^T; \\
S(0) & = & C_1^T\bar{R}(0)D_1^T + C_2^T\bar{S}(0)D_2^T; \\
P(0) & = & -R(0); \\
Q(0) & = & -S(0); \\
k & : & = 0;
\end{array}$$

2. *If $\|R(k)\| \le \varepsilon$, $\|S(k)\| \le \varepsilon$, then stop; else, $k := k+1$;*

3. *Set $k := k+1$. Calculate*

$$\begin{array}{rcl}
\alpha(k) & = & \dfrac{\mathrm{tr}\left[P^T(k)R(k)\right] + \mathrm{tr}\left[Q^T(k)S(k)\right]}{\|A_1P(k)B_1 + C_1Q(k)D_1\|^2 + \|A_2P(k)B_2 + C_2Q(k)D_2\|^2}; \\
X(k+1) & = & X(k) + \alpha(k)P(k) \in \mathbb{R}^{r\times s}; \\
Y(k+1) & = & Y(k) + \alpha(k)Q(k) \in \mathbb{R}^{p\times q}; \\
\bar{R}(k+1) & = & E_1 - A_1X(k+1)B_1 - C_1Y(k+1)D_1 \in \mathbb{R}^{m_1\times n_1}; \\
\bar{S}(k+1) & = & E_2 - A_2X(k+1)B_2 - C_2Y(k+1)D_2 \in \mathbb{R}^{m_2\times n_2}; \\
R(k+1) & = & A_1^T\bar{R}(k+1)B_1^T + A_2^T\bar{S}(k+1)B_2^T \in \mathbb{R}^{r\times s}; \\
S(k+1) & = & C_1^T\bar{R}(k+1)D_1^T + C_2^T\bar{S}(k+1)D_2^T \in \mathbb{R}^{p\times q}; \\
P(k+1) & = & -R(k+1) + \dfrac{\|R(k+1)\|^2 + \|S(k+1)\|^2}{\|R(k)\|^2 + \|S(k)\|^2}P(k) \in \mathbb{R}^{r\times s}; \\
Q(k+1) & = & -S(k+1) + \dfrac{\|R(k+1)\|^2 + \|S(k+1)\|^2}{\|R(k)\|^2 + \|S(k)\|^2}Q(k) \in \mathbb{R}^{p\times q};
\end{array}$$

4. *Go to Step 2.*

In the rest of this subsection, we analyze the convergence property of the proposed algorithm. Firstly, we give two lemmas, whose proofs are presented in the Appendix.

Lemma 1 *For the sequences $\{R(i)\}$, $\{S(i)\}$, $\{P(i)\}$ and $\{Q(i)\}$ generated by Algorithm 1, the following relation holds:*

$$\mathrm{tr}\left[R^T(k+1)P(k)\right] + \mathrm{tr}\left[S^T(k+1)Q(k)\right] = 0$$

for $k \ge 0$.

3

Lemma 2 *For the sequences $\{R(i)\}$, $\{S(i)\}$, $\{P(i)\}$ and $\{Q(i)\}$ generated by Algorithm 1, the following relation holds:*

$$\sum_{k\geq 0} \frac{\left(\|R(k)\|^2 + \|S(k)\|^2\right)^2}{\|P(k)\|^2 + \|Q(k)\|^2} < \infty.$$

With these two lemmas, the following result on the convergence property of Algorithm 1 can be obtained.

Theorem 1 *The sequences $\{R(i)\}$ and $\{S(i)\}$ generated by Algorithm 1 satisfy*

$$\lim_{k\to\infty} \|R(k)\| = 0, \ \lim_{k\to\infty} \|S(k)\| = 0.$$

Thus, $X(k)$ and $Y(k)$ given by Algorithm 1 converge to a least squares solution of (2).

Proof. By using Lemma 1 and the expressions of $P(k+1)$ and $Q(k+1)$ in Algorithm 1, one has

$$\begin{aligned}
& \|P(k+1)\|^2 + \|Q(k+1)\|^2 \\
= & \left\| -R(k+1) + \frac{\|R(k+1)\|^2 + \|S(k+1)\|^2}{\|R(k)\|^2 + \|S(k)\|^2} P(k) \right\|^2 + \left\| -S(k+1) + \frac{\|R(k+1)\|^2 + \|S(k+1)\|^2}{\|R(k)\|^2 + \|S(k)\|^2} Q(k) \right\|^2 \\
= & \left(\frac{\|R(k+1)\|^2 + \|S(k+1)\|^2}{\|R(k)\|^2 + \|S(k)\|^2} \right)^2 \left(\|P(k)\|^2 + \|Q(k)\|^2 \right) + \|R(k+1)\|^2 + \|S(k+1)\|^2.
\end{aligned}$$

Denote

$$t_k = \frac{\|P(k)\|^2 + \|Q(k)\|^2}{\left(\|R(k)\|^2 + \|S(k)\|^2\right)^2}.$$

Then the preceding relation can be equivalently written as

$$t_{k+1} = t_k + \frac{1}{\|R(k+1)\|^2 + \|S(k+1)\|^2}. \tag{4}$$

We now proceed by contradiction and assume that

$$\lim_{k\to\infty} \left(\|R(k)\|^2 + \|S(k)\|^2\right) \neq 0. \tag{5}$$

This relation implies that there exists a constant $\delta > 0$ such that

$$\|R(k)\|^2 + \|S(k)\|^2 > \delta$$

for all $k \geq 0$. It follows from (4) and (5) that

$$t_{k+1} < t_0 + \frac{k+1}{\delta}.$$

This implies that

$$\frac{1}{t_{k+1}} \geq \frac{\delta}{\delta t_0 + k + 1}.$$

Thus one has

$$\sum_{k=1}^{\infty} \frac{1}{t_k} \geq \sum_{k=1}^{\infty} \frac{\delta}{\delta t_0 + k} = \infty.$$

However, it follows from Lemma 2 that

$$\sum_{k=1}^{\infty} \frac{1}{t_k} < \infty.$$

This gives a contradiction. So the conclusion of this theorem is true. ■

3 General cases

In this section, we will extend the iterative methods in Section 2 to solve more general coupled Sylvester matrix equations of the form

$$A_{i1}X_1B_{i1} + A_{i2}X_2B_{i2} + \cdots + A_{ip}X_pB_{ip} = E_i, i \in I[1,N], \tag{6}$$

where $A_{ij} \in \mathbb{R}^{m_i \times r_j}$, $B_{ij} \in \mathbb{R}^{s_j \times n_i}$, $E_i \in \mathbb{R}^{m_i \times n_i}$, $i \in I[1,N]$, $j \in I[1,p]$ are known matrices and $X_j \in \mathbb{R}^{r_j \times s_j}$, $j \in I[1,p]$ are matrices to be determined. Such type of matrix matrix equations include the coupled matrix equation (2) as a special case. When $N = 2$ and $p = 2$, the matrix equation (6) becomes the coupled matrix equations (2). Similar to the idea of Section 2, we solve the least squares solutions. That is, we search X_j, $j \in I[1,p]$ such that the index function

$$J\left(X_i, i \in I[1,N]\right) = \frac{1}{2}\sum_{i=1}^{N}\left\|E_i - \sum_{j=1}^{p} A_{ij}X_jB_{ij}\right\|^2 \tag{7}$$

is minimized. It is easy to obtain that

$$\frac{\partial J}{\partial X_j} = \sum_{i=1}^{N} A_{ij}^T\left(E_i - \sum_{\omega=1}^{p} A_{i\omega}X_\omega B_{i\omega}\right)B_{ij}^T\, j \in I[1,p].$$

A least squares solution $(X_{1*}, X_{2*}, \cdots, X_{p*})$ should satisfy

$$\left.\frac{\partial J}{\partial X_j}\right|_{X_j = X_{j*}} = 0,\ j \in I[1,p].$$

By generalizing Algorithm 1 for the coupled Sylvester matrix equation (2) in Section 2.2, we give the following iteration method to solve the least squares solution group of the coupled matrix equation (6).

Algorithm 2 *(Iterative method for least squares solutions of (6))*

1. *Given initial values* $X_j(0)$, $j \in I[1,p]$, *calculate*

$$\begin{aligned}
\bar{R}_i\left(0\right) &= E_i - \sum_{j=1}^{p} A_{ij}X_j\left(0\right)B_{ij},\ i \in I[1,N];\\
R_j\left(0\right) &= \sum_{i=1}^{N} A_{ij}^T\bar{R}_i\left(0\right)B_{ij}^T,\ j \in I[1,p];\\
P_j\left(0\right) &= -R_j\left(0\right);\\
k &: = 0;
\end{aligned}$$

2. *If* $\|R_j(k)\| \le \varepsilon$, $j \in I[1,p]$, *then stop; else,* $k := k+1$;

3. *Set* $k := k+1$, *calculate*

$$\begin{aligned}
\alpha(k) &= \frac{\sum_{j=1}^{p}\operatorname{tr}\left[P_j^T\left(k\right)R_j\left(k\right)\right]}{\sum_{i=1}^{N}\left\|\sum_{j=1}^{p}A_{ij}P_j\left(k\right)B_{ij}\right\|^2};\\
X_j\left(k+1\right) &= X_j\left(k\right) + \alpha(k)P_j\left(k\right) \in \mathbb{R}^{r_j \times s_j}, j \in I[1,p];\\
\bar{R}_i\left(k+1\right) &= E_i - \sum_{j=1}^{p} A_{ij}X_j\left(k+1\right)B_{ij}, i \in I[1,N];\\
R_j\left(k+1\right) &= \sum_{i=1}^{N} A_{ij}^T\bar{R}_i\left(k+1\right)B_{ij}^T, j \in I[1,p];\\
P_j\left(k+1\right) &= -R_j\left(k+1\right) + \frac{\sum_{i=1}^{p}\left\|R_i\left(k+1\right)\right\|^2}{\sum_{i=1}^{p}\left\|R_i\left(k\right)\right\|^2}P_j\left(k\right), j \in I[1,p].
\end{aligned}$$

4. Go to Step 2.

Similar to the conclusions in Subsection 2.2, on the convergence property of this algorithm one has the following results. The proofs are all omitted.

Lemma 3 *For the sequences $\{R_i(k)\}$, $\{P_i(k)\}$, $j \in I[1,p]$ generated by Algorithm 2, the following relation holds:*

$$\sum_{j=1}^{p} \operatorname{tr}\left[R_j^T(k+1)P_j(k)\right] = 0$$

for $k \geq 0$.

Lemma 4 *For the sequences $\{R_i(k)\}$, $\{P_i(k)\}$, $j \in I[1,p]$ generated by Algorithm 2, the following relation holds:*

$$\sum_{k\geq 0} \frac{\left(\sum_{j=1}^{p} \|R_i(k)\|^2\right)^2}{\sum_{j=1}^{p} \|P_i(k)\|^2} < \infty.$$

Theorem 2 *The sequences $\{R_i(k)\}$, $j \in I[1,p]$ generated by Algorithm 2 satisfy*

$$\lim_{k\to\infty} \|R_j(k)\| = 0, j \in I[1,p].$$

Thus, $X_j(k)$, $j \in I[1,p]$ given by Algorithm 1 converge to a least squares solution of (6).

Remark 1 *The problem of solving least squares solutions of coupled matrix equations has been investigated in [9]. By using the hierarchical identification principle, an infinite iterative algorithm is constructed to solve the coupled linear matrix equation (6). However, the method in [9] require that some coefficient matrices must have full column or row ranks. In addition, the matrix inversion is required in each iteration, which may turn out to be numerically expensive. The proposed Algorithm 2 removes such restrictions, and thus is expected to have advantage over some existing methods.*

Remark 2 *The coupled matrix equations with the form of (6) have been investigated in [9], [10], [12]. In [9], [10] the considered coupled matrix equations are required to satisfy that $p = N$, and that all the unknown matrices X_i, $i \in I[1,N]$ have the same dimension. While in [12], it is required that*

$$\sum_{i=1}^{N} m_i n_i = \sum_{j=1}^{p} r_j s_j.$$

In this paer, the proposed Algorithm 2 removes these restrictions.

Remark 3 *Different from the methods in [10], [12], an exact solution of the coupled matrix equation can be obtained in finite iteration steps in the absence of round-off errors. It should also be mentioned that the methods in [10], [12] are only suitable for solving the coupled matrix equation (6) with unique solutions.*

Remark 4 *In [10] and [12], in order to guarantee the convergence of the proposed algorithms one must choose an appropriate step-size or convergence factor. In general, such convergence factor can be obtained by some complex computation. Different from these methods, the iterative algorithm given in this paper does not involve the problem of parameters choice, and thus is easier to carry out.*

Remark 5 *Due to round-off errors, Algorithm 2 may not terminate in a finite number of steps. In this case, one can use the algorithm to obtain an approximate solution in sufficient steps.*

4 A special case

In this section, we consider the following matrix equation, which is a special case of the coupled matrix equation (6)

$$A_i X B_i = E_i, i \in I[1, N] \tag{8}$$

where $A_i \in \mathbb{R}^{m_i \times r}$, $B_i \in \mathbb{R}^{s \times n_i}$, and $E_i \in \mathbb{R}^{m_i \times n_i}$, $i \in I[1, N]$ are the known coefficient matrices, and $X \in \mathbb{R}^{r \times s}$ is the matrix to be determined. Similarly, we search matrix X to minimize the index function

$$J(X) = \frac{1}{2} \sum_{i=1}^{N} \|E_i - A_i X B_i\|^2 . \tag{9}$$

One easily obtain that

$$\frac{\partial J}{\partial X} = \sum_{i=1}^{N} A_i^T (E_i - A_i X B_i) B_i^T .$$

The iterative method to obtain the least squares solution of the matrix equation (8) is constructed as the following algorithm.

Algorithm 3 *(Finite iterative algorithm for (8))*

1. *Given initial values X, calculate*

$$\begin{aligned} \bar{R}_i(0) &= E_i - A_i X(0) B_i, i \in I[1, N]; \\ R(0) &= \sum_{i=1}^{N} A_i^T \bar{R}_i(0) B_i^T; \\ P(0) &= -R(0); \\ k &:= 0; \end{aligned}$$

2. *If $\|R(k)\| \leq \varepsilon$, then stop; else, $k := k + 1$;*
3. *Set $k := k + 1$, calculate*

$$\begin{aligned} \alpha(k) &= \frac{\operatorname{tr}\left[P^T(k) R(k)\right]}{\sum_{i=1}^{N} \|A_i P(k) B_i\|^2}; \\ X(k+1) &= X(k) + \alpha(k) P(k) \in \mathbb{R}^{r \times s}; \\ \bar{R}_i(k+1) &= E_i - A_i X(k+1) B_i, i \in I[1, N]; \\ R(k+1) &= \sum_{i=1}^{N} A_i^T \bar{R}_i(k+1) B_i^T; \\ P(k+1) &= -R(k+1) + \frac{\|R(k+1)\|^2}{\|R(k)\|^2} P(k). \end{aligned}$$

4. *Go to Step 2.*

For the convergence properties one has the following results. The proofs are all omitted.

Lemma 5 *For the sequences $\{R(k)\}$ and $\{P(k)\}$ generated by Algorithm 3, the following relation holds:*

$$\operatorname{tr}\left[R^T(k+1) P(k)\right] = 0$$

for $k \geq 0$.

Lemma 6 *For the sequences $\{R(k)\}$ and $\{P_i(k)\}$ generated by Algorithm 3, the following relation holds:*

$$\sum_{k\geq 0} \frac{\|R(k)\|^4}{\|P(k)\|^2} < \infty.$$

Theorem 3 *The sequences $\{R(k)\}$ generated by Algorithm 3 satisfy*

$$\lim_{k\to\infty} \|R(k)\| = 0.$$

Thus, $X(k)$, $j \in I[1,p]$ given by Algorithm 3 converge to a least squares solution of (8).

5 A numerical example

In this section, we give an example to illustrate the effectiveness of the proposed methods. This example has been used in [9]. The matrix equation is in the form of (2) with

$$\begin{aligned} A_1 &= \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}, B_1 = C_1 = B_2 = C_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\ D_1 &= \begin{bmatrix} 1 & -0.2 \\ 0.2 & 1 \end{bmatrix}, E_1 = \begin{bmatrix} 13.2 & 10.6 \\ 0.6 & 8.4 \end{bmatrix}, \\ A_2 &= \begin{bmatrix} -2 & -0.5 \\ 0.5 & 2 \end{bmatrix}, D_2 = \begin{bmatrix} -1 & -3 \\ 2 & -4 \end{bmatrix}, E_2 = \begin{bmatrix} -9.5 & -18 \\ 16 & 3.5 \end{bmatrix}. \end{aligned}$$

The solution of X and Y to this equation is

$$X = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}, Y = \begin{bmatrix} 2 & 1 \\ -2 & 3 \end{bmatrix}.$$

Taking $X(0) = Y(0) = 10^{-6}\mathbf{1}(2)$, we apply Algorithm 1 to compute $X(k)$ and $Y(k)$. The iterative solutions of the matrix equation is shown in Table 1.

Table 1. The iteration solution by Algorithm 1

k	x_{11}	x_{12}	x_{21}	x_{22}	y_{12}	y_{22}	y_{21}	y_{22}
1	2.583305	2.473221	2.502577	2.123399	3.648917	3.240872	−1.349385	1.297523
2	4.087500	1.691281	3.804495	2.798320	3.067254	1.091186	−1.916194	2.593294
3	3.873979	2.146630	3.395190	3.986429	2.384152	1.057442	−2.671572	2.215298
4	3.678258	2.509525	3.149014	4.126780	2.016496	1.250040	−2.208587	3.192737
5	4.057452	2.909192	2.870312	4.005162	2.042451	1.029262	−2.056753	3.048898
6	4.002255	3.005361	2.999156	3.982074	1.991317	1.005123	−2.020698	3.006351
7	4.000130	2.999937	3.000034	4.000004	1.999854	1.000127	−1.999934	2.999947
8	4.000000	3.000000	3.000000	4.000000	2.000000	1.000000	−2.000000	3.000000
Solution	4	3	3	4	2	1	−2	3

We compare our proposed methods with those given in [10] and [12]. For this aim, as in [10] and [12] we define the relative iteration error as

$$\delta(k) = \sqrt{\frac{\|X(k) - X\|^2 + \|Y(k) - Y\|^2}{\|X\|^2 + \|Y\|^2}}.$$

The results computed by using these methods are plotted in Fig. 1. It is seen that the convergence performances of the proposed algorithms in this paper are better than those by using the methods given in [10] and [12].

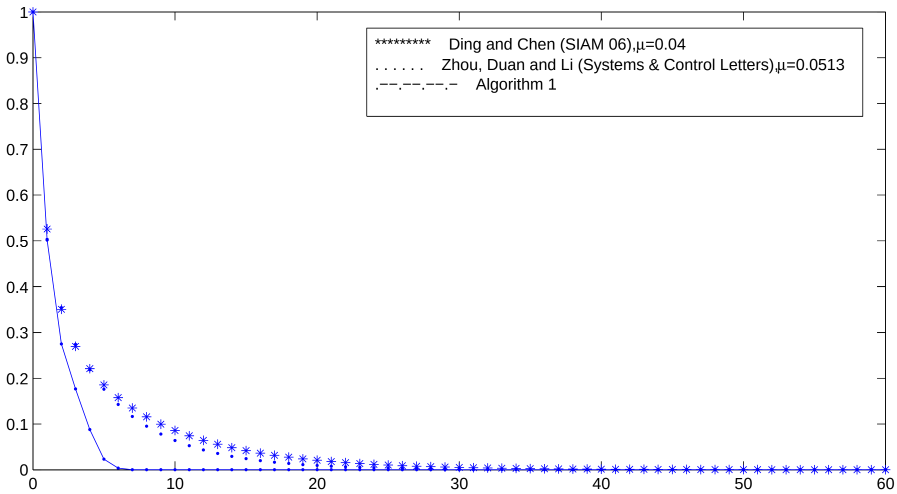

Figure 1: Comparison of different iterative algorithms

6 Conclusions

In this paper, iterative algorithms have been developed for solving a class of coupled linear matrix equations which have wide applications in linear system theory. For the case when the considered coupled matrix equation is consistent, the proposed iterative algorithm is proven to converge the exact solution in finite iteration steps in the absence of round-off errors. For the case when the equation is inconsistent, an iterative algorithm is also proposed to give a least squares solution. Compared with some existing methods, the proposed methods in this paper have some advantage. For example, they do not require that some coefficient matrices have full column or row ranks; they are suitable for solving the coupled linear matrix equations having more than one solution.

7 Appendix

This appendix provides the proofs of some lemmas.

7.1 Proof of Lemma 1

From the expressions of $X(k+1)$ and $Y(k+1)$ in Algorithm 2, one has

$$\begin{aligned}
R(k+1) &= A_1^T\left(E_1 - A_1X(k+1)B_1 - C_1Y(k+1)D_1\right)B_1^T \\
&\quad + A_2^T\left(E_2 - A_2X(k+1)B_2 - C_2Y(k+1)D_2\right)B_2^T \\
&= A_1^T\left(E_1 - A_1X(k)B_1 - C_1Y(k)D_1 - \alpha(k)A_1P(k)B_1 - \alpha(k)C_1Q(k)D_1\right)B_1^T \\
&\quad + A_2^T\left(E_2 - A_2X(k)B_2 - C_2Y(k)D_2 - \alpha(k)A_2P(k)B_2 - \alpha(k)C_2Q(k)D_2\right)B_2^T \\
&= R(k) - \alpha(k)A_1^T\left(A_1P(k)B_1 + C_1Q(k)D_1\right)B_1^T \\
&\quad - \alpha(k)A_2^T\left(A_2P(k)B_2 + C_2Q(k)D_2\right)B_2^T
\end{aligned}$$

and

$$\begin{aligned}
S(k+1) &= S(k) - \alpha(k)C_1^T\left(A_1P(k)B_1 + C_1Q(k)D_1\right)D_1^T \\
&\quad - \alpha(k)C_2^T\left(A_2P(k)B_2 + C_2Q(k)D_2\right)D_2^T.
\end{aligned}$$

In view of the expression of $\alpha(k)$, it is easily obtained that

$$\begin{aligned}
& \operatorname{tr}\left[R^T(k+1)P(k)\right]+\operatorname{tr}\left[S^T(k+1)Q(k)\right] \\
= \; & \operatorname{tr}\left[R^T(k)P(k)\right]+\operatorname{tr}\left[S^T(k)Q(k)\right]-\alpha(k)\operatorname{tr}\left[B_1\left(B_1^TP^T(k)A_1^T+D_1^TQ^T(k)C_1^T\right)A_1P(k)\right] \\
& -\alpha(k)\operatorname{tr}\left[B_2\left(B_2^TP^T(k)A_2^T+D_2^TQ^T(k)C_2^T\right)A_2P(k)\right] \\
& -\alpha(k)\operatorname{tr}\left[D_1\left(B_1^TP^T(k)A_1^T+D_1^TQ^T(k)C_1^T\right)C_1Q(k)\right] \\
& -\alpha(k)\operatorname{tr}\left[D_2\left(B_2^TP^T(k)A_2^T+D_2^TQ^T(k)C_2^T\right)C_2Q(k)\right] \\
= \; & \operatorname{tr}\left[R^T(k)P(k)\right]+\operatorname{tr}\left[S^T(k)Q(k)\right] \\
& -\alpha(k)\left[\left\|A_1P(k)B_1+C_1Q(k)D_1\right\|^2+\left\|A_2P(k)B_2+C_2Q(k)D_2\right\|^2\right] \\
= \; & 0.
\end{aligned}$$

The proof is thus completed.

7.2 Proof of Lemma 2

Before giving the proof of this lemma, we need a simple conclusion.

Lemma 7 *For the sequences $\{P(k)\}$, $\{Q(k)\}$, $\{R(k)\}$, $\{S(k)\}$ generated by Algorithm 2, for $k \geq 0$ the following relation holds:*

$$\operatorname{tr}\left[R^T(k)P(k)\right]+\operatorname{tr}\left[S^T(k)Q(k)\right]=-\left\|R(k)\right\|^2-\left\|S(k)\right\|^2. \tag{10}$$

Proof. Obviously, the relation (10) holds for $k=0$. In addition, from the expressions of $P(k+1)$ and $Q(k+1)$ in Algorithm 2 one has for $k \geq 0$

$$\begin{aligned}
& \operatorname{tr}\left[R^T(k+1)P(k+1)\right]+\operatorname{tr}\left[S^T(k+1)Q(k+1)\right] \\
= \; & -\left\|R(k+1)\right\|^2-\left\|S(k+1)\right\|^2 \\
& +\frac{\left\|R(k+1)\right\|^2+\left\|S(k+1)\right\|^2}{\left\|R(k)\right\|^2+\left\|S(k)\right\|^2}\left[\operatorname{tr}\left[R^T(k+1)P(k)\right]+\operatorname{tr}\left[S^T(k+1)Q(k)\right]\right].
\end{aligned}$$

By applying Lemma 1, it is known that

$$\operatorname{tr}\left[R^T(k+1)P(k+1)\right]+\operatorname{tr}\left[S^T(k+1)Q(k+1)\right]=-\left\|R(k+1)\right\|^2-\left\|S(k+1)\right\|^2.$$

With the above argument, the proof is thus completed. ■

Now we start the proof Lemma 2. Firstly, we denote

$$\pi=\left\|\begin{bmatrix} B_1^T\otimes A_1 & D_1^T\otimes C_1 \\ B_2^T\otimes A_2 & D_2^T\otimes C_2\end{bmatrix}\right\|_2^2. \tag{11}$$

By using Kronecker product, one has

$$\begin{aligned}
& \left\|A_1P(k)B_1+C_1Q(k)D_1\right\|^2+\left\|A_2P(k)B_2+C_2Q(k)D_2\right\|^2 \\
= \; & \left\|\left(B_1^T\otimes A_1\right)\operatorname{vec}(P(k))+\left(D_1^T\otimes C_1\right)\operatorname{vec}(Q(k))\right\|^2 \\
& +\left\|\left(B_2^T\otimes A_2\right)\operatorname{vec}(P(k))+\left(D_2^T\otimes C_2\right)\operatorname{vec}(Q(k))\right\|^2 \\
= \; & \left\|\begin{matrix}\left(B_1^T\otimes A_1\right)\operatorname{vec}(P(k))+\left(D_1^T\otimes C_1\right)\operatorname{vec}(Q(k)) \\ \left(B_2^T\otimes A_2\right)\operatorname{vec}(P(k))+\left(D_2^T\otimes C_2\right)\operatorname{vec}(Q(k))\end{matrix}\right\|^2 \\
= \; & \left\|\begin{bmatrix} B_1^T\otimes A_1 & D_1^T\otimes C_1 \\ B_2^T\otimes A_2 & D_2^T\otimes C_2\end{bmatrix}\begin{bmatrix}\operatorname{vec}(P(k)) \\ \operatorname{vec}(Q(k))\end{bmatrix}\right\|^2 \\
\leq \; & \left\|\begin{bmatrix} B_1^T\otimes A_1 & D_1^T\otimes C_1 \\ B_2^T\otimes A_2 & D_2^T\otimes C_2\end{bmatrix}\right\|_2^2\left\|\begin{bmatrix}\operatorname{vec}(P(k)) \\ \operatorname{vec}(Q(k))\end{bmatrix}\right\|^2 \\
= \; & \pi\left(\left\|P(k)\right\|^2+\left\|Q(k)\right\|^2\right).
\end{aligned}$$

With the notation of (11), one has

$$\begin{aligned} & \|A_1P(k)B_1 + C_1Q(k)D_1\|^2 + \|A_2P(k)B_2 + C_2Q(k)D_2\|^2 \\ \leq\ & \pi\left(\|P(k)\|^2 + \|Q(k)\|^2\right). \end{aligned} \tag{12}$$

Applying the expressions in Algorithm 1 and Lemma 7, for $k \geq 0$ one has

$$\begin{aligned} & J\left(X(k+1), Y(k+1)\right) \\ =\ & \frac{1}{2}\left\|\bar{R}\left(k\right) - \alpha(k)\left[A_1P(k)B_1 + C_1Q(k)D_1\right]\right\|^2 \\ & + \frac{1}{2}\left\|\bar{S}\left(k\right) - \alpha(k)\left[A_2P(k)B_2 + C_2Q(k)D_2\right]\right\|^2 \\ =\ & \frac{1}{2}\left(\left\|\bar{R}\left(k\right)\right\|^2 + \left\|\bar{S}\left(k\right)\right\|^2\right) - \alpha(k)\mathrm{tr}\left[\bar{R}^T\left(k\right)\left(A_1P(k)B_1 + C_1Q(k)D_1\right)\right] \\ & - \alpha(k)\mathrm{tr}\left[\bar{S}^T\left(k\right)\left(A_2P(k)B_2 + C_2Q(k)D_2\right)\right] \\ & + \frac{1}{2}\alpha^2\left(k\right)\|A_1P(k)B_1 + C_1Q(k)D_1\|^2 + \frac{1}{2}\alpha^2\left(k\right)\|A_2P(k)B_2 + C_2Q(k)D_2\|^2 \\ =\ & J\left(X(k), Y(k)\right) - \alpha(k)\mathrm{tr}\left[P^T\left(k\right)A_1^T\bar{R}(k)B_1^T\right] - \alpha(k)\mathrm{tr}\left[Q^T\left(k\right)C_1^T\bar{R}(k)D_1^T\right] \\ & - \alpha(k)\mathrm{tr}\left[P^T\left(k\right)A_2^T\bar{S}(k)B_2^T\right] - \alpha(k)\mathrm{tr}\left[Q^T\left(k\right)C_2^T\bar{S}(k)D_2^T\right] \\ & + \frac{1}{2}\alpha^2\left(k\right)\|A_1P(k)B_1 + C_1Q(k)D_1\|^2 + \frac{1}{2}\alpha^2\left(k\right)\|A_2P(k)B_2 + C_2Q(k)D_2\|^2 \\ =\ & J\left(X(k), Y(k)\right) - \alpha(k)\mathrm{tr}\left[P^T\left(k\right)R\left(k\right)\right] - \alpha(k)\mathrm{tr}\left[Q^T\left(k\right)S\left(k\right)\right] \\ & + \frac{1}{2}\alpha\left(k\right)\left[\mathrm{tr}\left[P^T\left(k\right)R\left(k\right)\right] + \mathrm{tr}\left[Q^T\left(k\right)S\left(k\right)\right]\right] \\ =\ & J\left(X(k), Y(k)\right) - \frac{1}{2}\alpha\left(k\right)\left[\mathrm{tr}\left[P^T\left(k\right)R\left(k\right)\right] + \mathrm{tr}\left[Q^T\left(k\right)S\left(k\right)\right]\right]. \end{aligned}$$

Again using the expression of $\alpha(k)$ one has

$$\begin{aligned} & J\left(X(k+1), Y(k+1)\right) - J\left(X(k), Y(k)\right) \\ =\ & -\frac{1}{2}\frac{\left(\|R(k)\|^2 + \|S(k)\|^2\right)^2}{\|A_1P(k)B_1 + C_1Q(k)D_1\|^2 + \|A_2P(k)B_2 + C_2Q(k)D_2\|^2} \\ \leq\ & 0. \end{aligned} \tag{13}$$

This implies that $\{J(X(k), Y(k))\}$ is a descent sequence. Therefore, for all $k \geq 1$ there holds

$$J(X(k), Y(k)) \leq J(X(0), Y(0)).$$

Then one has

$$\begin{aligned} & \sum_{k=0}^{\infty}\left[J(X(k), Y(k)) - J(X(k+1), Y(k+1))\right] \\ =\ & J(X(0), Y(0)) - \lim_{k\to\infty} J(X(k), Y(k)) \\ <\ & \infty. \end{aligned} \tag{14}$$

In addition, combining (12) with (13) gives

$$\begin{aligned} & \frac{\left(\|R(k)\|^2 + \|S(k)\|^2\right)^2}{\|P(k)\|^2 + \|Q(k)\|^2} \\ \leq\ & 2\pi\left(J(X(k), Y(k)) - J(X(k+1), Y(k+1))\right). \end{aligned}$$

This relation, together with (29), shows that the conclusion of this lemma is true.

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ESSENTIAL NORM OF WEIGHTED COMPOSITION OPERATOR FROM HARDY SPACE TO WEIGHTED BERGMAN SPACE ON THE UNIT BALL

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Abstract. In this paper, we estimate the essential norm of a weighted composition operator from Hardy space to weighted Bergman space on the unit ball in the complex N-dimensional Euclidean space. In our results, we use Carleson type measures and certain integral operators to estimate the essential norm.

1. Introduction

Throughout this paper, let $N \geq 1$ be a fixed integer and B denote the unit ball of the complex N-dimensional Euclidean space C^N. Denote $H(B)$ the class of all holomorphic functions on B and $S(B)$ the collection of all holomorphic self-maps of B. Let $d\nu$ denote the normalized Lebesgue measure on B. For each $\alpha > -1$, we set $c_\alpha = \Gamma(N+\alpha+1)/\{\Gamma(N+1)\Gamma(\alpha+1)\}$ and $d\nu_\alpha(z) = c_\alpha(1-|z|^2)^\alpha d\nu(z), z \in B$. It is obvious that $\nu_\alpha(B) = 1$.

For $0 < p < \infty$ and $\alpha > -1$, the weighted Bergman space $A^p_\alpha = A^p_\alpha(B)$ is defined by

$$A^p_\alpha = \{f \in H(B):\ \|f\|^p_{A^p_\alpha} = \int_B |f(z)|^p d\nu_\alpha(z) < \infty\}.$$

It is well-known that the weighted Bergman space A^p_α is a Banach space under the norm $\|.\|_{A^p_\alpha}$.

For $1 \leq q < \infty$, the Hardy space $H^q = H^q(B)$ is

$$H^q = \{f \in H(B):\ \|f\|^q_{H^q} = \sup_{0<r<1} \int_{\partial B} |f(r\zeta)|^q d\sigma(\zeta) < \infty\},$$

where $d\sigma$ is the normalized Lebesgue measure on the boundary ∂B of B. Analogously, the Hardy space H^q is a Banach space under the norm $\|.\|_{H^q}$.

Let $\varphi \in S(B)$, the composition operator C_φ induced by φ is defined as follows

$$(C_\varphi f)(z) = f(\varphi(z)),\ \ f \in H(B),\ z \in B.$$

This operator is well studied for many years, readers interested in this topic can refer to the books [6] by Shapiro, [2] by Cowen and MacCluer, and [10, 11] by K. H. Zhu, which are

The work was supported in part by the National Natural Science Foundation of China (Grant Nos. 11371276; 11301373; 11201331).
*Corresponding author.
2010 Mathematics Subject Classification. Primary: 47B38; Secondary: 32A36, 32H02, 47B33.
Key words and phrases. weighted composition operator, Hardy space, weighted Bergman space, essential norm, compact.

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excellent sources for the development of the theory of composition operators and function spaces.

For $u \in H(B)$ and $\varphi \in S(B)$, the weighted composition operator uC_φ is

$$uC_\varphi(f)(z) = u(z)f(\varphi(z)), \text{ for } f \in H(B).$$

In particular, if $u \equiv 1$, then uC_φ becomes the composition operator C_φ. In the special case that φ is the identity mapping of B, uC_φ is called the multiplication operator and is denoted by M_u.

Let X and Y be Banach spaces. For a bounded linear operator $T : X \to Y$, the essential norm $\|T\|_{e,X\to Y}$ is defined to be the distance from T to the set of the compact operators $\mathcal{K}$, namely,

$$\|T\|_{e,X\to Y} = \inf\{\|T - \mathcal{K}\| : \mathcal{K} \text{ is compact from } X \text{ into } Y\},$$

where $\|.\|$ denotes the usual operator norm. Clearly, T is compact if and only if $\|T\|_{e,X\to Y} = 0$. Thus, the essential norm is closely related to the compactness problem of concrete operators.

Recently, the essential norm of weighted composition operator between Hardy spaces is investigated in [8]. Moreover, the essential norm of weighted composition operator between weighted Bergman spaces is discussed in [5]. The paper [7] considered the boundedness and compactness of weighted composition operators from weighted Bergman space to weighted Hardy space on the unit ball. Very recently, the first author estimated the essential norms of weighted composition operator acting from weighted Bergman space to mixed-norm space in [9]. So by building on those foundations, the present paper continues this line of research, we mainly estimate the essential norm of the weighted composition operator acting from Hardy space H^q to weighted Bergman space A^p_α on the unit ball. The remainder is assembled as follows: In section 2, we state a couple of lemmas. Our main theorems are given in section 3.

Throughout the remainder of this paper, C will denote a positive constant, the exact value of which will vary from one appearance to the next. The notation $A \asymp B$ means that there is a positive constant C such that $B/C \le A \le CB$.

2. Some Lemmas

To begin the discussion, let us state a couple of lemmas, which are used in the proof of the main results.

For each $\zeta \in \partial B$, and $t > 0$, let

$$S(\zeta, t) = \{z \in \overline{B} : |1 - \langle z, \zeta\rangle| < t\}, \text{ where } \overline{B} = B \cup \partial B.$$

$$B(\zeta, t) = S(\zeta, t) \cap B, \;\; Q(\zeta, t) = S(\zeta, t) \cap \partial B.$$

It is well known that $\sigma(Q(\zeta, t))$ is comparable to t^N, see, e.g. [4, P. 67].

Lemma 1. [8, Lemma 2.3] *Let $1 < q \le p < \infty$, suppose that μ is a positive Borel measure on $\overline{B}$ such that*

$$\mu(S(\zeta, t)) \le Ct^{pN/q} \; (\zeta \in \partial B, \; t > 0),$$

for some constant $C > 0$. Then there exists a constant $K > 0$ such that

$$\int_{\overline{B}} |f^*|^p d\mu \le K\|f\|^p_{H^q}$$

for all $f \in H^q$. Here, the notation f^* denotes the function defined on $\overline{B}$ by $f^* = f$ in B and $f^* = \lim_{r\to 1} f_r$ a.e. $[\sigma]$ on ∂B.

For $f \in H_q$, recall that an $f \in H(B)$ has the homogeneous expansion $f(z) = \sum_{k=0}^{\infty} \sum_{|r|=k} c(r)z^r$, where $r = (r_1, ..., r_N)$ is a multi-index, $|r| = r_1 + ... + r_N$ and $z^r = z_1^{r_1}...z_N^{r_N}$. For the homogeneous expansion of f and any integer $n \geq 1$, let

$$R_n f(z) = \sum_{k=n}^{\infty} \sum_{|r|=k} c(r)z^r, \text{ and } K_n = I - R_n,$$

where $If = f$ is the identity operator.

Lemma 2. [8, Corollary 3.4] *If $1 < q < \infty$, then R_n converges to 0 pointwise in H_q as $n \to \infty$. Moreover,* $\sup\{\|R_n\| : n \geq 1\} < \infty$.

Lemma 3. [8, Lemma 3.5] *Let $1 < q < \infty$. For each $f \in H^q$ and $n \geq 1$,*

$$|R_n f(z)| \leq \|f\|_{H^q} \sum_{k=n}^{\infty} \frac{\Gamma(N+k)}{k!\Gamma(N)} |z|^k.$$

By a similar proof of [2, Lemma 3.16], we can obtain the following lemma.

Lemma 4. *Let $1 < p < \infty$. If uC_φ is bounded from H^q into A^p_α, then*

$$\|uC_\varphi\|_{e,H^q \to A^p_\alpha} \leq \liminf_{n \to \infty} \|uC_\varphi R_n\|_{H^q \to A^p_\alpha}.$$

3. Main Results

Before proving our main results, we need to prepare the notation about the measure. Let $1 < q < \infty$ and $\alpha > -1$. For $\varphi \in S(B)$ and $u \in A^p_\alpha$, we define a finite positive Borel measure $\mu_{\varphi,u}$ on B by

$$\mu_{\varphi,u}(E) = \int_{\varphi^{-1}(E)} |u|^q d\nu_\alpha,$$

for all Borel sets E of B. By a change of variables formula from measure theory, we can verify

$$\int_B g d\mu_{\varphi,u} = \int_B |u|^q (g \circ \varphi) d\nu_\alpha,$$

for each nonnegative measurable function g on B.

Theorem 1. *Let $1 < q \leq p < \infty$, and $\alpha > -1$. Suppose that uC_φ is bounded from H^q into A^p_α. Then*

$$\begin{aligned} \|uC_\varphi\|^p_{e,H^q \to A^p_\alpha} &\asymp \limsup_{|w| \to 1^-} \int_B |u(z)|^p \left(\frac{1 - |w|^2}{|1 - \langle \varphi(z), w \rangle|^2} \right)^{pN/q} d\nu_\alpha(z) \\ &\asymp \limsup_{t \to 0^+} \sup_{\zeta \in \partial B} \frac{\mu_{\varphi,u}(S(\zeta, t))}{t^{pN/q}}. \end{aligned} \tag{3.1}$$

Proof. The lower estimates. For each $w \in B$, define the function f_w on $\overline{B}$ by

$$f_w(z) = \left(\frac{1 - |w|^2}{(1 - \langle z, w \rangle)^2} \right)^{N/q}. \tag{3.2}$$

By [11, Theorem 1.12] we obtain that

$$\begin{aligned} \|f_w\|^q_{H^q} &= \sup_{0<r<1} \int_{\partial B} \left(\frac{1 - |w|^2}{|1 - \langle r\zeta, w \rangle|^2} \right)^N d\sigma(\zeta) \\ &\leq \sup_{0<r<1} \frac{(1 - |w|^2)^N}{(1 - r|w|^2)^N} \leq 1. \end{aligned}$$

Then the functions $\{f_w : w \in B\} \subset H^q$ belong to the ball algebra $A(B)$ and form a bounded sequence of H^q. For a compact operator $\mathcal{K} : H_q \to A^p_\alpha$ arbitrarily. Since the sequence $\{f_w\}$ converges to 0 uniformly on compact subsets of B as $|w| \to 1$, therefore $\|\mathcal{K}f_w\|_{A^p_\alpha} \to 0$ as $|w| \to 1^-$. Consequently, we infer that

$$C\|uC_\varphi - \mathcal{K}\|_{H^q\to A^p_\alpha} \geq \limsup_{|w|\to 1^-} \|(uC_\varphi - \mathcal{K})f_w\|_{A^p_\alpha} \geq \limsup_{|w|\to 1^-} \|uC_\varphi f_w\|_{A^p_\alpha}. \tag{3.3}$$

By the definition of f_w, it is evident that

$$\|uC_\varphi f_w\|^p_{A^p_\alpha} = \int_B |u(z)|^p \left(\frac{1-|w|^2}{|1-\langle \varphi(z), w\rangle|^2}\right)^{pN/q} d\nu_\alpha(z). \tag{3.4}$$

Combining (3.3) and (3.4), we get that

$$\|uC_\varphi - \mathcal{K}\|^p_{H^q\to A^p_\alpha} \geq C \limsup_{|w|\to 1^-} \int_B |u(z)|^p \left(\frac{1-|w|^2}{|1-\langle \varphi(z), w\rangle|^2}\right)^{pN/q} d\nu_\alpha(z). \tag{3.5}$$

Since this holds for every compact operator $\mathcal{K}$, it is clear that

$$\|uC_\varphi\|^p_{e,H^q\to A^p_\alpha} \geq C \limsup_{|w|\to 1^-} \int_B |u(z)|^p \left(\frac{1-|w|^2}{|1-\langle \varphi(z), w\rangle|^2}\right)^{pN/q} d\nu_\alpha(z). \tag{3.6}$$

Furthermore, putting $w = (1-t)\zeta$, for $0 < t < 1$ and $\zeta \in \partial B$, it is easy to check

$$\begin{aligned} |f_{(1-t)\zeta}(z)| &= \left(\frac{2t-t^2}{|1-(1-t)\langle z, \zeta\rangle|^2}\right)^{N/q} \\ &\geq \left(\frac{2t-t^2}{(|1-\langle z, \zeta\rangle| + t|\langle z, \zeta\rangle|)^2}\right)^{N/q} \\ &\geq \left(\frac{2t-t^2}{2t^2}\right)^{N/q} \geq Ct^{-N/q}. \end{aligned}$$

That is, $|f_{(1-t)\zeta}(z)|^p > Ct^{-pN/q}$ for all $z \in S(\zeta, t)$, we have

$$C \sup_{\zeta\in\partial B} \frac{\mu_{\varphi,u}(S(\zeta,t))}{t^{pN/q}} \leq \sup_{\zeta\in\partial B} \int_{S(\zeta,t)} |f_{(1-t)\zeta}|^p d\mu_{\varphi,u} \leq \sup_{\zeta\in\partial B} \|uC_\varphi f_{(1-t)\zeta}\|^p_{A^p_\alpha}. \tag{3.7}$$

Letting $t \to 0^+$, thus $|w| = |(1-t)\zeta| \to 1^-$, we obtain

$$C \limsup_{t\to 0^+} \sup_{\zeta\in\partial B} \frac{\mu_{\varphi,u}(S(\zeta,t))}{t^{pN/q}} \leq \limsup_{t\to 0^+} \sup_{\zeta\in\partial B} \|uC_\varphi f_{(1-t)\zeta}\|^p_{A^p_\alpha}. \tag{3.8}$$

Combining (3.8) with (3.6) and (3.4) we obtain

$$C \limsup_{t\to 0^+} \sup_{\zeta\in\partial B} \frac{\mu_{\varphi,u}(S(\zeta,t))}{t^{pN/q}} \leq \|uC_\varphi\|^p_{e,H^q\to A^p_\alpha}. \tag{3.9}$$

Then by (3.6) and (3.9), we complete the proof of the lower estimates.

The upper estimates. For the sake of convenience, set

$$M_1 := \limsup_{|w|\to 1^-} \int_B |u(z)|^p \left(\frac{1-|w|^2}{|1-\langle \varphi(z), w\rangle|^2}\right)^{pN/q} d\nu_\alpha(z) \tag{3.10}$$

and

$$M_2 := \limsup_{t\to 0^+} \sup_{\zeta\in\partial B} \frac{\mu_{\varphi,u}(S(\zeta,t))}{t^{pN/q}}. \tag{3.11}$$

$$D(\zeta, t) = \{z \in \overline{B} : \ |z| > 1-t, \ \frac{z}{|z|} \in Q(\zeta, t)\}. \tag{3.12}$$

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By (3.10), for a given $\varepsilon > 0$, choose an $R_1 \in (0,1)$ such that

$$\int_B |u(z)|^p \left(\frac{1-|w|^2}{|1-\langle\varphi(z),w\rangle|^2}\right)^{pN/q} d\nu_\alpha(z) < M_1 + \varepsilon, \tag{3.13}$$

for $w \in B$ with $|w| \geq R_1$. For each $\zeta \in \partial B$ and $0 < t \leq 1 - R_1 \equiv t_1$, we denote $w_1 = (1-t)\zeta \in B$. Since the function

$$f_{w_1}(z) = \left(\frac{1-|w_1|^2}{(1-\langle z,w\rangle)^2}\right)^{N/q}$$

satisfies $|f_{w_1}(z)|^q > 4^{-N}t^{-N}$ for all $z \in S(\zeta,t)$, the inequality (3.13) shows that

$$\begin{aligned}\frac{\mu_{\varphi,u}(S(\zeta,t))}{t^{pN/q}} &< C\int_{S(\zeta,t)} |f_{w_1}(z)|^p d\mu_{\varphi,u}(z)\\ &\leq \int_B |u(z)|^p (f_{w_1}\circ\varphi(z)) d\nu_\alpha(z)\\ &= \int_B |u(z)|^p \left(\frac{1-|w_1|^2}{|1-\langle\varphi(z),w_1\rangle|^2}\right)^{pN/q} d\nu_\alpha(z)\\ &\leq C(M_1+\varepsilon),\end{aligned} \tag{3.14}$$

for all $\zeta \in \partial B$ and $0 < t < t_1$.

By (3.11), choose $0 < t_2 < 1$ such that

$$\sup_{\zeta\in\partial B} \frac{\mu_{\varphi,u}(S(\zeta,t))}{t^{pN/q}} < M_2 + \varepsilon, \tag{3.15}$$

for all $0 < t < t_2$. Let μ_1 and μ_2 be the restrictions of $\mu_{\varphi,u}$ to $\overline{B}\setminus(1-t_1)\overline{B}$ and $\overline{B}\setminus(1-t_2)\overline{B}$, respectively. Using the same argument of the proof in [1, Theorem 1.1 (ii)], there exists constant $C > 0$ which depend only on p, q, α and N such that $\mu_j (j = 1,2)$ also satisfies the Carleson measure condition

$$\mu_j(S(\zeta,t)) \leq C(M_j+\varepsilon)t^{pN/q} \tag{3.16}$$

for all $\zeta \in \partial B$ and $t > 0$. By (3.14) and (3.15) we obtain that (3.16) is true for all $0 < t \leq t_j$. Hence, we assume that $t > t_j$. For a finite cover $\{Q(w_k, t_j/3)\}$, where $w_k \in Q(\zeta,t)$ of the set $\overline{Q}(\zeta,t) = \{z \in \partial B : |1-\langle z,\zeta\rangle| \leq t\}$, the covering property implies that there exists a disjoint subcollection Γ of $\{Q(w_k,t_j/3)\}$ satisfying

$$Q(\zeta,t) \subset \cup_\Gamma Q(w_k,t_j). \tag{3.17}$$

Moreover, we obtain card $(\Gamma) \leq C(t/t_j)^N$. By the notation (3.12), it follows that

$$\begin{aligned}\mu_j(S(\zeta,t)) &\leq \mu_j(D(\zeta,t)) \leq \sum_\Gamma \mu_j(D(w_k,t_j))\\ &\leq \sum_\Gamma \mu_j(S(w_k,2t_j)) \leq C(t/t_j)^N(M_j+\varepsilon)t_j^{pN/q}\\ &= C(M_j+\varepsilon)t^N t_j^{(p/q-1)N} \leq C(M_j+\varepsilon)t^{pN/q},\end{aligned} \tag{3.18}$$

where the constant C depends only on p, q, α and the dimension N. Now, taking a function $f \in H^q$ with $\|f\|_{H^q} \leq 1$, it follows that

$$\begin{aligned} \|uC_\varphi R_n f\|^p_{A^p_\alpha} &= \int_B |u(z)|^p |R_n f(\varphi(z))|^p d\nu_\alpha(z) \\ &= \int_B |R_n f(z)|^p d\mu_{\varphi,u}(z) \\ &= \left(\int_{B \setminus (1-t_j)\overline{B}} + \int_{(1-t_j)\overline{B}} \right) |R_n f(z)|^p d\mu_{\varphi,u}(z) \\ &= \int_B |R_n f(z)|^p d\mu_j(z) + \int_{(1-t_j)\overline{B}} |R_n f(z)|^p d\mu_{\varphi,u}(z) \end{aligned} \tag{3.19}$$

for integers $n \geq 1$. Combining (3.16) with lemma 1 it follows that

$$\int_B |R_n f(z)|^p d\mu_j(z) \leq C(M_j + \varepsilon) \|R_n f\|^p_{H^q} \leq C \sup_{n \geq 1} \|R_n\|^p (M_j + \varepsilon). \tag{3.20}$$

On the other hand, by lemma 3, we have

$$\int_{(1-t_j)\overline{B}} |R_n f(z)|^p d\mu_{\varphi,u}(z) \leq \|f\|^p_{H^q} \left(\sum_{k=n}^{\infty} \frac{\Gamma(N+k)}{k!\Gamma(N)} |1-t_j|^k \right)^p \|u\|^p_{A^p_\alpha}. \tag{3.21}$$

By the boundedness of uC_φ and taking $f(z) = 1 \in H^q$, it is evident that $u \in A^p_\alpha$. Furthermore, the convergence of the series $\sum (\Gamma(N+k))/(k!\Gamma(N))|1-t_j|^k$ implies

$$\sum_{k=n}^{\infty} \frac{\Gamma(N+k)}{k!\Gamma(N)} |1-t_j|^k \to 0, \ n \to \infty. \tag{3.22}$$

Thus combining (3.21) and (3.22) it is clear that

$$\int_{(1-t_j)\overline{B}} |R_n f(z)|^p d\mu_{\varphi,u}(z) \to 0, \ n \to \infty. \tag{3.23}$$

Combining (3.19), (3.20) and (3.23) with lemma 4, it follows that

$$\|uC_\varphi\|^p_{e,H^q \to A^p_\alpha} \leq \liminf_{n \to \infty} \|uC_\varphi R_n\|^p_{H^q \to A^p_\alpha} \leq C \sup_{n \geq 1} \|R_n\|^p (M_j + \varepsilon). \tag{3.24}$$

Since $\sup_{n \geq 1} \|R_n\| < \infty$ from lemma 2. Moreover, since $\varepsilon > 0$ is arbitrary, we conclude that

$$\|uC_\varphi\|^p_{e,H^q \to A^p_\alpha} \leq C \lim \sup_{|w| \to 1^-} \int_B |u(z)|^p \left(\frac{1-|w|^2}{|1-\langle \varphi(z), w \rangle|^2} \right)^{pN/q} d\nu_\alpha(z) \tag{3.25}$$

and

$$\|uC_\varphi\|^p_{e,H^q \to A^p_\alpha} \leq C \lim \sup_{t \to 0^+} \sup_{\zeta \in \partial B} \frac{\mu_{\varphi,u}(S(\zeta,t))}{t^{pN/q}}, \tag{3.26}$$

completing the upper estimates. This finishes the proof. □

The following corollary is an immediate consequence of the above theorem.

Corollary 1. *Let $1 < q \leq p < \infty$, and $\alpha > -1$. Suppose that uC_φ is bounded from H^q into A^p_α. The following conditions are equivalent:*
(a) $uC_\varphi : H^q \to A^p_\alpha$ is compact;
(b) u and φ satisfy

$$\limsup_{|w| \to 1^-} \int_B |u(z)|^p \left(\frac{1-|w|^2}{|1-\langle \varphi(z), w \rangle|^2} \right)^{pN/q} d\nu_\alpha(z) = 0;$$

(c) u and φ satisfy

$$\limsup_{t\to 0^+} \sup_{\zeta\in\partial B} \frac{\mu_{\varphi,u}(S(\zeta,t))}{t^{pN/q}} = 0.$$

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ON THE q-ANALOGUE OF λ-DAEHEE POLYNOMIALS

JIN-WOO PARK

Abstract. Recently, Daehee polynomials and λ-Daehee polynomials are introduced in [3, 4]. In this paper, we study the q-analogue of λ-Daehee polynomials of first kind or second kind. In addition, we investigate their properties arising from the p-adic q-integral equations.

1. Introduction

Let d be fixed positive integer and let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will respectively denote the ring of p-adic rational integers, the field of p-adic rational numbers and the completions of algebraic closure of $\mathbb{Q}_p$. the p-adic norm is defined $|p|_p = \frac{1}{p}$

We set

$$X_d = \lim_{\overleftarrow{N}} \mathbb{Z}/dp^N\mathbb{Z},\ X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} (a + dp\mathbb{Z}_p),$$

$$a + dp^N\mathbb{Z}_p = \left\{x \in X | x \equiv a \pmod{dp^N}\right\},$$

where $a \in \mathbb{Z}$ and $0 \leq a < dp^n$.

When one talks of q-extension, q is various considered as an indeterminate, a complex $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for each $x \in \mathbb{Z}_p$. Throughout this paper, we use the notation :

$$[x]_q = \frac{1-q^x}{1-q}.$$

Hence, $\lim_{q\to 1}[x]_q = x$ for each $x \in \mathbb{Z}_p$.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the *p-adic q-integral on $\mathbb{Z}_p$* is defined by Kim as follows :

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)q^x,\ \text{(see [5, 6])}. \tag{1.1}$$

Let $f_1(x) = f(x+1)$. Then, by (1.4), we can get the following well-known integral identity

$$-qI_q(f_1) + I_q(f) = (1-q)f(0) + \frac{1-q}{\log q} f'(0), \tag{1.2}$$

where $f'(0) = \frac{df(x)}{dx}|_{x=0}$.

1991 *Mathematics Subject Classification.* 11B68, 11S40, 11S80.

Key words and phrases. λ-Daehee polynomials, q-Daehee polynomials, q-extension of λ-Daehee polynomials first kind, q-extension of λ-Daehee polynomials second kind.

The *Stirling number of the first kind* is defined by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l)x^l,\ (n \geq 0), \tag{1.3}$$

and the *Stirling numbers of the second kind* is defined by

$$(e^t-1)^n = n!\sum_{l=n}^{\infty} S_2(l,n)\frac{t^l}{l!},$$

(see [2, 10]).

It is well known that the *q-Bernoulli polynomials of order k* are defined by the generating function to be

$$\left(\frac{q-1+\frac{(q-1)}{\log q}t}{qe^t-1}\right)^k e^{xt} = \sum_{n=0}^{\infty} B_{n,q}^{(k)}(x)\frac{t^n}{n!},\ (\text{see [1, 9]}). \tag{1.4}$$

When $x=0$, $B_{n,q}=B_{n,q}(0)$ are called the *$n-th$ q-Bernoulli numbers.*

The *Daehee polynomials of the first kind* are defined by the generating function to be

$$\frac{\log(1+t)}{x}(1+t)^x = \sum_{n=0}^{\infty} D_n(x)\frac{t^n}{n!},$$

and the *Daehee polynomials of the second kind* are given by

$$\frac{\log(1+t)}{x+1}(1+t)^x = \sum_{n=0}^{\infty} \widehat{D}_n(x)\frac{t^n}{n!},$$

(see [3, 7]), and in [1], authors defined the *q-Daehee polynomials* as follows.

$$\frac{1-q+\frac{1-q}{\log q}\log(1+t)}{1-q-qt}(1+t)^x = \sum_{n=0}^{\infty} D_{n,q}(x)\frac{t^n}{n!}.$$

Recently, Daehee numbers and polynomials are introduced by Kim et. al., and found interesting identities(see [1, 3, 4, 7-12]). In [4], D. S. Kim et. al. considered the λ-Daehee polynomials and investigate their properties. In this paper, we derive the q-analog of the λ-Daehee polynomials and found some interesting identities and properties.

2. q-analogue of λ-Daehee polynomials of the first kind

In this section, we assume that $t \in \mathbb{C}$ with $|t|_p < p^{-\frac{1}{p-1}}$ and $\lambda \in \mathbb{Z}_p$.

Now, we consider the *λ-q-Daehee polynomials* which are a generalization of Daehee polynomials as follows:

$$\frac{q-1+\frac{q-1}{\log q}\lambda\log(1+t)}{q(1+t)^\lambda-1}(1+t)^x = \sum_{n=0}^{\infty} D_{n,\lambda,q}(x)\frac{t^n}{n!}. \tag{2.1}$$

When $x=0$, $D_{n,\lambda,q}=D_{n,\lambda,q}(0)$ are called the *λ-q-Daehee numbers.* It is easy to see that $D_{n,q}(x)=D_{n,1,q}(x)$ and $D_{n,q}=D_{n,1,q}$.

Let us take $f(x)=(1+t)^{\lambda t}$. From (1.2), we have

$$\int_{\mathbb{Z}_p}(1+t)^{\lambda x}d\mu_q(x) = \frac{q-1+\frac{q-1}{\log q}\lambda\log(1+t)}{q(1+t)^\lambda-1}. \tag{2.2}$$

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By (2.2), we get

$$\begin{aligned}\sum_{n=0}^{\infty} D_{n,\lambda,q}(x)\frac{t^n}{n!} &= \frac{q-1+\frac{q-1}{\log q}\lambda\log(1+t)}{q(1+t)^{\lambda}-1}(1+t)^x \\ &= \int_{\mathbb{Z}_p}(1+t)^{\lambda y+x}d\mu_q(x),\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}\int_{\mathbb{Z}_p}(1+t)^{\lambda y+x}d\mu_q(y) &= \sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\binom{\lambda y+x}{n}d\mu_q(y)t^n \\ &= \sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}(\lambda y+x)_n d\mu_q(y)\frac{t^n}{n!}.\end{aligned} \tag{2.4}$$

Thus, by (2.3) and (2.4), we obtain the following theorem.

Theorem 2.1. *For $n \geq 0$, we have*

$$D_{n,\lambda,q}(x) = \int_{\mathbb{Z}_p}(\lambda y+x)_n d\mu_q(y).$$

By replacing t by e^t-1 in (2.1), we get

$$\begin{aligned}\frac{q-1+\frac{q-1}{\log q}\lambda t}{qe^{\lambda t}-1}e^{tx} &= \sum_{n=0}^{\infty} D_{n,\lambda,q}(x)\frac{1}{n!}\left(e^t-1\right)^n \\ &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} D_{n,\lambda,q}(x)S_2(m,n)\right)\frac{t^m}{m!}\end{aligned} \tag{2.5}$$

and

$$\frac{q-1+\frac{q-1}{\log q}\lambda t}{qe^{\lambda t}-1}e^{tx} = \frac{q-1+\frac{q-1}{\log q}\lambda t}{qe^{\lambda t}-1}e^{\lambda t\left(\frac{\lambda}{x}\right)} = \sum_{n=0}^{\infty}\lambda^n B_{n,q}\left(\frac{x}{\lambda}\right)\frac{t^n}{n!}. \tag{2.6}$$

By (1.3) and Theorem 2.1,

$$\begin{aligned}D_{n,\lambda,q} &= \int_{\mathbb{Z}_p}(\lambda y+x)_n d\mu_q(y) \\ &= \sum_{l=0}^{n} S_1(n,l)\lambda^l\int_{\mathbb{Z}_p}\left(y+\frac{x}{\lambda}\right)d\mu_q(y) \\ &= \sum_{l-0}^{n} S_1(n,l)\lambda^l B_l\left(\frac{x}{\lambda}\right).\end{aligned} \tag{2.7}$$

Therefore, by (2.5), (2.6) and (2.7), we obtain the following theorem.

Theorem 2.2. *For $n \geq 0$, we have*

$$D_{n,\lambda,q}(x) = \sum_{l=0}^{n} S_1(n,l)\lambda^l B_{l,q}\left(\frac{x}{\lambda}\right),$$

and

$$\lambda^m B_{m,q}\left(\frac{x}{\lambda}\right) = \sum_{n=0}^{m} S_2(m,n)D_{n,\lambda,q}(x).$$

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Let us consider the *λ-q-Daehee polynomials of the first kind* with order $k(\in \mathbb{N})$ as follows:

$$D_{n,\lambda,q}^{(k)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (\lambda x_1 + \cdots + \lambda x_k + x)_n d\mu_q(x_1) \cdots d\mu_q(x_k). \tag{2.8}$$

From (2.8), we can derive the generating function of $D_{n,\lambda,q}^{(k)}(x)$ as follows:

$$\begin{aligned}
&\sum_{n=0}^{\infty} D_{n,\lambda,q}^{(k)}(x)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (\lambda x_1 + \cdots + \lambda x_k + x)_n d\mu_q(x_1) \cdots d\mu_q(x_k)\frac{t^n}{n!}\\
&=\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{\lambda x_1 + \cdots + \lambda x_k + x} d\mu_q(x_1) \cdots d\mu_q(x_k)\\
&=\left(\frac{q-1+\frac{q-1}{\log q}\lambda\log(1+t)}{q(1+t)^{\lambda}-1}\right)^k (1+t)^x.
\end{aligned} \tag{2.9}$$

Note that by (2.8),

$$\begin{aligned}
D_{n,\lambda,q}(x) &= \sum_{l=0}^{n}\lambda^l \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left(x_1 + \cdots + x_k + \frac{x}{\lambda}\right)^l d\mu_q(x_1) \cdots d\mu_q(x_k)\\
&= \sum_{l=0}^{n} S_1(n,l)\lambda^l B_{l,q}^{(k)}\left(\frac{x}{\lambda}\right).
\end{aligned} \tag{2.10}$$

From (2.9), we have

$$\begin{aligned}
\left(\frac{q-1+\frac{q-1}{\log q}\lambda t}{qe^{\lambda t}-1}\right)^k e^{\lambda t\left(\frac{x}{\lambda}\right)} &= \sum_{n=0}^{\infty} D_{n,\lambda,q}^{(k)}(x)\frac{1}{n!}\left(e^t-1\right)^n\\
&= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} D_{n,\lambda,q}^{(k)}(x)S_2(m,n)\right)\frac{t^m}{m!}
\end{aligned} \tag{2.11}$$

and

$$\left(\frac{q-1+\frac{q-1}{\log q}\lambda t}{qe^{\lambda t}-1}\right)^k e^{\lambda t\left(\frac{\lambda}{x}\right)} = \sum_{n=0}^{\infty}\lambda^n B_{n,q}^{(k)}\left(\frac{x}{\lambda}\right)\frac{t^n}{n!}. \tag{2.12}$$

Thus, by (2.10), (2.11) and (2.12), we obtain the following theorem.

Theorem 2.3. *For $n \geq 0$, $k \in \mathbb{N}$, we have*

$$\lambda^n B_{n,q}^{(k)}\left(\frac{x}{\lambda}\right) = \sum_{m=0}^{n} S_2(n,m)D_{m,\lambda,q}^{(k)}(x),$$

and

$$\begin{aligned}
D_{n,\lambda,q}^{(k)}(x) &= \sum_{l=0}^{n} S_1(n,l)\lambda^l B_{l,q}^{(k)}\left(\frac{x}{\lambda}\right)\\
&= \sum_{l=0}^{n}\sum_{m=0}^{l} S_1(n,l)S_2(l,m)D_{m,\lambda,q}^{(k)}(x).
\end{aligned}$$

3. q-analogue of λ-Daehee polynomials of the second kind

For $n \geq 0$, the *rising factorial sequence* is defined by

$$\begin{aligned} x^{(n)} =& x(x+1)\cdots(x+n-1) = (-1)^n(-x)_n \\ =& \sum_{l=0}^{n}(-1)^{n-l}S_1(n,l)x^l. \end{aligned} \tag{3.1}$$

Let us define the *λ-q-Daehee polynomials of the second kind* as follows:

$$\frac{q-1-\frac{q-1}{\log q}\lambda\log(1+t)}{q-(1+t)^\lambda}(1+t)^{\lambda+x} = \sum_{n=0}^{\infty}\widehat{D}_{n,\lambda,q}(x)\frac{t^n}{n!}. \tag{3.2}$$

In the special case, $\lambda=1$, $\widehat{D}_{n,1,q}(x) = \widehat{D}_{n,q}(x)$ is called the *q-Daehee polynomials of the second kind*, and if $x=0$, then $\widehat{D}_{n,1,q}(0) = \widehat{D}_{n,q}(0) = \widehat{D}_{n,q}$ is called the *q-Daehee numbers of the second kind*.

Let us take $f(x) = (1+t)^{-\lambda x}$. Then, by (2.1), we have

$$\int_{\mathbb{Z}_p}(1+t)^{-\lambda x}d\mu_q(x) = \frac{q-1-\frac{q-1}{\log q}\lambda\log(1+t)}{q-(1+t)^\lambda}(1+t)^\lambda, \tag{3.3}$$

and so

$$\begin{aligned} \int_{\mathbb{Z}_p}(1+t)^{-\lambda y+x}d\mu_q(y) =& \frac{q-1-\frac{q-1}{\log q}\lambda\log(1+t)}{q-(1+t)^\lambda}(1+t)^{\lambda+x} \\ =& \sum_{n=0}^{\infty}\widehat{D}_{n,\lambda,q}(x)\frac{t^n}{n!}. \end{aligned} \tag{3.4}$$

By (3.4), we get

$$\begin{aligned} \int_{\mathbb{Z}_p}(1+t)^{-\lambda y+x}d\mu_q(y) =& \sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\binom{-\lambda y+x}{n}d\mu_q(y)t^n \\ =& \sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}(-\lambda y+x)_n\frac{t^n}{n!}. \end{aligned} \tag{3.5}$$

By (3.3) and (3.5), we get

$$\begin{aligned} \widehat{D}_{n,\lambda,q}(x) =& \int_{\mathbb{Z}_p}(-\lambda y+x)_n d\mu_q(y) \\ =& \sum_{l=0}^{n}S_1(n,l)(-\lambda)^l\int_{\mathbb{Z}_p}\left(y-\frac{x}{\lambda}\right)^l d\mu_q(y) \\ =& \sum_{l=0}^{n}S_1(n,l)(-\lambda)^l B_{l,q}\left(-\frac{x}{\lambda}\right). \end{aligned} \tag{3.6}$$

By (3.4), we get

$$\begin{aligned} \frac{q-1-\frac{q-1}{\log q}\lambda t}{q-e^{\lambda t}}e^{(\lambda+x)t} =& \sum_{n=0}^{\infty}\widehat{D}_{n,\lambda,q}(x)\frac{(e^t-1)^n}{n!} \\ =& \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}\widehat{D}_{n,\lambda,q}(x)S_2(m,n)\right)\frac{t^m}{m!}, \end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\frac{q-1-\frac{q-1}{\log q}\lambda t}{q-e^{\lambda t}}e^{(\lambda+x)t} =&\frac{q-1-\frac{q-1}{\log q}\lambda t}{q-e^{\lambda t}}e^{\left(1+\frac{x}{\lambda}\right)\lambda t} \\ =&\sum_{m=0}^{\infty}(-\lambda)^m B_{m,q}\left(-\frac{x}{\lambda}\right)\frac{t^m}{m!}.\end{aligned}\tag{3.8}$$

Therefore, by (3.6), (3.7) and (3.8), we obtain the following theorem.

Theorem 3.1. *For $m \geq 0$, we have*

$$\widehat{D}_{m,\lambda,q}(x) = \sum_{l=0}^{m} S_1(m,l)(-\lambda)^l B_{l,q}\left(-\frac{x}{\lambda}\right)$$

and

$$\begin{aligned}(-\lambda)^m B_{m,q}\left(-\frac{x}{\lambda}\right) =&\sum_{n=0}^{m}\widehat{D}_{n,\lambda,q}(x)S_2(m,n) \\ =&\sum_{n=0}^{m}\sum_{l=0}^{n}S_2(m,n)S_1(n,l)(-\lambda)^l B_{l,q}\left(-\frac{x}{\lambda}\right).\end{aligned}$$

By the Theorem 3.1, we obtain the following corollary.

Corollary 3.2. *For $n \geq 0$, we have*

$$\widehat{D}_{n,\lambda,q} = \sum_{l=0}^{m} S_1(m,l)(-\lambda)^l B_{l,q},$$

and

$$\begin{aligned}B_{m,q} =&(-\lambda)^m\sum_{n=0}^{m}\widehat{D}_{n,\lambda,q}S_2(m,n) \\ =&\sum_{n=0}^{m}\sum_{l=0}^{n}(-\lambda)^{m+l}S_2(m,n)S_1(n,l)B_{l,q}.\end{aligned}$$

Remark 3.3. As the special case of the Corollary 3.2, $\lambda = 1$, we have

$$\widehat{D}_{n,q} = \sum_{l=0}^{m}(-1)^l S_1(m,l)B_{l,q}$$

and

$$\begin{aligned}B_{m,q} =&(-1)^m\sum_{n-0}^{m}\widehat{D}_{n,q}S_2(m,n) \\ =&\sum_{n=0}^{m}\sum_{l=0}^{n}(-1)^{l+m}S_1(n,l)S_2(m,n)B_{l,q}.\end{aligned}$$

For $k \in \mathbb{N}$, we define the *λ-q-Daehee polynomials of the second kind with order k*:

$$\widehat{D}^{(k)}_{n,\lambda,q}(x) = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(-\lambda x_1 - \cdots - \lambda x_k + x)_n d\mu_q(x_1)\cdots d\mu_q(x_k).\tag{3.9}$$

From (3.9), we can derive the generating function of

$$\begin{aligned}
&\sum_{n=0}^{\infty} \widehat{D}_{n,\lambda,q}^{(k)}(x)\frac{t^n}{n!} \\
=&\sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (-\lambda x_1 - \cdots - \lambda x_k + x)_n d\mu_q(x_1)\cdots d\mu_q(x_k)\frac{t^n}{n!} \\
=&\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{-\lambda x_1 - \cdots - \lambda x_k + x} d\mu_q(x_1)\cdots d\mu_q(x_k) \\
=&\left(\frac{q-1-\frac{q-1}{\log q}\lambda \log(1+t)}{q-(1+t)^{\lambda}}\right)^k (1+t)^{\lambda k + x}.
\end{aligned} \tag{3.10}$$

Replacing t by $e^t - 1$ in (3.10), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} \widehat{D}_{n,\lambda,q}^{(k)}(x)\frac{(e^t-1)^n}{n!} &= \left(\frac{q-1-\frac{q-1}{\log q}\lambda t}{q-e^{\lambda t}}\right)^k e^{\lambda t\left(k+\frac{x}{\lambda}\right)} \\
&= \sum_{m=0}^{\infty} (-\lambda)^m B_{m,q}^{(k)}\left(-\frac{x}{\lambda}\right)\frac{t^m}{m!}
\end{aligned} \tag{3.11}$$

and

$$\sum_{n=0}^{\infty} \widehat{D}_{n,\lambda,q}^{(k)}(x)\frac{(e^t-1)^n}{n!} = \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} S_2(m,n)\widehat{D}_{n,\lambda,q}^{(k)}(x)\right)\frac{t^m}{m!}. \tag{3.12}$$

By (1.3) and (3.9), we get

$$\begin{aligned}
&\widehat{D}_{n,\lambda,q}^{(k)}(x) \\
=&\sum_{l=0}^{n} S_1(n,l) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (-\lambda x_1 - \cdots - \lambda x_k + x)^l d\mu_q(x_1)\cdots d\mu_q(x_k) \\
=&\sum_{l=0}^{n} S_1(n,l)(-\lambda)^l B_{l,q}^{(k)}\left(-\frac{x}{\lambda}\right).
\end{aligned} \tag{3.13}$$

Hence, by (3.11), (3.12) and (3.13), we obtain the following theorem.

Theorem 3.4. *For $n \geq 0$, we have*

$$\widehat{D}_{n,\lambda,q}^{(k)}(x) = \sum_{l=0}^{n} S_1(n,l)(-\lambda)^l B_{l,q}^{(k)}\left(-\frac{x}{\lambda}\right)$$

and

$$\begin{aligned}
(-\lambda)^n B_{n,q}^{(k)}\left(-\frac{x}{\lambda}\right) &= \sum_{m=0}^{n} S_2(n,m)\widehat{D}_{m,\lambda,q}^{(k)}(x) \\
&= \sum_{m=0}^{n}\sum_{l=0}^{m} S_1(m,l)S_2(n,m)(-\lambda)^l B_{l,q}^{(k)}\left(-\frac{x}{\lambda}\right).
\end{aligned}$$

As the special case of Theorem 3.4, if we put $x = 0$, then

$$\widehat{D}_{n,\lambda,q}^{(k)} = \sum_{l=0}^{n} S_1(n,l)(-\lambda)^l B_{l,q}^{(k)}$$

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and

$$\begin{aligned}(-\lambda)^n B_{n,q}^{(k)} &= \sum_{m=0}^{n} S_2(n,m)\widehat{D}_{m,\lambda,q}^{(k)}\\ &= \sum_{m=0}^{n}\sum_{l=0}^{m} S_1(m,l)S_2(n,m)(-\lambda)^l B_{l,q}^{(k)}.\end{aligned}$$

If we put $\lambda = 1$, then

$$\widehat{D}_{n,q}^{(k)} = \sum_{l=0}^{n} S_1(n,l)(-1)^l B_{l,q}^{(k)}$$

and

$$\begin{aligned}B_{n,q}^{(k)} =&(-1)^n \sum_{m=0}^{n} S_2(n,m)\widehat{D}_{m,q}^{(k)}\\ =&\sum_{m=0}^{n}\sum_{l=0}^{m}(-1)^{l+m} S_1(m,l)S_2(n,m)B_{l,q}^{(k)}.\end{aligned}$$

We observe that

$$\begin{aligned}(-1)^n\frac{D_{n,\lambda,q}}{n!} =&(-1)^n\int_{\mathbb{Z}_p}\frac{(\lambda y+x)_n}{n!}\\ =&\int_{\mathbb{Z}_p}\binom{\lambda y-x+n-1}{n}d\mu_q(y)\\ =&\sum_{m=0}^{n}\binom{n-1}{n-m}\int_{\mathbb{Z}_p}\binom{-\lambda y-x}{m}d\mu_q(y)\\ =&\sum_{m=1}^{n}\frac{\binom{n-1}{m-1}}{m!}m!\int_{\mathbb{Z}_p}\binom{-\lambda y-x}{m}d\mu_q(y)\\ =&\sum_{m=1}^{n}\binom{n-1}{m-1}\frac{\widehat{D}_{n,\lambda,q}(-x)}{m!},\end{aligned}\tag{3.14}$$

and

$$\begin{aligned}(-1)^n\frac{\widehat{D}_{n,\lambda,q}(x)}{n!} =&(-1)^n\int_{\mathbb{Z}_p}\binom{-\lambda y+x}{n}d\mu_q(y)\\ =&\int_{\mathbb{Z}_p}\binom{\lambda y-x+n-1}{n}d\mu_q(y)\\ =&\sum_{m=0}^{n}\binom{n-1}{n-m}\int_{\mathbb{Z}_p}\binom{\lambda y-x}{m}d\mu_q(y)\\ =&\sum_{m=1}^{n}\frac{\binom{n-1}{m-1}}{m!}m!\int_{\mathbb{Z}_p}\binom{\lambda y-x}{m}d\mu_q(y)\\ =&\sum_{m=1}^{n}\binom{n-1}{m-1}\frac{D_{n,\lambda,q}(-x)}{m!}.\end{aligned}\tag{3.15}$$

Therefore, by (3.14), (3.15), Theorem 2.2 and Theorem 3.1, we obtain the following theorem.

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Theorem 3.5. *For $n \geq 1$, we have*

$$\begin{aligned}(-1)^n \frac{D_{n,\lambda,q}(x)}{n!} &= \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{\widehat{D}_{m,\lambda,q}(-x)}{m!} \\ &= \sum_{m=1}^{n} \sum_{l=0}^{m} \frac{\binom{n-1}{m-1}}{m!} S_1(m,l)(-\lambda)^l B_{l,q}\left(\frac{x}{\lambda}\right)\end{aligned}$$

and

$$\begin{aligned}(-1)^n \frac{\widehat{D}_{n,\lambda,q}(x)}{n!} &= \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{D_{m,\lambda,q}(-x)}{m!} \\ &= \sum_{m=1}^{n} \sum_{l=0}^{m} \frac{\binom{n-1}{m-1}}{m!} S_1(m,l)\lambda^l B_{l,q}\left(-\frac{x}{\lambda}\right).\end{aligned}$$

By the similar way to Theorem 3.5, we also obtain the following remark.

Remark 3.6. For $n \geq 1$, we have

$$(-1)^n \frac{D^{(k)}_{n,\lambda,q}(x)}{n!} = \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{\widehat{D}^{(k)}_{m,\lambda,q}(-x)}{m!},$$

and

$$(-1)^n \frac{\widehat{D}^{(k)}_{n,\lambda,q}(x)}{n!} = \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{D^{(k)}_{m,\lambda,q}(-x)}{m!}.$$

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On fuzzy B-algebras over t-norm

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Abstract. In this paper, we introduce the notion of T-fuzzy subalgebra/normal of B-algebras using a t-norm T. Then some related properties of them are investigated. We discuss the direct product and T-product of T-fuzzy normal subalgebras of B-algebras. We also generalize the idea to the product of T-fuzzy normal subalgebras of B-algebras.

1. Introduction

The notion of B-algebras was introduced by J. Neggers and H. S. Kim ([6]) as a class of algebras which is related to several classes of algebras of interest such as $BCH/BCI/BCK$-algebras. J. R. Cho and H. S. Kim ([2]) proved that every B-algebra is a quasigroup, and M. Kondo and Y. B. Jun ([4]) showed that the class of all B-algebras is equivalent in one sense to the class of groups. A. Walendziak ([8]) obtained some systems of axioms defining a B-algebra, and also obtained a simplified axiomatization of 0-commutative B-algebras. Y. B. Jun et al.([3]) fuzzyfied (normal) B-algebras and gave a characterization of a fuzzy B-algebras. S. S. Ahn and K. Bang([1]) classified the subalgebras by their family of level subalgebras in B-algebras.

In this paper, we introduce the notion of T-fuzzy subalgebra/normal of B-algebras over a T-norm T, and then we investigate some related properties. We also discuss the direct product and T-product of T-fuzzy normal subalgebras of B-algebras.

2. Preliminaries

A B-*algebra* ([6]) is a non-empty set A with a constant 0 and a binary operation "$*$" satisfying the following axioms:

$(a1)$ $x * x = 0$,
$(a2)$ $x * 0 = x$,
$(a3)$ $(x * y) * z = x * (z * (0 * y))$
for all x, y, z in A.

In any B-algebra X, we define a relation "$\leq$" by putting $x \leq y$ if and only if $x * y = 0$.

[0] **2010 Mathematics Subject Classification**: 06F35; 03G25.

[0] **Keywords**: B-algebra; T-fuzzy subalgebra/normal; T-product.

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A B-algebra X has the following properties (see [2,6]):

(b1) $(\forall x, y, z \in X)(x * (y * z) = (x * (0 * z)) * y)$.
(b2) $(\forall x, y \in X)((x * y) * (0 * y) = x)$.
(b3) $(\forall x, y, z \in X)(x * z = y * z \Rightarrow x = y)$.
(b4) $(\forall x, y \in X)(x * y = x * (0 * (0 * y)))$.
(b5) $(\forall x, y \in X)(x * y = 0 \Rightarrow x = y)$.
(b6) $(\forall x, y \in X)(0 * x = 0 * y \Rightarrow x = y)$.
(b7) $(\forall x \in X)(0 * (0 * x) = x)$.

A non-empty subset N of a B-algebra X is called a *B-subalgebra* of X if $x * y \in N$ for any $x, y \in N$. A non-empty subset N of a B-algebra X is said to be *normal* if $(x * a) * (y * b) \in N$ whenever $x * y \in N$ and $a * b \in N$. Note that any normal subset N of a B-algebra X is a B-subalgebra of X, but the converse need not be true (see [7]). A non-empty subset N of a B-algebra X is called a *normal B-subalgebra* of X if it is both a B-subalgebra and normal. A mapping $f : X \to Y$ of B-algebras is called a *B-homomorphism* if $f(x * y) = f(x) * f(y)$ for any $x, y \in X$. A mapping $f : X \to Y$ of B-algebras is called an *epimorphism* if it is an onto B-homomorphism. Note that if f is a B-homomorphism, then $f(0) = 0$.

We now review some fuzzy logic concepts. Let X be a non-empty set. A fuzzy set μ in X is a function $\mu : X \to [0, 1]$. For any fuzzy sets α and β of a set X, we define $\alpha \cap \beta(x) := \min\{\alpha(x), \beta(x)\}$ for all $x \in X$

Definition 2.1.([3]) Let μ be a fuzzy set in a B-algebra. Then μ is called a *fuzzy subalgebra* of X if $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$.

Definition 2.2.([5]) A binary operation T on $[0, 1]$ is called a *triangular norm* (briefly, *t-norm*) if

(T1) boundary condition: $T(x, 1) = x$;
(T2) commutativity: $T(x, y) = T(y, x)$;
(T3) associativity: $T(x, T(y, z)) = T(T(x, y), z)$;
(T4) monotonicity: $T(x, y) \leq T(x, z)$ whenever $y \leq z$, for all $x, y, z \in [0, 1]$.

Note that $T(x, y) \leq \min\{x, y\}$ for all $x, y \in [0, 1]$ and

$$T(T(x, y), T(z, t)) = T(T(x, z), T(y, t))$$

for all $x, y, z, t \in [0, 1]$.

Definition 2.3. Let P be a t-norm. Denote by $\triangle_P$ the set of elements $x \in [0, 1]$ such that $P(x, x) = x$, i.e., $\triangle_P = \{x \in [0,]|P(x, x) = x\}$. If $Im(\mu) \subseteq \triangle_P$, then the fuzzy set μ is said to be *imaginable*.

3. T-fuzzy subalgebras/normal of B-algebras

In what follows, let T and X denote a t-norm and a B-algebras respectively, unless otherwise specified.

Definition 3.1. A fuzzy set μ in X is called a *fuzzy subalgebra* of X over T (briefly, a T*-fuzzy subalgebra* of X) if it satisfies
(TF_0) $\mu(x * y) \geq T\{\mu(x), \mu(y)\}$
for all $x, y \in X$.

Example 3.2. Let $X := \{0, 1, 2, 3, 4, 5\}$ be a B-algebra([3]) with the following table:

$*$	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	0	1

Let $T_m : [0, 1] \times [0, 1] \to [0, 1]$ be a function defined by $T_m(x, y) = \max(x + y - 1, 0)$ for all $x, y \in [0, 1]$. Then T_m is a t-norm. Define a fuzzy set μ in X by $\mu(0) = 0.8, \mu(3) = 0.65$ and $\mu(x) = 0.25$ for all $x \in X \setminus \{0, 3\}$. Then μ is a T_m-fuzzy subalgebra of X.

Definition 3.3. A T-fuzzy subalgebra μ is called an *imaginable* T*-fuzzy subalgebra* of X if it satisfies the imaginable property with respect to T.

Example 3.4. Let T_m be a t-norm and let $X = \{0, 1, 2, 3, 4, 5\}$ be a B-algebra as in Example 3.2. Define a fuzzy set ν in X by $\nu(x) = 1$ if $x \in \{0, 3\}$ and $\nu(x) = 0$ if $x \in X \setminus \{0, 3\}$. It is easy to show that $\nu(x * y) \geq T\{\nu(x), \nu(y)\}$ for all $x, y \in X$. Also $Im\ \nu \subseteq \triangle_{T_m}$. Hence ν is an imaginable T_m-fuzzy subalgebra of X.

Proposition 3.5. *If μ is an imaginable T-fuzzy subalgebra of X, then $\mu(0) \geq \mu(x)$ for all $x \in X$.*

Proof. Let $x \in X$. Then $\mu(0) = \mu(x * x) \geq T\{\mu(x), \mu(x)\} = \mu(x)$. □

For any element x and y of X, let us write $\prod^n x * y$ for $x * (\cdots * (x * (x * y)))$ where x occurs n times.

Proposition 3.6. *Let μ be an imaginable T-fuzzy subalgebra of X and let $n \in \mathbb{N}$. Then*

(i) $\mu(\prod^n x * x) \geq \mu(x)$ *whenever n is odd,*
(ii) $\mu(\prod^n x * x) \geq \mu(x)$ *whenever n is even*

for all $x \in X$.

Proof. (i) Let $x \in X$ and assume that n is odd. Then $n = 2k - 1$ for some positive integer k. Observe that $\mu(x * x) = \mu(0) \geq \mu(x)$ by Proposition 3.5. Suppose that $\mu(\prod^{2k-1} x * x) \geq \mu(x)$ for

a positive integer k. Then

$$\begin{aligned}\mu(\prod^{2(k+1)-1} x*x) =&\mu(\prod^{2k+1} x*x)\\ =&\mu(\prod^{2k-1} x*(x*(x*x)))\\ =&\mu(\prod^{2k-1} x*x)\ (\because (a1) \text{ and } (a2))\\ \geq&\mu(x).\end{aligned}$$

Hence (i) holds. (ii) is similar to (i). □

Proposition 3.7. *If μ is an imaginable T-subalgebra of X, then*
(fB1) $\mu(0*x)\geq\mu(x)$,
(fB2) $\mu(x*(0*y))\geq T\{\mu(x),\mu(y)\}$
for all $x,y\in X$.

Proof. (fB1) For all $x\in X$, we have

$$\mu(0*x)\geq T\{\mu(0),\mu(x)\}=T\{\mu(x*x),\mu(x)\}\geq T\{T\{\mu(x),\mu(x)\},\mu(x)\}=\mu(x)$$

since μ is imaginable.
(fB2) Let $x,y\in X$. Then we get

$$\begin{aligned}\mu(x*(0*y))\geq&T\{\mu(x),\mu(0*y)\}\\ \geq&T\{\mu(x),T\{\mu(0),\mu(y)\}\}\\ \geq&T\{\mu(x),T\{\mu(y),\mu(y)\}\}\ (\because \text{Proposition 3.5 and (T4)})\\ =&T\{\mu(x),\mu(y)\}.\end{aligned}$$

□

Using (b7), Proposition 3.5 and (T4), if μ is an imaginable T-subalgebra of X, then $\mu(x)=\mu(0*(0*x))\geq T\{\mu(0),\mu(0*x)\}\geq T\{\mu(0*x),\mu(0*x)\}=\mu(0*x)$, i.e., $\mu(x)=\mu(0*x)$, for any $x\in X$.

Proposition 3.8. *If a fuzzy set μ in X satisfies (fB1) and (fB2), then μ is a T-fuzzy subalgebra of X.*

Proof. Assume that a fuzzy set μ in X satisfies the condition (fB1) and (fB2) and let $x,y\in X$. Using (b4), (fB2), (fB1) and (T4), we obtain $\mu(x*y)=\mu(x*(0*(0*y)))\geq T\{\mu(x),\mu(0*y)\}\geq T\{\mu(x),\mu(y)\}$. Hence μ is a T-fuzzy subalgebra of X. □

Theorem 3.9. *Let μ be an imaginable T-fuzzy subalgebra of X. If there exists a sequence in X such that $lim_{n\to\infty}\mu(x_n)=1$, then $\mu(0)=1$.*

Proof. By Proposition 3.5, $\mu(0)\geq\mu(x)$ for all $x\in X$. Hence $\mu(0)\geq\mu(x_n)$ for any positive integer n. Since $1\geq\mu(0)\geq lim_{n\to\infty}\mu(x_n)=1$, we have $\mu(0)=1$. □

Definition 3.10. A fuzzy set μ in X is called a *T-fuzzy normal* of X if it satisfies the inequality

(TF_1) $\mu((x * a) * (y * b)) \geq T\{\mu(x * y), \mu(a * b)\}$

for all $a, b, x, y \in X$. A T-fuzzy normal μ is called an *imaginable T-fuzzy normal* of X if it satisfies the imaginable property with respect to T.

Example 3.11. Let $T_m(x, y) = \max(x + y - 1, 0)$ for all $x, y \in [0, 1]$ be a t-norm and let $X = \{0, 1, 2, 3, 4, 5\}$ be a B-algebras as in Example 3.2. If we define a fuzzy set $\nu : X \to [0, 1]$ by $\nu(0) = \nu(1) = \nu(2) = 0.8$ and $\nu(3) = \nu(4) = \nu(5) = 0.3$, then ν is a T-fuzzy normal of X.

Theorem 3.12. *Every T-fuzzy normal set μ in X is a T-fuzzy subalgebra of X.*

Proof. For any $x, y \in X$, since μ is a T-fuzzy normal of X, $\mu(x * y) = \mu((x * y) * (0 * 0)) \geq T\{\mu(x * 0), \mu(y * 0)\} = T\{\mu(x), \mu(y)\}$. Hence μ is a T-fuzzy subalgebra of X. $\square$

Remark 3.13. The converse of Theorem 3.12 is not true. For example, the T_m-fuzzy subalgebra μ of X in Example 3.2 is not a T_m-fuzzy normal of X, since $\mu((2 * 5) * (4 * 1)) = \mu(2) = 0.25 < 0.3 = T_m\{\mu(2 * 4), \mu(5 * 1)\}$.

Definition 3.14. A fuzzy set μ in X is called a *T-fuzzy normal subalgebra* of X if it is a T-fuzzy subalgebra which is a T-fuzzy normal.

Proposition 3.15. *If a fuzzy set μ in X is an imaginable T-fuzzy normal subalgebra of X, then $\mu(x * y) = \mu(y * x)$ for all $x, y \in X$.*

Proof. Let $x, y \in X$. Then we have

$$\begin{aligned}
\mu(x * y) =& \mu((x * y) * (x * x)) \ (\because (a1) \text{ and } (a2)) \\
\geq& T\{\mu(x * x), \mu(y * x)\} \ (\because \mu : \ T\text{-fuzzy normal}) \\
=& T\{\mu(0), \mu(y * x)\} \\
\geq& T\{\mu(y * x), \mu(y * x)\} \ (\because (T4) \text{ and Proposition 3.5}) \\
=& \mu(y * x) \ (\because \mu : \text{imaginable}).
\end{aligned}$$

Interchanging x with y, we obtain $\mu(y * x) \geq \mu(x * y)$, which proves the proposition. $\square$

Theorem 3.16. *Let μ be an imaginable T-fuzzy normal subalgebra of X. Then the set*

$$X_\mu := \{x \in X | \mu(x) = \mu(0)\}$$

is a normal B-subalgebra of X.

Proof. It is sufficient to show that X_μ is normal. Let $a, b, x, y \in X$ be such that $x * y, a * b \in X_\mu$. Then $\mu(x * y) = \mu(0) = \mu(a * b)$. Since μ is a T-fuzzy normal of X, it follows that

$$\begin{aligned}
\mu((x * a) * (y * b)) \geq& T\{\mu(x * y), \mu(a * b)\} \\
=& T\{\mu(0), \mu(0)\} \\
=& \mu(0) \ (\because \mu : \text{imaginable}).
\end{aligned}$$

Combining Proposition 3.5, we conclude that $\mu((x*a)*(y*b))=\mu(0)$ which shows that $(x*a)*(y*b)\in X_\mu$. This completes the proof. □

Theorem 3.17. *The intersection of any set of T-fuzzy normal subalgebras of X is also a T-fuzzy normal subalgebra of X.*

Proof. Let $\{\mu_\alpha|\alpha\in\Lambda\}$ be a family of T-fuzzy normal subalgebras of X and let $a,b,x,y\in X$. Then

$$\begin{aligned}(\cap_{\alpha\in\Lambda}\mu_\alpha)((x*a)*(y*b)) &= \inf_{\alpha\in\Lambda}\mu_\alpha((x*a)*(y*b))\\ &\geq \inf_{\alpha\in\Lambda}\{T\{\mu_\alpha(x*y),\mu_\alpha(a*b)\}\}\\ &\geq T\{\inf_{\alpha\in\Lambda}\{\mu_\alpha(x*y),\inf_{\alpha\in\Lambda}\{\mu_\alpha(a*b)\}\}\\ &= T\{\cap_{\alpha\in\Lambda}\mu_\alpha(x*y),\cap_{\alpha\in\Lambda}\mu_\alpha(a*b)\}\end{aligned}$$

which shows that $\cap_{\alpha\in\Lambda}\mu_\alpha$ is a T-fuzzy normal of X. By Theorem 3.12, $\cap_{\alpha\in\Lambda}\mu_\alpha$ is a T-fuzzy subalgebra of X. Thus it is a T-fuzzy normal subalgebra of X. □

The union of any set of T-fuzzy normal subalgebras of X need not be a T-fuzzy normal subalgebra of X. For example, if we define a fuzzy set $\sigma: X\to[0,1]$ by $\sigma(0)=\sigma(4)=0.8>0.2=\sigma(1)=\sigma(2)=\sigma(3)=\sigma(5)$ in Example 3.2, then it is also a T_m-fuzzy normal subalgebra of X. Since $(\mu\cup\sigma)(3*4)=(\mu\cup\sigma)(2)=0.25<0.45=T_m\{(\mu\cup\sigma)(3),(\mu\cup\sigma)(4)\}$, $\mu\cup\sigma$ is not a T_m-fuzzy subalgebra. Since every T-fuzzy normal of X is a T-fuzzy subalgebra of X, the union of T-fuzzy normal subalgebra of X need not be a T-fuzzy normal subalgebra of X.

4. Direct products and t-normed products of B-algebras

Definition 4.1. Let μ and ν be fuzzy sets of a B-algebra X and let T be a t-norm of X. Then the *T-product* of μ and ν is defined by

$$[\mu\cdot\nu]_T(x):=T(\mu(x),\nu(x))$$

for all $x\in X$ and denoted by $[\mu\cdot\nu]_T$.

Theorem 4.2. *Let μ,ν be two T-fuzzy normal of X and let T^* be a t-norm which dominates T, i.e.,*

$$T^*(T(a,b),T(c,d))\geq T(T^*(a,c),T^*(b,d))$$

for all a,b,c and $d\in[0,1]$. Then T^*-product of μ and ν, $[\mu\cdot\nu]_{T^*}$ is a T-fuzzy normal of X.

Proof. For any $x,y,a,b\in X$, we have

$$\begin{aligned}[\mu\cdot\nu]_{T^*}((x*a)*(y*b)) &= T^*\{\mu((x*a)*(y*b)),\nu((x*a)*(y*b))\}\\ &\geq T^*\{T\{\mu(x*y),\mu(a*b)\},T\{\nu(x*y),\nu(a*b)\}\}\\ &\geq T\{T^*\{\mu(x*y),\nu(x*y)\},T^*\{\mu(a*b),\nu(a*b)\}\}\\ &= T\{[\mu\cdot\nu]_{T^*}(x*y),[\mu\cdot\nu]_{T^*}(a*b)\}.\end{aligned}$$

Hence $[\mu\cdot\nu]_{T^*}$ is a T-fuzzy normal of X. □

Corollary 4.3. *Let μ, ν be two T-fuzzy subalgebra of X and let T^* be a t-norm which dominates T, i.e.,*

$$T^*(T(a,b), T(c,d)) \geq T(T^*(a,c), T^*(b,d))$$

for all a, b, c and $d \in [0,1]$. Then T^*-product of μ and ν, $[\mu \cdot \nu]_{T^*}$ is a T-fuzzy subalgebra of X.

Proof. Put $a := 0, b := 0$ in proof of Theorem 4.2. □

Let f be a mapping defined on X and let μ be a fuzzy set in $f(X)$. The fuzzy set $f^{-1}(\mu)$ in X defined by $[f^{-1}(\mu)](x) := \mu(f(x))$ for all $x \in X$ is called the *preimage* of μ under f.

Lemma 4.4. *Let $f : X \to Y$ be an epimorphism of B-algebras and let μ be a T-fuzzy subalgebra(normal) of X. Then the pre-image $f^{-1}(\mu)$ of μ of μ under f is a T-fuzzy subalgebra(normal) of X.*

Proof. Assume that μ is a T-fuzzy subalgebra of Y. Let $x, y \in X$. Then

$$\begin{aligned}[f^{-1}(\mu)](x * y)) &= \mu(f(x * y)) \\ &= \mu(f(x) * f(y)) \\ &\geq T\{\mu(f(x)), \mu(f(y))\} \\ &= T\{[f^{-1}(\mu)](x), [f^{-1}(\mu)](y))\}.\end{aligned}$$

Hence $f^{-1}(\mu)$ is a T-fuzzy subalgebra of X.

Assume that μ is a T-fuzzy normal of Y. Let $x, y, a, b \in X$. Then

$$\begin{aligned}[f^{-1}(\mu)]((x * a) * (y * b)) &= \mu(f((x * a) * (y * b))) \\ &= \mu((f(x) * f(a)) * (f(y) * f(b))) \\ &\geq T\{\mu(f(x) * f(y)), \mu(f(a) * f(b))\} \\ &= T\{\mu(f(x * y)), \mu(f(a * b))\} \\ &= T\{[f^{-1}(\mu)](x * y), [f^{-1}(\mu)](a * b)\}.\end{aligned}$$

Therefore $f^{-1}(\mu)$ is a T-fuzzy normal of X. □

Let $f : X \to Y$ be an epimorphism of B-algebras. If μ and ν are T-fuzzy subalgebras of Y, then the T^*-product $[\mu \cdot \nu]_{T^*}$ of μ and ν is a T-fuzzy subalgebra of Y whenever T^* dominates T. Since every epimorphic pre-image of a T-fuzzy subalgebra is a T-fuzzy subalgebra, the pre-images $f^{-1}(\mu), f^{-1}(\nu)$ and $f^{-1}([\mu \cdot \nu]_{T^*})$ are T-fuzzy subalgebras. The next theorem provides the relation between $f^{-1}([\mu \cdot \nu]_{T^*})$ and the product $[f^{-1}(\mu) \cdot f^{-1}(\nu)]_{T^*}$ of $f^{-1}(\mu)$ and $f^{-1}(\nu)$.

Proposition 4.5. *Assume that $f : X \to Y$ is an epimorphism of B-algebras and T, T^* are t-norms such that T^* dominates T. For any T-fuzzy subalgebra μ and ν of Y, we have*

$$f^{-1}([\mu \cdot \nu]_{T^*}) = [f^{-1}(\mu) \cdot f^{-1}(\nu)]_{T^*}.$$

Proof. For any $x \in X$, we get

$$\begin{aligned}\{f^{-1}([\mu\cdot\nu]_{T^*})\}(x) =&[\mu\cdot\nu]_{T^*}(f(x))\\ =&T^*\{\mu(f(x)),\nu(f(x))\}\\ =&T^*([f^{-1}(\mu)](x),[f^{-1}(\nu)](x))\\ =&[f^{-1}(\mu)\cdot f^{-1}(\nu)]_{T^*}(x),\end{aligned}$$

completing the proof. □

Let $(X_1, *_1, 0_1)$ and $(X_2, *_2, 0_2)$ be B-algebras. Define a binary operation $*$ on $X_1 \times X_2$ by

$$(x_1, x_2) * (y_1, y_2) := (x_1 *_1 x_2, y_1 *_2 y_2)$$

for all $(x_1, x_2), (y_1, y_2) \in X$. Then $(X, *, 0)$ is a B-algebra, where $0 = (0_1, 0_2)$.

Theorem 4.6. *Let $X = X_1 \times X_2$ be the direct product of B-algebras X_1 and X_2. If μ_1 (resp., μ_2) is a T-fuzzy normal subalgebra of X_1(resp., X_2), then $\mu := \mu_1 \times \mu_2$ is a T-fuzzy normal subalgebra of X defined by*

$$\mu(x_1, x_2) = (\mu_1 \times \mu_2)(x_1, x_2) = T(\mu_1(x_1), \mu_2(x_2))$$

for all $(x_1, x_2) \in X_1 \times X_2$.

Proof. Let $x = (x_1, x_2), y = (y_1, y_2), a = (a_1, a_2), b = (b_1, b_2) \in X$. Then we have

$$\begin{aligned}\mu((x*a)*(y*b)) =&\mu[((x_1,x_2)*(a_1,a_2))*((y_1,y_2)*(b_1,b_2))]\\ =&\mu((x_1*a_1)*(y_1*b_1),(x_2*a_2)*(y_2*b_2))\\ =&(\mu_1\times\mu_2)((x_1*a_1)*(y_1*b_1),(x_2*a_2)*(y_2*b_2))\\ =&T(\mu_1((x_1*a_1)*(y_1*b_1)),\mu_2((x_2*a_2)*(y_2*b_2))\\ \geq&T(T(\mu_1(x_1*y_1),\mu_1(a_1*b_1)),T(\mu_2(x_2*y_2),\mu_2(a_2*b_2)))\\ =&T(T(\mu_1(x_1*y_1),\mu_2(x_2*y_2)),T(\mu_1(a_1*b_1),\mu_2(a_2*b_2))\\ =&T((\mu_1\times\mu_2)(x_1*y_1,x_2*y_2),(\mu_1\times\mu_2)(a_1*b_1,a_2*b_2))\\ =&T(\mu((x_1,x_2)*(y_1,y_2)),\mu((a_1,a_2)*(b_1,b_2)))\\ =&T(\mu(x*y),\mu(a*b)).\end{aligned}$$

Hence $\mu = \mu_1 \times \mu_2$ is a T-fuzzy normal of X. By Theorem 3.12, $\mu = \mu_1 \times \mu_2$ is a T-fuzzy subalgebra of X. Therefore $\mu = \mu_1 \times \mu_2$ is a T-fuzzy normal subalgebra of X. □

Now, we generalize the idea to the product of T-fuzzy normal subalgebra. We first need to generalize the domain of t-norm to $\prod_{i=1}^{n}[0, 1]$ as follows.

Definition 4.7. The function $T_n : \prod_{i=1}^{n}[0, 1] \to [0, 1]$ is defined by $T_n(\alpha_1, \alpha_2, \cdots, \alpha_n) := T(\alpha_i, T_{n-1}(\alpha_1, \cdots, \alpha_{i-1}, \alpha_{i+1}, \cdots, \alpha_n))$ for all $1 \leq i \leq n$, where $T_2 = T$ and $T_1 = id$.

Using the induction on n, we have following two lemmas:

Lemma 4.8. *For a t-norm T and every α_i, β_i, where $1 \le i \le n$ and $n \ge 2$, we have*

$$\begin{aligned} T_n(T(\alpha_1, \beta_1), &T(\alpha_2, \beta_2), \cdots, T(\alpha_n, \beta_n)) \\ &= T(T_n(\alpha_1, \alpha_2, \cdots, \alpha_n), T_n(\beta_1, \beta_2, \cdots, \beta_n)). \end{aligned}$$

Lemma 4.9. *For a t-norm T and every $\alpha_1, \alpha_2, \cdots, \alpha_n \in [0,1]$, where $n \ge 2$, we have*

$$\begin{aligned} T_n(\alpha_1, \alpha_2, \cdots, \alpha_n) &= T(\cdots, T(T(T(\alpha_1, \alpha_2), \alpha_3), \alpha_4), \cdots, \alpha_n) \\ &= T(\alpha_1, T(\alpha_2, T(\alpha_3, \cdots, T(\alpha_{n-1}, \alpha_n) \cdots))). \end{aligned}$$

Theorem 4.10. *Let $X := \prod_{i=1}^n X_i$ be the direct product of B-algebras $\{X_i\}_{i=1}^n$. If μ_i is a T-fuzzy normal subalgebra of X_i, where $1 \le i \le n$, then $\mu = \prod_{i=1}^n \mu_i$ defined by*

$$\mu(x_1, \cdots, x_n) = (\prod_{i=1}^n \mu_i)(x_1, \cdots, x_n) = T(\mu(x_1), \cdots, \mu(x_n))$$

is a T-fuzzy normal subalgebra of X.

Proof. Let $x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n), a = (a_1, \cdots, a_n), b = (b_1, \cdots, b_n)$ be any elements of X. Using Lemmas 4.8 and 4.9, we have

$$\begin{aligned} &\mu((x*y) * (a * b)) \\ &\quad= \mu((x_1 * y_1) * (a_1 * b_1), (x_2 * y_2) * (a_2 * b_2), \cdots, (x_n * y_n) * (a_n * b_n)) \\ &\quad= T_n(\mu_1((x_1 * y_1) * (a_1 * b_1)), \cdots, \mu_n((x_n * y_n) * (a_n * b_n))) \\ &\quad\ge T_n[T(\mu_1(x_1 * a_1), \mu_1(y_1 * b_1)), \cdots, T(\mu_n(x_n * a_n), \mu_n(y_n * b_n))] \\ &\quad= T[T_n(\mu_1(x_1 * a_1), \mu_2(x_2 * a_2), \cdots, \mu_n(x_n * a_n)), \\ &\qquad T_n(\mu_1(y_1 * b_1), \mu_2(y_2 * b_2), \cdots, \mu_n(y_n * b_n))] \\ &\quad= T[\mu(((x_1, \cdots, x_n) * (a_1, \cdots, a_n)) * ((y_1, \cdots, y_n) * (b_1, \cdots, b_n))] \\ &\quad= T[\mu((x * a) * (y * b))]. \end{aligned}$$

Hence $\mu = \prod_{i=1}^n \mu_i$ is a T-fuzzy normal of X. By Theorem 3.12, $\mu = \prod_{i=1}^n \mu_i$ is a T-fuzzy subalgebra of X. Therefore $\mu = \prod_{i=1}^n \mu_i$ is a T-fuzzy normal subalgebra of X. □

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Hybrid Boundary value problems of q-difference equations and inclusions

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Abstract

This paper investigates the existence of solutions for boundary value problems of nonlinear q-difference hybrid equations and inclusions. Our results are new in the given setting and rely on appropriate fixed point theorems.

Key words and phrases: q-difference equations and inclusions; boundary value problems; fixed point theorem; existence.
AMS (MOS) Subject Classifications: 34A60, 34A08, 34B18.

1 Introduction

In this paper, we study the existence of solutions continuous at 0 for boundary value problems of nonlinear q-difference hybrid equations and inclusions. First we consider the single-valued problem given by

$$\begin{cases} D_q^2\left(\dfrac{x(t)}{f(t,x(t))}\right) = g(t,x(t)), \quad t \in I_q, \\ \qquad x(0)=0, \quad x(1)=0, \end{cases} \tag{1}$$

where $f \in C(I_q \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ $g : C(I_q \times \mathbb{R}, \mathbb{R})$, are such that $f(t,x(t)), g(t,x(t))$ are continuous at $t=0$, $I_q = \{q^n : n \in \mathbb{N}\} \cup \{0,1\}$, $q \in (0,1)$ is a fixed constant

Next we consider the corresponding multi-valued problem

$$\begin{cases} D_q^2\left(\dfrac{x(t)}{f(t,x(t))}\right) \in F(t,x(t)), \quad 0<t<1, \quad 1<\alpha \le 2 \\ \qquad x(0)=x(1)=0, \end{cases} \tag{2}$$

where $f \in C([0,1] \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$.

$q-$difference equations are found to be quite useful in the theory of quantum groups [17]. For historical notes and development of the subject, we refer the reader to [14, 15, 16] while some recent results on $q-$difference equations can be found in [1, 2, 3, 4, 6, 10, 13].

The present paper is motivated by a recent paper of Sun *et al.* [18] dealing with the following boundary value problem:

$$\begin{cases} D_{0+}^{\alpha}\left(\dfrac{x(t)}{f(t,x(t))}\right) = g(t,x(t)), \quad 0 \le t \le 1, \\ \qquad x(0)=0, \quad x(1)=0, \end{cases} \tag{3}$$

where $1 < \alpha \leq 2$ is a real number and D^{α}_{0+} is the standard Riemann-Liouville fractional derivative.

Here we establish the existence results for the problems (1) and (2) by using the fixed point theorems due to Dhage [8, 9] under Lipschitz and Carathéodory conditions. For some recent results on hybrid fractional differential equations, we refer to [5, 11, 19] and the references cited therein.

The paper is organized as follows: in Section 2 we recall some preliminary facts that we need in the sequel. Section 3 contains our main result for the single-valued problem (1), while we prove the corresponding results for the multi-valued problem (2) in Section 4.

2 Preliminaries

Let us recall some basic concepts of q-calculus [7, 14, 17].

Let $0 < q < 1$, and f a function defined on a q-geometric set A, i.e. $qt \in A$ for all $t \in A$. The q-difference operator is defined by

$$D_q f(t) = \begin{cases} \dfrac{f(t) - f(qt)}{(1-q)t}, & t \in A \setminus \{0\}, \\[2ex] \lim\limits_{n\to\infty} \dfrac{f(tq^n) - f(0)}{tq^n}, & t = 0, \end{cases}$$

provided that the limit exists and does not depend on t. The higher order q-derivatives are given by

$$D_q^0 f(t) = f(t), \quad D_q^n f(t) = D_q D_q^{n-1} f(t), \quad n \in \mathbb{N}.$$

The Jackson q-integration [15] is

$$\int_0^a f(t) d_q t = a(1-q)\sum_{n=0}^{\infty} q^n f(aq^n), \int_a^b f(t) d_q(t) = \int_0^b f(t) d_q(t) - \int_0^a f(t) d_q(t),$$

where $a, b \in A$, provided that the series converge. Here we remark that the integral $\int_a^b f(t) d_q(t)$ is understood as a right inverse of the q-derivative.

For $0 \in A$, f is called q-regular at zero if $\lim_{n\to\infty} f(tq^n) = f(0)$ for every $t \in A, t \neq 0$. It is important to note that continuity at zero implies q-regularity at zero but the converse is not true (see an example on page 7 in [7]).

Definition 2.1 *Let f be a function defined on a q-geometric set A. Then f is q-integrable on A if and only if $\int_0^t f(\mu) d_q\mu$ exists for all $t \in A$.*

The q-integration by parts rule is

$$\int_a^b u(qt) D_q v(t) d_q t = u(b)v(b) - u(a)v(a) + \int_a^b D_q u(t) v(t) d_q t,$$

provided that u and v are q-regular at zero functions.

Let f be a q-regular at zero function defined on a q-geometric set A containing zero. Then

$$F(z) = \int_c^z f(s) d_q s, \quad z \in A,$$

is q-regular at zero, where c is a fixed point in A. Furthermore, $D_q F(z)$ exists for every $z \in A$ and

$$D_q F(z) = f(z), \quad z \in A.$$

Conversely, if a and b are two points in A, then

$$\int_a^b D_q f(s) d_q s = f(b) - f(a).$$

In the sequel, we denote by $\mathcal{C} = C(I_q, \mathbb{R})$ the space of functions from $I_q \to \mathbb{R}$ which are continuous at 0.

Lemma 2.2 *Given $y \in \mathcal{C}$ then the boundary value problem*

$$\begin{cases} D_q^2\left(\dfrac{x(t)}{f(t,x(t))}\right) = y(t), \quad t \in I_q, \\ \qquad x(0) = x(1) = 0, \end{cases} \tag{4}$$

is equivalent to a q-integral equation

$$x(t) = f(t,x(t))\left(\int_0^t (t-qs)y(s)d_qs - t\int_0^1 (1-qs)y(s)d_qs\right), \quad t \in I_q. \tag{5}$$

Proof. As argued in [7], a solution continuous at zero of the equation $D_q^2x(t) = y(t)$ can be written as

$$x\,(t) = \int_0^t (t-qs)\,y(s)d_qs + c_0t + c_1, \quad t \in I_q, \tag{6}$$

where c_0, c_1 are constants (for functions not necessarily continuous at zero, the constants c_0, c_1 are q-periodic functions [7]). Using the boundary conditions in (6), we have

$$c_1 = 0, \quad c_0 = -\int_0^1 (1-qs)y(s)d_qs.$$

Substituting c_0, c_1 in (6) we have

$$x(t) \quad = \quad f(t,x(t))\left(\int_0^t (t-qs)y(s)d_qs - t\int_0^1 (1-qs)y(s)d_qs\right), \quad t \in I_q.$$

The converse follows by applying the operator D_q on (5) and using the q-integration by parts formula. □

3 Existence result-the single-valued case

The following fixed point theorem due to Dhage [8] is fundamental in the proof of our main result in this section.

Lemma 3.1 *Let S be a non-empty, closed convex and bounded subset of the Banach algebra X and $A: X \to X$, $B: S \to X$ two operators such that (a) A is Lipschitzian with a Lipschitz constant k; (b) B is completely continuous; (c) $x = AxBy \Rightarrow x \in S$ for all $y \in S$ and (d) $Mk < 1$, where $M = \|B(S)\| = \sup\{\|B(x)\| : x \in S\}$. Then the operator equation $x = AxBx$ has a solution.*

Theorem 3.2 *Assume that:*

(H_1) *The function $f: I_q \times \mathbb{R} \to \mathbb{R}\setminus\{0\}$ is continuous and there exists a bounded function ϕ, with bound $\|\phi\|$, such that $\phi(t) > 0$, for $t \in I_q$ and*

$$|f(t,x(t)) - f(t,y(t))| \le \phi(t)|x(t)-y(t)|, \quad \text{for } t \in I_q \text{ and for all } x,y \in \mathbb{R};$$

(H_2) *there exists a continuous nondecreasing function $\psi: [0,\infty) \to (0,\infty)$ and a function $p \in C(I_q, \mathbb{R}^+)$ such that $|g(t,x)| \le p(t)\psi(\|x\|)$ for each $(t,x) \in I_q \times \mathbb{R}$;*

(H_3) *There exists a constant $r > 0$ such that*

$$r > \frac{\dfrac{2F_0}{1+q}\|p\|\psi(r)}{1 - \dfrac{2\|\phi\|}{1+q}\|p\|\psi(r)}, \text{ where } \frac{2\|\phi\|}{1+q}\|p\|\psi(r) < 1, \text{ and } F_0 = \sup_{t\in[0,1]}|f(t,0)|. \tag{7}$$

Then the boundary value problem (1) has at least one solution on I_q.

Proof. Set $X = C(I_q, \mathbb{R})$ and define a subset S of X defined by

$$S = \{x \in X : \|x\| \leq r\},$$

where r is satisfies the inequality (7).

Clearly S is closed, convex and bounded subset of the Banach space X. By Lemma 2.2 the boundary value problem (1) is equivalent to the equation

$$x(t) = f(t, x(t)) \left(\int_0^t (t-qs)g(s, x(s))d_q s - t\int_0^1 (1-qs)g(s, x(s))d_q s \right), \quad t \in I_q. \tag{8}$$

Define two operators $\mathcal{A} : X \to X$ by

$$\mathcal{A}x(t) = f(t, x(t)), \ t \in I_q, \tag{9}$$

and $\mathcal{B} : S \to X$ by

$$\mathcal{B}x(t) = \int_0^t (t-qs)g(s, x(s))d_q s - t\int_0^1 (1-qs)g(s, x(s))d_q s, \quad t \in I_q. \tag{10}$$

Then $x = \mathcal{A}x\mathcal{B}x$. We shall show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Lemma 3.1. For the sake of convenience, we break the proof into a sequence of steps.

Step 1. *We first show that $\mathcal{A}$ is a Lipschitz on X, i.e. (a) of Lemma 3.1 holds.*

Let $x, y \in X$. Then by (H_1) we have

$$|\mathcal{A}x(t) - \mathcal{A}y(t)| = |f(t, x(t)) - f(t, y(t))| \leq \phi(t)|x(t) - y(t)| \leq \|\phi\|\|x - y\|,$$

for all $t \in I_q$. Taking the supremum over t we obtain

$$\|\mathcal{A}x - \mathcal{A}y\| \leq \|\phi\|\|x - y\|$$

for all $x, y \in X$. So $\mathcal{A}$ is a Lipschitz on X with Lipschitz constant $\|\phi\|$.

Step 2. *Now we show that the multi-valued operator $\mathcal{B}$ is completely continuous on S, i.e. (b) of Lemma 3.1 holds.*

First we show that $\mathcal{B}$ is continuous on S. Let $\{x_n\}$ be a sequence in S coverging to a pont $x \in S$. Then by Lebesque dominated convergence theorem,

$$\begin{aligned}
\lim_{n\to\infty} \mathcal{B}x_n(t) &= \lim_{n\to\infty} \int_0^t (t-qs)g(s, x_n(s))d_q s - \lim_{n\to\infty} t\int_0^1 (1-qs)g(s, x_n(s))d_q s \\
&= \int_0^t (t-qs) \lim_{n\to\infty} g(s, x_n(s))d_q s - t\int_0^1 (1-qs) \lim_{n\to\infty} g(s, x_n(s))d_q s \\
&= \int_0^t (t-qs)g(s, x(s))d_q s - t\int_0^1 (1-qs)g(s, x(s))d_q s \\
&= \mathcal{B}x(t),
\end{aligned}$$

for all $t \in I_q$. This shows that $\mathcal{B}$ is continuous os S. It is enough to show that $\mathcal{B}(S)$ is a uniformly bounded and equicontinuous set in X. First we note that

$$\begin{aligned}
|\mathcal{B}x(t)| &= \left| \int_0^t (t-qs)g(s, x(s))d_q s - t\int_0^1 (1-qs)g(s, x(s))d_q s \right| \\
&\leq \int_0^t (t-qs)|g(s, x(s))|d_q s + \int_0^1 (1-qs)|g(s, x(s))|d_q s
\end{aligned}$$

$$\leq \quad 2\int_0^1 (1-qs)p(s)\psi(\|x\|)d_qs \leq 2\|p\|\psi(\|x\|)\frac{1}{1+q},$$

for all $t \in I_q$. Taking supremum over $t \in I_q$, we get

$$\|\mathcal{B}x\| \leq \frac{2}{1+q}\|p\|\psi(r),$$

for all $x \in S$. This shows that $\mathcal{B}$ is uniformly bounded on S.

Next we show that $\mathcal{B}$ is an equicontinuous set in X. Let $t_1, t_2 \in I_q$ with $t_1 < t_2$. Then we have

$$\begin{aligned}
|\mathcal{B}x(t_2) - \mathcal{B}x(t_1)| &\leq \Big|\int_0^{t_2}(t_2-qs)g(s,x(s))d_qs - \int_0^{t_1}(t_1-qs)g(s,x(s))d_qs\Big| \\
&\quad +(t_2-t_1)\int_0^1 (1-qs)g(s,x(s))d_qs \\
&\leq \|p\|\psi(r)\int_0^{t_1}|t_2-t_1|d_qs + \|p\|\psi(r)\int_{t_1}^{t_2}|t_2-qs|d_qs \\
&\quad +\|p\|\psi(r)(t_2-t_1)\int_0^1 (1-qs)d_qs.
\end{aligned}$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in S$ as $t_2 - t_1 \to 0$. Therefore, it follows from the Arzelá-Ascoli theorem that $\mathcal{B}$ is a completely continuous operator on S.

Step 3. *Next we show that hypothesis (c) of Lemma 3.1 is satisfied.* Let $x \in X$ and $y \in S$ be arbitrary such that $x = \mathcal{A}x\mathcal{B}y$. Then we have

$$\begin{aligned}
|x(t)| &= |\mathcal{A}x(t)||\mathcal{B}y(t)| = |f(t,x(t))|\left|\int_0^t (t-qs)g(s,y(s))d_qs - t\int_0^1 (1-qs)g(s,y(s))d_qs\right| \\
&\leq [|f(t,x(t)) - f(t,0)| + |f(t,0)|]\Big|\int_0^t (t-qs)g(s,y(s))d_qs - t\int_0^1 (1-qs)g(s,y(s))d_qs\Big| \\
&\leq [\phi(t)x(t) + F_0]2\int_0^1 (1-qs)p(s)\psi(\|y\|)d_qs \\
&\leq [\|\phi\|x(t) + F_0]\frac{2}{1+q}\|p\|\psi(\|y\|) \leq [\|\phi\|x(t) + F_0]\frac{2}{1+q}\|p\|\psi(r).
\end{aligned}$$

Taking supremum over $t \in I_q$ and solving for $\|x\|$, we get

$$\|x\| \leq \frac{\dfrac{2F_0}{1+q}\|p\|\psi(r)}{1 - \dfrac{2\|\phi\|}{1+q}\|p\|\psi(r)} \leq r,$$

that is, $x \in S$.

Step 4. *Now we show that $Mk < 1$, i.e. (d) of Lemma 3.1 holds.*

This is obvious by (H_3) since we have $M = \|B(S)\| = \sup\{\|\mathcal{B}x\| : x \in S\} \leq 2\|h\|_{L^1}$ and $k = \|\phi\|$.

Thus all the conditions of Lemma 3.1 are satisfied and hence the operator equation $x = \mathcal{A}x\mathcal{B}x$ has a solution in S. As a result, the boundary value problem (1) has a solution defined on I_q. This completes the proof. □

Example 3.3 *Consider the boundary value problem*

$$\begin{cases} D_{1/2}^2\left(\dfrac{x(t)}{\sin x + 2}\right) = \dfrac{1}{4}\cos x(t), \quad 0 < t < 1, \\ \qquad x(0) = x(1) = 0, \end{cases} \tag{11}$$

Let $f(t,x) = \sin x + 2, g(t,x) = \dfrac{1}{4}\cos x$. Then (H_1) and (H_2) hold with $\phi(t) = 1$ and $p(t) = \dfrac{1}{4}, \psi(x) = 1$ respectively. Since $\dfrac{2\|\phi\|}{1+q}\|p\|\psi(r) = \dfrac{1}{3} < 1$, the boundary value problem (11) has a solution.

4 Existence result-the multi-valued case

Let us first outline the basic notions of multi-valued maps and fix our terminology [12, 20].

For a normed space $(X, \|\cdot\|)$, we define $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$ and

$$\mathcal{P}_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}.$$

A multi-valued map $\mathcal{G} : X \to \mathcal{P}(X)$ is convex (closed) valued if $\mathcal{G}(x)$ is convex (closed) for all $x \in X$. The map $\mathcal{G}$ is bounded on bounded sets if $\mathcal{G}(\mathbb{B}) = \cup_{x\in\mathbb{B}}\mathcal{G}(x)$ is bounded in X for all $\mathbb{B} \in \mathcal{P}_b(X)$ (i.e. $\sup_{x\in\mathbb{B}}\{\sup\{|y| : y \in \mathcal{G}(x)\}\} < \infty$). $\mathcal{G}$ is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $\mathcal{G}(x_0)$ is a nonempty closed subset of X, and for each open set N of X containing $\mathcal{G}(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $\mathcal{G}(\mathcal{N}_0) \subseteq N$. $\mathcal{G}$ is said to be completely continuous if $\mathcal{G}(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in P_b(X)$. If the multi-valued map $\mathcal{G}$ is completely continuous with nonempty compact values, then $\mathcal{G}$ is u.s.c. if and only if $\mathcal{G}$ has a closed graph, i.e., $x_n \to x_*,\ y_n \to y_*$, $y_n \in \mathcal{G}(x_n)$ imply $y_* \in \mathcal{G}(x_*)$.

Let $L^1(I_q)$ denote the space of all functions f defined on I_q such that $\|x\|_{L^1_q} = \int_0^1 |x(t)|d_qt < \infty$.

Definition 4.1 *A multivalued map $F : I_q \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory (in the sense of q-calculus) if $x \longmapsto F(t,x)$ is upper semicontinuous on I_q. Further a Carathéodory function F is called L^1-Carathéodory if there exists $\varphi_\alpha \in L^1(I_q, \mathbb{R}^+)$ such that $\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \le \varphi_\alpha(t)$ for all $\|x\| \le \alpha$ on I_q for each $\alpha > 0$.*

Definition 4.2 *A function $x \in \mathcal{C}$ is a solution of problem (2) if there exists a function $v \in L^1([0,1],\mathbb{R})$ such that it is continuous at $t = 0$ and $v(t) \in F(t,x(t))$ on I_q, and*

$$x(t) = f(t,x(t))\left(\int_0^t (t-qs)v(s)d_qs - t\int_0^1 (1-qs)v(s)d_qs\right), \quad t \in I_q.$$

For each $y \in \mathcal{C}$, define the set of selections of F by

$$S_{F,y} := \{v \in \mathcal{C} : v(t) \in F(t,y(t)) \text{ on } I_q\}.$$

The following lemma will be used in the sequel.

Lemma 4.3 *([21]) Let X be a Banach space. Let $F : J \times X \to \mathcal{P}_{cp,c}(X)$ be an L^1- Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1(J,X)$ to $C(J,X)$. Then the operator*

$$\Theta \circ S_F : C(J,X) \to \mathcal{P}_{cp,c}(C(J,X)), \quad x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$$

is a closed graph operator in $C(J,X) \times C(J,X)$.

The following fixed point theorem due to Dhage [9] is fundamental in the proof of our main result in this section.

Lemma 4.4 *Let X be a Banach algebra and let $A : X \to X$ be a single valued and $B : X \to \mathcal{P}_{cp,c}(X)$ be a multi-valued operator satisfying the conditions: (a) A is single-valued Lipschitz with a Lipschitz constant k; (b) B is compact and upper semi-continuous; (c) $2Mk < 1$, where $M = \|B(X)\|$. Then either (i) the operator inclusion $x \in AxBx$ has a solution, or (ii) the set $\mathcal{E} = \{u \in X | \mu u \in AuBu,\ \mu > 1\}$ is unbounded.*

Theorem 4.5 *Assume that (H_1) holds. In addition we suppose that*

(A_1) $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is* L^1*-Carathéodory and has nonempty compact and convex values;*

(A_2) $2\|\phi\|\|\varphi_\alpha\|_{L^1_q} < \frac{1}{2}$, *(*$\varphi_\alpha$ *is the function appeared in Definition 4.1).*

Then the boundary value problem (2) has at least one solution on $[0,1]$.

Proof. Set $X = C([0,1],\mathbb{R})$. Transform the problem (2) into a fixed point problem. Consider the operator $\mathcal{N}_1 : C([0,1],\mathbb{R}) \to \mathcal{P}(C([0,1],\mathbb{R}))$ defined by

$$\mathcal{N}_1(x) = \left\{ h \in C([0,1],\mathbb{R}) : h(t) = f(t,x(t))\left(\int_0^t (t-qs)v(s)d_qs - t\int_0^1 (1-qs)v(s)d_qs\right),\ v \in S_{F,x} \right\}.$$

Now we define two operators $\mathcal{A}_1 : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ by

$$\mathcal{A}_1x(t) = f(t,x(t)),\ t \in [0,1], \tag{12}$$

and $\mathcal{B}_1 : C([0,1],\mathbb{R}) \to \mathcal{P}(C([0,1],\mathbb{R}))$ by

$$\mathcal{B}_1(x) = \left\{ h \in C([0,1],\mathbb{R}) : h(t) = \textstyle\int_0^t (t-qs)v(s)d_qs - t\int_0^1 (1-qs)v(s)d_qs,\ v \in S_{F,x} \right\}. \tag{13}$$

Then $\mathcal{N}_1(x) = \mathcal{A}_1x\mathcal{B}_1x$. We shall show that the operators $\mathcal{A}_1$ and $\mathcal{B}_1$ satisfy all the conditions of Lemma 4.4. For better readability we break the proof into a sequence of steps.

Step 1. *We first show that* $\mathcal{A}_1$ *is a Lipschitz on* X, *i.e. (a) of Lemma 4.4 holds.*

The proof is similar to the one for the operator $\mathcal{A}$ in Step 1 of Theorem 3.2.

Step 2. *Now we show that the multi-valued operator* $\mathcal{B}_1$ *is compact and upper semi- continuous on* X, *i.e. (b) of Lemma 4.4 holds.*

First we show that $\mathcal{B}_1$ has convex values.
Let $u_1, u_2 \in \mathcal{B}_1x$. Then there are $v_1, v_2 \in S_{F,x}$ such that

$$u_i(t) = \int_0^t (t-qs)v_i(s)d_qs - t\int_0^1 (1-qs)v_i(s)d_qs, \quad t \in [0,1],\ i = 1,2.$$

For any $\theta \in [0,1]$, we have

$$\begin{aligned}\theta u_1(t) + (1-\theta)u_2(t) &= \int_0^t (t-qs)[\theta u_1(s) + (1-\theta)u_2(s)] - t\int_0^1 (1-qs)[\theta u_1(s) + (1-\theta)u_2(s)]d_qs \\ &= \int_0^t (t-qs)\overline{v}(s)d_qs - t\int_0^1 (1-qs)\overline{v}(s)d_qs,\end{aligned}$$

where $\overline{v}(t) = \theta v_1(t) + (1-\theta)v_2(t) \in F(t,x(t))$ for all $t \in [0,1]$. Hence $\theta u_1(t) + (1-\theta)u_2(t) \in \mathcal{B}_1x$ and consequently $\mathcal{B}_1x$ is convex for each $x \in X$. As a result $\mathcal{B}_1$ defines a multi valued operator $\mathcal{B}_1 : X \to \mathcal{P}_{cv}(X)$.

Next we show that $\mathcal{B}_1$ maps bounded sets into bounded sets in X. To see this, let Q be a bounded set in X. Then there exists a real number $r > 0$ such that $\|x\| \le r, \forall x \in Q$.
Now for each $h \in \mathcal{B}_1x$, there exists a $v \in S_{F,x}$ such that

$$h(t) = \int_0^t (t-qs)v(s)d_qs - t\int_0^1 (1-qs)v(s)d_qs.$$

Then for each $t \in [0,1]$,

$$|\mathcal{B}_1x(t)| = \left|\int_0^t (t-qs)v(s)d_qs - t\int_0^1 (1-qs)v(s)d_qs\right| \le \int_0^t (t-qs)g(s)d_qs + \int_0^1 (1-qs)g(s)d_qs$$

$$\leq \quad 2\int_0^1 (1-qs)\varphi_\alpha(s)d_qs = 2\|\varphi_\alpha\|_{L_q^1}.$$

This implies that $\|h\| \leq 2\|\varphi_\alpha\|_{L_q^1}$ and so $\mathcal{B}_1(X)$ is uniformly bounded.

Next we show that $\mathcal{B}_1$ maps bounded sets into equicontinuous sets. Let Q be, as above, a bounded set and $h \in \mathcal{B}_1 x$ for some $x \in Q$. Then there exists a $v \in S_{F,x}$ such that

$$h(t) = \int_0^t (t-qs)v(s)d_qs - t\int_0^1 (1-qs)v(s)d_qs.$$

Then for any $t_1, t_2 \in [0,1]$ with $t_1 \leq t_2$ we have

$$|h(t_2)-h(t_1)| \leq \Big| \int_0^{t_2} (t_2-qs)v(s)d_qs - \int_0^{t_1} (t_1-qs)v(s)d_qs \Big| + (t_2-t_1)\int_0^1 (1-qs)\varphi_\alpha(s)d_qs$$
$$\leq \quad \int_0^{t_1} [(t_2-qs)-(t_1-qs)]\,\varphi_\alpha(s)d_qs + \int_{t_1}^{t_2} (t_2-qs)\varphi_\alpha(s)d_qs + (t_2-t_1)\int_0^1 (1-qs)\varphi_\alpha(s)d_qs.$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in Q$ as $t_2 - t_1 \to 0$. Therefore it follows by the Arzelá-Ascoli theorem that $\mathcal{B}_1 : C([0,1],\mathbb{R}) \to \mathcal{P}(C([0,1],\mathbb{R}))$ is completely continuous.

In our next step, we show that $\mathcal{B}_1$ *has a closed graph.* Let $x_n \to x_*, h_n \in \mathcal{B}_1(x_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in \mathcal{B}_1$. Associated with $h_n \in \mathcal{B}_1(x_n)$, there exists $v_n \in S_{F,x_n}$ such that for each $t \in [0,1]$,

$$h_n(t) = \int_0^t (t-qs)v_n(s)d_qs - t\int_0^1 (1-qs)v_n(s)d_qs.$$

Thus it suffices to show that there exists $v_* \in S_{F,x_*}$ such that for each $t \in [0,1]$,

$$h_*(t) = \int_0^t (t-qs)v_*(s)d_qs - t\int_0^1 (1-qs)v_*(s)d_qs.$$

Let us consider the linear operator $\Theta : L^1([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ given by

$$f \mapsto \Theta(v)(t) = \int_0^t (t-qs)v(s)d_qs - t\int_0^1 (1-qs)v(s)d_qs.$$

Observe that

$$\|h_n(t)-h_*(t)\| = \left\| \int_0^t (t-qs)(v_n(s)-v_*(s))d_qs - t\int_0^1 (1-qs)(v_n(s)-v_*(s))d_qs \right\| \to 0, \text{as } n \to \infty.$$

Thus, it follows by Lemma 4.3 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, therefore, we have

$$h_*(t) = \int_0^t (t-qs)v_*(s)d_qs - t\int_0^1 (1-qs)v_*(s)d_qs.$$

for some $v_* \in S_{F,x_*}$.

As a result we have that the operator $\mathcal{B}_1$ is compact and upper semicontinuous operator on X.

Step 3. *Now we show that* $2Mk < 1$, *i.e. (c) of Lemma 4.4 holds.*

This is obvious by (H_3) since we have $M = \|B(X)\| = \sup\{|\mathcal{B}_1 x : x \in X\} \leq 2\|\varphi_\alpha\|_{L_q^1}$ and $k = \|\phi\|$.

Thus all the conditions of Lemma 4.4 are satisfied and a direct application of it yields that either the conclusion (i) or the conclusion (ii) holds. We show that the conclusion (ii) is not possible.

Let $u \in \mathcal{E}$ be arbitrary. Then we have for $\lambda > 1$, $\lambda u \in \mathcal{A}_1 u(t)\mathcal{B}_1 u(t)$. Then there exists $v \in S_{F,x}$ such that for any $\lambda > 1$, one has

$$u(t) = \lambda^{-1}[f(t,u(t)]\left(\int_0^t (t-qs)v(s)d_qs - t\int_0^1 (1-qs)v(s)d_qs\right),$$

for all $t \in [0,1]$. Then we have

$$\begin{aligned} |u(t)| &\leq \lambda^{-1}|f(t,u(t)|\left(\int_0^t (t-qs)|v(s)|d_qs + \int_0^1 (1-qs)|v(s)|d_qs\right) \\ &\leq [|f(t,u(t)-f(t,0)|+|f(t,0)|]\left(\int_0^t (t-qs)|v(s)|d_qs + \int_0^1 (1-qs)|v(s)|d_qs\right) \\ &\leq [\|\phi\|\|u\|+F_0]2\int_0^1 (1-qs)\varphi_\alpha(s)d_qs \leq [\|\phi\|\|u\|+F_0]2\|\varphi_\alpha\|_{L_q^1}, \end{aligned}$$

where we have used $F_0 = \sup_{t\in[0,1]} |f(t,0)|$. In consequence, we have

$$\|u\| \leq \frac{2F_0\|\varphi_\alpha\|_{L_q^1}}{1-2\|\phi\|\|\varphi_\alpha\|_{L_q^1}}.$$

Thus the condition (ii) of Thorem 4.4 does not hold. Therefore the operator equation $\mathcal{A}_1 x\mathcal{B}_1 x$ and consequently problem (2) has a solution on $[0,1]$. This completes the preginoof. □

Example 4.6 *Consider the problem*

$$\begin{cases} D^{3/2}\Big[\dfrac{x(t)}{(t\tan^{-1}x)/8+2}\Big] \in F(t,x(t)), \quad 0<t<1, \\ x(0)=x(1)=0, \end{cases} \tag{14}$$

where $F : [0,1]\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is a multivalued map given by*

$$t \to F(t,x) = \left[\frac{|x|^5}{5(|x|^5+4)}, \frac{|\sin x|}{7(|\sin x|+1)} + \frac{6}{7}\right].$$

For $f \in F$ we have

$$|f| \leq \max\left(\frac{|x|^5}{5(|x|^5+4)}, \frac{|\sin x|}{7(|\sin x|+1)} + \frac{6}{7}\right) \leq 1, \ x \in \mathbb{R}.$$

Thus

$$\|F(t,x)\| = \sup\{|y| : y \in F(t,x)\} \leq 1 = \varphi_\alpha(t), \ \ x \in \mathbb{R}.$$

Further, $\|\phi\| = 1/8$, and

$$2\|\phi\|\|\varphi_\alpha\|_{L_q^1} = \frac{1}{4} < \frac{1}{2}.$$

Thus all the conditions of Theorem 4.5 are satisfied and consequently, the problem (14) has a solution on $[0,1]$.

Acknowledgment. This paper was supported by Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, Saudi Arabia. The authors, therefore, acknowledge technical and financial support of KAU.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 19, NO.6, 2015, COPYRIGHT 2015 EUDOXUS PRESS, LLC

$q-$Wallis inequality

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Abstract

In the paper, we present the following double inequality

$$\frac{1}{\Gamma_{q^2}(1/2)\sqrt{[n+\nu]_{q^2}}} < W_{n,q} < \frac{1}{\Gamma_{q^2}(1/2)\sqrt{[n+\mu]_{q^2}}}, \qquad 0 < \mu \leq \frac{1}{4};\ \nu \geq \frac{1}{2}$$

where $n \in N$, $0 < q < 1$, $\Gamma_q(x)$ is the $q-$gamma function and $W_{n,q}$ is the $q-$Wallis ratio.

2010 Mathematics Subject Classification: 05A10, 33D05, 26D20.

Key Words: $q-$numbers, $q-$double factorial, $q-$Wallis inequality, $q-$gamma function.

1 Introduction.

The double factorial is defined for $n = 1, 2, 3, ...$ by

$$(2n)!! = \prod_{k=1}^{n}(2k) = 2^n\Gamma(n+1)$$

and

$$(2n-1)!! = \prod_{k=1}^{n}(2k-1) = \frac{2^n}{\sqrt{\pi}}\Gamma(n+1/2),$$

where $\Gamma(x)$ is the ordinary gamma function. The formula

$$\frac{\pi}{2} = \lim_{n\to\infty}\left(\frac{1}{2n+1}\left[\frac{(2n)!!}{(2n-1)!!}\right]^2\right) = \prod_{n=1}^{\infty}\frac{4n^2}{(2n-1)(2n+1)} \tag{1}$$

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was first proved by the British mathematician John Wallis (1616 - 1703) [23] and he also showed that

$$\frac{1}{\sqrt{\pi(n+1/2)}} < \frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{\pi n}}; \qquad n = 1, 2, \dots , \tag{2}$$

which is called Wallis inequality and the ratio

$$W_n = \frac{(2n-1)!!}{(2n)!!} = \frac{\Gamma(n+1/2)}{\sqrt{\pi}\,\Gamma(n+1)}, \qquad n \in N \tag{3}$$

is called also Wallis ratio. These results of Wallis have wide applications in mathematics formulae specially in graph theory, combinatorics and special functions. For more information on inequality and formula of Wallis, see [6], [8], [10], [12], [13], [15], [17], [18], [21], [22].

N. D. Kazarinoff [11] presented the following refinement of the upper bound of the Wallis inequality:

$$\frac{1}{\sqrt{\pi(n+1/2)}} < W_n < \frac{1}{\sqrt{\pi(n+1/4)}}, \qquad n = 1, 2, \dots . \tag{4}$$

C.-P. Chen and F. Qi [3] improved the lower bound of the inequality (4) and proved the sharpness of the inequality

$$\frac{1}{\sqrt{\pi(n+\frac{4}{\pi}-1)}} < W_n < \frac{1}{\sqrt{\pi(n+1/4)}}, \qquad n = 1, 2, \dots . \tag{5}$$

For the same result with different handling see [4] and [5].

Also, C. Mortici [16] improved the estimates in [24] and [25] by presenting the following upper and lower estimates

$$\frac{1}{\sqrt{\pi n\left(1+\frac{1}{4n-\frac{1}{2}+\frac{3}{16n+\frac{15}{4n}}}\right)}} < W_n < \frac{1}{\sqrt{\pi n\left(1+\frac{1}{4n-\frac{1}{2}+\frac{3}{16n}}\right)}}.$$

Recently, F. Qi an C. Mortici [20] improved the formulas and inequalities for the ratio W_n presented in [9] and they deduce the following new approximation formula as $n \to \infty$

$$W_n \sim \sqrt{\frac{e}{\pi\, n}}\left[1-\frac{1}{2(n+1/3)}\right]^{n+1/3}$$

and the double inequality for $n \geq 1$

$$\sqrt{\frac{e}{\pi\, n}}\left[1-\frac{1}{2(n+1/3)}\right]^{n+1/3} < W_n < \sqrt{\frac{e}{\pi\, n}}\left[1-\frac{1}{2(n+1/3)}\right]^{n+1/3} e^{\frac{1}{144n^3}}.$$

In this paper, we define the $q-$Wallis ratio by

$$W_{n,q} = \frac{[2n-1]_q!!}{[2n]_q!!}, \qquad n \in N;\ 0 < q < 1 \tag{6}$$

where the $q-$number $[n]_q = \frac{1-q^n}{1-q}$ and the $q-$double factorial is defined by

$$[2n]_q!! = \prod_{k=1}^{n} [2k]_q$$

and

$$[2n-1]_q!! = \prod_{k=1}^{n} [2k-1]_q.$$

Using the relation $\lim_{q\to 1^-}[n]_q = n$ (see [7]), we can conclude that

$$\lim_{q\to 1^-} W_{n,q} = W_n.$$

Also, by direct calculations, we have

$$(1-q^{2n+2})W_{n+1,q} - (1-q^{2n+1})W_{n,q} = 0$$

and

$$W_{n+1,q} < W_{n,q},$$

where $1-q^{2n+1} < 1-q^{2n+2}$ for $n \in N$ and $0 < q < 1$. Also, we will present some new double inequalities of $q-$Wallis ratio, which include some double inequalities of Wallis ratio as $q \to 1^-$.

2 Main results

Lemma 2.1. *For $n \in N$ and $0 < q < 1$, we have*

$$\frac{1}{\sqrt{[2]_q[2n]_q}} < W_{n,q} < \frac{1}{\sqrt{[2n+1]_q}}. \tag{7}$$

Proof.

$$W_{n,q} = \frac{[2n-1]_q!!}{[2n]_q!!} = \frac{[1]_q[3]_q[5]_q...[2n-3]_q[2n-1]_q}{[2]_q[4]_q[6]_q...[2n-2]_q[2n]_q}.$$

For $0 < q < 1$ and $k = 1, 2, ...,$ we have

$$0 < \frac{[2k-1]_q[2k+1]_q}{[2k]_q^2} = 1 - \frac{(1-q)^2q^{2k}}{q(1-q^{2k})^2}$$

and hence

$$\frac{[2k-1]_q[2k+1]_q}{[2k]_q^2} < 1.$$

Then
$$\frac{[2k-1]_q}{[2k]_q} < \frac{[2k]_q}{[2k+1]_q}, \qquad 0 < q < 1;\ k = 1, 2, \dots . \tag{8}$$
Now using the inequality (8), we obtain
$$\begin{aligned} W_{n,q} &< \frac{[2]_q}{[3]_q}\frac{[4]_q}{[5]_q}\cdots\frac{[2n]_q}{[2n+1]_q} \\ &< \frac{1}{\frac{[1]_q[3]_q\dots[2n-1]_q[2n+1]_q}{[2]_q[4]_q\dots[2n]_q}} \\ &< \frac{1}{[2n+1]_q W_{n,q}}. \end{aligned}$$

Then
$$W_{n,q} < \frac{1}{\sqrt{[2n+1]_q}}, \qquad n \in N;\ 0 < q < 1. \tag{9}$$
For $0 < q < 1$ and $k = 2, 3, \dots$, we have
$$0 < \frac{[2k-1]_q^2}{[2k]_q[2k-2]_q} = 1 + \frac{(1-q)^2 q^{2k}}{q^2(1-q^{2k})(1-q^{2k-2})}$$

and hence
$$\frac{[2k-1]_q^2}{[2k]_q[2k-2]_q} > 1.$$
Then
$$\frac{[2k-1]_q}{[2k]_q} > \frac{[2k-2]_q}{[2k-1]_q}, \qquad 0 < q < 1;\ k = 2, 3, \dots . \tag{10}$$
Now using the inequality (10), we obtain
$$\begin{aligned} W_{n,q} &> \frac{1}{[2]_q}\frac{[2]_q}{[3]_q}\frac{[4]_q}{[5]_q}\cdots\frac{[2n-2]_q}{[2n-1]_q} \\ &> \frac{1}{[2]_q[2n]_q}\left(\frac{[2]_q[4]_q\dots[2n]_q}{[3]_q[5]_q\dots[2n-1]_q}\right) \\ &> \frac{1}{[2]_q[2n]_q W_{n,q}}. \end{aligned}$$

Then
$$W_{n,q} > \frac{1}{\sqrt{[2]_q[2n]_q}}, \qquad n \in N;\ 0 < q < 1. \tag{11}$$

□

When $q \to 1^-$, we get the following result [3]:

Corollary 2.2. *For $n \in N$, we have*
$$\frac{1}{2\sqrt{n}} < W_n < \frac{1}{\sqrt{2n+1}}. \tag{12}$$

The $q-$gamma function $\Gamma_q(x)$ is defined by [7]

$$\Gamma_q(z) = \frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z}; \qquad z \neq 0, -1, -2, ..., \tag{13}$$

where $0 < q < 1$. Here we use the $q-$Pochhammer ($q-$shifted) symbol [14]:
$(a;q)_0 = 1$,
$(a;q)_k = \prod_{j=0}^{k-1}(1-aq^j); \qquad k = 1, 2, ...$
This function is a $q-$analogue of the ordinary gamma function since we have

$$\lim_{q\to 1}\Gamma_q(z) = \Gamma(z).$$

Also, it satisfies the functional equation

$$\Gamma_q(z+1) = [z]_q\Gamma_q(z); \quad \Gamma_q(1) = 1, \tag{14}$$

which is a $q-$extension of the well-known functional equation

$$\Gamma(z+1) = z\Gamma(z); \quad \Gamma(1) = 1.$$

In the next reult, we will present a $q-$analogue of the relation (3).

Lemma 2.3. *For $n \in N$ and $0 < q < 1$, we have*

$$W_{n,q} = \frac{\Gamma_{q^2}(n+1/2)}{\Gamma_{q^2}(1/2)\Gamma_{q^2}(n+1)} \tag{15}$$

Proof. Using the relation (14), we have

$$\begin{aligned}\Gamma_q(n+1/2) &= \frac{1-q^{n-1/2}}{1-q}\frac{1-q^{n-3/2}}{1-q}...\frac{1-q^{1/2}}{1-q}\Gamma_q(1/2)\\ &= ([1/2]_q)^n\,[2n-1]_{q^{1/2}}\Gamma_q(1/2).\end{aligned}$$

But

$$[1/2]_q = \frac{1}{[2]_{q^{1/2}}},$$

then

$$[2n-1]_q!! = \frac{([2]_q)^n\Gamma_q(n+1/2)}{\Gamma_{q^2}(1/2)}. \tag{16}$$

Also,

$$\frac{\Gamma_q(n+1)}{([1/2]_q)^n} = \frac{(1-q^n)(1-q^{n-1})...(1-q)}{(1-q^{1/2})^n} = [2n]_{q^{1/2}}.$$

Then

$$[2n]_q!! = ([2]_q)^n\Gamma_{q^2}(n+1). \tag{17}$$

Using the relations (16) and (17), we obtain our result (15). $\square$

Our next result will present a general $q-$analogue of the inequality (4).

Theorem 1. *For $n \in N$ and $0<q<1$, we have*

$$\frac{1}{\Gamma_{q^2}(1/2)\sqrt{[n+\nu]_{q^2}}} < W_{n,q} < \frac{1}{\Gamma_{q^2}(1/2)\sqrt{[n+\mu]_{q^2}}}, \qquad 0<\mu\le\frac{1}{4};\ \nu\ge\frac{1}{2}. \tag{18}$$

Proof. Let

$$H_{n,q,h} = \Gamma_{q^2}(1/2)W_{n,q}\sqrt{[n+h]_{q^2}}.$$

Alzer presented the double inequality [1] (see also [19])

$$\left[\frac{1-q^{x+\alpha(q,s)}}{1-q}\right]^{1-s} < \frac{\Gamma_q(x+1)}{\Gamma_q(x+s)} < \left[\frac{1-q^{x+\beta(q,s)}}{1-q}\right]^{1-s}, \quad x>0;\ s\in(0,1);\ 0<q<1 \tag{19}$$

with the best possible values $\alpha(q,s) = \frac{\ln[(q^s-q)/(1-s)(1-q)]}{\ln q}$ and $\beta(q,s) = \frac{\ln\left[1-(1-q)[\Gamma_q(s)]^{1/(s-1)}\right]}{\ln q}$. Using the inequality (19) at $s=1/2$, $x=n\in N$ and by replacing q by q^2, we get

$$\left[\frac{1-q^2}{1-q^{2n+2\beta(q,1/2)}}\right]^{1/2} < \frac{\Gamma_{q^2}(n+1/2)}{\Gamma_{q^2}(n+1)} < \left[\frac{1-q^2}{1-q^{2n+2\alpha(q,1/2)}}\right]^{1/2}, \quad 0<q<1. \tag{20}$$

Now multiply (20) by $\sqrt{[n+h]_{q^2}}$ with $h>0$ to obtain

$$\left[\frac{1-q^{2n+2h}}{1-q^{2n+2\beta(q,1/2)}}\right]^{1/2} < \frac{\Gamma_{q^2}(n+1/2)}{\Gamma_{q^2}(n+1)}\sqrt{[n+h]_{q^2}} < \left[\frac{1-q^{2n+2h}}{1-q^{2n+2\alpha(q,1/2)}}\right]^{1/2}, \quad h>0;\ 0<q<1. \tag{21}$$

Then

$$\lim_{n\to\infty}\frac{\Gamma_{q^2}(n+1/2)}{\Gamma_{q^2}(n+1)}\sqrt{[n+h]_{q^2}} = 1, \qquad h>0;\ 0<q<1.$$

Hence

$$\lim_{n\to\infty} H_{n,q,h} = 1, \qquad h>0;\ 0<q<1.$$

Now let

$$\begin{aligned} M_{n,q,h} &= (1-q^{2n+2h})(1-q^{2n+2})^2 - (1-q^{2n+2h+2})(1-q^{2n+1})^2 \\ &= (1-q)q^{2n}\left(q\left(2-q^{2n+1}-q^{2n+2}\right) + q^{2h}\left(-1-q+2q^{2n+2}\right)\right). \end{aligned} \tag{22}$$

If we suppose that $h=\mu$ with

$$0<\mu\le\frac{1}{4},$$

then we get

$$\begin{aligned} M_{n,q,\mu} &< (1-q)q^{2n}\left(q^{2\mu+1/2}\left(2-q^{2n+1}-q^{2n+2}\right) + q^{2\mu}\left(-1-q+2q^{2n+2}\right)\right) \\ &< -(1-q)q^{2n+2\mu+1/2}(1-\sqrt{q})^2(1+q^{2n+3/2}) \\ &< 0, \qquad 0<\mu\le\frac{1}{4}. \end{aligned}$$

Then

$$\frac{H_{n,q,\mu}}{H_{n+1,q,\mu}} = \frac{1-q^{2n+2}}{1-q^{2n+1}}\sqrt{\frac{1-q^{2n+2\mu}}{1-q^{2n+2\mu+2}}} < 1$$

and hence the $\{H_{n,q,\mu}\}$; $0<\mu\le 1/4$; is increasing sequence and its limit is 1 as n tends to ∞, hence

$$H_{n,q,\mu}<1 \qquad 0<\mu\le 1/4.$$

Then we have

$$W_{n,q}<\frac{1}{\Gamma_{q^2}(1/2)\sqrt{[n+\mu]_{q^2}}}, \qquad 0<\mu\le\frac{1}{4};\ 0<q<1;\ n\in N.$$

If we suppose that $h=\nu$ with

$$\nu\ge\frac{1}{2},$$

we get

$$\begin{aligned} M_{n,q,\nu} &> (1-q)q^{2n}\left(q\left(2-q^{2n+1}-q^{2n+2}\right)+q\left(-1-q+2q^{2n+2}\right)\right)\\ &> (1-q)^2q^{2n+1}(1-q^{2n+1})\\ &> 0, \qquad \nu\ge\frac{1}{2}. \end{aligned}$$

Then

$$\frac{H_{n,q,\nu}}{H_{n+1,q,\nu}}=\frac{1-q^{2n+2}}{1-q^{2n+1}}\sqrt{\frac{1-q^{2n+2\nu}}{1-q^{2n+2\nu+2}}}>1$$

and hence the $\{H_{n,q,\nu}\}$; $\nu\ge 1/2$; is decreasing sequence and its limit is 1 as n tends to ∞, hence

$$H_{n,q,\nu}>1 \qquad \nu\ge 1/2.$$

Then we have

$$W_{n,q}>\frac{1}{\Gamma_{q^2}(1/2)\sqrt{[n+\nu]_{q^2}}}, \qquad \nu\ge 1/2;\ 0<q<1;\ n\in N.$$

□

2.1 Special cases

1− If we let $\mu_1=\frac{1}{4}$ and $\nu_1=\frac{1}{2}$ and take the limit as $q\to 1^-$ for the inequality (18), we get the inequality (4).

2− Let $\mu_2=\frac{1}{4}$ and $\nu_2=\frac{4}{\Gamma^2_{q^2}(1/2)}-1$, then μ_2 satisfies the condition of (18) and will prove that ν_2 also satisfies it. Using the inequality [2]

$$\Gamma_q(x)\le 1 \quad \text{iff} \quad 1\le x\le 2, \qquad\qquad 0<q<1$$

we get

$$[1/2]_q\Gamma_q(1/2)\le 1$$

and

$$\Gamma_{q^2}(1/2)\le 1+q.$$

Then

$$2\nu_2 - 1 > \frac{8 - 3q - 3}{1 + q} > 0$$

and we obtain the following result from the inequality (18):

Lemma 2.4. *For $n \in N$ and $0 < q < 1$, we have*

$$\frac{1}{\Gamma_{q^2}(1/2)\sqrt{\left[n + \frac{4}{\Gamma^2_{q^2}(1/2)} - 1\right]_{q^2}}} < W_{n,q} < \frac{1}{\Gamma_{q^2}(1/2)\sqrt{[n + 1/4]_{q^2}}}. \tag{23}$$

Remark 1. The inequality (5) is a limiting case of the inequality (23) as $q \to 1^-$.

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ON A μ-MIXED TRIGONOMETRIC FUNCTIONAL EQUATION

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ABSTRACT. In this paper, we study the superstability for the μ-mixed trigonometric functional equation:

$$\int_G f(xty)d\mu(t) - \int_G f(xt\sigma(y))d\mu(t) = 2f(x)g(y), \quad x, y \in G,$$

where G is a locally compact group, $f, g : G \to \mathbb{C}$ are continuous functions, μ is *a complex bounded measure* and σ is a continuous involution of G. We also give some applications.

1. Introduction

The stability problem for the functional equations

$$f(xy) + f(x\sigma(y)) = 2f(x)g(y), \quad x, y \in G, \tag{W}$$

$$f(xy) - f(x\sigma(y)) = 2f(x)g(y), \quad x, y \in G, \tag{T}$$

where σ is an involution of the group G, i.e., $\sigma(\sigma(x)) = x$ and $\sigma(xy) = \sigma(y)\sigma(x)$, for all $x, y \in G$, has been considered in [16, 21]. It has been proved in [21] that the functional equation (T) is superstable in the class of functions f, g defined on an arbitrary group G and take its values in $\mathbb{C}$, i.e., every such function satisfying the inequality

$$|f(xy) - f(x\sigma(y)) - 2f(x)g(y)| \leq \varepsilon, \quad x, y \in G, \tag{I}$$

where ε is a fixed positive real number, either is bounded or satisfies (T). The first results of that kind have been established in [3] for the exponential equation, in [2] for the equation (W) on an abelian group and in [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 18, 19, 20, 22, 23, 24] for trigonmetric functional equations on any groups. For more information concerning superstability we refer to [4].

In this paper, let G be a locally compact group, e denotes its neutral element. $C(G)$ (resp. $C_b(G)$) designates the space of continuous (resp. continuous and bounded) complex valued functions. Let σ be a continuous involution of G, $\mathbb{C}$ the field of complex numbers and let μ be a complex bounded measure which is σ-invariant, i.e.,

$$\int_G f(\sigma(t))d\mu(t) = \int_G f(t)d\mu(t) \textit{ for all } \ f \in C(G).$$

2010 *Mathematics Subject Classification.* Primary 39B72, 39B82.

Key words and phrases. Hyers-Ulam stability; Superstability; d'Alembert's equation; μ-Trigonometric functional equation.

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We recall that the convolution $\mu*\mu$ is the measure given by $\langle \mu * \mu, f\rangle = \int_G \int_G f(ts)d\mu(t)d\mu(s)$ for all $f \in C_b(G)$.

The aim of this paper is to investigate the superstability problem for the μ-mixed trigonometric functional equations

$$\int_G f(xty)d\mu(t) - \int_G f(xt\sigma(y))d\mu(t) = 2f(x)f(y), \qquad x, y \in G, \qquad (T(\mu))$$

$$\int_G f(xty)d\mu(t) - \int_G f(xt\sigma(y))d\mu(t) = 2f(x)g(y), \qquad x, y \in G, \qquad (T_{f,g}(\mu))$$

on any locally compact group under the condition that μ is a compactly supported and σ-invariant measure on G. We do not impose any conditions on the continuous functions f, g.

2. Prelimenary results

In this section we collect some auxiliary results on solutions of the functional equation $(T_{f,g}(\mu))$.

Lemma 1. *Assume that $f, g \in C_b(G)$ are solutions of the functional equation*

$$\int_G f(xty)d\mu(t) - \int_G f(xt\sigma(y))d\mu(t) = 2f(x)g(y), \quad x, y \in G,$$

such that $f \neq 0$. Then

(i) g is odd, that is, $g(\sigma(x)) = -g(x)$ for all $x \in G$.

(ii) g satisfies the identity

$$\int_G g(xty)d\mu(t) - \int_G g(xt\sigma(y))d\mu(t) - \int_G g(ytx)d\mu(t) + \int_G g(\sigma(y)tx)d\mu(t) = 0, \qquad (2.1)$$

for all $x, y \in G$.

Proof. (i) Since f, g are solutions of $(T_{f,g}(\mu))$, we have

$$\int_G f(xt\sigma(y))d\mu(t) - \int_G f(xty)d\mu(t) = 2f(x)g(\sigma(y)), \qquad (2.2)$$

for all $x, y \in G$. By adding the identities (2.2) and $(T_{f,g}(\mu))$, we get

$$2f(x)(g(y) + g(\sigma(y))) = 0, \; x, y \in G.$$

Therefore, $g(\sigma(x)) = -g(x)$ for all $x \in G$.

(ii) Choosing $a \in X$ such that $f(a) \neq 0$, we get from the equation $(T_{f,g}(\mu))$ that

$$g(x) = \frac{1}{2f(a)} \left(\int_G f(xta)d\mu(t) - \int_G f(xt\sigma(a))d\mu(t) \right). \qquad (2.3)$$

Using repeatedly (2.3), we get

$$\begin{aligned}
&2f(a)\left\{\int_G g(xty)d\mu(t)-\int_G g(xt\sigma(y))d\mu(t)-\int_G g(ytx)d\mu(t)+\int_G g(\sigma(y)tx)d\mu(t)\right\}\\
&=\int_G\int_G f(atxsy)d\mu(s)d\mu(t)-\int_G\int_G f(at\sigma(y)s\sigma(x))d\mu(s)d\mu(t)\\
&\quad-\int_G\int_G f(atysx)d\mu(s)d\mu(t)+\int_G\int_G f(at\sigma(x)s\sigma(y))d\mu(s)d\mu(t)\\
&\quad-\int_G\int_G f(atxs\sigma(y))d\mu(s)d\mu(t)+\int_G\int_G f(atys\sigma(x))d\mu(s)d\mu(t)\\
&\quad+\int_G\int_G f(at\sigma(y)sx)d\mu(s)d\mu(t)-\int_G\int_G f(at\sigma(x)sy)d\mu(s)d\mu(t)\\
&=2g(y)\left\{\int_G f(atx)d\mu(t)-\int_G f(at\sigma(x))d\mu(t)\right\}\\
&\quad+2g(x)\left\{-\int_G f(aty)d\mu(t)+\int_G f(at\sigma(y))d\mu(t)\right\}\\
&=4g(y)f(a)g(x)-4g(x)f(a)g(y)=0.
\end{aligned}$$

Since $f(a)\neq 0$, we get that g is a solution of the functional equation (2.1). □

Lemma 2. *Let the pair $f,g\in C_b(G)$ be a solution of $(T_{f,g}(\mu))$ such that $g\neq 0$ and μ is idempotent, i.e., $\mu*\mu=\mu$. Then f is right μ-invariant, that is, $\int_G f(xt)d\mu(t)=f(x)$ for all $x\in G$.*

Proof. Using $\mu*\mu=\mu$, we get that

$$\begin{aligned}
\int_G f(xt)d\mu(t)g(y) &= \int_G\int_G f(xtsy)d\mu(s)d\mu(t)-\int_G\int_G f(xts\sigma(y))d\mu(s)d\mu(t)\\
&= \int_G f(xty)d\mu(t)-\int_G f(xt\sigma(y)t)d\mu(t)\\
&= 2f(x)g(y).
\end{aligned}$$

Since $g\neq 0$, we obtain

$$\int_G f(xt)d\mu(t)=f(x)$$

for all $x\in G$. □

It will be convenient to solve the functional equation $(T(\mu))$.

Lemma 3. *Assume that $\mu*\mu=\mu$. Then the solution of the functional equation $(T(\mu))$ on any locally compact group G is the zero function $f\equiv 0$.*

Proof. Assume that f is a solution of $(T(\mu))$. By virtue of Lemma 2, we have

$$\int_G f(xt)d\mu(t)=f(x), \tag{2.4}$$

for all $x \in G$ and

$$\begin{aligned}\int_G f(tx)d\mu(t) &= \int_G f(\sigma(x)t)d\mu(t) \\ &= f(\sigma(x)) = -f(x), \end{aligned} \tag{2.5}$$

for all $x \in G$. Using Lemma 1 (i), we have

$$f(\sigma(y)) = -f(y), \quad y \in G.$$

Taking $y = e$ in $((T(\mu))$, we obtain

$$\int_G f(t\sigma(y))d\mu(t) - \int_G f(ty)d\mu(t) = 2f(e)f(\sigma(y)), \qquad x, y \in G.$$

Using (2.4), (2.5) and the fact that $f(e) = 0$, one can obtain that the last equality implies

$$-f(\sigma(y)) + f(y) = 0,$$

for all $y \in G$. Consequently, we have

$$f(\sigma(y)) = f(y) = -f(y), \; y \in G,$$

which shows that $f \equiv 0$. ☐

3. Superstability of the equation $(T_{f,g}(\mu))$

In this section, μ is assumed to be a compactly supported measure on G which is σ-invariant. $\varphi, \psi : G \to \mathbb{R}$ are continuous functions and ε is a nonnegative real constant. We will study the superstability of the functional equation $(T_{f,g}(\mu))$ in the sense of Badora and Ger [1], by starting with Lemma 4 where the unboundeness of f is assumed.

Lemma 4. *Assume that $f, g \in C(G)$ and a continuous function $\psi : G \to \mathbb{R}$ satisfy the inequality*

$$\left| \int_G f(xty)d\mu(t) - \int_G f(xt\sigma(y))d\mu(t) - 2f(x)g(y) \right| \leq \psi(y), \tag{3.1}$$

for all $x, y \in G$. Then either f is bounded or we have the following identity

$$\int_G g(xty)d\mu(t) - \int_G g(xt\sigma(y))d\mu(t) - \int_G g(ytx))d\mu(t) + \int_G g(\sigma(y)tx)d\mu(t) = 0, \tag{3.2}$$

for all $x, y \in G$.

Proof. Assume that f, g satisfy the inequality (3.1) such that f is unbounded. For all $x, y, z \in G$, we have

$$2\,|f(z)| \left| \int_G g(xty)d\mu(t) - \int_G g(xt\sigma(y))d\mu(t) - \int_G g(ytx))d\mu(t) + \int_G g(\sigma(y)tx)d\mu(t) \right|$$

$$
\begin{aligned}
&= \left|2f(z)\int_G g(xty)d\mu(t) - 2f(z)\int_G g(xt\sigma(y))d\mu(t) - 2f(z)\int_G g(ytx))d\mu(t) + 2f(z)\int_G g(\sigma(y)tx)d\mu(t)\right| \\
&\le \left|-\int_G\int_G f(ztxsy)d\mu(t)d\mu(s) + \int_G\int_G f(zt\sigma(y)s\sigma(x))d\mu(t)d\mu(s) + 2f(z)\int_G g(xty)d\mu(t)\right| \\
&+ \left|\int_G\int_G f(ztxs\sigma(y))d\mu(t)d\mu(s) - \int_G\int_G f(ztys\sigma(x))d\mu(t)d\mu(s) - 2f(z)\int_G g(xt\sigma(y))d\mu(t)\right| \\
&+ \left|\int_G\int_G f(ztysx)d\mu(t)d\mu(s) - \int_G\int_G f(zt\sigma(x)s\sigma(y))d\mu(t)d\mu(s) - 2f(z)\int_G g(ytx)d\mu(t)\right| \\
&+ \left|-\int_G\int_G f(zt\sigma(y)sx)d\mu(t)d\mu(s) + \int_G\int_G f(zt\sigma(x)sy)d\mu(t)d\mu(s) + 2f(z)\int_G g(\sigma(y)tx)d\mu(t)\right| \\
&+ \left|\int_G\int_G f(ztxsy)d\mu(t)d\mu(s) - \int_G\int_G f(ztxs\sigma(y))d\mu(t)d\mu(s) - 2g(y)\int_G f(ztx)d\mu(t)\right| \\
&+ \left|-\int_G\int_G f(zt\sigma(x)sy)d\mu(t)d\mu(s) + \int_G\int_G f(zt\sigma(x)s\sigma(y))d\mu(t)d\mu(s) + 2g(y)\int_G f(zt\sigma(x))d\mu(t)\right| \\
&+ \left|-\int_G\int_G f(ztysx)d\mu(t)d\mu(s) + \int_G\int_G f(ztys\sigma(x))d\mu(t)d\mu(s) + 2g(x)\int_G f(zty)d\mu(t)\right| \\
&+ \left|\int_G\int_G f(zt\sigma(y)sx)d\mu(t)d\mu(s) - \int_G\int_G f(zt\sigma(y)s\sigma(x))d\mu(t)d\mu(s) - 2g(x)\int_G f(zt\sigma(y))d\mu(t)\right| \\
&+2\left|g(y)\right|\left|\int_G f(ztx)d\mu(t) - \int_G f(zt\sigma(x))d\mu(t) - 2f(z)g(x)\right| \\
&+2\left|g(x)\right|\left|-\int_G f(zty)d\mu(t) + \int_G f(zt\sigma(y))d\mu(t) + 2f(z)g(y)\right|.
\end{aligned}
$$

By virtue of the inequality (3.1), we have

$$
\begin{aligned}
&2f(z)\left|\int_G g(xty)d\mu(t) - \int_G g(xt\sigma(y))d\mu(t) - \int_G g(ytx)d\mu(t) + \int_G g(\sigma(y)tx)d\mu(t)\right| \\
\le\ &\int_G \psi(xty)d\mu(t) + \int_G \psi(xt\sigma(y))d\mu(t) + \int_G \psi(ytx)d\mu(t) + \int_G \psi(\sigma(y)tx)d\mu(t) \\
&+2\left\|\mu\right\|(\psi(y)+\psi(x)) + 2(\left|g(y)\right|\psi(x) + \left|g(x)\right|\psi(y)).
\end{aligned}
$$

If we fix x, y, then the right hand side of the above inequality is bounded function of z. So (3.2) follows immediately since f is unbounded. This ends this proof. □

Lemma 5. *Assume that $f, g \in C(G)$ and $\varphi : G \to \mathbb{R}$ satisfy the inequality*

$$\left|\int_G f(xty)d\mu(t) - \int_G f(xt\sigma(y))d\mu(t) - 2f(x)g(y)\right| \le \varphi(x),\ \ x, y \in G, \tag{3.3}$$

such that $f \neq 0$. If g is unbounded then so is f.

Proof. Assume that g is an unbounded function satisfying the inequality (3.3). Let $M = \sup|f|$ and choose $a \in G$ such that $f(a) \neq 0$. If the nonzero function f is bounded, then we get by using (3.3) the inequality

$$\left|\int_G f(aty)d\mu(t) - \int_G f(at\sigma(y))d\mu(t) - 2f(a)g(y)\right| \le \varphi(a),\quad y \in G,$$

and so $|g(y)| \leq \frac{1}{2|f(a)|}(\varphi(a) + 2\|\mu\| M)$ for all $y \in G$. Then g is bounded which contradicts our assumption. $\square$

In the following theorem, the superstability of the equation $(T_{f,g}(\mu))$ will be investigated on any locally compact group G. We will look at the case g is unbounded.

Theorem 1. *Assume that $f, g \in C(G)$ and continuous functions $\varphi, \psi : G \to \mathbb{R}$ satisfy the inequality*

$$\left| \int_G f(xty)d\mu(t) - \int_G f(xt\sigma(y))d\mu(t) - 2f(x)g(y) \right| \leq \min(\varphi(x), \psi(y)), \tag{3.4}$$

for all $x, y \in G$. Then:

(i) f, g are bounded or

(ii) f is unbounded and g satisfies the functional equation (3.2), or

(iii) g is unbounded and the pair (f, g) satisfies the equation $(T_{f,g}(\mu))$. (if $f \neq 0$, then g satisfies the identity (3.2)).

Proof. (ii) follows from Lemma 4.

(iii) If g is unbounded, then for $f = 0$ the pair (f, g) is a trivial solution of the equation $(T_{f,g}(\mu))$and so we obtain (iii). Now assume that $f \neq 0$. For all $x, y, z \in G$ we have

$$\begin{aligned}
&|2g(z)| \left| \int_G f(xty)d\mu(t) - \int_G f(xt\sigma(y))d\mu(t) - 2f(x)g(y) \right| \\
&= \left| 2\int_G f(xty)d\mu(t)g(z) - 2\int_G f(xt\sigma(y))d\mu(t)g(z) - 4f(x)g(y)g(z) \right| \\
&\leq \left| -\int_G\int_G f(xtysz)d\mu(t)d\mu(s) + \int_G\int_G f(xtys\sigma(z))d\mu(t)d\mu(s) + 2\int_G f(xty)d\mu(t)g(z) \right| \\
&+ \left| \int_G\int_G f(xt\sigma(y)sz)d\mu(t)d\mu(s) - \int_G\int_G f(xt\sigma(y)s\sigma(z))d\mu(t)d\mu(s) - 2\int_G f(xt\sigma(y))d\mu(t)g(z) \right| \\
&+ \left| \int_G\int_G f(xtysz)d\mu(t)d\mu(s) - \int_G\int_G f(xt\sigma(z)s\sigma(y))d\mu(t)d\mu(s) - 2f(x)\int_G g(ytz)d\mu(t) \right|
\end{aligned}$$

$$\begin{aligned}
&+ \left| -\int_G\int_G f(xtys\sigma(z))d\mu(t)d\mu(s) + \int_G\int_G f(xtzs\sigma(y))d\mu(t)d\mu(s) + 2f(x)\int_G g(yt\sigma(z))d\mu(t) \right| \\
&+ \left| \int_G\int_G f(xt\sigma(z)sy)d\mu(t)d\mu(s) - \int_G\int_G f(xt\sigma(y)sz)d\mu(t)d\mu(s) - 2f(x)\int_G g(\sigma(z)ty)d\mu(t) \right| \\
&+ \left| -\int_G\int_G f(xtzsy)d\mu(t)d\mu(s) + \int_G\int_G f(xt\sigma(y)s\sigma(z))d\mu(t)d\mu(s) + 2f(x)\int_G g(zty)d\mu(t) \right| \\
&+ \left| -\int_G\int_G f(xt\sigma(z)sy)d\mu(t)d\mu(s) + \int_G\int_G f(xt\sigma(z)s\sigma(y))d\mu(t)d\mu(s) + 2g(y)\int_G f(xt\sigma(z))d\mu(t) \right|
\end{aligned}$$

$$+\left|\int_G\int_G f(xtzsy)d\mu(t)d\mu(s)-\int_G\int_G f(xtzs\sigma(y))d\mu(t)d\mu(s)-2g(y)\int_G f(xtz)d\mu(t)\right|$$
$$+|2f(x)|\left|\int_G g(ytz)d\mu(t)-\int_G g(yt\sigma(z))d\mu(t)-\int_G g(zty)d\mu(t)+\int_G g(\sigma(z)ty)d\mu(t)\right|$$
$$+\left|2\int_G f(xtz)d\mu(t)g(y)-2\int_G f(xt\sigma(z))d\mu(t)g(y)-4f(x)g(y)g(z)\right|.$$

In virtue of the inequality (3.4), we obtain

$$|2g(z)|\left|\int_G f(xty)d\mu(t)-\int_G f(xt\sigma(y))d\mu(t)-2f(x)g(y)\right|$$
$$\leq\int_G\varphi(xty)d\mu(t)+\int_G\varphi(xt\sigma(y))d\mu(t)+2\|\mu\|(2\varphi(x)+\psi(y))+2|g(y)|\varphi(x)$$
$$+|2f(x)|\left|\int_G g(ytz)d\mu(t)-\int_G g(yt\sigma(z))d\mu(t)-\int_G g(zty)d\mu(t)+\int_G g(\sigma(z)ty)d\mu(t)\right|.$$

Using Lemma 5, we obtain that the unboundedness of g implies necessarily that f is unbounded and hence according to Lemma 4 g is a solution of the equation (3.2). So we conclude that

$$|2g(z)|\left|\int_G f(xty)d\mu(t)-\int_G f(xt\sigma(y))d\mu(t)-2f(x)g(y)\right| \tag{3.5}$$
$$\leq\int_G\varphi(xty)d\mu(t)+\int_G\varphi(xt\sigma(y))d\mu(t)+2\|\mu\|(2\varphi(x)+\psi(y))+2|g(y)|\varphi(x).$$

Again the right hand side of (3.5) as a function of z is bounded for every fixed x,y. Since g is unbounded, from (3.5), we see that the pair (f,g) satisfies $(T_{f,g}(\mu))$ and therefore, by using Lemma 1 (ii), we get that if $f\neq 0$, then g satisfies (3.2), which completes the proof of the theorem. □

As consequences of Theorem 1, we have the following results.

Corollary 1. *Assume that functions $f,g\in C(G)$ satisfy the inequality*

$$\left|\int_G f(xty)d\mu(t)-\int_G f(xt\sigma(y))d\mu(t)-2f(x)g(y)\right|\leq\varepsilon,$$

for all $x,y\in G$. Then:

(i) f,g are bounded or

(ii) f is unbounded and g satisfies the functional equation (3.2), or

(iii) g is unbounded and the pair (f,g) satisfies the equation $(T_{f,g}(\mu))$.

Corollary 2. ([21]) *Assume that functions $f,g:G\to\mathbb{C}$ satisfy the inequality*

$$|f(xy)-f(x\sigma(y)-2f(x)g(y)|\leq\varepsilon,$$

for all $x,y\in G$. Then:

(i) f,g are bounded or

(ii) f is unbounded and $g(xy)-g(x\sigma(y)-g(yx)+g(\sigma(y)x)=0$, $x,y\in G$, or

(iii) g is unbounded and the pair (f,g) satisfies the equation (T).

Corollary 3. *Assume that a function $f \in C(G)$ satisfies the inequality*

$$\left| \int_G f(xty)d\mu(t) - \int_G f(xt\sigma(y))d\mu(t) - 2f(x)f(y) \right| \leq \varepsilon,$$

for all $x, y \in G$. Then:

(*i*) *Either f is bounded or f is a solution the functional equation ($T(\mu)$).*

(*ii*) *If, additionally, $\mu * \mu = \mu$, then*

$$|f(x)| \leq \frac{\|\mu\| + \sqrt{\|\mu\|^2 + 2\varepsilon}}{2}, \quad \text{for all } x \in G,$$

Proof. (i) Letting $g = f$ in Theorem 1. we get the desired result.

(ii) It is an immediate consequence of Theorem 1 and Lemma 3. □

Corollary 4. ([21]) *Assume that a function $f : G \to \mathbb{C}$ satisfies the inequality*

$$|f(xy) - f(x\sigma(y)) - 2f(x)f(y)| \leq \varepsilon,$$

for all $x, y \in G$. Then f is bounded.

Acknowledgments

C. Park was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2012R1A1A2004299).

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Barnes-type Narumi of the first kind and Barnes-type Peters of the first kind hybrid polynomials

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MR Subject Classifications: 05A15, 05A40, 11B68, 11B75, 11B83

Abstract

In this paper, by considering Barnes-type Narumi polynomials of the first kind and Barnes-type Peters polynomials of the first kind, we define and investigate the hybrid polynomials of these polynomials. From the properties of Sheffer sequences of these polynomials arising from umbral calculus, we derive new and interesting identities.

1 Introduction

In this paper, we consider the polynomials

$$\mathrm{NS}_n(x) = \mathrm{NS}_n(x|a;\lambda;\mu) = \mathrm{NS}_n(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)$$

called the Barnes-type Narumi of the first kind and Barnes-type Peters of the first kind hybrid polynomials, whose generating function is given by

$$\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\prod_{j=1}^{s}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}(1+t)^x = \sum_{n=0}^{\infty}\mathrm{NS}_n(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\frac{t^n}{n!}, \quad (1)$$

where $a_1,\dots,a_r,\lambda_1,\dots,\lambda_s,\mu_1,\dots,\mu_s\in\mathbb{C}$ with $a_1,\dots,a_r,\lambda_1,\dots,\lambda_s\neq 0$. When $x=0$, $\mathrm{NS}_n=\mathrm{NS}_n(0)=\mathrm{NS}_n(0|a;\lambda;\mu)=\mathrm{NS}_n(0|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)$ are called the Barnes-type Narumi of the first kind and Barnes-type Peters of the first kind hybrid numbers.

Recall that the Barnes-type Narumi polynomials of the first kind, denoted by $\mathrm{N}_n(x|a_1,\dots,a_r)$, are given by the generating function as

$$\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)(1+t)^x=\sum_{n=0}^{\infty}\mathrm{N}_n(x|a_1,\dots,a_r)\frac{t^n}{n!}.$$

If $x=0$, we write $\mathrm{N}_n(a_1,\dots,a_r)=\mathrm{N}_n(0|a_1,\dots,a_r)$. Narumi polynomials $\mathrm{N}_n(x|\underbrace{1,\dots,1}_{r})$ were mentioned in [8, p.127]. In addition, the Barnes-type Peters polynomials of the first kind, denoted by $S_n(x|\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)$, are given by the generating function as

$$\prod_{j=1}^{s}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}(1+t)^x=\sum_{n=0}^{\infty}S_n(x|\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\frac{t^n}{n!}.$$

If $s=1$, then $S_n(x|\lambda;\mu)$ are the Peters polynomials of the first kind. Peters polynomials were mentioned in [8, p.128] and have been investigated in e.g. [5, 7].

In this paper, by considering Barnes-type Narumi polynomials of the first kind and Barnes-type Peters polynomials of the first kind, we define and investigate the hybrid polynomials of these polynomials. From the properties of Sheffer sequences of these polynomials arising from umbral calculus, we derive new and interesting identities.

2 Umbral calculus

Let $\mathbb{C}$ be the complex number field and let $\mathcal{F}$ be the set of all formal power series in the variable t:

$$\mathcal{F}=\left\{f(t)=\sum_{k=0}^{\infty}\frac{a_k}{k!}t^k\,\middle|\,a_k\in\mathbb{C}\right\}. \quad (2)$$

Let $\mathbb{P}=\mathbb{C}[x]$ and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. $\langle L|p(x)\rangle$ is the action of the linear functional L on the polynomial $p(x)$, and we recall that the vector

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space operations on $\mathbb{P}^*$ are defined by $\langle L+M|p(x)\rangle = \langle L|p(x)\rangle + \langle M|p(x)\rangle$, $\langle cL|p(x)\rangle = c\langle L|p(x)\rangle$, where c is a complex constant in $\mathbb{C}$. For $f(t) \in \mathcal{F}$, let us define the linear functional on $\mathbb{P}$ by setting

$$\langle f(t)|x^n\rangle = a_n, \quad (n \geq 0). \tag{3}$$

In particular,

$$\left\langle t^k|x^n\right\rangle = n!\delta_{n,k} \quad (n,k \geq 0), \tag{4}$$

where $\delta_{n,k}$ is the Kronecker's symbol.

For $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^k\rangle}{k!} t^k$, we have $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$. That is, $L = f_L(t)$. The map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ denotes both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power series and a linear functional. We call $\mathcal{F}$ the *umbral algebra* and the *umbral calculus* is the study of umbral algebra. The order $O\big(f(t)\big)$ of a power series $f(t)(\neq 0)$ is the smallest integer k for which the coefficient of t^k does not vanish. If $O\big(f(t)\big) = 1$, then $f(t)$ is called a *delta series*; if $O\big(f(t)\big) = 0$, then $f(t)$ is called an *invertible series*. For $f(t), g(t) \in \mathcal{F}$ with $O\big(f(t)\big) = 1$ and $O\big(g(t)\big) = 0$, there exists a unique sequence $s_n(x)$ ($\deg s_n(x) = n$) such that $\left\langle g(t)f(t)^k|s_n(x)\right\rangle = n!\delta_{n,k}$, for $n,k \geq 0$. Such a sequence $s_n(x)$ is called the *Sheffer sequence* for $\big(g(t), f(t)\big)$ which is denoted by $s_n(x) \sim \big(g(t), f(t)\big)$.

For $f(t), g(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, we have

$$\langle f(t)g(t)|p(x)\rangle = \langle f(t)|g(t)p(x)\rangle = \langle g(t)|f(t)p(x)\rangle \tag{5}$$

and

$$f(t) = \sum_{k=0}^{\infty} \left\langle f(t)|x^k\right\rangle \frac{t^k}{k!}, \quad p(x) = \sum_{k=0}^{\infty} \left\langle t^k|p(x)\right\rangle \frac{x^k}{k!} \tag{6}$$

([8, Theorem 2.2.5]). Thus, by (6), we get

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k} \quad \text{and} \quad e^{yt}p(x) = p(x+y). \tag{7}$$

Sheffer sequences are characterized in the generating function ([8, Theorem 2.3.4]).

Lemma 1 *The sequence $s_n(x)$ is Sheffer for $\big(g(t), f(t)\big)$ if and only if*

$$\frac{1}{g\big(\bar{f}(t)\big)} e^{y\bar{f}(t)} = \sum_{k=0}^{\infty} \frac{s_k(y)}{k!} t^k \quad (y \in \mathbb{C}),$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$.

3

For $s_n(x) \sim \big(g(t), f(t)\big)$, we have the following equations ([8, Theorem 2.3.7, Theorem 2.3.5, Theorem 2.3.9]):

$$f(t)s_n(x) = ns_{n-1}(x) \quad (n \geq 0), \tag{8}$$

$$s_n(x) = \sum_{j=0}^{n} \frac{1}{j!} \left\langle g\big(\bar{f}(t)\big)^{-1} \bar{f}(t)^j | x^n \right\rangle x^j, \tag{9}$$

$$s_n(x+y) = \sum_{j=0}^{n} \binom{n}{j} s_j(x) p_{n-j}(y)\,, \tag{10}$$

where $p_n(x) = g(t)s_n(x)$.

Assume that $p_n(x) \sim \big(1, f(t)\big)$ and $q_n(x) \sim \big(1, g(t)\big)$. Then the transfer formula ([8, Corollary 3.8.2]) is given by

$$q_n(x) = x \left(\frac{f(t)}{g(t)}\right)^n x^{-1} p_n(x) \quad (n \geq 1).$$

For $s_n(x) \sim \big(g(t), f(t)\big)$ and $r_n(x) \sim \big(h(t), l(t)\big)$, assume that

$$s_n(x) = \sum_{m=0}^{n} C_{n,m} r_m(x) \quad (n \geq 0)\,.$$

Then we have ([8, p.132])

$$C_{n,m} = \frac{1}{m!} \left\langle \frac{h\big(\bar{f}(t)\big)}{g\big(\bar{f}(t)\big)} l\big(\bar{f}(t)\big)^m \middle| x^n \right\rangle\,. \tag{11}$$

3 Main results

For convenience, we introduce an Appell sequence of polynomials $F_n(x|a_1, \ldots, a_r)$ $(a_1, \ldots, a_r \neq 0)$, defined by

$$\prod_{i=1}^{r} \left(\frac{e^{a_i t} - 1}{t}\right) e^{xt} = \sum_{n=0}^{\infty} F_n(x|a_1, \ldots, a_r) \frac{t^n}{n!}\,.$$

If $x = 0$, we write $F_n(a_1, \ldots, a_r) = F_n(0|a_1, \ldots, a_r)$. By [8, Theorem 2.5.8] with $y = 0$, we have

$$F_n(x|a_1, \ldots, a_r) = \sum_{m=0}^{n} \binom{n}{m} F_{n-m}(a_1, \ldots, a_r) x^m\,.$$

More precisely, one can show that

$$F_n(x|a_1, \ldots, a_r) = \sum_{i=0}^{n} \sum_{l_1+\cdots+l_r=i} \frac{i!}{(i+r)!} \binom{n}{i} \binom{i+r}{l_1+1, \ldots, l_r+1} a_1^{l_1+1} \ldots a_r^{l_r+1} x^{n-i}\,,$$

where
$$\binom{i+r}{l_1+1,\dots,l_r+1}=\frac{(i+r)!}{(l_1+1)!\cdots(l_r+1)!}\,.$$

Remember that the generalized Barnes-type Euler polynomials $E_n(x|\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)$ are defined by the generating function

$$\prod_{j=1}^{s}\left(\frac{2}{1+e^{\lambda_j t}}\right)^{\mu_j}e^{xt}=\sum_{n=0}^{\infty}E_n(x|\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\frac{t^n}{n!}\,.$$

If $\mu_1=\cdots=\mu_s=1$, then $E_n(x|\lambda_1,\dots,\lambda_s)=E_n(x|\lambda_1,\dots,\lambda_s;1,\dots,1)$ are called the Barnes-type Euler polynomials. If further $\lambda_1=\cdots=\lambda_s=1$, then $E_n^{(r)}(x)=E_n(x|1,\dots,1;1,\dots,1)$ are called the Euler polynomials of order s. If $x=0$, we write $E_n(\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)=E_n(0|\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)$. By [8, Theorem 2.5.8] with $y=0$, we have

$$E_n(x|\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)=\sum_{m=0}^{n}\binom{n}{m}E_{n-m}(\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)x^m\,.$$

We note that

$$\begin{aligned}tF_n(x|a_1,\dots,a_r)&=\frac{d}{dx}F_n(x|a_1,\dots,a_r)=nF_{n-1}(x|a_1,\dots,a_r)\,,\\ tE_n(x|\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)&=\frac{d}{dx}E_n(x|\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\\ &=nE_{n-1}(x|\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\,.\end{aligned}$$

From the definition (1), $\mathrm{NS}_n(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)$ is the Sheffer sequence for the pair

$$g(t)=\prod_{i=1}^{r}\left(\frac{t}{e^{a_i t}-1}\right)\prod_{j=1}^{s}(1+e^{\lambda_j t})^{\mu_j}\quad\text{and}\quad f(t)=e^t-1.$$

So,

$$\mathrm{NS}_n(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\sim\left(\prod_{i=1}^{r}\left(\frac{t}{e^{a_i t}-1}\right)\prod_{j=1}^{s}(1+e^{\lambda_j t})^{\mu_j},e^t-1\right)\,. \tag{12}$$

3.1 Explicit expressions

Let $(n)_j=n(n-1)\cdots(n-j+1)$ $(j\ge 1)$ with $(n)_0=1$. The (signed) Stirling numbers of the first kind $S_1(n,m)$ are defined by

$$(x)_n=\sum_{m=0}^{n}S_1(n,m)x^m\,.$$

Theorem 1

$$\begin{aligned}
&\mathrm{NS}_n(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\\
&=2^{-\sum_{j=1}^{s}\mu_j}\sum_{m=0}^{n}\sum_{l=0}^{m}S_1(n,m)\sum_{l=0}^{m}\binom{m}{l}E_{m-l}(\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)F_l(x|a_1,\dots,a_r) \quad (13)\\
&=\sum_{j=0}^{n}\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\mathrm{NS}_{n-l}x^j \quad (14)\\
&=2^{-\sum_{j=1}^{s}\mu_j}\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{m}{l}S_1(n,m)F_{m-l}(a_1,\dots,a_r)E_l(x|\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s) \quad (15)\\
&=\sum_{l=0}^{n}\binom{n}{l}S_{n-l}(\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)N_l(x|a_1,\dots,a_r)\,, \quad (16)\\
&=\sum_{l=0}^{n}\binom{n}{l}N_{n-l}(a_1,\dots,a_r)S_l(x|\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\,. \quad (17)
\end{aligned}$$

Proof. Since

$$\prod_{i=1}^{r}\left(\frac{t}{e^{a_it}-1}\right)\prod_{j=1}^{s}(1+e^{\lambda_jt})^{\mu_j}\mathrm{NS}_n(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\sim(1,e^t-1) \quad (18)$$

and

$$(x)_n\sim(1,e^t-1)\,, \quad (19)$$

we have

$$
\begin{aligned}
&\mathrm{NS}_n(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\\
&=\prod_{i=1}^{r}\left(\frac{e^{a_it}-1}{t}\right)\prod_{j=1}^{s}(1+e^{\lambda_jt})^{-\mu_j}(x)_n\\
&=\sum_{m=0}^{n}S_1(n,m)\prod_{i=1}^{r}\left(\frac{e^{a_it}-1}{t}\right)\prod_{j=1}^{s}(1+e^{\lambda_jt})^{-\mu_j}x^m\\
&=2^{-\sum_{j=1}^{s}\mu_j}\sum_{m=0}^{n}S_1(n,m)\prod_{i=1}^{r}\left(\frac{e^{a_it}-1}{t}\right)\prod_{j=1}^{s}\left(\frac{2}{1+e^{\lambda_jt}}\right)^{\mu_j}x^m\\
&=2^{-\sum_{j=1}^{s}\mu_j}\sum_{m=0}^{n}S_1(n,m)\prod_{i=1}^{r}\left(\frac{e^{a_it}-1}{t}\right)E_m(x|\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\\
&=2^{-\sum_{j=1}^{s}\mu_j}\sum_{m=0}^{n}S_1(n,m)\prod_{i=1}^{r}\left(\frac{e^{a_it}-1}{t}\right)\sum_{l=0}^{m}\binom{m}{l}E_{m-l}(\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)x^l\\
&=2^{-\sum_{j=1}^{s}\mu_j}\sum_{m=0}^{n}S_1(n,m)\sum_{l=0}^{m}\binom{m}{l}E_{m-l}(\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\prod_{i=1}^{r}\left(\frac{e^{a_it}-1}{t}\right)x^l\\
&=2^{-\sum_{j=1}^{s}\mu_j}\sum_{m=0}^{n}\sum_{l=0}^{m}S_1(n,m)\sum_{l=0}^{m}\binom{m}{l}E_{m-l}(\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)F_l(x|a_1,\dots,a_r)\,.
\end{aligned}
$$

So, we get (13).
By (12), we have

$$
\begin{aligned}
&\left\langle g\big(\bar f(t)\big)^{-1}\bar f(t)^j|x^n\right\rangle\\
&=\left\langle\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\prod_{j=1}^{s}\big(1+(1+t)^{\lambda_j}\big)^{-\mu_j}\big(\ln(1+t)\big)^j\Big|x^n\right\rangle\\
&=\left\langle\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\prod_{j=1}^{s}\big(1+(1+t)^{\lambda_j}\big)^{-\mu_j}\Big|j!\sum_{l=j}^{\infty}S_1(l,j)\frac{t^l}{l!}x^n\right\rangle\\
&=j!\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\left\langle\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\prod_{j=1}^{s}\big(1+(1+t)^{\lambda_j}\big)^{-\mu_j}\Big|x^{n-l}\right\rangle\\
&=j!\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\left\langle\sum_{i=0}^{\infty}\mathrm{NS}_i\frac{t^i}{i!}\Big|x^{n-l}\right\rangle\\
&=j!\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\mathrm{NS}_{n-l}\,.
\end{aligned}
$$

By (9), we get the identity (14)

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From the proof in (13),

$$\begin{aligned}
&\mathrm{NS}_n(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\\
&=\sum_{m=0}^{n} S_1(n,m)\prod_{i=1}^{r}\left(\frac{e^{a_it}-1}{t}\right)\prod_{j=1}^{s}(1+e^{\lambda_jt})^{-\mu_j}x^m\\
&=\sum_{m=0}^{n} S_1(n,m)\prod_{j=1}^{s}(1+e^{\lambda_jt})^{-\mu_j}F_m(x|a_1,\dots,a_r)\\
&=\sum_{m=0}^{n} S_1(n,m)\prod_{j=1}^{s}(1+e^{\lambda_jt})^{-\mu_j}\sum_{l=0}^{m}\binom{m}{l}F_{m-l}(a_1,\dots,a_r)x^l\\
&=\sum_{m=0}^{n} S_1(n,m)\sum_{l=0}^{m}\binom{m}{l}F_{m-l}(a_1,\dots,a_r)\prod_{j=1}^{s}(1+e^{\lambda_jt})^{-\mu_j}x^l\\
&=2^{-\sum_{j=1}^{s}\mu_j}\sum_{m=0}^{n} S_1(n,m)\sum_{l=0}^{m}\binom{m}{l}F_{m-l}(a_1,\dots,a_r)\prod_{j=1}^{s}\left(\frac{2}{1+e^{\lambda_jt}}\right)^{\mu_j}x^l\\
&=2^{-\sum_{j=1}^{s}\mu_j}\sum_{m=0}^{n} S_1(n,m)\sum_{l=0}^{m}\binom{m}{l}F_{m-l}(a_1,\dots,a_r)E_l(x|\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\\
&=2^{-\sum_{j=1}^{s}\mu_j}\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{m}{l}S_1(n,m)F_{m-l}(a_1,\dots,a_r)E_l(x|\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\,.
\end{aligned}$$

which is the identity (15).

Next,

$$\begin{aligned}
&\mathrm{NS}_n(y|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s)\\
&=\left\langle \sum_{i=0}^{\infty}\mathrm{NS}_i(y|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s)\frac{t^i}{i!}\middle| x^n\right\rangle\\
&=\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\prod_{j=1}^{s}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}(1+t)^y\middle| x^n\right\rangle\\
&=\left\langle \prod_{j=1}^{s}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\middle| \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)(1+t)^y x^n\right\rangle\\
&=\left\langle \prod_{j=1}^{s}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\middle| \sum_{l=0}^{\infty}N_l(y|a_1,\ldots,a_r)\frac{t^l}{l!}x^n\right\rangle\\
&=\sum_{l=0}^{n}\binom{n}{l}N_l(y|a_1,\ldots,a_r)\left\langle \prod_{j=1}^{s}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\middle| x^{n-l}\right\rangle\\
&=\sum_{l=0}^{n}\binom{n}{l}N_l(y|a_1,\ldots,a_r)\left\langle \sum_{i=0}^{\infty}S_i(\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s)\frac{t^i}{i!}\middle| x^{n-l}\right\rangle\\
&=\sum_{l=0}^{n}\binom{n}{l}N_l(y|a_1,\ldots,a_r)S_{n-l}(\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s)\,.
\end{aligned}$$

Thus, we obtain (16).

Finally, we obtain that

$$
\begin{aligned}
&\mathrm{NS}_n(y|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\\
&=\left\langle \sum_{i=0}^{\infty}\mathrm{NS}_i(y|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\frac{t^i}{i!}\Big|x^n\right\rangle\\
&=\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\prod_{j=1}^{s}\big(1+(1+t)^{\lambda_j}\big)^{-\mu_j}(1+t)^y\Big|x^n\right\rangle\\
&=\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\Big|\prod_{j=1}^{s}\big(1+(1+t)^{\lambda_j}\big)^{-\mu_j}(1+t)^y x^n\right\rangle\\
&=\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\Big|\sum_{l=0}^{\infty}s_l(y|\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\frac{t^l}{l!}x^n\right\rangle\\
&=\sum_{l=0}^{n}S_l(y|\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\binom{n}{l}\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\Big|x^{n-l}\right\rangle\\
&=\sum_{l=0}^{n}S_l(y|\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\binom{n}{l}\left\langle \sum_{i=0}^{\infty}N_i(a_1,\dots,a_r)\frac{t^i}{i!}\Big|x^{n-l}\right\rangle\\
&=\sum_{l=0}^{n}\binom{n}{l}S_l(y|\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)N_{n-l}(a_1,\dots,a_r)\,.
\end{aligned}
$$

Thus, we get the identity (17). ∎

3.2 Sheffer identity

Theorem 2

$$
\begin{aligned}
&\mathrm{NS}_n(x+y|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\\
&\qquad=\sum_{l=0}^{n}\binom{n}{l}\mathrm{NS}_l(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)(y)_{n-l}\,. \qquad (20)
\end{aligned}
$$

Proof. By (12) with

$$
\begin{aligned}
p_n(x)&=\prod_{i=1}^{r}\left(\frac{t}{e^{a_i t}-1}\right)\prod_{j=1}^{s}(1+e^{\lambda_j t})^{\mu_j}\mathrm{NS}_n(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\\
&=(x)_n\sim(1,e^t-1)\,,
\end{aligned}
$$

using (10), we have (20). ∎

3.3 Difference relations

Theorem 3

$$\begin{aligned}&\mathrm{NS}_n(x+1|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)-\mathrm{NS}_n(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\\&\qquad=n\mathrm{NS}_{n-1}(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\,.\end{aligned}\tag{21}$$

Proof. By (8) with (12), we get

$$\begin{aligned}&(e^t-1)\mathrm{NS}_n(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\\&\qquad=n\mathrm{NS}_{n-1}(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\,.\end{aligned}$$

By (7), we have (21). ∎

3.4 Recurrence

Theorem 4

$$\begin{aligned}&\mathrm{NS}_{n+1}(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\\&=x\mathrm{NS}_n(x-1|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\\&+2^{-\sum_{j=1}^s\mu_j}\sum_{m=0}^n\sum_{l=0}^m\frac{\binom{m}{l}}{l+1}S_1(n,m)E_{m-l}(\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\\&\quad\times\left(\sum_{i=1}^r a_iF_{l+1}(x+a_i-1|a_1,\dots,a_{i-1},a_{i+1},\dots,a_r)-rF_{l+1}(x-1|a_1,\dots,a_r)\right)\\&-2^{-1-\sum_{j=1}^s\mu_j}\sum_{i=1}^s\mu_i\lambda_i\sum_{m=0}^n\sum_{l=0}^m\binom{m}{l}S_1(n,m)F_{m-l}(a_1,\dots,a_r)E_l(x+\lambda_i-1|\lambda;\mu+e_i)\,.\end{aligned}\tag{22}$$

Proof. By applying

$$s_{n+1}(x)=\left(x-\frac{g'(t)}{g(t)}\right)\frac{1}{f'(t)}s_n(x)\tag{23}$$

([8, Corollary 3.7.2]) with (12), we get

$$\mathrm{NS}_{n+1}(x)=x\mathrm{NS}_n(x-1)-e^{-t}\frac{g'(t)}{g(t)}\mathrm{NS}_n(x)\,.$$

Observe that

$$\begin{aligned}\frac{g'(t)}{g(t)} &= \bigl(\ln g(t)\bigr)' \\ &= \left(r\ln t - \sum_{i=1}^{r}\ln(e^{a_i t}-1) + \sum_{j=1}^{s}\mu_j \ln(1+e^{\lambda_j t})\right)' \\ &= \frac{r}{t} - \sum_{i=1}^{r}\frac{a_i e^{a_i t}}{e^{a_i t}-1} + \sum_{j=1}^{s}\frac{\mu_j\lambda_j e^{\lambda_j t}}{1+e^{\lambda_j t}} \\ &= \frac{r-\sum_{i=1}^{r}\frac{a_i t e^{a_i t}}{e^{a_i t}-1}}{t} + \sum_{j=1}^{s}\frac{\mu_j\lambda_j e^{\lambda_j t}}{1+e^{\lambda_j t}}\,,\end{aligned}$$

where

$$r-\sum_{i=1}^{r}\frac{a_i t e^{a_i t}}{e^{a_i t}-1} = -\frac{1}{2}\left(\sum_{i=1}^{r}a_i\right)t+\cdots$$

has the order at least one. Since from the proofs of (13) and (15)

$$\begin{aligned}\mathrm{NS}_n(x) &= 2^{-\sum_{j=1}^{s}\mu_j}\sum_{m=0}^{n}S_1(n,m)\sum_{l=0}^{m}\binom{m}{l}E_{m-l}(\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\prod_{i=1}^{r}\left(\frac{e^{a_i t}-1}{t}\right)x^l \\ &= 2^{-\sum_{j=1}^{s}\mu_j}\sum_{m=0}^{n}S_1(n,m)\sum_{l=0}^{m}\binom{m}{l}F_{m-l}(a_1,\dots,a_r)\prod_{j=1}^{s}\left(\frac{2}{1+e^{\lambda_j t}}\right)^{\mu_j}x^l\,,\end{aligned}$$

we have

$$\begin{aligned}&\frac{g'(t)}{g(t)}\mathrm{NS}_n(x) \\ &= \left(\frac{r-\sum_{i=1}^{r}\frac{a_i t e^{a_i t}}{e^{a_i t}-1}}{t} + \sum_{j=1}^{s}\frac{\mu_j\lambda_j e^{\lambda_j t}}{1+e^{\lambda_j t}}\right)\mathrm{NS}_n(x) \\ &= 2^{-\sum_{j=1}^{s}\mu_j}\sum_{m=0}^{n}S_1(n,m)\sum_{l=0}^{m}\binom{m}{l}E_{m-l}(\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s) \\ &\quad\times\prod_{i=1}^{r}\left(\frac{e^{a_i t}-1}{t}\right)\frac{r-\sum_{i=1}^{r}\frac{a_i t e^{a_i t}}{e^{a_i t}-1}}{t}x^l \\ &\quad + 2^{-\sum_{j=1}^{s}\mu_j}\sum_{m=0}^{n}S_1(n,m)\sum_{l=0}^{m}\binom{m}{l}F_{m-l}(a_1,\dots,a_r)\sum_{i=1}^{s}\frac{\mu_i\lambda_i e^{\lambda_i t}}{1+e^{\lambda_i t}}\prod_{j=1}^{s}\left(\frac{2}{1+e^{\lambda_j t}}\right)^{\mu_j}x^l\,.\end{aligned}$$

The second term is equal to

$$2^{-\sum_{j=1}^{s}\mu_j}\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{m}{l}S_1(n,m)F_{m-l}(a_1,\dots,a_r)\sum_{i=1}^{s}\frac{\mu_i\lambda_i}{2}E_l(x+\lambda_i|\lambda;\mu+e_i)\,,$$

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where $\lambda=(\lambda_1,\dots,\lambda_s)$, $\mu=(\mu_1,\dots,\mu_s)$ and $e_i=(\underbrace{0,\dots,0}_{i-1},1,\underbrace{0,\dots,0}_{s-i})$. The first term is

$$
\begin{aligned}
&2^{-\sum_{j=1}^s\mu_j}\sum_{m=0}^n\sum_{l=0}^m\binom{m}{l}S_1(n,m)E_{m-l}(\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\\
&\quad\times\prod_{i=1}^r\left(\frac{e^{a_it}-1}{t}\right)\left(r-\sum_{i=1}^r\frac{a_ite^{a_it}}{e^{a_it}-1}\right)\frac{x^{l+1}}{l+1}\\
&=r2^{-\sum_{j=1}^s\mu_j}\sum_{m=0}^n\sum_{l=0}^m\frac{\binom{m}{l}}{l+1}S_1(n,m)E_{m-l}(\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\prod_{i=1}^r\left(\frac{e^{a_it}-1}{t}\right)x^{l+1}\\
&\quad-2^{-\sum_{j=1}^s\mu_j}\sum_{m=0}^n\sum_{l=0}^m\frac{\binom{m}{l}}{l+1}S_1(n,m)E_{m-l}(\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\\
&\quad\times\sum_{i=1}^r a_ie^{a_it}\frac{t}{e^{a_it}-1}\prod_{i=1}^r\left(\frac{e^{a_it}-1}{t}\right)x^{l+1}\\
&=r2^{-\sum_{j=1}^s\mu_j}\sum_{m=0}^n\sum_{l=0}^m\frac{\binom{m}{l}}{l+1}S_1(n,m)E_{m-l}(\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)F_{l+1}(x|a_1,\dots,a_r)\\
&\quad-2^{-\sum_{j=1}^s\mu_j}\sum_{m=0}^n\sum_{l=0}^m\frac{\binom{m}{l}}{l+1}S_1(n,m)E_{m-l}(\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\\
&\quad\times\sum_{i=1}^r a_iF_{l+1}(x+a_i|a_1,\dots,a_{i-1},a_{i+1},\dots,a_r)\\
&=2^{-\sum_{j=1}^s\mu_j}\sum_{m=0}^n\sum_{l=0}^m\frac{\binom{m}{l}}{l+1}S_1(n,m)E_{m-l}(\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\\
&\quad\times\left(rF_{l+1}(x|a_1,\dots,a_r)-\sum_{i=1}^r a_iF_{l+1}(x+a_i|a_1,\dots,a_{i-1},a_{i+1},\dots,a_r)\right).
\end{aligned}
$$

Therefore, we obtain the identity (22). ∎

3.5 Differentiation

Theorem 5

$$
\begin{aligned}
&\frac{d}{dx}\mathrm{NS}_n(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\\
&\qquad=n!\sum_{l=0}^{n-1}\frac{(-1)^{n-l-1}}{l!(n-l)}\mathrm{NS}_l(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\,. \qquad (24)
\end{aligned}
$$

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Proof. We shall use

$$\frac{d}{dx}s_n(x) = \sum_{l=0}^{n-1}\binom{n}{l}\left\langle \bar{f}(t)|x^{n-l}\right\rangle s_l(x)$$

(*Cf.* [8, Theorem 2.3.12]). Since

$$\begin{aligned}\left\langle \bar{f}(t)|x^{n-l}\right\rangle &= \left\langle \ln(1+t)|x^{n-l}\right\rangle \\ &= \left\langle \sum_{m=1}^{\infty}\frac{(-1)^{m-1}t^m}{m}\middle|x^{n-l}\right\rangle \\ &= \sum_{m=1}^{n-l}\frac{(-1)^{m-1}}{m}\left\langle t^m|x^{n-l}\right\rangle \\ &= \sum_{m=1}^{n-l}\frac{(-1)^{m-1}}{m}(n-l)!\delta_{m,n-l} \\ &= (-1)^{n-l-1}(n-l-1)!\end{aligned}$$

with (12), we have

$$\begin{aligned}&\frac{d}{dx}\mathrm{NS}_n(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s) \\ &= \sum_{l=0}^{n-1}\binom{n}{l}(-1)^{n-l-1}(n-l-1)!\mathrm{NS}_l(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s) \\ &= n!\sum_{l=0}^{n-1}\frac{(-1)^{n-l-1}}{l!(n-l)}\mathrm{NS}_l(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\,,\end{aligned}$$

which is the identity (24). ∎

3.6 A more relation

The classical Cauchy numbers of the first kind c_n are defined by

$$\frac{t}{\ln(1+t)} = \sum_{n=0}^{\infty}c_n\frac{t^n}{n!}$$

(see e.g. [2, 6]).

Theorem 6

$$\begin{aligned}
&\mathrm{NS}_n(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\\
&= x\mathrm{NS}_{n-1}(x-1|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s) \qquad (25)\\
&\quad -\sum_{i=1}^{s}\mu_i\lambda_i\mathrm{NS}_{n-1}(x+\lambda_i-1|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu+e_i)\\
&\quad +\frac{1}{n}\left(\sum_{i=1}^{r}\sum_{l=0}^{n}\binom{n}{l}a_ic_l\mathrm{NS}_{n-l}(x+a_i-1|a_1,\dots,a_{i-1},a_{i+1},\dots,a_r;\lambda;\mu)\right.\\
&\quad \left. -r\sum_{l=0}^{n}\binom{n}{l}c_l\mathrm{NS}_{n-l}(x-1|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\right). \qquad (26)
\end{aligned}$$

Proof. For $n \ge 1$, we have

$$\begin{aligned}
&\mathrm{NS}_n(y|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\\
&= \left\langle \sum_{l=0}^{\infty}\mathrm{NS}_l(y|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\frac{t^l}{l!}\middle| x^n\right\rangle\\
&= \left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\prod_{j=1}^{s}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}(1+t)^y\middle| x^n\right\rangle\\
&= \left\langle \partial_t\left(\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\prod_{j=1}^{s}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}(1+t)^y\right)\middle| x^{n-1}\right\rangle\\
&= \left\langle \left(\partial_t\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\right)\prod_{j=1}^{s}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}(1+t)^y\middle| x^{n-1}\right\rangle\\
&\quad + \left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\left(\partial_t\prod_{j=1}^{s}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\right)(1+t)^y\middle| x^{n-1}\right\rangle\\
&\quad + \left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\prod_{j=1}^{s}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\left(\partial_t(1+t)^y\right)\middle| x^{n-1}\right\rangle .
\end{aligned}$$

The third term is

$$\begin{aligned}
&y\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\prod_{j=1}^{s}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}(1+t)^{y-1}\middle| x^{n-1}\right\rangle\\
&= y\mathrm{NS}_{n-1}(y-1|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s).
\end{aligned}$$

Since

$$\partial_t\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)=\frac{1}{1+t}\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\frac{\sum_{\nu=1}^{r}\left(\frac{a_\nu t(1+t)^{a_\nu}}{(1+t)^{a_\nu}-1}-\frac{t}{\ln(1+t)}\right)}{t},$$

with
$$\sum_{\nu=1}^{r}\left(\frac{a_\nu t(1+t)^{a_\nu}}{(1+t)^{a_\nu}-1}-\frac{t}{\ln(1+t)}\right)$$
having order≥ 1, the first term is
$$\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\prod_{j=1}^{s}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}(1+t)^{y-1}\middle|\frac{\sum_{\nu=1}^{r}\left(\frac{a_\nu t(1+t)^{a_\nu}}{(1+t)^{a_\nu}-1}-\frac{t}{\ln(1+t)}\right)}{t}x^{n-1}\right\rangle$$
$$=\frac{1}{n}\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\prod_{j=1}^{s}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}(1+t)^{y-1}\right|$$
$$\left.\sum_{\nu=1}^{r}\left(\frac{a_\nu t(1+t)^{a_\nu}}{(1+t)^{a_\nu}-1}-\frac{t}{\ln(1+t)}\right)x^n\right\rangle$$
$$=\frac{1}{n}\left(\sum_{\nu=1}^{r}a_\nu\left\langle\frac{\ln(1+t)}{(1+t)^{a_\nu}-1}\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\prod_{j=1}^{s}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\right.\right.$$
$$\left.(1+t)^{y+a_\nu-1}\middle|\frac{t}{\ln(1+t)}x^n\right\rangle$$
$$\left.-r\left\langle\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\prod_{j=1}^{s}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}(1+t)^{y-1}\middle|\frac{t}{\ln(1+t)}x^n\right\rangle\right)$$
$$=\frac{1}{n}\left(\sum_{\nu=1}^{r}a_\nu\left\langle\frac{\ln(1+t)}{(1+t)^{a_\nu}-1}\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\prod_{j=1}^{s}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\right.\right.$$
$$\left.(1+t)^{y+a_\nu-1}\middle|\sum_{l=0}^{\infty}c_l\frac{t^l}{l!}x^n\right\rangle$$
$$\left.-r\left\langle\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\prod_{j=1}^{s}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}(1+t)^{y-1}\middle|\sum_{l=0}^{\infty}c_l\frac{t^l}{l!}x^n\right\rangle\right)$$
$$=\frac{1}{n}\left(\sum_{\nu=1}^{r}\sum_{l=0}^{n}\binom{n}{l}a_\nu c_l \mathrm{NS}_{n-l}(y+a_\nu-1|a_1,\dots,a_{\nu-1},a_{\nu+1},\dots,a_r;\lambda;\mu)\right.$$
$$\left.-r\sum_{l=0}^{n}\binom{n}{l}c_l\mathrm{NS}_{n-l}(y-1|a;\lambda;\mu)\right).$$

Since
$$\partial_t\prod_{j=1}^{s}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}$$
$$=-\sum_{i=1}^{s}\mu_i\lambda_i(1+t)^{\lambda_i-1}\left(1+(1+t)\lambda_i\right)^{-1}\prod_{j=1}^{s}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j},$$

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the second term is

$$-\sum_{i=1}^{s}\mu_i\lambda_i\left\langle\left(1+(1+t)\lambda_i\right)^{-1}\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\prod_{j=1}^{s}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}(1+t)^{y+\lambda_i-1}|x^{n-1}\right\rangle$$
$$=-\sum_{i=1}^{s}\mu_i\lambda_i\mathrm{NS}_{n-1}(y+\lambda_i-1|a;\lambda;\mu+e_i)\,.$$

Therefore, we obtain

$$\begin{aligned}&\mathrm{NS}_n(x|a;\lambda;\mu)\\&=x\mathrm{NS}_{n-1}(x-1|a;\lambda;\mu)-\sum_{i=1}^{s}\mu_i\lambda_i\mathrm{NS}_{n-1}(x+\lambda_i-1|a;\lambda;\mu+e_i)\\&\quad+\frac{1}{n}\left(\sum_{i=1}^{r}\sum_{l=0}^{n}\binom{n}{l}a_ic_l\mathrm{NS}_{n-l}(x+a_i-1|a_1,\dots,a_{i-1},a_{i+1},\dots,a_r;\lambda;\mu)\right.\\&\quad\left.-r\sum_{l=0}^{n}\binom{n}{l}c_l\mathrm{NS}_{n-l}(x-1|a;\lambda;\mu)\right)\,,\end{aligned}$$

which is the identity (26). ∎

3.7 A relation including the Stirling numbers of the first kind

Theorem 7 *For $n-1\ge m\ge 1$, we have*

$$\begin{aligned}&\sum_{l=0}^{n-m}\binom{n}{l}S_1(n-l,m)\mathrm{NS}_l(a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\\&=\frac{1}{n}\sum_{k=0}^{n-m}\sum_{l=k}^{n-m}\binom{n}{l}\binom{l}{k}S_1(n-l,m)c_{l-k}\\&\quad\times\left(\sum_{i=1}^{r}a_i\mathrm{NS}_k(a_i-1|a_1,\dots,a_{i-1},a_{i+1},\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\right.\\&\quad\quad\left.-r\mathrm{NS}_k(-1|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\right)\\&\quad-\sum_{j=1}^{s}\mu_j\lambda_j\sum_{l=0}^{n-m-1}\binom{n-1}{l}S_1(n-l-1,m)\\&\quad\times\mathrm{NS}_l(\lambda_j-1|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu+e_j)\\&\quad+\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)\mathrm{NS}_l(-1|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\,.\end{aligned}\tag{27}$$

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Proof. We shall compute

$$\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\prod_{j=1}^{s}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\left(\ln(1+t)\right)^m \middle| x^n\right\rangle$$

in two different ways. On the one hand, it is equal to

$$\begin{aligned}
&\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\prod_{j=1}^{s}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\middle|\left(\ln(1+t)\right)^m x^n\right\rangle\\
&=\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\prod_{j=1}^{s}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\middle| m!\sum_{l=m}^{\infty}S_1(l,m)\frac{t^l}{l!}x^n\right\rangle\\
&=m!\sum_{l=m}^{n}\binom{n}{l}S_1(l,m)\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\prod_{j=1}^{s}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\middle| x^{n-l}\right\rangle\\
&=m!\sum_{l=m}^{n}\binom{n}{l}S_1(l,m)\left\langle \sum_{i=0}^{\infty}\mathrm{NS}_i(a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\frac{t^i}{i!}\middle| x^{n-l}\right\rangle\\
&=m!\sum_{l=m}^{n}\binom{n}{l}S_1(l,m)\mathrm{NS}_{n-l}(a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\\
&=m!\sum_{l=0}^{n-m}\binom{n}{l}S_1(n-l,m)\mathrm{NS}_{l}(a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\,.
\end{aligned}$$

On the other hand, it is equal to

$$\begin{aligned}
&\left\langle \partial_t\left(\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\prod_{j=1}^{s}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\left(\ln(1+t)\right)^m\right)\middle| x^{n-1}\right\rangle\\
&=\left\langle\left(\partial_t\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\right)\prod_{j=1}^{s}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\left(\ln(1+t)\right)^m\middle| x^{n-1}\right\rangle\\
&\quad+\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\left(\partial_t\prod_{j=1}^{s}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\right)\left(\ln(1+t)\right)^m\middle| x^{n-1}\right\rangle\\
&\quad+\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\prod_{j=1}^{s}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\left(\partial_t\left(\ln(1+t)\right)^m\right)\middle| x^{n-1}\right\rangle\,.
\end{aligned}\tag{28}$$

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The third term of (28) is equal to

$$
\begin{aligned}
& m\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\prod_{j=1}^{s}\bigl(1+(1+t)^{\lambda_j}\bigr)^{-\mu_j}(1+t)^{-1}\middle|\bigl(\ln(1+t)\bigr)^{m-1}x^{n-1}\right\rangle \\
&= m\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\prod_{j=1}^{s}\bigl(1+(1+t)^{\lambda_j}\bigr)^{-\mu_j}(1+t)^{-1}\middle| \right. \\
&\qquad \left. (m-1)!\sum_{l=m-1}^{\infty}S_1(l,m-1)\frac{t^l}{l!}x^{n-1}\right\rangle \\
&= m!\sum_{l=m-1}^{n-1}\binom{n-1}{l}S_1(l,m-1) \\
&\quad \times\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\prod_{j=1}^{s}\bigl(1+(1+t)^{\lambda_j}\bigr)^{-\mu_j}(1+t)^{-1}\middle| x^{n-1-l}\right\rangle \\
&= m!\sum_{l=m-1}^{n-1}\binom{n-1}{l}S_1(l,m-1)\mathrm{NS}_{n-1-l}(-1|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s) \\
&= m!\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)\mathrm{NS}_{l}(-1|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\,.
\end{aligned}
$$

Since

$$
\partial_t\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)=\frac{1}{1+t}\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\frac{\sum_{\nu=1}^{r}\left(\frac{a_\nu t(1+t)^{a_\nu}}{(1+t)^{a_\nu}-1}-\frac{t}{\ln(1+t)}\right)}{t}\,,
$$

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the first term of (28) is equal to

$$\begin{aligned}
&\left\langle \frac{1}{1+t}\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\frac{\sum_{\nu=1}^{r}\left(\frac{a_\nu t(1+t)^{a_\nu}}{(1+t)^{a_\nu}-1}-\frac{t}{\ln(1+t)}\right)}{t}\right.\\
&\qquad\left.\prod_{j=1}^{s}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\left(\ln(1+t)\right)^m\middle| x^{n-1}\right\rangle\\
&=\frac{1}{n}\left\langle\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\prod_{j=1}^{s}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}(1+t)^{-1}\right.\\
&\qquad\left.\sum_{\nu=1}^{r}\left(\frac{a_\nu t(1+t)^{a_\nu}}{(1+t)^{a_\nu}-1}-\frac{t}{\ln(1+t)}\right)\middle|\left(\ln(1+t)\right)^m x^n\right\rangle\\
&=\frac{1}{n}\left\langle\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\prod_{j=1}^{s}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}(1+t)^{-1}\right.\\
&\qquad\left.\sum_{\nu=1}^{r}\left(\frac{a_\nu t(1+t)^{a_\nu}}{(1+t)^{a_\nu}-1}-\frac{t}{\ln(1+t)}\right)\middle| m!\sum_{l=m}^{\infty}S_1(l,m)\frac{t^l}{l!}x^n\right\rangle\\
&=\frac{m!}{n}\sum_{l=m}^{n}\binom{n}{l}S_1(l,m)\left(\sum_{\nu=1}^{r}a_\nu\left\langle\frac{\ln(1+t)}{(1+t)^{a_\nu}-1}\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\right.\right.\\
&\qquad\left.\prod_{j=1}^{s}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}(1+t)^{a_\nu-1}\middle|\frac{t}{\ln(1+t)}x^{n-l}\right\rangle\\
&\qquad\left.-r\left\langle\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\prod_{j=1}^{s}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}(1+t)^{-1}\middle|\frac{t}{\ln(1+t)}x^{n-l}\right\rangle\right)\\
&=\frac{m!}{n}\sum_{l=m}^{n}\binom{n}{l}S_1(l,m)\sum_{k=0}^{n-l}\binom{n-l}{k}c_k\\
&\quad\times\left(\sum_{\nu=1}^{r}a_\nu\mathrm{NS}_{n-l-k}(a_\nu-1|a_1,\dots,a_{\nu-1},a_{\nu+1},\dots,a_r;\lambda;\mu)-r\mathrm{NS}_{n-l-k}(-1|a;\lambda;\mu)\right)\\
&=\frac{m!}{n}\sum_{k=0}^{n-m}\sum_{l=k}^{n-m}\binom{n}{l}\binom{l}{k}S_1(n-l,m)c_{l-k}\\
&\quad\times\left(\sum_{\nu=1}^{r}a_\nu\mathrm{NS}_{k}(a_\nu-1|a_1,\dots,a_{\nu-1},a_{\nu+1},\dots,a_r;\lambda;\mu)-r\mathrm{NS}_{k}(-1|a;\lambda;\mu)\right).
\end{aligned}$$

Since

$$\begin{aligned}&\partial_t \prod_{j=1}^{s}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\\&=-\sum_{j=1}^{s}\mu_j\lambda_j(1+t)^{\lambda_j-1}\left(1+(1+t)^{\lambda_j}\right)^{-1}\prod_{j=1}^{s}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j},\end{aligned}$$

the second term of (28) is equal to

$$\begin{aligned}&=-\sum_{\nu=1}^{s}\mu_\nu\lambda_\nu\left\langle\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\left(1+(1+t)^{\lambda_\nu}\right)^{-1}\right.\\&\qquad\left.\prod_{j=1}^{s}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}(1+t)^{\lambda_\nu-1}\right|\left(\ln(1+t)\right)^m x^{n-1}\Big\rangle\\&=-\sum_{\nu=1}^{s}\mu_\nu\lambda_\nu\left\langle\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\left(1+(1+t)^{\lambda_\nu}\right)^{-1}\right.\\&\qquad\left.\prod_{j=1}^{s}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}(1+t)^{\lambda_\nu-1}\right|m!\sum_{l=m}^{\infty}S_1(l,m)\frac{t^l}{l!}x^{n-1}\Big\rangle\\&=-m!\sum_{\nu=1}^{s}\mu_\nu\lambda_\nu\sum_{l=m}^{n-1}\binom{n-1}{l}S_1(l,m)\\&\quad\times\left\langle\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\left(1+(1+t)^{\lambda_\nu}\right)^{-1}\prod_{j=1}^{s}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}(1+t)^{\lambda_\nu-1}\right|x^{n-l-1}\Big\rangle\\&=-m!\sum_{\nu=1}^{s}\mu_\nu\lambda_\nu\sum_{l=m}^{n-1}\binom{n-1}{l}S_1(l,m)\mathrm{NS}_{n-l-1}(\lambda_\nu-1|a;\lambda;\mu+e_\nu)\\&=-m!\sum_{\nu=1}^{s}\mu_\nu\lambda_\nu\sum_{l=0}^{n-m-1}\binom{n-1}{l}S_1(n-l-1,m)\mathrm{NS}_l(\lambda_\nu-1|a;\lambda;\mu+e_\nu)\,.\end{aligned}$$

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Therefore, we get, for $n-1 \geq m \geq 1$,

$$
\begin{aligned}
& m! \sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l,m) \mathrm{NS}_l(a;\lambda;\mu) \\
&= \frac{m!}{n} \sum_{k=0}^{n-m} \sum_{l=k}^{n-m} \binom{n}{l} \binom{l}{k} S_1(n-l,m) c_{l-k} \\
&\quad \times \left(\sum_{i=1}^{r} a_i \mathrm{NS}_k(a_i - 1 | a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_r; \lambda; \mu) - r \mathrm{NS}_k(-1|a;\lambda;\mu) \right) \\
&\quad - m! \sum_{j=1}^{s} \mu_j \lambda_j \sum_{l=0}^{n-m-1} \binom{n-1}{l} S_1(n-l-1,m) \mathrm{NS}_l(\lambda_j - 1|a;\lambda;\mu + e_j) \\
&\quad + m! \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1,m-1) \mathrm{NS}_l(-1|a;\lambda;\mu)\,.
\end{aligned}
$$

Dividing both sides by $m!$, we obtain for $n-1 \geq m \geq 1$

$$
\begin{aligned}
& \sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l,m) \mathrm{NS}_l(a;\lambda;\mu) \\
&= \frac{1}{n} \sum_{k=0}^{n-m} \sum_{l=k}^{n-m} \binom{n}{l} \binom{l}{k} S_1(n-l,m) c_{l-k} \\
&\quad \times \left(\sum_{i=1}^{r} a_i \mathrm{NS}_k(a_i - 1 | a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_r; \lambda; \mu) - r \mathrm{NS}_k(-1|a;\lambda;\mu) \right) \\
&\quad - \sum_{j=1}^{s} \mu_j \lambda_j \sum_{l=0}^{n-m-1} \binom{n-1}{l} S_1(n-l-1,m) \mathrm{NS}_l(\lambda_j - 1|a;\lambda;\mu + e_j) \\
&\quad + \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1,m-1) \mathrm{NS}_l(-1|a;\lambda;\mu)\,.
\end{aligned}
$$

Thus, we get (27). ∎

3.8 A relation with the falling factorials

Theorem 8

$$
\begin{aligned}
& \mathrm{NS}_n(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s) \\
&\qquad = \sum_{m=0}^{n} \binom{n}{m} \mathrm{NS}_{n-m}(a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s)(x)_m\,. \qquad (29)
\end{aligned}
$$

Proof. For (12) and (19), assume that $\mathrm{NS}_n(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s) = \sum_{m=0}^{n} C_{n,m}(x)_m$. By (11), we have

$$\begin{aligned}
C_{n,m} &= \frac{1}{m!}\left\langle \frac{1}{\prod_{i=1}^{r}\left(\frac{\ln(1+t)}{e^{a_i\ln(1+t)}-1}\right)\prod_{j=1}^{s}\left(1+e^{\lambda_j\ln(1+t)}\right)^{\mu_j}}t^m \middle| x^n\right\rangle \\
&= \frac{1}{m!}\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\prod_{j=1}^{s}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j} \middle| t^m x^n\right\rangle \\
&= \binom{n}{m}\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\prod_{j=1}^{s}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j} \middle| x^{n-m}\right\rangle \\
&= \binom{n}{m}\mathrm{NS}_{n-m}\,.
\end{aligned}$$

Thus, we get the identity (29). ∎

3.9 A relation with higher-order Frobenius-Euler polynomials

For $\alpha \in \mathbb{C}$ with $\alpha \neq 1$, the Frobenius-Euler polynomials of order r, $H_n^{(r)}(x|\alpha)$ are defined by the generating function

$$\left(\frac{1-\alpha}{e^t-\alpha}\right)^r e^{xt} = \sum_{n=0}^{\infty} H_n^{(r)}(x|\alpha)\frac{t^n}{n!}$$

(see e.g. [4]).

Theorem 9

$$\begin{aligned}
\mathrm{NS}_n(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s) &= \sum_{m=0}^{n}\left(\sum_{j=0}^{n-m}\sum_{l=0}^{n-m-j}\binom{\sigma}{j}\binom{n-j}{l}(n)_j\right. \\
&\quad \left.\times(1-\alpha)^{-j}S_1(n-j-l,m)\mathrm{NS}_l\right)H_m^{(\sigma)}(x|\alpha)\,. \qquad (30)
\end{aligned}$$

Proof. For (12) and

$$H_n^{(\sigma)}(x|\alpha) \sim \left(\left(\frac{e^t-\alpha}{1-\alpha}\right)^{\sigma}, t\right), \qquad (31)$$

assume that $\mathrm{NS}_n(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s) = \sum_{m=0}^{n} C_{n,m}H_m^{(\sigma)}(x|\alpha)$. By (11),

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similarly to the proof of (27), we have

$$
\begin{aligned}
C_{n,m} &= \frac{1}{m!}\left\langle \frac{\left(\frac{e^{\ln(1+t)}-\alpha}{1-\alpha}\right)^{\sigma}}{\prod_{i=1}^{r}\left(\frac{\ln(1+t)}{e^{a_i\ln(1+t)}-1}\right)\prod_{j=1}^{s}\left(1+e^{\lambda_j\ln(1+t)}\right)^{\mu_j}}\big(\ln(1+t)\big)^m \middle| x^n\right\rangle\\
&= \frac{1}{m!(1-\alpha)^{\sigma}}\\
&\quad\times\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\prod_{j=1}^{s}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\big(\ln(1+t)\big)^m(1-\alpha+t)^{\sigma}\middle| x^n\right\rangle\\
&= \frac{1}{m!(1-\alpha)^{\sigma}}\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\prod_{j=1}^{s}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\right.\\
&\qquad \left.\big(\ln(1+t)\big)^m\middle|\sum_{\nu=0}^{\min\{\sigma,n\}}\binom{\sigma}{\nu}(1-\alpha)^{\sigma-\nu}t^{\nu}x^n\right\rangle\\
&= \frac{1}{m!(1-\alpha)^{\sigma}}\sum_{\nu=0}^{n-m}\binom{\sigma}{\nu}(1-\alpha)^{\sigma-\nu}(n)_{\nu}\\
&\quad\times\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\prod_{j=1}^{s}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\big(\ln(1+t)\big)^m\middle| x^{n-\nu}\right\rangle\\
&= \frac{1}{m!(1-\alpha)^{\sigma}}\sum_{\nu=0}^{n-m}\binom{\sigma}{\nu}(1-\alpha)^{\sigma-\nu}(n)_{\nu}\sum_{l=0}^{n-m-\nu}m!\binom{n-\nu}{l}S_1(n-\nu-l,m)\mathrm{NS}_l\\
&= \sum_{\nu=0}^{n-m}\sum_{l=0}^{n-m-\nu}\binom{\sigma}{\nu}\binom{n-\nu}{l}(n)_{\nu}(1-\alpha)^{-\nu}S_1(n-\nu-l,m)\mathrm{NS}_l\,.
\end{aligned}
$$

Thus, we get the identity (30). ∎

3.10 A relation with higher-order Bernoulli polynomials

Bernoulli polynomials $\mathfrak{B}_n^{(r)}(x)$ of order r are defined by

$$\left(\frac{t}{e^t-1}\right)^r e^{xt}=\sum_{n=0}^{\infty}\frac{\mathfrak{B}_n^{(r)}(x)}{n!}t^n$$

(see e.g. [8, Section 2.2]). In addition, Cauchy numbers of the first kind $\mathfrak{C}_n^{(r)}$ of order r are defined by

$$\left(\frac{t}{\ln(1+t)}\right)^r=\sum_{n=0}^{\infty}\frac{\mathfrak{C}_n^{(r)}}{n!}t^n$$

(see e.g. [1, (2.1)], [3, (6)]).

Theorem 10

$$\begin{aligned}&\mathrm{NS}_n(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\\&=\sum_{m=0}^{n}\left(\sum_{i=0}^{n-m}\sum_{l=0}^{n-m-i}\binom{n}{i}\binom{n-i}{l}\mathfrak{C}_i^{(\sigma)}S_1(n-i-l,m)\mathrm{NS}_l\right)\mathfrak{B}_m^{(\sigma)}(x)\,.\end{aligned}\tag{32}$$

Proof. For (12) and

$$\mathfrak{B}_n^{(\sigma)}(x)\sim\left(\left(\frac{e^t-1}{t}\right)^{\sigma},t\right)\,,\tag{33}$$

assume that $\mathrm{NS}_n(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)=\sum_{m=0}^{n}C_{n,m}\mathfrak{B}_m^{(s)}(x)$. By (11), similarly to the proof of (27), we have

$$\begin{aligned}C_{n,m}&=\frac{1}{m!}\left\langle\frac{\left(\frac{e^{\ln(1+t)}-1}{\ln(1+t)}\right)^{\sigma}}{\prod_{i=1}^{r}\left(\frac{\ln(1+t)}{e^{a_i\ln(1+t)}-1}\right)\prod_{j=1}^{s}\left(1+e^{\lambda_j\ln(1+t)}\right)^{\mu_j}}\big(\ln(1+t)\big)^m\middle|x^n\right\rangle\\&=\frac{1}{m!}\left\langle\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\prod_{j=1}^{s}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\big(\ln(1+t)\big)^m\middle|\left(\frac{t}{\ln(1+t)}\right)^{\sigma}x^n\right\rangle\\&=\frac{1}{m!}\left\langle\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\prod_{j=1}^{s}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\big(\ln(1+t)\big)^m\middle|\sum_{i=0}^{\infty}\mathfrak{C}_i^{(\sigma)}\frac{t^i}{i!}x^n\right\rangle\\&=\frac{1}{m!}\sum_{i=0}^{n-m}\mathfrak{C}_i^{(\sigma)}\binom{n}{i}\left\langle\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\ln(1+t)}\right)\prod_{j=1}^{s}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\big(\ln(1+t)\big)^m\middle|x^{n-i}\right\rangle\\&=\frac{1}{m!}\sum_{i=0}^{n-m}\mathfrak{C}_i^{(\sigma)}\binom{n}{i}\sum_{l=0}^{n-m-i}m!\binom{n-i}{l}S_1(n-i-l,m)\mathrm{NS}_l\\&=\sum_{i=0}^{n-m}\sum_{l=0}^{n-m-i}\binom{n}{i}\binom{n-i}{l}\mathfrak{C}_i^{(\sigma)}S_1(n-i-l,m)\mathrm{NS}_l\,.\end{aligned}$$

Thus, we get the identity (32). ∎

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A New Algorithm Based on Lagrangian Relaxation for Solving Indefinite Quadratic Integer Programming*

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Abstract: In this paper we propose a new algorithm for finding a global solution of indefinite quadratic integer programming. We first derive Lagrangian dual bounds by D.C. decomposition and Lagrangian relaxation. And then a new branch-and-bound based on Lagrangian dual bounds and integral hyper-rectangular bisection is presented. Finally, preliminary numerical results are reported.

Key words. indefinite quadratic integer programming, branch-and-bound, Lagrangian relaxation, Lagrangian dual bound

AMS subject classification 90C, 49M.

1 Introduction

Consider the following indefinite quadratic integer programming problem:

$$(QIP) \qquad \begin{aligned} \min\ & f(x) = x^TQx + c^Tx \\ \text{s.t.}\ & Ax \le b, \\ & x \ge 0,\ x \in \mathbb{Z}^n, \end{aligned}$$

where $\mathbb{Z}^n$ is the set of integer vectors in $\mathbb{R}^n$, Q is an indefinite symmetric $n \times n$ matrix, $c \in \mathbb{R}^n$, A is an $m \times n$ matrix, $b \in \mathbb{R}^m$. Let S_{QIP} denote the feasible set of (QIP) and $S \triangleq \{x \in \mathbb{R}^n \mid Ax \le b,\ x \ge 0\}$. We assume that S is bounded and $S_{QIP} \subseteq X_I \triangleq \{x \in \mathbb{Z}^n \mid l \le x \le u\}$, where l and u are integer vectors.

Quadratic integer programming plays an important role both in optimization and in the real world. Several other important optimization models either can be formulated as quadratic integer programs or are special cases of (QIP). These include the quadratic 0-1 problem([1, 2, 3, 4]), the quadratic assignment problem [5], the quadratic knapsack problem ([6, 7, 8, 9]), and the discrete version of the bilinear programming problem ([10]). And many practical problems can be solved as quadratic integer programs. For example,

*This work was supported by National Natural Science Foundation (No.11271086), the National Science Foundation of Guangxi (No.2012GXNSFAA053007) and Science Foundation of Guangxi Education Department (No.201102ZD002).

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portfolio selection problems ([11, 12]), computer-aided layout design [13], site selection for electric message systems [14].

The indefiniteness of any global optimization leads to difficulty in solving it. Tawarmalani and Sahinidis [15] proposed a polyhedral branch-and-cut approach to nonconvex global optimization. Saxena et. al. [16] address the problem of generating strong convex relaxations of mixed integer quadratically constrained programming problems. Problem (QIP) is NP-hard. Some special cases of (QIP), for example, 0-1 quadratic problems and quadratic knapsack problems, have widely been investigated by many authors (see [17, 18, 3, 4, 8, 9]). But only a few algorithms, to our knowledge, have been developed in the literature for problem (QIP). Erenguc and Benson [19] proposed a branch-and-bound method which lower bounds were derived by orthogonal transformation and convex envelope. Thoai [20] proposed a branch-and-bound method with the help of concave programming for lower bounding and integral hyper-rectangular partition. Sun et al. [21] derived several kinds of lower bounds and their relationship.

Motivated from the ideas of [21] and partition methods of region, in this paper we further investigate problem (QIP) and propose a new branch-and-bound method for solving problem (QIP). We, first, convert equivalently problem (QIP) into a separable form by means of D.C. decomposition and Cholesky factorization of a positive definite matrix. We then construct a separable convex relaxation of problem (QIP) and Lagrangian bounds are derived by applying Lagrangian decomposition schemes to the separable convex relaxation. We incorporate our algorithm with integral hyper-rectangular bisection.

The rest of this paper is organizaed as follows. In the next section, we describe an equivalent reformulation of problem (QIP). In Section 3, we derive Lagrangian dual bounds. In Section 4, we present a new branch and bound method based on Lagrangian dual bounds and integral hyper-rectangular bisection. An illustrative example is given to demonstrate how the new algorithm works in Section 5. Finally, preliminary computational results are reported and some concluding remarks are given in Section 6.

2 Reformulation

The indefiniteness of $f(x)$ leads to the difficulty in solving (QIP). In this section, we convert $f(x)$ into a separable form by D. C. decomposition.

Let $\lambda_1, \dots, \lambda_n$ be the real eigenvalues of the symmetric matrix Q. Without loss of generality, we assume that $\lambda_1, \dots, \lambda_n$ are ranked in nondecreasing order. Obviously, we have $\lambda_1 < 0,\ \lambda_n > 0$ due to the indefiniteness of Q.

Let $\xi = (\xi_1, \dots, \xi_n)^T > 0$ be such that $diag(\xi) - Q$ is positive definite, where $diag(\xi)$ denotes the diagonal matrix with diagonal elements $\xi_1, \dots, \xi_n$. For example, we can choose

$\xi_j > \lambda_n,\ j = 1, \ldots, n$. Then $f(x)$ has the following D.C. decomposition:

$$x^T Qx + c^T x = x^T diag(\xi)x - x^T[diag(\xi) - Q]x + c^T x, \tag{1}$$

where $diag(\xi) - Q$ is positive definite.

We have the Cholesky factorization for the positive definite matrix $diag(\xi) - Q$:

$$diag(\xi) - Q = U^T U, \tag{2}$$

where U is an upper triangular matrix. Substituting (2) into (1), we obtain

$$x^T Qx + c^T x = -x^T U^T Ux + x^T diag(\xi)x + c^T x \tag{3}$$

Let $y = Ux$, then

$$x^T Qx + c^T x = -y^T y + x^T diag(\xi)x + c^T x, \tag{4}$$

So problem (QIP) is transformed equivalently into the following problem

$$\begin{aligned} (\widetilde{P}) \qquad\qquad & \min \quad -y^T y + x^T diag(\xi)x + c^T x \\ & \text{s.t.} \ \ y = Ux, \quad Ax \le b, \\ & \qquad l \le x \le u, \quad \widetilde{l} \le y \le \widetilde{u}, \\ & \qquad x_j \ integer, \ j = 1, 2, \cdots, n, \end{aligned}$$

where $\widetilde{l} = (\widetilde{l}_1, \ldots, \widetilde{l}_n)^T,\ \widetilde{u} = (\widetilde{u}_1, \ldots, \widetilde{u}_n)^T,\ \widetilde{l}_j$ and $\widetilde{u}_j$ $(j = 1, 2, \cdots, n)$ are the optimal value of the following linear program, respectively:

$$\begin{aligned} \widetilde{l}_j := \min \ & U_j x \\ \text{s.t.} \ & Ax \le b, \\ & l \le x \le u, \end{aligned} \tag{5}$$

and

$$\begin{aligned} \widetilde{u}_j := \max \ & U_j x \\ \text{s.t.} \ & Ax \le b, \\ & l \le x \le u, \end{aligned} \tag{6}$$

where U_j is the jth row of matrix U, $j = 1, \ldots, n$.

3 Lagrangian Relaxation

Lagrangian dual method has been a powerful methodology in integer programming. It can often provide tight bounds by exploiting the separable structures of the problem.

Problem $(\widetilde{P})$ is a mixed integer quadratic programming problem. Let $L_j(y_j) = \alpha_j y_j + \beta_j$ be the linear underestimation of the concave function $-y_j^2$ on $[\widetilde{l_j}, \widetilde{u_j}]$, $(j = 1, 2, \cdots, n)$ satisfying the following conditions:

$$L_j(\widetilde{l_j}) = \alpha_j \widetilde{l_j} + \beta_j = -\widetilde{l_j}^2, \quad L_j(\widetilde{u_j}) = \alpha_j \widetilde{u_j} + \beta_j = -\widetilde{u_j}^2, \quad j = 1, 2, \cdots, n.$$

So we obtain

$$\alpha_j = -(\widetilde{l_j} + \widetilde{u_j}), \quad \beta_j = \widetilde{l_j}\widetilde{u_j}, \quad j = 1, 2, \cdots, n.$$

Hence

$$L_j(y_j) = -(\widetilde{l_j} + \widetilde{u_j})y_j + \widetilde{l_j}\widetilde{u_j}, \quad j = 1, 2, \cdots, n. \tag{7}$$

Replacing $-y_j^2$ by $L_j(y_j)$ in problem $(\widetilde{P})$, we can get a convex integer relaxation of problem $(\widetilde{P})$, i.e.,

$$\begin{aligned}
(CIP_1) \qquad\qquad & \min \ \sum_{j=1}^{n} L_j(y_j) + x^T diag(\xi)x + c^T x \\
& \text{s.t.} \ \ y = Ux, \\
& \qquad Ax \leq b, \\
& \qquad l \leq x \leq u, \quad \widetilde{l} \leq y \leq \widetilde{u}, \\
& \qquad x_j \ integer, j = 1, 2, \cdots, n.
\end{aligned}$$

For reducing computation, we eliminate y by $y = Ux$ in (CIP_1) and obtain

$$\begin{aligned}
(CIP_2) \qquad\qquad & \min \ \ \rho + (U^T\phi + c)^T x + x^T diag(\xi)x \\
& \text{s.t.} \ \ Ax \leq b, \\
& \qquad l \leq x \leq u,, \\
& \qquad x_j \ integer, j = 1, 2, \cdots, n,
\end{aligned}$$

where $\rho = \sum_{j=1}^{n} \widetilde{l_j}\widetilde{u_j}$ and $\phi = (-\widetilde{l_1} - \widetilde{u}_1, \cdots, -\widetilde{l_n} - \widetilde{u}_n)^T$. Obviously, problem (CIP_2) is also a convex integer relaxation of problem (QIP).

Problem (CIP_2) is separable and we can consider the Lagrangian dual scheme. For $\nu \in \mathbb{R}^m_+$, the dual function of (CIP_2) is defined as follows:

$$d(\nu) \ = \ \min_{x \in X_I} \left\{ \rho + (U^T\phi + c)^T x + x^T diag(\xi)x + \nu^T(Ax - b) \right\}$$

$$\begin{aligned} &= \rho - \nu^T b + \sum_{j=1}^{n} \min\left\{\xi_j x_j^2 + \tau_j(\nu) x_j \mid l_j \le x_j \le u_j,\ x_j \text{ integer}\right\}, \\ &= \rho - \nu^T b + \sum_{j=1}^{n} \psi_j(\nu), \end{aligned}$$

where $\tau(\nu) = (U^T\phi + c + A^T\nu)^T$ and

$$\psi_j(\nu) = \begin{cases} \xi_j l_j^2 + \tau_j(\nu) l_j, & t_j < l_j \\ \min\{\xi_j \lfloor t_j \rfloor^2 + \tau_j(\nu)\lfloor t_j \rfloor, \xi_j(\lfloor t_j \rfloor + 1)^2 + \tau_j(\nu)(\lfloor t_j \rfloor + 1)\}, & l_j \le t_j \le u_j \\ \xi_j u_j^2 + \tau_j(\nu) u_j, & t_j > u_j \end{cases}$$

with $t_j = -\frac{\tau_j(\nu)}{2\xi_j}$ and $\lfloor t_j \rfloor$ being the maximum integer less than or equal to t_j.

From the weak duality, $d(\nu)$ is always a lower bound of (CIP_2), further, (QIP). The dual problem of (CIP_2) is then to search for the maximum lower bound yielded by $d(\nu)$:

$$(D) \qquad \max_{\nu \in \mathbb{R}^m_+} d(\nu).$$

From the above discussion, we obtain the following conclusion

Theorem 1 $v(D) \le v(CIP_2) \le v(QIP)$, *where* $v(\cdot)$ *denotes the optimal value of problem* $(\cdot)$.

4 Branch-and-Bound Algorithm

In this section we will give a new branch-and-bound algorithm. The algorithm consists of two key procedures, i.e., partition of region and lower bounding. Lower bounding is discussed in the previous section, so in this section we first restate briefly the method of region partition given by Thoai in [20], and then describe the new algorithm in detail.

4.1 Integral hyper-rectangular bisection

Let $P = \langle a, b \rangle \subset \mathbb{R}^n$ be a hyper-rectangle, which has integral bound vectors, that is

$$P = \{x \in \mathbb{R}^n \mid a \le x \le b\} = \{x \in \mathbb{R}^n \mid \ a_j \le x_j \le b_j,\ j = 1, 2, \cdots, n\},$$

where a_j and b_j are integers satisfying $a_j \le b_j$ $(j = 1, 2, \cdots, n)$. We assume that $\widehat{i}$ is the branch index at the current iteration. We define two subregions P^1 and P^2 of P as:

$$P^1 = \{x \in P \mid a_{\widehat{i}} \le x_{\widehat{i}} \le \eta\}, \quad P^2 = \{x \in P \mid \eta + 1 \le x_{\widehat{i}} \le b_{\widehat{i}}\},$$

where $\eta = \lfloor (a_{\widehat{i}} + b_{\widehat{i}})/2 \rfloor$. It is obviously

$$P^1 \cup P^2 \subset P, \quad P^1 \cap P^2 = \emptyset,$$

$$(P^1 \cap Z^n) \cup (P^2 \cap Z^n) = P \cap Z^n.$$

Remark 1 *For convenience, we call a, b a lower corner vertex and an upper corner vertex of box $\langle a, b\rangle$, respectively.*

4.2 The Algorithm

We are now in a position to present a new branch-and-bound algorithm for solving problem (QIP) based on Lagrangian dual bounds and integral hyper-rectangle bisection.

The proposed method is a branch-and-bound method. Each subproblem (subbox) corresponds to a *node* in the search tree. The lower bound on each node is computed by using the Lagrangian dual search scheme. The new subproblems are branched by integral hyper-rectangle bisection. The algorithm starts by computing a lower bound on the initial box $\langle l, u\rangle$. The initial upper bound is set to be infinity if any initial feasible point is not found. At the k-th iteration, the algorithm maintains a list of subboxes. The subbox with the minimum lower bound is chosen from the list and partitioned into two new subboxes. For each new subbox, a lower bound is calculated by the Lagrangian dual search scheme. If the lower bound is greater than or equal to the upper bound of the optimal value of problem (QIP), then the new subbox is fathomed. All the subboxes in the list with lower bound greater than or equal to the upper bound are fathomed. The process repeats until the list is empty.

Algorithm A

Step 0. (Initialization). Compute the eigenvalues $\lambda_1, \dots, \lambda_n$ of the matrix Q. Set $l^1 = l, u^1 = u, P_1 = \langle l^1, u^1\rangle$. Let S^0 be a set containing some feasible points of problem (QIP). Let the initial lower bound ζ_1 be the optimal value of problem (D) on P_1. If $S^0 \neq \emptyset$, then set the initial upper bound $\gamma_1 = \min\{f(x) \mid x \in S^0\}$ and choose $x^1 \in S^0$ such that $f(x^1) = \gamma_1$; otherwise, set $\gamma_1 = +\infty$. Set $\Omega = \{P_1\},\ k := 1$.

Step 1. (Partition). Partition P_k into two subboxes P_k^1, P_k^2 by integral hyper-rectangular bisection. Set $i = 1, \Omega := \Omega \backslash P_k$.

Step 2. (Bounding). Solve dual problem (D) on P_k^i and obtain the optimal value d^{ki}. Set $\zeta(P_k^i) = d^{ki}$. If $\zeta(P_k^i) < \gamma_k$, then $\Omega := \Omega \cup \{P_k^i\}, i := i + 1$. If $i = 2$, goto Step 3, otherwise, repeat Step 2.

Step 3. (Fathoming). Update S^0 by means of the feasible points yielded by the procedure. Set $\gamma_{k+1} = \min\ \{f(x) \mid x \in S^0\}$ and choose $x^{k+1} \in S^0$ such that $\gamma_{k+1} = f(x^{k+1})$. Set $\Omega := \Omega \backslash \{P \mid \zeta(P) \geq \gamma_{k+1}\}$. If $\Omega = \emptyset$, stop, x^{k+1} is the optimal solution of problem

(QIP). Otherwise, set $\zeta_{k+1} = \min\ \{\zeta(P) \mid P \in \Omega\}$. Choose P (denote by P_{k+1}) from Ω such that $\zeta(P_{k+1}) = \zeta_{k+1}$. Set $k := k+1$, go back to Step 1.

Remark 2 *We choose the index* $\widehat{i}$ *of the edge with the maximum length to branch in implementing integral hyper-rectangular bisection.*

The following conclusion holds true clearly due to the boundedness of the feasible region and partition method.

Theorem 2 *Algorithm A terminates after a finite number of iterations.*

5 An Illustrative Example

In this section we demonstrate Algorithm A by a small example.

Example 1

$$\begin{aligned} \min \quad & f(x) = x_2^2 - x_1 x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \le 6, \\ & 0 \le x_1 \le 5, \\ & 0 \le x_2 \le 3, \\ & x = (x_1, x_2)^T \in \mathbb{Z}^2. \end{aligned}$$

The optimal solution of this example is $x^* = (4,1)^T$ with $f(x^*) = -3$. The feasible region of the example is shown in Figure 1 and the iterations of the algorithm are described as follows.

Iteration 1

Step 0. The eigenvalue $\lambda_1 = -0.207, \lambda_2 = 1.207$. $l = (0,0)^T,\ u = (5,3)^T$. $S^0 = \{(3,1)^T\},\ P_1 = \langle l, u\rangle$. $\zeta_1 = -10.229,\ x^1 = (3,1)^T,\ \gamma_1 = f(x^1) = -2,\ \Omega = \{P_1\}$.

Step 1. The edge of the box P_1 paralleling x_1-axis is the maximum length edge(its length is 5). By the integral midpoint $\lfloor \eta \rfloor = \lfloor \frac{0+5}{2} \rfloor = 2$, we partition P_1 into two new subboxes:

$$P_1^1 = \langle (0,0)^T, (2,3)^T \rangle, \quad P_1^2 = \langle (3,0)^T, (5,3)^T \rangle,$$

and remove P_1 from Ω:

$$\Omega := \Omega \backslash \{P_1\} = \emptyset.$$

Step 2. Compute the lower bound $\zeta(P_1^1) = -2.307,\ \zeta(P_1^2) = -8.893$. The new subboxes P_1^1, P_1^2 are added into Ω due to $\zeta(P_1^1) < \gamma_1,\ \zeta(P_1^2) < \gamma_1$:

$$\Omega := \Omega \cup \{P_1^1,\ P_1^2\} = \{\ P_1^1,\ P_1^2\ \} = \{\langle (0,0)^T, (2,3)^T \rangle, \langle (3,0)^T, (5,3)^T \rangle\}.$$

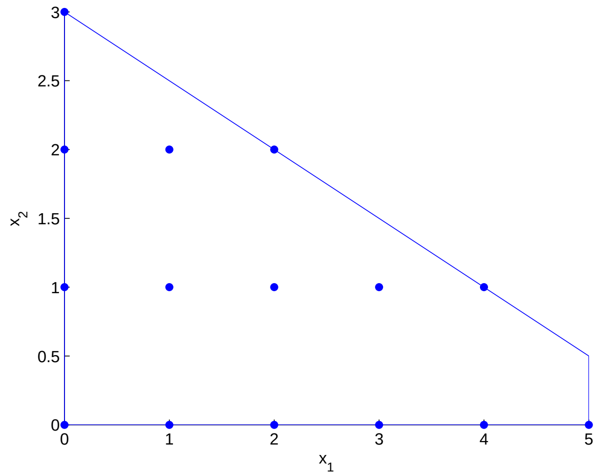

Figure 1: Feasible region of Example 1.

Step 3. The upper bound γ_1 is not updated in this process, i.e., $\gamma_2 = \gamma_1 = -2$. The new lower bound $\zeta_2 = \zeta(P_1^2) = -8.893$. Take $P_2 = P_1^2 = \langle (3,0)^T, (5,3)^T \rangle$ from Ω with the minimum lower bound.

The process of Iteration 1 is illustrated in Figure 2.

Iteration 2

Step 1. The edge of the box P_2 paralleling x_2-axis is the maximum length edge(its length is 3). By the integral midpoint $\lfloor \eta \rfloor = \lfloor \frac{0+3}{2} \rfloor = 1$, we partition P_2 into two new subboxes:

$$P_2^1 = \langle (3,0)^T, (5,1)^T \rangle, \quad P_2^2 = \langle (3,2)^T, (5,3)^T \rangle.$$

The subbox P_2^2 is deleted directly since it does not contain any feasible solutions of this example. And remove P_2 from Ω:

$$\Omega := \Omega \backslash \{P_2\} = \{P_1^1\} = \{\langle (0,0)^T, (2,3)^T \rangle\}.$$

Step 2. Compute the lower bound $\zeta(P_2^1) = -2.4$. Add P_2^1 into Ω due to $\zeta(P_2^1) < \gamma_2$:

$$\Omega := \Omega \cup \{P_2^1\} = \{P_1^1, P_2^1\} = \{\langle (0,0)^T, (2,3)^T \rangle, \langle (3,0)^T, (5,1)^T \rangle\}.$$

Step 3. In this process we obtained a feasible point $(4,1)^T$, so we have

$$S^0 = S^0 \cup \{(4,1)^T\} = \{(3,1)^T,\ (4,1)^T\}.$$

The new upper bound $\gamma_3 = \min\{f(x) \mid x \in S^0\} = -3$. Choose $x^3 = (4,1)^T \in S^0$ such that $f(x^3) = \gamma_3 = -3$. In view of $\zeta(P_1^1) > \gamma_3$, $\zeta(P_2^1) > \gamma_3$, we remove P_1^1, P_2^1 from Ω. Hence,we obtain

$$\Omega = \emptyset,$$

Therefore the algorithm stops and $x^3 = (4,1)^T$ is the optimal solution. The process of Iteration 2 is illustrated in Figure 3.

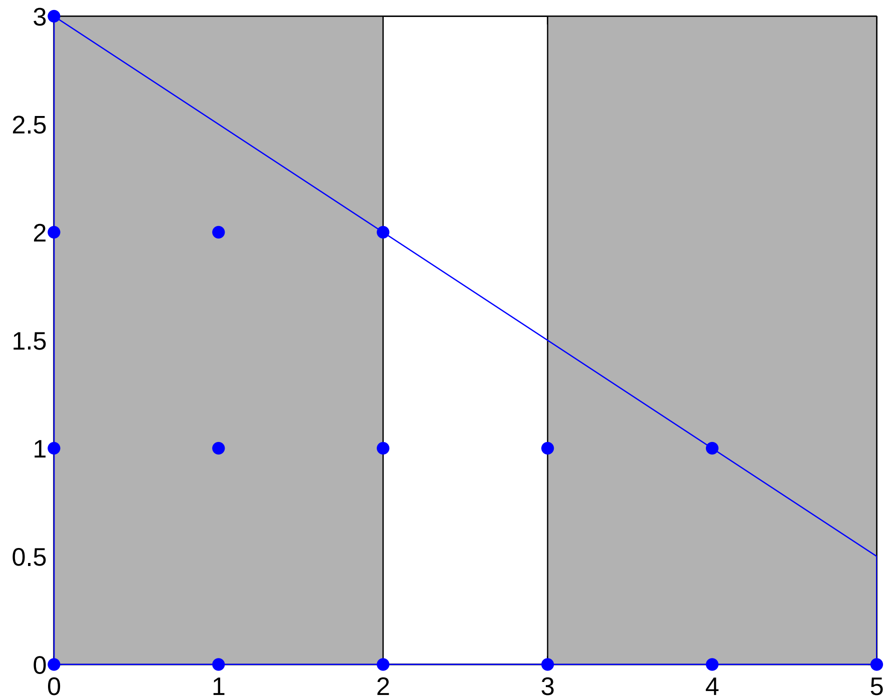

Figure 2: Illustration of Iteration 1 of Algorithm A for Example 1.

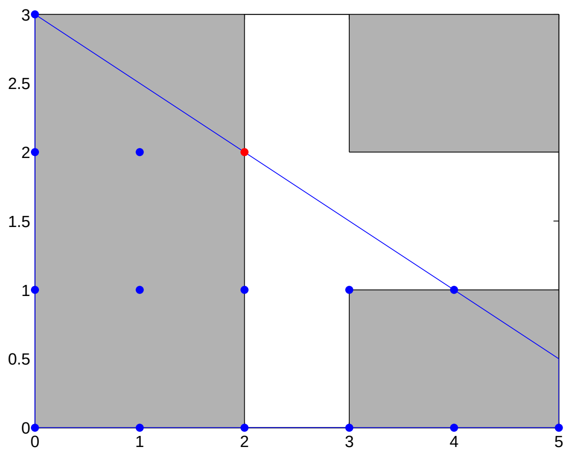

Figure 3: Illustration of Iteration 2 of Algorithm A for Example 1.

6 Numerical results and Concluding Remarks

Algorithm A was coded by Fortran 90 and run on a Pentium IV PC with 2.0 GHz and 1.0G RAM. The test problems are generated randomly and the indefinite matrix is generated by the way in [22]. We take $l = 1,\ u = 50$. In order to test the efficiency of Algorithm A, we compare Algorithm A with the algorithm in [20] (denoted by Algorithm B). The numerical results are summarized in Table 1 , where the average CPU time and the average number of iterations are obtained by testing 10 problems. The "fail" means that the corresponding algorithm doesn't find the solution of the test problem in reasonable time.

Table 1: Numerical results

n	m	$AverageCPU\ \ Time\ (Seconds)$		$AverageNumberofIterations$	
		Algorithm A	Algorithm B	Algorithm A	Algorithm B
10	4	0.094	0.087	77	52
20	10	11.856	13.221	283	326
30	10	60.894	33.733	1879	413
40	10	61.991	fail	1515	fail
50	10	615.463	fail	3254	fail
60	9	44.731	fail	275	fail
70	10	80.331	fail	874	fail
80	9	137.225	fail	1135	fail

The numerical results show that Algorithm A is capable of finding an optimal solution to medium-scale indefinite integer quadratic programming in reasonable time. Algorithm A is more efficient than Algorithm B: the scale of problem (QIP) solved by Algorithm A is larger than Algorithm B. Algorithm A can solve test problems with n up to 80, while the largest size of test problems solved by Algorithm B is 30. The outstanding performance of Algorithm A is largely due to the tightness and efficiency of the Lagrangian bounds.

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A NOTE ON AN INTERVAL-VALUED PSEUDO-CONVOLUTION OF INTERVAL-VALUED FUNCTIONS AND ITS APPLICATIONS

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Abstract. Markova-Stupnanova (1999) investigated the idempotent with respect to a pseudo-convolution. Jang (2013) defined the interval-valued generalized fuzzy integral and investigate their properties.

In this paper, we define the interval-valued pseudo-integral and an interval-valued pseudo-convolution of interval-valued functions. We also investigate an interval-valued idempotent function with respect to an interval-valued pseudo-convolution. Furthermore, we define the interval-valued pseudo-Laplace transform and discuss its characterizations.

1. Introduction

Many researchers [1,2,4,9-15,20,22,23] have studied various integrals of measurable multi-valued functions which are used for representing uncertain functions, for examples, the Aumann integral, the fuzzy integral, and the Choquet integral of measurable interval-valued functions in many different mathematical theories and their applications. Benvenuti-Mesiar [3], Deschrijver [4], Fang [6,7], Grbic-Stajner and Papuga-Strboja [8], Mesiar et al. [18], and Wu et al. [24, 25] have studied pseudo-multiplications and the pseudo-integrals.

Markova-Stupnanova [16] introduced the pseudo-convolution of functions based on pseudo-mathematical operations and investigated the idempotent with respect to a pseudo-convolution. Recently, Jang [14] defined the interval-valued generalized fuzzy integral and discuss some properties of them. The purpose of this study is to define the interval-valued pseudo-integral and to investigate interval-valued pseudo-convolution of the interval-valued functions and the interval-valued idempotent functions with respect to an the interval-valued pseudo-convolution.

The paper is organized in six sections. In section 2, we list definitions and some properties of the pseudo-integral, and a pseudo-convolution of functions. In section 3, we define an interval-valued pseudo-addition, an interval-valued pseudo-multiplication, the interval-valued pseudo-integral, and an interval-valued pseudo-convolution of interval-valued functions. In section 4, we define an interval-valued idempotent function with respect to interval-valued pseudo-integrals and investigate some properties of them. In section 5, we define the interval-valued pseudo-Laplace transform and investigate the interval-valued pseudo-exchange formula. In section 6, we give a brief summary results and some conclusions.

1991 *Mathematics Subject Classification.* 28E10, 28E20, 03E72, 26E50 11B68.

Key words and phrases. pseudo-integral, pseudo-addition, pseudo-multiplication, interval-valued function, pseudo-convolution, pseudo-Laplace transform.

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2. Definitions and Preliminaries

In this section, we introduce a pseudo-addition and a pseudo-multiplication as follows.

Definition 2.1. ([4,6,16-19]) (1) A binary operation $\oplus : [0,\infty]^2 \longrightarrow [0,\infty]$ is called a pseudo-addition if it satisfies the following axioms:
(i) $x \oplus y = y \oplus x$ for all $x, y \in [0,\infty]$,
(ii) $x \leq y \Longrightarrow x \oplus z \leq y \oplus z$ for all $x, y, z \in [0,\infty]$,
(iii) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ for all $x, y, z \in [0,\infty]$,
(iv) $\exists \mathbf{0} \in [0,\infty]$ such that $x \oplus \mathbf{0} = x$ for all $x \in [0,\infty]$,
(v) $x_n \longrightarrow x,\ y_n \longrightarrow y \implies x_n \oplus y_n \longrightarrow x \oplus y$.
(2) A binary operation $\odot : [0,\infty]^2 \longrightarrow [0,\infty]$ is called a pseudo-multiplication with respect to $\oplus$ if it satisfies the following axioms:
(i) $x \odot y = y \odot x$ for all $x, y \in [0,\infty]$,
(ii) $(x \odot y) \odot z = x \odot (y \odot z)$ for all $x, y, z \in [0,\infty]$,
(iii) $\exists \mathbf{1} \in [0,\infty]$ such that $x \odot \mathbf{1} = x$ for all $x \in [0,\infty]$,
(iv) $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$ for all $x, y, z \in [0,\infty]$,
(v) $x \odot \mathbf{0} = \mathbf{0}$ for all $x \in [0,\infty]$,
(vi) $x \leq y \Longrightarrow x \odot z \leq y \odot z$ if $z \geq \mathbf{0}$ and $x \odot z \geq y \odot z$ if $z \leq \mathbf{0}$ for all $x, y, z \in [0,\infty]$.

Let $\mathfrak{F}([0,\infty))$ be the set of all functions $f : [0,\infty) \longrightarrow [0,\infty)$ and $([0,\infty], \otimes)$ be a continuous semiring, and $\mathcal{A}$ be a σ-algebra of $[0,\infty)$.

Definition 2.2. ([4, 17, 19]) (1) Let $f \in \mathfrak{F}([0,\infty))$. The pseudo-integral of f with respect to a sup-measure m_ψ is defined by

$$\int_A^{\oplus} f \odot dm_\psi = \sup_{x \in A}\{f(x) \odot \psi(x)\}, \tag{1}$$

where $A \in \mathcal{A}$ and a function ψ defines sup-measure m_ψ, that is, $m_\psi(A) = \sup_{u \in A} \psi(u)$.
(2) f is said to be integrable if $\int_A^{\oplus} f \odot dm_\psi$ is finite.

Let $\mathfrak{F}^*([0,\infty))$ be the set of all integrable functions. We also consider the pseudo-convolution of functions in $\mathfrak{F}([0,\infty))$(see[4,17,19]).

Definition 2.3. Let $f, h \in \mathfrak{F}([0,\infty))$. The pseudo-convolution of f and h is defined by

$$(f * h)(t) = \int_{[0,t]}^{\oplus} f(t-u) \odot dm_h(u), \ \ \forall\, t \in [0,\infty) \tag{2}$$

where $m_h(A) = \sup_{x \in A} h(x)$ for all $A \in \mathcal{A}$.

By Definition 2.3, we directly obtain the following basic characterizations.

Theorem 2.1. (1) *If $x \oplus y = \max\{x, y\}$ and $x \odot y = x \wedge y$ for all $x, y \in [0,\infty)$, and $f, h \in \mathfrak{F}([0,\infty))$, then for all $t \in [0,\infty)$,*

$$f * h = \sup_{u \in [0,t]} f(t-u) \wedge h(u). \tag{3}$$

(2) *If* $x \oplus y = \max\{x, y\}$ *and* $x \odot y = x + y$ *for all* $x, y \in [0, \infty)$*, and* $f, h \in \mathfrak{F}([0, \infty))$*, then for all* $t \in [0, \infty)$*,*

$$f * h = \sup_{u \in [0,t]} (f(t-u) + h(u)). \tag{4}$$

Proof. (1) If $x \oplus y = \max\{x, y\}$ and $x \odot y = x \wedge y$ for all $x, y \in [0, \infty)$, and $f, h \in \mathfrak{F}([0, \infty))$, and $t \in [0, \infty)$, by Definition 2.2 and Definition 2.3, then we have

$$\begin{aligned} f * h &= \int_{[0,t]}^{\max} f(t-u) \wedge dm_h \\ &= \sup_{u \in [0,t]} \{f(t-u) \wedge h(u)\}. \end{aligned} \tag{5}$$

(2) The proof is similar to the proof of (1).

Now, we easily obtain commutativity and associativity of a pseudo-convolution.

Theorem 2.2. (1) *If* $x \oplus y = \max\{x, y\}$ *and* $x \odot y = x \wedge y$ *for all* $x, y \in [0, \infty)$*, then we have*

$$f * h = h * f \tag{6}$$

and

$$(f * h) * k = f * (h * k). \tag{7}$$

(2) *If* $x \oplus y = \max\{x, y\}$ *and* $x \odot y = x + y$ *for all* $x, y \in [0, \infty)$*, then we have*

$$f * h = h * f \tag{8}$$

and

$$(f * h) * k = f * (h * k). \tag{9}$$

Proof. (1) By Theorem 2.1 (1) and changing $w = t - u$,

$$\begin{aligned} (f * h)(t) &= \sup_{u \in [0,t]} f(t-u) \wedge h(u) \\ &= \sup_{w \in [0,t]} f(w) \wedge h(t-w) \\ &= (h * f)(t) \end{aligned} \tag{10}$$

By Theorem 2.1 (1) and changing $w = u - z$,

$$\begin{aligned} ((f * h) * k)(t) &= \sup_{u \in [0,t]} [(f * h)(t-u) \wedge k(u)] \\ &= \sup_{u \in [0,t]} \left[\sup_{z \in [0,t-u]} (f(t-u-z) \wedge h(z)) \wedge k(u) \right] \\ &= \sup_{u \in [0,t]} \sup_{w \in [u,t]} [(f(t-w) \wedge h(w-u)) \wedge k(u)] \\ &= \sup_{u \in [0,t]} \sup_{w \in [u,t]} [f(t-w) \wedge (h(w-u) \wedge k(u))] \\ &= \sup_{w \in [0,t]} f(t-w) \wedge \left[\sup_{u \in [0,w]} (h(w-u) \wedge k(u)) \right] \\ &= \sup_{w \in [0,t]} f(t-w) \wedge (h * k)(w) \\ &= (f * (h * k))(u). \end{aligned} \tag{11}$$

(2) By Theorem 2.1 (2) and changing $w = t - u$,

$$(f * h)(t) = \sup_{u \in [0,t]} f(t-u) + h(u)$$

$$\begin{aligned} &= \sup_{w\in[0,t]} f(w)+h(t-w) \\ &= (h*f)(t) \end{aligned} \tag{12}$$

By Theorem 2.1 (2) and changing $w=u-z$,

$$\begin{aligned} ((f*h)*k)(t) &= \sup_{u\in[0,t]}[(f*h)(t-u)+k(u)] \\ &= \sup_{u\in[0,t]}\left[\sup_{z\in[0,t-u]}(f(t-u-z)+h(z))+k(u)\right] \\ &= \sup_{u\in[0,t]}\sup_{w\in[u,t]}[(f(t-w)+h(w-u))+k(u)] \\ &= \sup_{u\in[0,t]}\sup_{w\in[u,t]}[f(t-w)+(h(w-u)+k(u))] \\ &= \sup_{w\in[0,t]} f(t-w)+\left[\sup_{u\in[0,w]}(h(w-u)+k(u))\right] \\ &= \sup_{w\in[0,t]} f(t-w)+(h*k)(w) \\ &= (f*(h*k))(u). \end{aligned} \tag{13}$$

3. The interval-representable pseudo-convolutions

Let $I(Y)$ be the set of all closed intervals(for short, intervals) in Y as follows:

$$I(Y)=\{\overline{a}=[a_l,a_r]|a_l,a_r\in Y \text{ and } a_l\leq a_r\}, \tag{14}$$

where Y is $[0,\infty)$ or $[0,\infty]$. For any $a\in Y$, we define $a=[a,a]$. Obviously, $a\in I(Y)$ (see [1,2,9-15, 22, 23]).

Definition 3.1. ([13-15]) If $\overline{a}=[a_l,a_r],\overline{b}=[b_l,b_r],\overline{a}_n=[a_{nl},a_{nr}],\overline{a}_\alpha=[a_{\alpha l},a_{\alpha r}]\in I(Y)$ for all $n\in\mathbb{N}$ and $\alpha\in[0,\infty)$, and $k\in[0,\infty)$, then we define arithmetic, maximum, minimum, order, inclusion, superior, inferior operations as follows:

(1) $\overline{a}+\overline{b}=[a_l+b_l,a_r+b_r]$,
(2) $k\overline{a}=[ka_l,ka_r]$,
(3) $\overline{a}\overline{b}=[a_lb_l,a_rb_r]$,
(4) $\overline{a}\vee\overline{b}=[a_l\vee b_l,a_r\vee b_r]$,
(5) $\overline{a}\wedge\overline{b}=[a_l\wedge b_l,a_r\wedge b_r]$,
(6) $\overline{a}\leq\overline{b}$ if and only if $a_l\leq b_l$ and $a_r\leq b_r$,
(7) $\overline{a}<\overline{b}$ if and only if $\overline{a}\leq\overline{b}$ and $\overline{a}\neq\overline{b}$,
(8) $\overline{a}\subset\overline{b}$ if and only if $b_l\leq a_l$ and $a_r\leq b_r$,
(9) $\sup_n\overline{a}_n=[\sup_n a_{nl},\sup_n a_{nr}]$,
(10) $\inf_n\overline{a}_n=[\inf_n a_{nl},\inf_n a_{nr}]$,
(11) $\sup_\alpha\overline{a}_\alpha=[\sup_\alpha a_{\alpha l},\sup_\alpha a_{\alpha r}]$, and
(12) $\inf_\alpha\overline{a}_\alpha=[\inf_\alpha a_{\alpha l},\inf_\alpha a_{\alpha r}]$.

We also consider an interval-valued pseudo-addition and an interval-valued pseudo-multiplication as follows.

Definition 3.2. ([15])(1) A binary operation $\bigoplus : I([0,\infty])^2 \longrightarrow I([0,\infty])$ is called an interval-valued pseudo-addition if there exist pseudo-additions $\oplus_l$ and $\oplus_r$ such that $x \oplus_l y \leq x \oplus_r y$ for all $x, y \in [0,\infty]$, and such that for all $\overline{a} = [a_l, a_r], \overline{b} = [b_l, b_r] \in I([0,\infty])$,

$$\overline{a} \bigoplus \overline{b} = [a_l \oplus_l b_l, a_r \oplus_r b_r]. \tag{15}$$

Then $\oplus_l$ and $\oplus_r$ are called the represents of $\bigoplus_I$.

(2) A binary operation $\bigodot : I([0,\infty])^2 \longrightarrow I([0,\infty])$ is called an interval-valued pseudo-multiplication if there exist pseudo-multiplications $\odot_l$ and $\odot_r$ such that $x \odot_l y \leq x \odot_r y$ for all $x, y \in [0,\infty]$, and such that for all $\overline{a} = [a_l, a_r], \overline{b} = [b_l, b_r] \in I([0,\infty])$,

$$\overline{a} \bigodot \overline{b} = [a_l \odot_l b_l, a_r \odot_r b_r]. \tag{16}$$

Then $\odot_l$ and $\odot_r$ are called the representants of $\bigodot$.

Let $\mathfrak{IF}([0,\infty))$ be the set of all interval-valued functions. By using an interval-valued addition and an interval-valued multiplication, we can define the interval-valued pseudo-integral.

Definition 3.3. (1) An interval-valued function $\overline{f} : [0,\infty) \to I([0,\infty) \setminus \{\emptyset\}$ is said to be measurable if for any open set $O \subset [0,\infty)$,

$$\overline{f}^{-1}(O) = \{x \in [0,\infty) \mid \overline{f}(x) \cap O \neq \emptyset\} \in \mathcal{A}. \tag{17}$$

(2)Let $x \oplus y = \max\{x, y\}$ for all $x, y \in [0,\infty)$ and $\bigodot$ be an interval-representable pseudo-multiplication and $\overline{f} \in \mathfrak{IF}(X)$. The interval-valued pseudo-integral with respect to an interval-valued sup-measure $m_{\overline{\psi}}$ is defined by

$$\int_A^{\oplus} \overline{f} \bigodot dm_{\overline{\psi}} = \sup_{x \in A}\{\overline{f}(x) \bigodot \overline{\psi}(x)\} \tag{18}$$

where $A \in \mathcal{A}$, $\overline{\psi} = [\psi_l, \psi_r]$, and ψ_s defines a sup-measure m_{ψ_s} for $s = l, r$.

(3) $\overline{f}$ is said to be integrable on $A \in \mathcal{A}$ if

$$\int_A^{\oplus} \overline{f} \bigodot dm_{\overline{\psi}} \in \mathcal{A} \setminus \{\emptyset\}. \tag{19}$$

Let $\mathfrak{IF}^*([0,\infty))$ be the set of all integrable interval-valued functions. By Definition 3.3, we directly obtain the following basic characterizations.

Theorem 3.1. *Let $\odot_l$ and $\odot_r$ be pseudo-multiplications on $[0,\infty]$ corresponding to a pseudo-addition $\oplus = \max$. If $\bigodot$ is a standard interval-valued pseudo-multiplication, $A \in \mathcal{A}$, and $f \in \mathfrak{IF}^*([0,\infty))$, then we have*

$$\int_A^{\oplus} \overline{f} \bigodot dm_{\overline{\psi}} = \left[\int_A^{\oplus} f_l \odot_l dm_{\psi_1}, \int_A^{\oplus} f_r \odot_r dm_{\psi_r}\right], \tag{20}$$

where $\overline{\psi} = [\psi_l, \psi_r]$ and ψ_s defines a sup-measure m_{ψ_s} for $s = l, r$.

Proof. By Definition 3.2 (2) , Theorem 2.1 (1), and Definition 3.1 (11),

$$\begin{aligned}\int_A^{\oplus} \overline{f}(x) \bigodot dm_{\overline{\psi}} &= \sup_{x \in A}\{\overline{f}(x) \bigodot \overline{\psi}(x)\} \\ &= \sup_{x \in A}[f_l(x), f_r(x)] \bigodot [\psi_l(x), \psi_r(x)]\end{aligned}$$

$$\begin{aligned} &= \sup_{x\in A}[f_l(x)\odot_l \psi_l(x), f_r(x)\odot_r \psi_r(x)] \\ &= [\sup_{x\in A} f_l(x)\odot_l \psi_l(x), \sup_{x\in A} f_r(x)\odot_r \psi_r(x)] \\ &= \left[\int_A^{\oplus} f_l \odot_l dm_{\psi_1}, \int_A^{\oplus} f_r \odot_r dm_{\psi_r}\right]. \end{aligned} \tag{21}$$

By using the interval-valued pseudo-integral with respect to an interval-valued sup-measure, we define the interval-valued pseudo-convolutions of interval-valued functions as follows.

Definition 3.4. Let $\oplus$ be a pseudo-addition, $\bigodot$ be an interval-valued pseudo-multiplication, and $\overline{f}=[f_l,f_r],\overline{h}=[h_l,h_r]\in \mathfrak{IF}^*([0,\infty))$. The interval-valued pseudo-convolution of $\overline{f}$ and $\overline{h}$ is defined by

$$(\overline{f}\star\overline{h}((t)=\int_{[0,t]}^{\oplus}\overline{f}(t-u)\bigodot dm_{\overline{h}}(u),\ \ \forall t\in[0,\infty) \tag{22}$$

where h_s defines a sup-measure m_{h_s} for $s=l,r$.

From Definition 3.4, we can obtain some characterizations of interval-valued pseudo-convolutions of interval-valued functions.

Theorem 3.2. *If $x\oplus y=\max\{x,y\}$ for all $x,y\in[0,\infty)$, $\overline{f}=[f_l,f_r],\overline{h}=[h_l,h_r]\in \mathfrak{IF}^*([0,\infty))$, and if $\odot_l$ and $\odot_r$ are the represents of an interval-valued pseudo-multiplication $\bigodot$, then we have*

$$\overline{f}\star\overline{h}=[f_l *_l h_l, f_r *_r h_r], \tag{23}$$

where $(f_s *_s h_s)(t)=\int_{[0,t]}^{\oplus} f_s(t-u)\odot_s dm_s$ for $s=l,r$.

Proof. By Theorem 3.2(1) and Definition 3.1 (11),

$$\begin{aligned} \overline{f}\star\overline{h} &= \int_{[0,t]}^{\oplus}\overline{f}(t-u)\bigodot dm_{\overline{h}}(u) \\ &= \sup_{u\in[0,t]}\overline{f}(t-u)\bigodot\overline{h}(u) \\ &= \sup_{u\in[0,t]}[f_l(t-u),f_r(t-u)]\bigodot[h_l(u),h_r(u)] \\ &= \sup_{u\in[0,t]}[f_l(t-u)\odot_l h_l(u), f_r(t-u)\odot_r h_r(u)] \\ &= [\sup_{u\in[0,t]} f_l(t-u)\odot_l h_l(u), \sup_{u\in[0,t]} f_r(t-u)\odot_r h_r(u)] \\ &= [f_l *_l h_l, f_r *_r h_r]. \end{aligned} \tag{24}$$

Now, we investigate commutativity and associativity of interval-valued pseudo-convolution.

Theorem 3.3. (1) *If $x\oplus y=\max\{x,y\}$ and $x\odot_l y=x\odot_r y=x\wedge y$ for all $x,t\in[0,\infty)$ are the represents of $\bigodot$, then we have*

$$\overline{f}\star\overline{h}=\overline{h}\star\overline{f} \tag{25}$$

and

$$(\overline{f} \star \overline{h}) \star \overline{k} = \overline{f} \star (\overline{h}) \star \overline{k}). \tag{26}$$

(2) *If* $x \oplus y = \max\{x, y\}$ *and* $x \odot_l y = x \odot_r y = x + y$ *for all* $x, t \in [0, \infty)$ *are the represents of* $\bigodot$, *then we have*

$$\overline{f} \star \overline{h} = \overline{h} \star \overline{f} \tag{27}$$

and

$$(\overline{f} \star \overline{h}) \star \overline{k} = \overline{f} \star (\overline{h}) \star \overline{k}). \tag{28}$$

Proof. (1) Let $\overline{f} = [f_l, f_r], \overline{h} = [h_l, h_r], \overline{k} = [k_l, k_r] \in \mathfrak{I}\mathfrak{F}^*([0, \infty))$. By Theorem 2.2(1) with $*_l = *_r = *$,

$$f_l * h_l = h_l * f_l \text{ and } f_r * h_r = h_r * f_r \tag{29}$$

and

$$(f_l * h_l) * k_l = f_l * (h_l * k_l) \text{ and } (f_r * h_r) * k_r = f_r * (h_r * k_r). \tag{30}$$

By Theorem 3.3(1) and (29),

$$\begin{aligned} \overline{f} \star \overline{h} &= [f_l * h_l, f_r * h_r] \\ &= [h_l * f_l, h_r * f_r] \\ &= \overline{h} \star \overline{f}. \end{aligned} \tag{31}$$

By Theorem 3.3(1) and (30),

$$\begin{aligned} (\overline{f} \star \overline{h}) \star \overline{k} &= [(f_l * h_l) * k_l, (f_r * h_r) * k_r] \\ &= [f_l * (h_l * k_l), f_r * (h_r * k_r)] \\ &= \overline{f} \star (\overline{h} \star \overline{k}). \end{aligned} \tag{32}$$

(2) By Theorem 2.2(2) and the same method of the proof of (1), we can obtain the result.

4. An interval-valued idempotent function with respect to an interval-valued pseudo-convolution

In this section, we consider an idempotent function with respect to a pseudo-convolution $*$ and define an interval-valued idempotent function with respect to an interval-representable pseudo-convolution $\star$ and investigate some properties of an interval-valued idempotent function.

Definition 4.1. (1) A function $f : [0, \infty) \longrightarrow [0, \infty)$ is called an idempotent function with respect to a pseudo-convolution $*$ if $f * f = f$.

(2) An interval-valued function $\overline{f} : [0, \infty) \longrightarrow I([0, \infty))$ is called an interval-valued idempotent function with respect to an interval-valued pseudo-convolution $\star$ if $\overline{f} \star \overline{f} = \overline{f}$.

Theorem 4.1. ([16]) *If* $x \oplus y = \max\{x, y\}$ *for all* $x, y \in [0, \infty)$, *and* $\odot$ *is a pseudo-multiplication, then there exists a neutral element* e *of the pseudo-convolution* $*$ *based on* $([0, \infty], \vee, \odot)$ *and* $e(0) = \mathbf{1}, e(x) = \mathbf{0}$ *for all* $x > 0$, *where* $x \odot \mathbf{1} = x$ *and* $x \odot \mathbf{0} = \mathbf{0}$ *for all* $x \in [0, \infty)$.

Theorem 4.2. *If $x \oplus y = \max\{x, y\}$ for all $x, y \in [0, \infty)$, and $\odot_l, \odot_r$ are the represents of $\bigodot$ then there exists an interval-valued neutral element $\overline{e}$ of the a extended interval-valued pseudo-convolution $\star$ based on $(I([0, \infty]), \vee, \bigodot)$ and $\overline{e}(0) = [\mathbf{1}_l, \mathbf{1}_r], \overline{e}(x) = [\mathbf{0}_l, \mathbf{0}_r]$ for all $x > 0$, where $x \odot_s \mathbf{1}_s = x$ and $x \odot_s \mathbf{0}_s = \mathbf{0}_s$ for all $x \in [0, \infty)$ and for $s = l, r$.*

Proof. By Theorem 4.1, there exists a neutral element e_s of the pseudo-convolution $*_s$ based on $([0, \infty], \vee, \odot_s)$ and $e(0) = \mathbf{1}_s, e(x) = \mathbf{0}_s$ for all $x > 0$, where $x \odot_s \mathbf{1}_s = x$ and $x \odot_s \mathbf{0}_s = \mathbf{0}_s$ for all $x \in [0, \infty)$ and for $s = l, r$. Thus, by Theorem 3.2, for any $t \in [0, \infty)$,

$$\begin{aligned}
(\overline{f} \star \overline{e})(t) &= \sup_{u\in[0,t]} [\overline{f}(t-u) \bigodot \overline{e}(u)] \\
&= \sup_{u\in[0,t]} [f_l(t-u) \odot_l e_l(u), f_r(t-u) \odot_r e_r(u)] \\
&= [\sup_{u\in[0,t]} f_l(t-u) \odot_l e_l(u), \sup_{u\in[0,t]} f_r(t-u) \odot_r e_r(u)] \\
&= [(f_l * e_l)(t), (f_r * e_r)(t)] \\
&= [f_l(t), f_r(t)] = \overline{f}(t).
\end{aligned} \tag{33}$$

Theorem 4.3. ([21]) *Let $x \oplus y = \max\{x, y\}$ and $x \odot y = \min\{x, y\}$ for all $x, y \in [0, \infty]$. If f is an idempotent function with respect to a pseudo-convolution $*$, then for each $c \in [0, \infty]$, $f^{(c)} + f^{(c)} \subset f^{(c)}$, where $f^{(c)} = \{x \in [0, \infty) | f(x) \geq c\}$ is so called c-cut of f. Conversely, if for each $c \in [0, \infty]$, $f^{(c)} + f^{(c)} = f^{(c)}$, then f is an idempotent function with respect to a pseudo-convolution $*$.*

Theorem 4.4. *Let $x \oplus y = \max\{x, y\}$ and $x \odot_l y = x \odot_r y = \min\{x, y\}$ for all $x, y \in [0, \infty]$ are the represents of $\bigodot$. If $\overline{f} = [f_l, f_r]$ is an interval-valued idempotent with respect to an interval-valued pseudo-convolution $\star$, then for each $c \in [0, \infty]$, $f_s^{(c)} + f_s^{(c)} \subset f_s^{(c)}$, where $f_s^{(c)} = \{x \in [0, \infty) | f_s(x) \geq c\}$ is so called c-cut of f_s for $s = l, r$.*

Conversely, if for each $c \in [0, \infty]$, $f_s^{(c)} + f_s^{(c)} = f_s^{(c)}$ for $s = l, r$, then $\overline{f}$ is an interval-valued idempotent function with respect to an interval-valued pseudo-convolution $\star$.

Proof. ($\Longrightarrow$) Since $(\overline{f} \star \overline{e})(t) = \overline{f}(t)$ for all $t \in [0, \infty)$, by Theorem 3.3(1),

$$(f_l *_l e_l)(t) = f_l(t) \text{ and } (f_r *_r e_r)(t) = f_r(t). \tag{34}$$

By Theorem 4.3, for each $c \in [0, \infty]$, we obtain $f_s^{(c)} + f_s^{(c)} \subset f_s^{(c)}$ for $s = l, r$.

($\Longleftarrow$) Suppose that for each $c \in [0, \infty]$, $f_s^{(c)} + f_s^{(c)} = f_s^{(c)}$ for $s = l, r$. By Theorem 4.3, f_s is an idempotent function with respect to a pseudo-convolution $*_s$ induced by $\odot_s$ for $s = l, r$. Therefore, by Theorem 3.3(1),

$$\begin{aligned}
(\overline{f} \star_1 \overline{f})(t) &= [(f_l *_l f_l)(t), (f_r *_r f_r)(t)] \\
&= [f_l(t), f_r(t)] = \overline{f}(t).
\end{aligned} \tag{35}$$

That is, $\overline{f}$ is an interval-valued idempotent function with respect to an interval-valued pseudo-convolution $\star$ induced by $\bigodot$.

5. The interval-valued pseudo-Laplace transform

In this section, we introduce the pseudo-Laplace transform and the pseudo-exchange formula as follows.

Definition 5.1. ([19]) Let $\odot$ be a pseudo-multiplication on $[0,\infty]$ corresponding to a pseudo-addition $\oplus = \max$.

(1) The pseudo-character of the group $([0,\infty),+)$ is a continuous map $\xi : [0,\infty) \longrightarrow [0,\infty)$ of the group $[0,\infty),+)$ into the semigroup $([0,\infty),\oplus,\odot)$ with property

$$\xi(x+y) = \xi(x) \odot \xi(y), \quad \text{for } x,y \in [0,\infty). \tag{36}$$

(2) The pseudo-Laplace transform $\mathfrak{L}^{\oplus}(f)$ of a function $f \in \mathfrak{F}^*([0,\infty)$ is defined by

$$(\mathfrak{L}^{\oplus} f)(\xi)(z) = \int_{[0,\infty)}^{\oplus} \xi(x,z) \odot dm_f(x), \tag{37}$$

where ξ is the pseudo-character.

Theorem 5.1. ([19]) *Let $\odot = +$ be a pseudo-multiplication on $[0,\infty]$ corresponding to a pseudo-addition $\oplus = \max$. If functions $f_1, f_2 \in \mathfrak{F}^*([0,\infty))$, then we have*

$$\mathfrak{L}^{\oplus}(f_1 * f_2) = \mathfrak{L}^{\oplus}(f_1) \odot \mathfrak{L}^{\oplus}(f_2). \tag{38}$$

Now, we define an interval-valued pseudo-character and the interval-valued pseudo-Laplace transform of interval-valued functions in $\mathfrak{IF}^*([0,\infty))$ as follows.

Definition 5.2. (1) Let $\odot_l$ and $\odot_r$ be the represents of $\bigodot$. The interval-valued pseudo-character $\overline{\xi}$ of the group $([0,\infty),+)$ is a continuous map $\overline{\xi} : [0,\infty) \longrightarrow [0,\infty)$ of the group $([0,\infty),+)$ into the semigroup $([0,\infty),\bigoplus,\bigodot)$ with property

$$\overline{\xi}(x+y) = \overline{\xi}(x) \bigodot \overline{\xi}(y), \quad \text{for } x,y \in [0,\infty). \tag{39}$$

(2) Let $\odot_l$ and $\odot_r$ be pseudo-multiplications on $[0,\infty]$ corresponding to a pseudo-addition $\oplus = \max$. If $\bigodot$ is a standard interval-valued pseudo-multiplication and $f \in \mathfrak{IF}^*([0,\infty))$, then the interval-valued pseudo-Laplace transform $\mathfrak{L}^{\oplus}(\overline{f})$ of a function $\overline{f} \in \mathfrak{IF}^*([0,\infty))$ is defined by

$$(\mathfrak{L}^{\oplus}\overline{f})(\overline{\xi})(z) = \int_{[0,\infty)}^{\oplus} \overline{\xi}(x,z) \bigodot dm_{\overline{f}(x)}, \tag{40}$$

where $\overline{\xi} = [\psi_l, \psi_r]$ is the interval-valued pseudo-character.

We discuss the pseudo-exchange formula for the interval-valued pseudo-Laplace transform. as follows:

Theorem 5.2. *Let $\odot_l$ and $\odot_r$ be pseudo-multiplications on $[0,\infty]$ corresponding to a pseudo-addition $\oplus = \max$. If $\bigodot$ is an interval-valued pseudo-multiplication and $f \in \mathfrak{IF}^*([0,\infty))$, then we have*

$$(\mathfrak{L}^{\oplus}\overline{f})(\overline{\xi})(z) = \left[\mathfrak{L}_l^{\oplus}(f_l), \mathfrak{L}_r^{\oplus}(f_r)\right], \tag{41}$$

where $\mathfrak{L}_s^{\oplus}(f_s) = \int^{\oplus} \xi_s(x,z) \odot_s dm_{f_s}(x)$ for $s = l, r$.

Proof. By Theorem 3.2(1), we have

$$(\mathfrak{L}^{\oplus}\overline{f})(\overline{\xi})(z) \quad = \quad \int^{\oplus} \overline{\xi}(x,z) \bigodot dm_{\overline{f}}(x)$$

$$\begin{aligned} &= \left[\int^{\oplus} \xi_l(x,z) \odot_l dm_{f_l}(x), \int^{\oplus} \xi_r(x,z) \odot_r dm_{f_r}(x)\right] \\ &= \left[\mathfrak{L}_l^{\oplus}(f_l), \mathfrak{L}_r^{\oplus}(f_r)\right], \end{aligned} \tag{42}$$

Theorem 5.3. *Let $\odot_l$ and $\odot_r$ be pseudo-multiplications on $[0,\infty]$ corresponding to a pseudo-addition $\oplus = \max$. If $\bigodot$ is an interval-valued pseudo-multiplication a nd $f_1 = [f_{1l}, f_{1r}], f_2 = [f_{2l}, f_{2r}] \in \mathfrak{IF}^*([0,\infty))$, then we have*

$$\mathfrak{L}^{\oplus}(f_1 \star f_2) = \mathfrak{L}^{\oplus}(\overline{f}_1) \bigodot \mathfrak{L}^{\oplus}(\overline{f}_2), \tag{43}$$

where $\star$ is an interval-valued pseudo-convolution.

Proof. By Theorem 3.3(1), we have

$$f_1 \star f_2 = [f_{1l} *_l f_{2l}, f_{1r} *_r f_{2r}]. \tag{44}$$

By (44), Theorem 5.2, and Theorem 5.1 , we have

$$\begin{aligned} \mathfrak{L}^{\oplus}(f_1 \star f_2) &= \left[\mathfrak{L}_l^{\oplus}(f_{1l} *_l f_{2l}), \mathfrak{L}_r^{\oplus}(f_{1r} *_r f_{2r})\right] \\ &= \left[\mathfrak{L}_l^{\oplus}(f_{1l}) \odot_l \mathfrak{L}_l^{\oplus}(f_{2l}), \mathfrak{L}_r^{\oplus}(f_{1r}) \odot_r \mathfrak{L}_r^{\oplus}(f_{2r})\right] \\ &= \left[\mathfrak{L}_l^{\oplus}(f_{1l}), \mathfrak{L}_r^{\oplus}(f_{1r})\right] \bigodot \left[\mathfrak{L}_l^{\oplus}(f_{2l}), \mathfrak{L}_r^{\oplus}(f_{2r})\right] \\ &= \mathfrak{L}^{\oplus}(\overline{f}_1) \bigodot \mathfrak{L}^{\oplus}(\overline{f}_2), \end{aligned} \tag{45}$$

6. Conclusions

In this paper, we introduced four general notions of an interval-valued pseudo-addition and an interval-valued pseudo-multiplication (see Definition 3.2), an interval-valued pseudo-integral(see Definition 3.3(2)), an interval-valued pseudo-convolution(see Definition 3.4), an interval-valued idempotent function(see Definition 4.1(2)), and the interval-valued pseudo-Laplace transform(see Definition 5.1(2)). From Definitions 2.1, 2.2, 2.3, and Theorems 2.1, 2.2, we defined an interval-valued pseudo-convolution of interval-valued functions by means of the interval-valued pseudo-integral and obtained some characterizations of them including commutativity and associativity of an interval-valued convolution (see Theorems 3.1, 3.2).

In particular, we defined an interval-valued idempotent function with respect to an interval-valued pseudo-convolution $\star$ induced by $\bigodot$ (see Definition 4.1 (2)) and investigated an equivalence relation between the existence of an interval-valued idempotent function and an interval-valued pseudo-convolution(see Theorem 4.4). We also obtain the pseudo-exchange formula for the standard interval-valued pseudo-Laplace transform(see Theorem 5.3).

In the future, we can expect that the interval-valued pseudo-convolutions are used (i) to generalize pseudo-Laplace transform, Hamilton-Jacobi equation on the space of functions, such as in nonlinearity and optimization(see [20]) (ii) to generalize the Skolasky type inequality for the pseudo-integral of functions such as in economics, finance, decision making(see [4]), etc.

Acknowledgement: This paper was supported by Konkuk University in 2014.

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